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On the Riemannian Positive Mass Theorem

Abstract

In this article, we discuss the Riemannian positive mass theorem (without initial data sets or spin assumptions) while providing a physical intuition to the mathematical formulation of this theorem. Lastly, we present three special cases of this theorem: the spherically symmetric case, the conformally flat case, and the 2-dimensional "toy model."

1 Introduction

Physically and intuitively, we know that the mass density function that describes the distribution of matter in the universe should be a nonnegative function. In Newton's theory of gravity, the divergence theorem tells us that if the mass density function is nonnegative everywhere and positive somewhere, then the total mass of the system is positive. That is, the gravitational potential asymptotically looks like a potential created by positive point masses. Physically, this means that test particles far away from the points where the mass density is positive are "attracted" to the source masses. In this sense, the positive mass theorem is concerned with the question: why is mass always attractive? In more precise terms, does nonnegative scalar curvature (i.e., nonnegative mass density) always imply nonnegative mass? A counterexample to this would indicate that matter could somehow "gravitationally repel" other matter.

While the positive mass theorem seems somewhat intuitive, it is not obvious at all why it should be true mathematically. Even to state the theorem, we must take great care and ask ourselves: what *actually* is mass? Thus, the goal of this article is to investigate the simplest setting of the positive mass theorem (as studied by Schoen and Yau in 1979 in [3]), and to verify its validity in some special configurations.

2 Preliminary definitions

In this section, we define a few objects that will be used throughout this article (a broader discussion related to these concepts can be found in do Carmo [1]). A *Riemannian manifold* (M^n, g) is a smooth manifold M^n equipped with a Riemannian metric $\mathrm{g} \in C^\infty(T^*M \odot T^*M)$; i.e., g is a positive-definite symmetric $(0,2)$-tensor. Since this metric defines an inner product on each tangent space T_pM, we oftentimes write $\mathrm{g}(v,w) = \langle v, w\rangle$, for $v, w \in T_pM$. If $(e_1, \dots, e_n)$ is an orthonormal basis of T_pM and $(e^1, \dots, e^n)$ is its corresponding dual basis (i.e., a basis of T_p^*M), we can write

$$\mathrm{g} = g_{ij} e^i \otimes e^j,$$

where $g_{ij} \coloneqq \langle e_i, e_j\rangle$ is a symmetric matrix of functions. It is worth noting that, from a Riemannian metric g, we can naturally define a volume form

$$\mathrm{vol} \coloneqq \sqrt{\det(g_{ij})}\, e^1 \wedge \cdots \wedge e^n. \tag{1}$$

As described in the first section of [2], we can use this volume form to construct a volume measure on M (sometimes called *Riemannian measure*). More precisely, in each coordinate patch, the measurable sets are those that correspond to Lebesgue measurable sets in $\mathbb{R}^n$ via the coordinate chart. Then, we take measurable sets in M to be countable union of those sets. Finally, the volume of a measurable set U in M is defined to be

$$\mu(U) \coloneqq \int_U \mathrm{vol}\,. \tag{2}$$

If M is non-orientable, we need to be a bit more careful when defining a volume measure since there are no globally defined volume forms. In this case, we simply push forward the volume measure on the (orientable) double cover of M and divide by 2. When integrating over a manifold M, we denote the volume measure by $\mathrm{d}\mu$ or $\mathrm{d}\mu_M$ or $\mathrm{d}\mu_{\mathrm{g}}$ if specificity is needed.

Now, given the Levi-Civita connection ∇ on a Riemannian manifold (M, g), we define the *divergence* $\operatorname{div} X\colon M \to \mathbb{R}$ of $X \in \mathfrak{X}(M)$ as

$$\operatorname{div} X(p) \coloneqq \operatorname{tr}\left(Y(p) \mapsto \nabla_Y X(p)\right), \quad p \in M.$$

Moreover, we define the *gradient* of a smooth function f as the vector field $\operatorname{grad} f$ given by

$$\langle \operatorname{grad} f(p), v\rangle = \mathrm{d}f_p(v), \quad p \in M,\ v \in T_pM.$$

Using these notions, we define the *Laplacian* $\Delta\colon C^\infty(M) \to C^\infty(M)$ as

$$\Delta f \coloneqq \operatorname{div} \operatorname{grad} f.$$

Finally, a crucial notion from Riemannian geometry used to study general relativity is *curvature*. The *Riemann curvature tensor* on (M, g) is defined here[1] as the $(1, 3)$-tensor given by

$$R(X, Y)Z \coloneqq -\nabla_X \nabla_Y Z + \nabla_Y \nabla_X Z + \nabla_{[X,Y]} Z,$$

for all $X, Y, Z \in \mathfrak{X}(M)$. We can use the metric g to view this curvature tensor as a $(0, 4)$-tensor given by

$$R(X, Y, Z, W) \coloneqq \mathrm{g}(R(X, Y)Z, W),$$

for all $X, Y, Z, W \in \mathfrak{X}(M)$. It is important to note that, for each $X, Y, Z, W \in \mathfrak{X}(M)$, R is symmetric in the first and last pairs of entries, skew-symmetric in the first two and last two entries, and R satisfies the First Bianchi Identity

$$R(X, Y)Z + R(Z, X)Y + R(Y, Z)X = 0.$$

The *Ricci tensor* is a $(0, 2)$-tensor defined as the trace of the Riemann curvature tensor over the second and fourth components. More precisely, if $(e_1, \ldots, e_n)$ is an orthonormal basis for T_pM, then

$$\operatorname{Ric}(X, Y) \coloneqq \operatorname{tr}(R(X, \cdot)Y) = \sum_{i=1}^{n} R(X, e_i, Y, e_i).$$

The *scalar curvature* is defined as the function $\operatorname{scal}\colon M \to \mathbb{R}$ given by the trace of Ric; namely,

$$\operatorname{scal} \coloneqq \operatorname{tr}(\operatorname{Ric}) = \sum_{j=1}^{n} \operatorname{Ric}(e_j, e_j) = 2 \sum_{1 \le i < j \le n} R(e_i, e_j, e_i, e_j).$$

These "simpler" tensors scal and Ric are very useful to study geometry since they carry information about the volume defect and volume distortion, respectively, of small balls in (M, g) compared to a space form. As discussed in the next sections, it turns out that scalar curvature is closely related to the mass density in the universe. Lastly, we define the divergence-free (as shown in [2]) *Einstein tensor* as

$$\mathrm{G} \coloneqq \operatorname{Ric} - \frac{1}{2} \operatorname{scal} \mathrm{g}.$$

3 The Riemannian positive mass theorem

3.1 Schwarzschild metrics

In preparation for stating the Riemannian positive mass theorem, we first investigate spherically symmetric metrics in search of a "fundamental metric" that describes spaces of constant scalar curvature. In this sense, one can think of a spherically symmetric metric g as the warped product of a line with the $(n-1)$-sphere S^{n-1} [2]. In particular, we can write such g as

$$\mathrm{g} = \mathrm{d}s^2 + r(s)^2 \mathrm{d}\Omega^2,$$

where $\mathrm{d}\Omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\, \mathrm{d}\phi^2$ is the standard round metric on S^{n-1}, and r is a positive function [2]. As noted in [2], the regions in which $r'(s) = 0$ correspond to the regions where the symmetric spheres are totally geodesic, and the regions where $r(s)$ is constant correspond to g being cylindrical (i.e., the

[1] The sign convention in do Carmo [1] is the opposite as the one used here.

Riemann product of a sphere and an interval). Now, in the regions where $r'(s) \neq 0$, we may take r as a new coordinate and express the metric g as

$$\mathrm{g} = \frac{\mathrm{d}r^2}{U(r)} + r^2 \mathrm{d}\Omega^2, \tag{3}$$

where $U(r)$ is again some positive function [2]. As described in [2], the scalar curvature of g above is

$$\mathrm{scal}_{\mathrm{g}} = \frac{n-1}{r^2}\left[(n-2)(1-U(r)) - rU'(r)\right]. \tag{4}$$

Note that, if we require the scalar curvature above to be constant, say $\mathrm{scal}_{\mathrm{g}} \equiv \kappa$, we obtain an inhomogeneous linear ODE in U. As shown in [2], all spherically symmetric metrics with constant scalar curvature $\mathrm{scal}_{\mathrm{g}} \equiv \kappa$ can in fact be written as in Equation (3) (up to diffeomorphism) with

$$U(r) = 1 - \frac{2m}{r^{n-2}} - \frac{\kappa}{n(n-1)} r^2, \tag{5}$$

where $m \in \mathbb{R}$ is a constant and the factor of 2 multiplying it is a convenient convention. When $\kappa > 0$, these metrics are called *Schwarzschild-de Sitter*; when $\kappa = 0$, these metrics are called *Schwarzschild*; and when $\kappa < 0$, these metrics are called *Schwarzschild-anti-de Sitter* [2].

Definition 1. Define the Schwarzschild metric of mass m as

$$\mathrm{g}_m := \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} \mathrm{d}r^2 + r^2 \mathrm{d}\Omega^2. \tag{6}$$

Note that if $m = 0$, then g_m is simply the Euclidean metric (in "hyperspherical" coordinates). Because of this, we call the Euclidean space $\mathbb{R}^n$ the *Schwarzschild space of mass zero* [2].

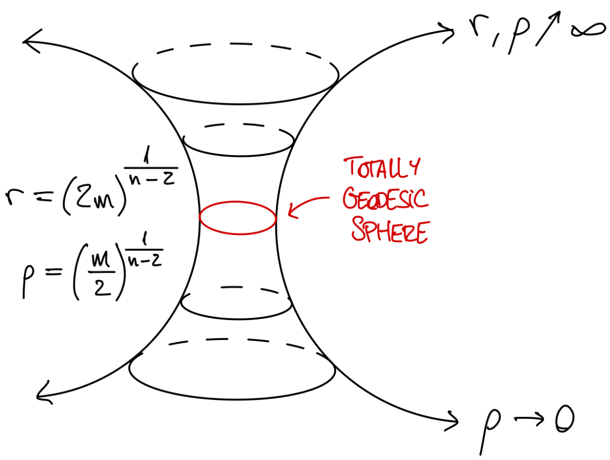

Figure 1: Schwarzschild space for $m > 0$.

For $m > 0$, the metric g_m is clearly complete in the limit $r \nearrow \infty$. On the other hand, there is a singularity at $r = (2m)^{\frac{1}{n-2}}$. Luckily, this is *not* a "geometric singularity" [2]. Rather, it is a singularity in the "coordinate choice," much like $r = 0$ in spherical coordinates in $\mathbb{R}^3$. More precisely, as described in [2], to see that $r = (2m)^{\frac{1}{n-2}}$ is not a geometric singularity, we can change the radial coordinates and write

$$\mathrm{g}_m = [u(\rho)]^{\frac{4}{n-2}} \left(\mathrm{d}\rho^2 + \rho^2 \mathrm{d}\Omega^2\right), \tag{7}$$

where

$$u(\rho) = 1 + \frac{m}{2\rho^{n-2}}.$$

Remark 1. By writing the metric g_m as in Equation (7), we can clearly see the conformal factor that relates g_m to the Euclidean metric. Switching to the standard coordinates $(x^1, \ldots, x^n)$ on $\mathbb{R}^n$ with $\rho = |x|$, we can write g_m on $\mathbb{R}^n \setminus \{0\}$ with

$$(g_m)_{ij} = [u(x)]^{\frac{4}{n-2}} \delta_{ij},$$

where

$$u(x) = 1 + \frac{m}{2|x|^{n-2}}.$$

In Physics, these coordinates are sometimes called *isotropic coordinates* for the Schwarzschild metric.

The new expression for g_m (Equation (7)) defines a metric on all of $(0,\infty) \times S^{n-1}$ since g_m is well-defined for all $\rho > 0$. Moreover, as argued in [2], one can show that $((0,\infty) \times S^{n-1}, g_m)$ is complete.

Finally, we note that for $m < 0$, the geometry of the metric g_m becomes singular as r goes to zero. This means that we cannot extend g_m to a smooth Riemannian metric on a bigger space [2].

3.2 Asymptotically flat manifolds

Note that for any Schwarzschild metric, as ρ goes to zero or infinity, the metric asymptotically looks like the Euclidean metric. In particular, for large enough ρ, the metric g_m differs from the Euclidean metric by a factor of order $O(\rho^{2-n})$. In the discussion that follows, these Schwarzschild spaces are going to serve as model spaces for the study of asymptotically flat manifolds with nonnegative scalar curvature. But first, following [2], we define the class of manifolds that asymptotically look flat.

Definition 2 (Asymptotically flat)**.** A Riemannian manifold (M^n, g), $n \geq 3$, is *asymptotically flat* if there exists a bounded set K such that $M \setminus K$ is a finite union of ends $M_1, \ldots, M_l$ so that, for each M_j, there is a diffeomorphism

$$\phi_j \colon M_j \longrightarrow \mathbb{R}^n \setminus \overline{B_1(0)},$$

where $\overline{B_1(0)}$ is the closure of the unit ball, with the property that, if we view each ϕ_j as a coordinate chart with coordinates $(x^1, \ldots, x^n)$, then, in that chart, we have

$$g_{ij}(x) = \delta_{ij} + O_2(|x|^{-q}),$$

for some $q > \frac{n-2}{2}$. We note that $O_2(|x|^{-q})$ is some (unspecified) function in the weighted space C^2_{-q} [2]. The coordinate charts ϕ_j are called *asymptotically flat coordinate charts* and q is referred to as the *asymptotic decay rate of* g.

Example 1. The Schwarzschild space of mass $m > 0$ (see Equation (7)) is asymptotically flat [2].

Remark 2. In addition, for reasons that will soon become clear, we ought to require the scalar curvature to be integrable over (M^n, g). For instance, as explained in [2], if have an asymptotic decay of $q = n-2$, we have that $\mathrm{scal}_g = O(|x|^{-n})$, which does not guarantee that $\mathrm{scal}_g \in L^1$.

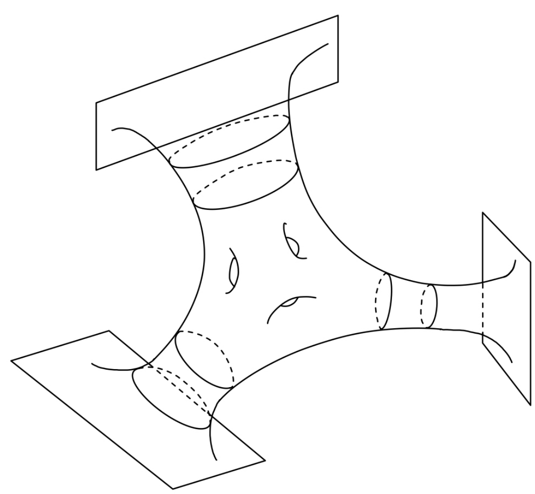

Figure 2: Asymptotically flat manifold with 3 ends.

Finally, we define a class of manifolds that lie within the class of asymptotically flat manifolds [2].

[2] A function $f \in C^2_{-q}$ if $|f(x)| + |x||\partial f(x)| + |x|^2|\partial^2 f(x)| < C|x|^{-q}$ for some $C \in \mathbb{R}$. In this case, ∂ represents the derivatives with respect to the background Euclidean metric.

Definition 3 (Asymptotically Schwarzschild)**.** A Riemannian manifold (M^n, g), $n \geq 3$, is *asymptotically Schwarzschild* if there exists a bounded set K such that $M \setminus K$ is a finite union of ends $M_1, \ldots, M_l$ so that, for each M_j, there exists $m_j \in \mathbb{R}$ and a diffeomorphism

$$\phi_j \colon M_j \longrightarrow \mathbb{R}^n \setminus \overline{B_1(0)},$$

such that, viewing each ϕ_j as a coordinate chart with coordinates $(x^1, \ldots, x^n)$, we have

$$g_{ij}(x) = \left(1 + \frac{2m_j}{n-2}|x|^{2-n}\right)\delta_{ij} + O_2(|x|^{1-n}).$$

The real number m_j is called the *mass* of the end M_j.

It is straightforward to see that the Schwarzschild space of mass m has 2 ends and is asymptotically Schwarzschild according to the definition above.

3.3 ADM mass

Note that, in the definition of asymptotically Schwarzschild manifolds (Definition 3), the mass parameter captures the deviation of g from being Euclidean. We would like to somehow extend the notion of "mass" so that we can study the larger class of asymptotically flat manifolds in the same way; namely, we want to explicitly find by how much asymptotically flat metrics differ from the Euclidean metric via a mass parameter. This generalization of the concept of mass is called the *ADM mass* (named after Arnowitt, Deser, and Misner) [2].

In order to understand what "mass" truly is, we need to go back to Newtonian gravitation. In this theory of gravitation, the effects of gravity are commanded by the gravitational potential $U \colon \mathbb{R}^3 \to \mathbb{R}$. In particular, gravity affects test particles by making their acceleration equal to $-\operatorname{grad} U$. Now, the gravitational potential U is determined by how matter is distributed in the universe. This distribution of matter is described by the mass density function $\rho \colon \mathbb{R}^3 \to \mathbb{R}$ via Poisson's equation

$$\Delta U = 4\pi\rho,$$

where we set Newton's gravitational constant G to be 1. We also require $U(x)$ to vanish as $|x| \to \infty$. So, if ρ has a fast enough decay, the solution to Poisson's equation can be written as

$$U(x) = -\int_{\mathbb{R}^3} \frac{\rho(y)}{|x-y|}\,\mathrm{d}y. \tag{8}$$

For instance, we can represent a "point mass" m located at $x_0 \in \mathbb{R}^3$ by taking $\rho(x)$ to be the Dirac delta function $m\delta(x - x_0)$[3]. In this case, the gravitational potential created by this point mass is given by

$$U_{m,x_0}(x) = -\frac{m}{|x-x_0|}. \tag{9}$$

Another classical result from Newtonian gravity is the following [2, Theorem 3.8].

Theorem 1 (Newtonian shell theorem)**.** *If ρ is compactly spported and purely radial, then $U(x) = -\frac{m}{|x|}$ for all x outside the support of ρ, where $m = \int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x$.*

Proof. Note that U must also be purely radial. So, U is a purely radial harmonic function outside the support of ρ. Every such function that decays at infinity can be written as $U(x) = -\frac{m}{|x|}$, for some $m \in \mathbb{R}$. Lastly,

$$\int_{\mathbb{R}^3} \rho(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} \frac{1}{4\pi}\Delta U\,\mathrm{d}x = \lim_{r\to\infty}\int_{S_r} \frac{1}{4\pi}\frac{\partial U}{\partial r}\,\mathrm{d}\mu_{S_r} = \lim_{r\to\infty}\int_{S_r} \frac{1}{4\pi}\frac{m}{|x|^2}\,\mathrm{d}\mu_{S_r} = m,$$

where S_r denotes the sphere of radius r centered at the origin. □

[3] If we allow ρ to be a distribution, the same results still hold [2].

Now, consider some density ρ that is supported in $|x| < r_1$ but is not spherically symmetric. Again, $U(x)$ is harmonic on $|x| \geq r_1$. So, by expanding U in spherical harmonics, we obtain that

$$U(x) = -\frac{m}{|x|} + O_1(|x|^{-2}),$$

for some $m \in \mathbb{R}$ [2]. So,

$$\int_{\mathbb{R}^3} \rho(x)\, \mathrm{d}x = \int_{\mathbb{R}^3} \frac{1}{4\pi} \Delta U\, \mathrm{d}x = \lim_{r\to\infty} \int_{S_r} \frac{1}{4\pi} \frac{\partial U}{\partial r}\, \mathrm{d}\mu_{S_r} = \lim_{r\to\infty} \int_{S_r} \frac{1}{4\pi} \left(\frac{m}{|x|^2} + O\left(|x|^{-3}\right) \right) \mathrm{d}\mu_{S_r} = m.$$

Thus, $m = \int_{\mathbb{R}^3} \rho(x)\, \mathrm{d}x$ once more. Historically, this integral was defined as the total mass of the system. However, note that the behavior of the highest-order term in $U(x)$ for large enough x is precisely the same as that of a point mass m at the origin. That is, the total mass of the system encodes information about the asymptotic behavior of U. It so happens that this mass is equal to the integral of ρ, but the integral itself has no clear physical meaning. For instance, test particles very close to the support of ρ are completely indifferent to the total mass. That is, only when a test particle is close enough to infinity it can feel the total mass. So, it is much more natural to *define* the mass of a system as

$$m \coloneqq \lim_{r\to\infty} \frac{1}{4\pi} \int_{S_r} \frac{\partial U}{\partial r}\, \mathrm{d}\mu_{S_r}. \tag{10}$$

The equality $m = \int_{\mathbb{R}^3} \rho(x)\, \mathrm{d}x$ is just a useful consequence of the linearity of the Laplacian [2].

Now, we steer away from Newtonian gravity and into a simple model of a 3-dimensional (isolated) system in general relativity. Consider, at an instant in time, a complete asymptotically flat manifold (M^3, g). Similar to the discussion above, we can consider the distribution of matter to be a mass density function $\rho\colon M \to \mathbb{R}$ [2]. But, in general relativity, ρ only *constraints* the metric g (instead of defining it) via

$$\mathrm{scal}_{\mathrm{g}} = 16\pi\rho;$$

i.e., the mass density is the scalar curvature (up to a constant) [2]. Here, the asymptotically flatness is a sort of boundary condition [2]. We are still interested in deriving what the total mass is (by the previous discussion, we know that it should not be $\int_M \rho\, \mathrm{d}\mu_{\mathrm{g}}$) [2]. Following the derivation in [2], if $\mathrm{scal}_{\mathrm{g}}$ were a "purely" linear operator of g, then the total mass of the system would simply be $\frac{1}{16\pi}\int_M \mathrm{scal}_{\mathrm{g}}\, \mathrm{d}\mu_{\mathrm{g}}$ (note that we are using that the scalar curvature is integrable over M, cf. Remark 2). If the scalar curvature is approximately linear (i.e., when (M^3, g) is very close to the background Euclidean space $(\mathbb{R}^3, \overline{\mathrm{g}})$, then it can be approximated by its linearization at $\overline{g}_{ij} = \delta_{ij}$ [2]. Then, using results from [2, 1.18], we define the mass to be

$$\begin{aligned} m &\coloneqq \frac{1}{16\pi} \int_{\mathbb{R}^3} D\,\mathrm{scal}\,|_{\overline{\mathrm{g}}}(\mathrm{g} - \overline{\mathrm{g}})\, \overline{\mathrm{d}\mu} \\ &= \frac{1}{16\pi} \int_{\mathbb{R}^3} \left[-\overline{\Delta}\left(\overline{\mathrm{tr}}\,\mathrm{g}\right) + \overline{\mathrm{div}}\left(\overline{\mathrm{div}}\,\mathrm{g}\right)\right] \overline{\mathrm{d}\mu} \\ &= \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \left[\overline{\mathrm{div}}\,\mathrm{g} - \mathrm{d}(\overline{\mathrm{tr}}\,\mathrm{g})\right] \overline{\nu}\, \overline{\mathrm{d}\mu}_{S_r} \\ &= \lim_{r\to\infty} \frac{1}{16\pi} \int_{S_r} \sum_{i,j=1}^{3} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \frac{x^j}{|x|}\, \overline{\mathrm{d}\mu}_{S_r}, \end{aligned} \tag{11}$$

where ν is the Euclidean outward pointing normal to S_r [2]. The formula above is precisely what we needed (i.e., the total mass as an asymptotic integral involving g), and it is true for any asymptotically flat (M^3, g) since all such metrics are close to Euclidean at infinity [2].

Now, we are ready to define the ADM mass of an asymptotically flat manifold [2, 3.9].

Definition 4 (ADM mass)**.** Given an asymptotically flat manifold (M^n, g) with ends $M_1, \ldots, M_l$, the ADM mass of the end M_j is

$$m_{\mathrm{ADM}}(M_j, \mathrm{g}) \coloneqq \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\rho} \left[\overline{\mathrm{div}}\,\mathrm{g} - \mathrm{d}(\overline{\mathrm{tr}}\,\mathrm{g})\right] \overline{\nu}\, \overline{\mathrm{d}\mu}_{S_\rho}, \tag{12}$$

where the barred quantities are all computed in the background Euclidean metric determined by the asymptotically flat coordinate chart ϕ_j (see Definition 2); S_ρ is the sphere of radius ρ and $\overline{\mathrm{d}\mu}_{S_\rho}$ is its volume measure induced by the Euclidean metric; $\overline{\nu}$ is the Euclidean outward pointing normal to S_ρ; and ω_{n-1} is the volume of the unit $(n-1)$-sphere.

We can write the ADM mass formula (Equation (12)) explicitly in coordinates as

$$m_{\mathrm{ADM}}(M_j, \mathrm{g}) = \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\rho} \sum_{i,j=1}^{3} \left(\partial_i g_{ij} - \partial_j g_{ii}\right) \frac{x^j}{|x|}\, \overline{\mathrm{d}\mu}_{S_\rho}. \tag{13}$$

Remark 3. It is shown in [2] that, even though the ADM mass seems to depend on the choice of coordinates, the ADM mass is a geometric invariant given Definition 2 of asymptotic flatness.

As proven in [2, 3.14], the ADM mass can also be written in terms of curvature.

Theorem 2. *Given an asymptotically flat manifold (M^n, g) with ends $M_1, \dots, M_l$, the ADM mass of the end M_j can be written as*

$$m_{ADM}(M_j, \mathrm{g}) = \lim_{i\to\infty} \frac{-1}{(n-1)(n-2)\omega_{n-1}} \int_{\Sigma_i} \mathrm{G}(X, \overline{\nu})\, \overline{\mathrm{d}\mu}_{\Sigma_i}, \tag{14}$$

where $\mathrm{G} \coloneqq \mathrm{Ric} - \frac{1}{2}\,\mathrm{scal}\,\mathrm{g}$ is the Einstein tensor; X is the vector field $x^i\partial_i$ on $M_j \simeq \mathbb{R}^n \setminus \overline{B_1(0)}$; and Σ_i is any sequence that exhausts M_j and is such that $\overline{\mu}_{\Sigma_i}(\Sigma_i) \le C(\inf_{x\in\Sigma_i}|x|)^{n-1}$ for some $C \in \mathbb{R}$ that does not depend on i. As before, the barred quantities are computed in the background Euclidean metric.

Remark 4. Since $g_{ij} - \delta_{ij}$ decays at infinity, we can replace $\overline{\nu}$ and $\overline{\mathrm{d}\mu}_{\Sigma_i}$ with ν and $\mathrm{d}\mu_{\Sigma_i}$ in Theorem 2 [2].

Remark 5. If (M, g) has an asymptotic decay rate $q = n-2$, then Theorem 2 gives a coordinate-independent formula for the ADM mass:

$$m_{\mathrm{ADM}}(M_j, \mathrm{g}) = \lim_{\rho\to\infty} \frac{-1}{(n-1)(n-2)\omega_{n-1}} \int_{S_\rho(p)\cap M_j} \rho\, \mathrm{G}(\nu,\nu)\, \mathrm{d}\mu_{S_\rho(p)},$$

for all $p \in M$, where $S_\rho(p)$ is the geodesic ball around p [2].

Finally, we may state the positive mass theorem which was conjectured in 1961 as soon as the concept of ADM mass was formulated [2]. Here, we refer to this version of the positive mass theorem as *Riemannian* so there is no confusion with more general formulations of the same theorem using initial data sets [2].

Theorem 3 (Riemannian positive mass theorem)**.** *Let (M, g) be a complete asymptotically flat manifold with nonnegative scalar curvature. Then, the ADM mass of each end of M is nonnegative.*

The 3-dimensional version of the theorem above was proven by Schoen and Yau in 1979 in [3]. They soon generalized the result to dimension less that 8. Nowadays, we know much more general results to be true, but we do not investigate these (much more involved) generalizations in this article. Schoen's and Yau's paper [3] also proved a rigidity result in positive mass.

Theorem 4 (Positive mass rigidity)**.** *Let (M, g) be a complete asymptotically flat manifold with nonnegative scalar curvature. If the ADM mass of any end of (M, g) is zero, then (M, g) is isometric to Euclidean space.*

Instead of presenting the (beautiful yet lengthy) proofs for Theorems 3 and 4, we study particular cases of these theorems in the following section.

4 Special cases of the positive mass theorem

4.1 Spherically symmetric case

The first case we consider is the spherically symmetric case as in [2, 3.20].

Proposition 1. *Let g be a complete asymptotically flat metric on $\mathbb{R}^n$ that is spherically symmetric in the sense that, under the (diffeomorphic) identification $\mathbb{R}^n \setminus \{0\} \simeq (0,\infty) \times S^{n-1}$, we can write*

$$\mathrm{g} = \frac{\mathrm{d}r^2}{U(r)} + r^2\mathrm{d}\Omega^2,$$

for some positive smooth function U.

If g has nonnegative scalar curvature, then it has nonnegative ADM mass. Moreover, if the ADM mass is zero, then g is Euclidean.

Proof. As noted in [2], when we write g as above, we implicitly assume that there are no symmetric minimal spheres around the origin. Now, from Equations (4) and (5), we have that

$$\mathrm{scal}_{\mathrm{g}} = \frac{n-1}{r^{n-1}} \frac{\mathrm{d}}{\mathrm{d}r} \left[r^{n-2}(1-U(r)) \right]$$

So, if $\mathrm{scal}_{\mathrm{g}} \geq 0$ everywhere, then $\frac{1}{2}r^{n-2}(1-U(r))$ is nondecreasing for all $r > 0$. Moreover, since we are assuming that g can be extended to a complete metric on $\mathbb{R}^n$, we must have that U is bounded as r approaches zero. Thus, since $\frac{1}{2}r^{n-2}(1-U(r))$ is nondecreasing for all $r > 0$,

$$0 = \lim_{r\to 0} \frac{r^{n-2}}{2}(1-U(r)) \leq \lim_{r\to\infty} \frac{r^{n-2}}{2}(1-U(r)) = m_{\mathrm{ADM}}(\mathrm{g}),$$

as desired. Finally, if $m_{\mathrm{ADM}} = 0$, then $r^{n-2}(1-U(r))$ must vanish (since it is nondecreasing), hence $U(r) \equiv 1$; i.e., g is Euclidean. □

4.2 Conformally flat case

Now, we consider the globally conformally Euclidean case [2, 3.22].

Proposition 2. *Let u be a positive smooth function on $\mathbb{R}^n$ such that*

$$u(x) = 1 + O_2(|x|^{-q}),$$

for some $q > \frac{n-2}{2}$, and such that $\Delta_{\mathrm{g}} u$ is integrable. Observe that [2, 3.12] *implies that $\left(\mathbb{R}^n, g_{ij} = u^{\frac{4}{n-2}}\delta_{ij}\right)$ is a complete asymptotically flat manifold.*

If g *has nonnegative scalar curvature, then it has nonnegative ADM mass. Moreover, if the ADM mass is zero, then* g *is Euclidean.*

Proof. From [2, 1.8], we have that

$$\mathrm{scal}_{\mathrm{g}} = -\frac{4(n-1)}{n-2} u^{-\frac{n+2}{n-2}} \overline{\Delta} u,$$

where the bar once again indicates the Euclidean background metric δ_{ij}. Moreover, from [2, 3.12],

$$\begin{aligned} m_{\mathrm{ADM}}(\mathrm{g}) &= m_{\mathrm{ADM}}(\delta) + \lim_{\rho\to\infty} \frac{2}{(n-2)\omega_{n-1}} \int_{S_\rho} -\frac{\partial u}{\partial r}\, \overline{\mathrm{d}\mu}_{S_\rho} \\ &= \frac{2}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} -\overline{\Delta} u\, \overline{\mathrm{d}\mu} \\ &= \frac{2}{(n-2)\omega_{n-1}} \int_{\mathbb{R}^n} \frac{n-2}{4(n-1)} u^{\frac{n+2}{n-2}} \,\mathrm{scal}_{\mathrm{g}}\, \overline{\mathrm{d}\mu}. \end{aligned} \tag{15}$$

Thus, clearly, if $\mathrm{scal}_{\mathrm{g}} \geq 0$, then $m_{\mathrm{ADM}} \geq 0$. Finally, for the rigidity, if $m_{\mathrm{ADM}} = 0$, then $\overline{\Delta} u$ must vanish by the equation above; i.e., u is harmonic on $\mathbb{R}^n$. Since u approaches 1 at infinity (by definition), $u \equiv 1$ by the maximum principle. □

Remark 6. If we linearize the the partial differential operator that scal represents here at $u = 1$, we find that, for $u \approx 1$, $\mathrm{scal}_{\mathrm{g}} \approx -\frac{4(n-1)}{n-2}\overline{\Delta} u$. Thus, setting $U := 2(1-u)$, we obtain a Newtonian limit as $u \to 1$. In particular, if $n = 3$, then $\mathrm{scal}_{\mathrm{g}} = 16\pi\rho$ with $u(\infty) = 1$ (from Section 3.3) "reduces" to $\Delta U = 4\pi\rho$ with $U(\infty) = 0$, as $u \to 1$.

4.3 Two-dimensional model

Here we study the 2-dimensional incarnation of the positive mass theorem, as described in [4]. As noted in [2], this is not a special case of the positive mass theorem, but rather a "toy model." As it so happens, asymptotic flatness is too strong of an assumption for surfaces with nonnegative Gauss curvature to be interesting. In turn, we consider asymptotically conical surfaces [4].

Definition 5 (Asymptotically conical surfaces)**.** A Riemannian surface (M^2, g) is asymptotically conical if there exists a bounded set K such that $M \setminus K$ is a finite union of ends $M_1, \ldots, M_l$ with the property that, for each M_j, there exists a diffeomorphism

$$\phi_j : M_j \longrightarrow \mathbb{R}^2 \setminus \overline{B_1(0)} \simeq (1, \infty) \times S^1,$$

so that, under ϕ_j,

$$\mathrm{g} = \mathrm{d}r^2 + r^2 \mathrm{d}\theta^2 + O_1(r^{-q}),$$

for some $q > 0$; where r is a coordinate for $(1, \infty)$, and $\mathrm{d}\theta^2$ is the metric on S^1 with length $2\pi\alpha$, for some $\alpha > 0$.

Remark 7. The value α is called the *cone angle* of that end [4].

Remark 8. By $O_1(r^{-q})$, we mean a 2-tensor τ such that $|\tau|_{\overline{\mathrm{g}}} + r|\overline{\mathrm{grad}}\,\tau|_{\overline{\mathrm{g}}} = O(r^{-q})$, where the barred quantities are computed with respect to the background metric $\overline{\mathrm{g}} = \mathrm{d}r^2 + r^2\mathrm{d}\theta^2$ [2].

Theorem 5 (Two-dimensional counterpart of the positive mass theorem)**.** *Let (M^2, g) be a connected complete asymptotically conical surface with nonnegative Gauss curvature. Then, each end of M has cone angle at most* 1*. Finally, if any cone angle is equal to* 1*, then (M, g) is the Euclidean plane.*

Remark 9. The cone angle somewhat plays an analogous role to the mass in higher dimensions [4].

Proof. Denote by M_ρ the (compact) region whose boundary ∂M_ρ is the union of the spheres $\{r = \rho\}$ in each end of M. By Gauss-Bonnet,

$$\int_{M_\rho} K_\mathrm{g} \, \mathrm{d}\mu_\mathrm{g} = 2\pi\chi(M_\rho) - \int_{\partial M_\rho} \kappa_\mathrm{g} \, \mathrm{d}s.$$

Since we assume M to be connected, we have that $2\pi\chi(M_\rho) \leq 2\pi$. Moreover, since (M, g) is asymptotically conical, for ∂M_ρ, $\kappa_\mathrm{g} = \frac{1}{\rho} + O(\rho^{-q-1})$. We also have that the length of ∂M_ρ is $2\pi\rho \sum_{j=1}^{l} \alpha_j + O(\rho^{1-q})$. Thus,

$$\int_{M_\rho} K_\mathrm{g} \, \mathrm{d}\mu_\mathrm{g} \leq 2\pi \left(1 - \sum_{j=1}^{l} \alpha_j \right) + O(\rho^{-q}).$$

In the limit $\rho \to \infty$, we find that $\sum_{j=1}^{l} \alpha_j \leq 1$, as desired.

As for rigidity, if the cone angle of one the ends is equal to 1, then this must be the *only* end. Moreover, we must have that $\int_M K_\mathrm{g} \, \mathrm{d}\mu_\mathrm{g} = 0$. This implies that $K_\mathrm{g} \equiv 0$; i.e., M is flat. Therefore, (M, g) has one planar end and is flat, hence (M, g) must be Euclidean [2, 2.33]. □

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