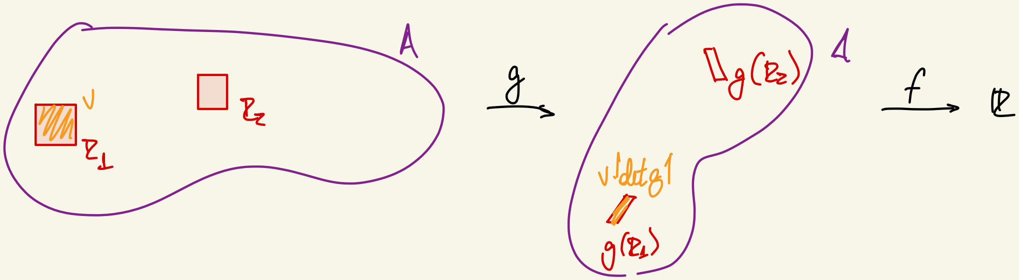
LECTURE 37: PROOF OF THE CHANGE OF VARIABLES THEOREM

Thm: (CHANGE OF VARIABLES) Let $A \subset \mathbb{R}^n$ be open, and $g: A \to \mathbb{R}^n$ be continuously differentiable and 1-1 and such that $\forall x \in A$ $g'(x)$ is invertible. If $f: g(A) \to \mathbb{R}$ is integrable, then

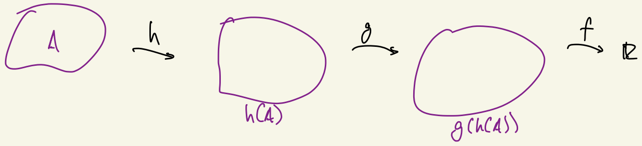
$$\int_{g(A)} f = \int_A (f \circ g)\,|\det g'| .$$



"Integration: summation of function values times a bit of 'chunk' that contains it". (but it's more intricate...)

→ Goal: write g as a composition of "simpler maps".

LEMMA: "$\operatorname{COV}(g), \operatorname{COV}(h) \Rightarrow \operatorname{COV}(g \circ h)$".



(Suppose we know this)

SIMPLER MAPS: LAYER PRESERVING MAPS (L.P. MAPS)

Def: (LAYER PRESERVING MAPS) A function $g:\mathbb{R}^n\to\mathbb{R}^n$ is layer preserving if

$$g\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} = \begin{pmatrix}\vdots\\ \vdots\\ x_n\end{pmatrix}.$$

Equivalently, if $g=\begin{pmatrix}g_1\\ \vdots\\ g_n\end{pmatrix}$, then $g_n(x_1,\dots,x_n)=x_n$.

* Try using COV$(n-1)$ ⇒ COV(n) for l.p. maps. (Fubini)

↳ But need to prove that every g is a composition of layer preserving maps.* (only locally, b/c IVT is "very local")

Suppose $g:\mathbb{R}^n_{x_j}\to\mathbb{R}^n_{y_i}$. We can write

$$\begin{aligned} y_1 &= g_1(x_1,\dots,x_n)\\ &\ \ \vdots \\ y_n &= g_n(x_1,\dots,x_n)\end{aligned}$$

So,

layer preserving

layer preserving

$$\begin{tikzcd} \begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} \arrow[r, "\alpha"] & \begin{pmatrix}x_1\\ \vdots\\ x_{n-1}\\ y_n\end{pmatrix} \arrow[r, "\beta"] & \begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix} \end{tikzcd}$$

α is invertible by the Inverse Function Thm.* ⟶ Need local ↔ global; i.e., COV$\left(\substack{\text{SMALL}\\ \text{SETS}}\right)$ ⇒ COV$\left(\substack{\text{LARGE}\\ \text{SETS}}\right)$ (PO1)

Lemma 1: "COV(h), COV(g) $\Rightarrow$ COV($g\circ h$)".

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend left, "g\circ h"] & h(A)=B \arrow[r, "g"] \arrow[d, "\bar{f}"] & g(B) \arrow[r, "f"] & \mathbb{R} \\ & \mathbb{R} & & \end{tikzcd}$$

WTS: $$\int_{(g\circ h)(A)} f = \int_A (f\circ g\circ h)\,|\det(D(g\circ h))|.$$

LECTURE 38: CHANGE OF VARIABLES THEOREM PROOF (continued)

Idea: Show for composition of maps so we can use "simpler" (layer preserving maps) to prove COV (using Fubini).

Proof of Lemma 1: We have

* Assumptions

$$\int_{(g\circ h)(A)} f \overset{cov(g)}{=} \int_{h(A)} \overbrace{(f\circ g)\,|\det g'|}^{:=\bar{f}} \overset{cov(h)}{=} \int_A (\bar{f}\circ h)\,|\det h'|$$

$$= \int_A (f\circ g\circ h)\,|\det g'\circ h|\,|\det h'|$$

$$= \int_A (f\circ g\circ h)\,|\det g'\circ h\; h'| = \int_A (f\circ g\circ h)\,|\det (g\circ h)'|.$$

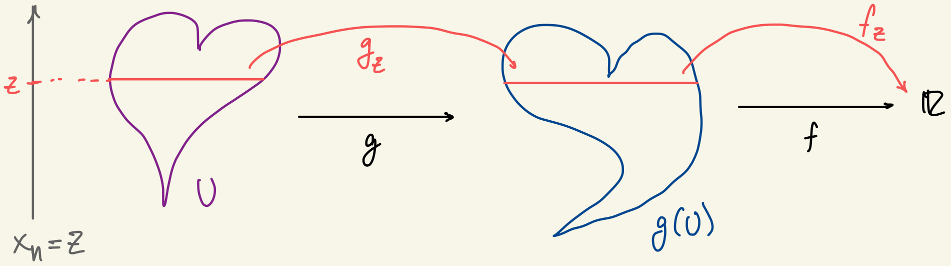
□

LEMMA 2: Assume COV$(n-1)$. Let $g: U \to \mathbb{R}^n$, where U is an open and bounded set, be a layer preserving map such that $g(U)$ is also bounded (meaning that $g(x_1, \dots, x_n) = (\_, \dots, x_n)$ or $g_n(x_1, \dots, x_n) = x_n$). Then, if $f: g(U) \to \mathbb{R}$ is continuous and $\operatorname{supp} f \subset g(U)$, then

$$\int_{g(U)} f = \int_U (f \circ g)\,|\det g'|.$$

Can replace $\{g(U)$ by anything (e.g., $\mathbb{R}^n$)

same



Pf: For $z \in \mathbb{R}$, define $g_z : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$g_z(x) = \begin{pmatrix} g_1(x, z) \\ \vdots \\ g_{n-1}(x, z) \end{pmatrix},$$

and $f_z : \mathbb{R}^{n-1} \to \mathbb{R}$ by

$$f_z(x) = f(x, z).$$

So, by Fubini

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx\, f(x, z) = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx\, f_z(x)$$

$$\overset{\text{COV}(n-1)}{=} \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (f_z \circ g_z)\,|\det g_z'| = (*)$$

LECTURE 39: CHANGE OF VARIABLES THEOREM PROOF (continued)

Aside: from the lemma previously,

$$g' = \left(\begin{array}{ccc|c} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial z} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial g_{n-1}}{\partial x_1} & \cdots & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial z} \\ \hline \cancel{\frac{\partial g_n}{\partial x_1}} & \cdots & \cancel{\frac{\partial g_n}{\partial x_{n-1}}} & \frac{\partial g_n}{\partial z} = 1 \end{array}\right) = \left(\begin{array}{ccc|c} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial z} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial g_{n-1}}{\partial x_1} & \cdots & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial z} \\ \hline 0 & \cdots & 0 & 1 \end{array}\right)$$

g_z' $\quad$ g_z'

all 0 $\quad$ "$g_n = z = x_n$"

$$= \left(\begin{array}{c|c} g_z' & * \\ \hline 0 & 1 \end{array}\right)$$

$$\Rightarrow \det g' = \det g_z'$$

So,

$$(*) = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (f \circ g)\,|\det g'| \overset{\text{Fubini}}{=} \int (f \circ g)\,|\det g'|.$$

Fubini (everything is continuous)

□

LEMMA 3: For every $a \in A$, there is some neighborhood U (open $U \ni a$) such that, on U, g is a composition of layer preserving maps and coördinate swaps.

↳ $\tau_{ij} : \mathbb{R}^n \to \mathbb{R}^n$

Now, also need to show COV for coördinate swaps }

$$\tau_{ij}(x_1, \dots, x_i, \dots, x_j, \dots, x_n) := (x_1, \dots, x_j, \dots, x_i, \dots, x_n)$$

Pf: Let $y_i = g_i(x_1, \dots, x_n)$. Idea:

$$\begin{tikzcd} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \arrow[r, "\alpha_k"] \arrow[rr, bend right, "g"'] & \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_k \end{pmatrix} \arrow[r, "\beta_k"] & \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \end{tikzcd}$$

Check if IVT applies ↓ α_k

l.p. for other entry

l.p. up to coörd. swap ↓ β_k

So, let $\alpha_k : \mathbb{R}^n \to \mathbb{R}^n$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_k \end{pmatrix} = \begin{pmatrix} x_1 \\ \vdots \\ g_k(x_1, \dots, x_n) \end{pmatrix}.$$

Now, compute α_k' to apply IVT:

$$\alpha_k' = \begin{pmatrix} \boxed{\begin{matrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{matrix}}^{\,n-1} & \begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix} \\ \frac{\partial g_k}{\partial x_1} \quad \cdots\cdots\cdots\cdots & \frac{\partial g_k}{\partial x_n} \end{pmatrix}$$

is invertible iff $\frac{\partial g_k}{\partial x_n} \neq 0$. So, assume not: $\underline{\forall k \ \frac{\partial g_k}{\partial x_n} = 0}$.

Then

$$g' = \left(\qquad \boxed{\begin{matrix} \frac{\partial g_1}{\partial x_n} \\ \vdots \\ \frac{\partial g_n}{\partial x_n} \end{matrix}} \right)^{\text{all zero}}$$

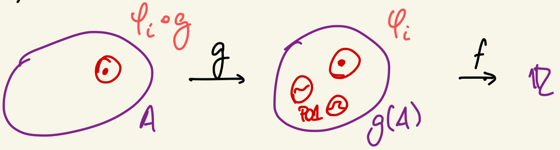
which contradicts the assumption that g' is invertible. So, for some k, $\frac{\partial g_k}{\partial x_n} \neq 0$. Fix such value for k. Now, α_k' is invertible at a, so, by the Inverse Function Theorem, α_k is invertible near a. Hence, near a, define $\beta_k := g \circ \alpha_k^{-1}$, and, now (all this near a)

$$\begin{aligned} g &= \beta_k \circ \alpha_k \\ &= \overbrace{\tau_{kn} \circ \underline{\tau_{kn}}}^{I} \underline{\circ \beta_k} \circ \overbrace{\tau_{1n} \circ \underline{\tau_{1n}}}^{I} \underline{\circ \alpha_k \circ} \overbrace{\underline{\tau_{1n}} \circ \tau_{1n}}^{I} \\ &= \underbrace{\tau_{kn}}_{\text{c.s.}} \circ \underbrace{(\tau_{kn} \circ \beta_k)}_{\text{layer preserving}} \circ \underbrace{\tau_{1n}}_{\text{c.s.}} \circ \underbrace{(\tau_{1n} \circ \alpha_k \circ \tau_{1n})}_{\text{layer preserving}} \circ \underbrace{\tau_{1n}}_{\text{c.s.}} \end{aligned}$$

□

Now, we are done w/ proving that every g can be written as a composition of l.p. maps and c.s. So, we have to prove "local ⇒ global".

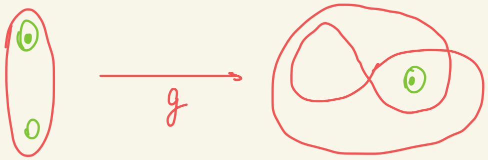
LEMMA 4: Local cov ⇒ global cov for continuous functions f.



small issue: PoI's must be compatible; i.e.,

$$\operatorname{supp}(\varphi_i \circ g) = g^{-1}(\operatorname{supp} \varphi_i).$$

E.g.



"We don't know if the green set is small, i.e., whether g is a composition of l.p. maps."

But, g is 1-1! So, everything works ☺

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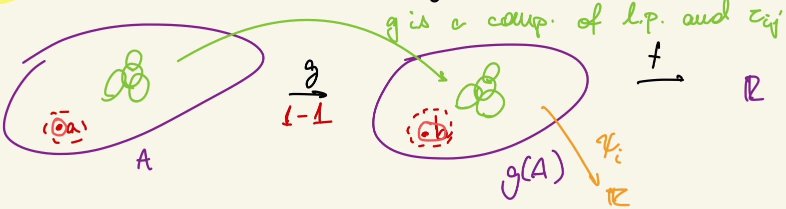
LECTURE 40: CHANGE OF VARIABLES THEOREM PROOF (continued)

For continuous functions only

Pf: (LEMMA 4: local cov ⇒ global cov)



Let $\mathcal{V} = \left\{ \underset{\text{open}}{V \subset} g(A) : \begin{array}{l} V \text{ is bounded; } g^{-1}(A) \text{ is bounded} \\ \text{and on } g^{-1}(V),\ g \text{ is a comp. of l.p. \& } \Sigma_{ij} \end{array} \right\}$

$\mathcal{V}$ is an open cover of $g(A)$. So, find a PO1 $\Psi = \{\psi_i\}$ subordinate to $\mathcal{V}$. Let $\varphi_i = \psi_i \circ g$. $\{\varphi_i\}$ is a PO1 for A subordinate to $\mathcal{U} = \{g^{-1}(V) : V \in \mathcal{V}\}$. Thus,

$$\int_{g(A)} f = \sum_i \int_{\underset{(\text{some } V \in \mathcal{V})}{\cancel{g(A)}}} \psi_i f = \sum_i \int_{\underset{(\text{some } U \in \mathcal{U})}{\cancel{A}}} (\psi_i \circ g)(f \circ g) |\det g'|$$

$$= \sum_i \int_{\text{some } U \in \mathcal{U}} \varphi_i (f \circ g) |\det g'| = \int_A (f \circ g) |\det g'|.$$

□

LEMMA 5: COV holds if $n = 1$ (base case of induction).

WLOG, $A = (a, b)$ and $g : (a, b) \to \mathbb{R}$ is 1-1 and continuous. So, g is monotone (by the Intermediate Value Theorem) either increasing or decreasing:

$$g(A) = g((a,b)) = \begin{cases} (g(a), g(b)), & g \text{ increasing} \\ (g(b), g(a)), & g \text{ decreasing} \end{cases}.$$

Pf: Take the case where g is decreasing:

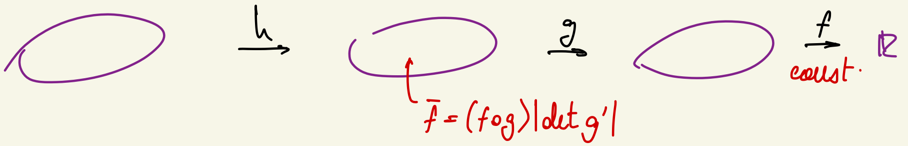
$$\int_{g(A)} f = \int_{g(b)}^{g(a)} f \overset{157}{=} \int_b^a (f \circ g) \underset{-|\det g'|}{\underset{\parallel}{g'}} = -\int_a^b (f \circ g)(-|\det g'|)$$

$$= \int_a^b (f \circ g) |\det g'| = \int_A (f \circ g) |\det g'|.$$

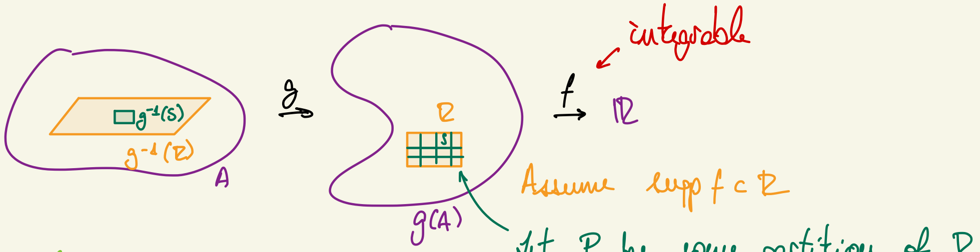
Other case is similar.

□

LEMMA 6: Suppose COV holds for continuous functions f's, then it holds for any integrable f.



Pf: Only prove a local version of COV for integrable functions, b/c this local version can be globalized as before (PO1).



goal: $L(f,P) \leq \ldots \leq U(f,P)$

$L(f\circ g)$; $U(f\circ g)$

First,

$$L(f,P) = \sum_{S\in P} \operatorname{vol}(S) \inf_{x\in S} f(x) = \sum_{S\in P} \int_S \overset{m_S(f)}{\inf_{x\in S} f(x)}$$

COV for constant function

$$= \sum_{S\in P} \int_{g^{-1}(S)} m_S(f)\,|\det g'|$$

$\inf_{x\in S} f(x) \leq f(x)$ $\forall x\in S$

$$\leq \sum_{S\in P} \int_{g^{-1}(S)} (f\circ g)\,|\det g'|$$

$$= \sum_{S\in P} \int_{g^{-1}(R)} \chi_{g^{-1}(S)} (f\circ g)\,|\det g'|$$

$\int h_1 + \int h_2 \le \int (h_1+h_2)$

$$\le \int_{g^{-1}(R)} \sum_{S\in P} \chi_{g^{-1}(S)} (f\circ g)\,|\det g'|$$

$$= \int_{g^{-1}(R)} (f\circ g)\,|\det g'| \le \int_{g^{-1}(R)} (f\circ g)\,|\det g'|$$

$$\le U(f,P).$$

Thus, by the integrability of f, we can make $L(f,P)$ and $U(f,P)$ as close to each other as we want, so all numbers above are close to each other.

□

LECTURE 41: COÖRDINATE SWAPS AND SARD'S THEOREM

Read along: p. 66-78

LEMMA 7. COV holds for coördinate swaps τ_{ij}.

Pf: Need to show

$$\int_{\tau_{ij}(A)} f = \int_A f\circ\tau_{ij}\,; \text{ e.g., } \int_{\tau(A)} f\circ\tau = \int_A f.$$

↳ so disturbingly obvious.

$\boxed{\text{COV}}$

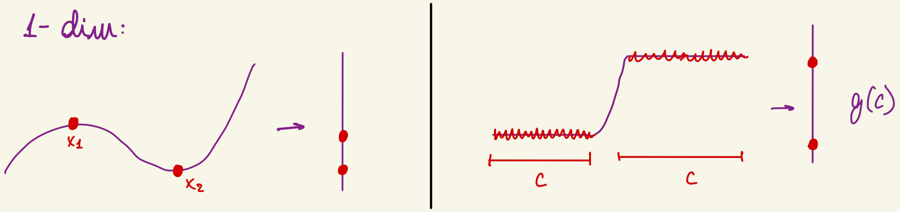
Baby version

Thm (SARD'S THEOREM) Let $A \subset \mathbb{R}^n$ be open and $g : A \to \mathbb{R}^n$ be continuously differentiable. Define

"Critical set of g" ← $C := \{x \in A : \det g'(x) = 0\}$.

Then, $g(C)$ is of measure 0.

In 1-dim:

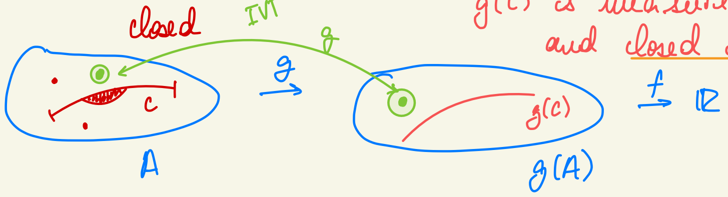


Claim: C is always closed ↘ sets defined by equations with continuous functions are always closed.

Pf: Let $h(x) = \det g'(x)$. Then $C = h^{-1}(\{0\})$. So $A \setminus C$ is open.

COROLLARY (SARD): In the COV, can drop the condition that "g' is invertible".

Pf:



$g(C)$ is measure 0 and closed in $g(A)$

$\xrightarrow{f} \mathbb{R}$

$g(A) \setminus g(C) = g(\underbrace{A \setminus C})$ → open set on which g' is invertible (so IVT applies to every point)

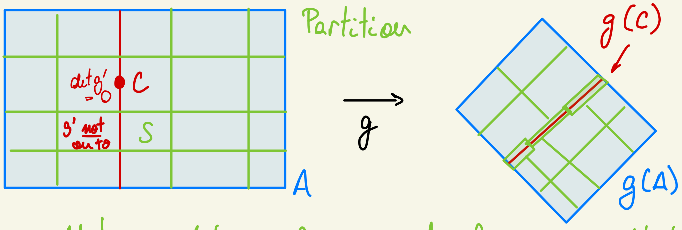
WTS:

$$\begin{array}{ccc} \int_A (f\circ g)\,|\det g'| & \overset{?}{=} & \int_{g(A)} f\,. \\ \| & & \| \\ \int_{A\setminus C} (f\circ g)\,|\det g'| & \overset{\text{cov}}{=} & \int_{g(A)\setminus g(C)} f \end{array}$$

Can ignore C b/c $\det g' = 0$ on C ⟵

$\square$

Baby proof of baby Sard:



Make partition finer and finer so that the volumes are smaller than $\varepsilon > 0$ so that $g(C)$ has measure 0.

Adult Sard's Theorem:

$g: \underset{\text{open}}{A} \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ and

$$C := \{x \in A : \operatorname{rank} g' < m\}$$

and g is C^k (k-times continuously differentiable), where $k = \max\{1, n-m+1\}$. Then $g(C)$ is measure 0.

Way harder !

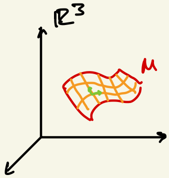
Now, we start a new chapter towards $\int_M d\omega = \int_{\partial M} \omega$

LECTURE 42: k- TENSORS

(Linear algebra)

Remember $\int_M d\omega = \int_{\partial M} \omega$. For example

To integrate over a manifold, we need a machine that inputs k vectors in $\mathbb{R}^n$ and output a number.



Def: Let V be a vector space over $\mathbb{R}$ and $k \in \mathbb{N} = \mathbb{Z}_{>0}$. A function $T: V^k \to \mathbb{R}$ is called "multilinear" or "k-linear" if

$$T(u_1, \dots, \alpha u_i' + \beta u_i'', u_k) = \alpha T(u_1, \dots, u_i', \dots, u_k) + \beta T(u_1, \dots, u_i'', \dots, u_k).$$

Ex: An inner product is a 2-linear map: $\langle u_1, u_2 \rangle \in \mathbb{R}$. On $\mathbb{R}^n$ $(x_1, \dots, x_n), (y_1, \dots, y_n)$ Then $T(x,y) = \sum x_i y_i$.

Ex:

$$\begin{tikzcd}(\mathbb{R}^n)^n = \mathbb{R}^{n^2} = M_{n\times n}(\mathbb{R}) \arrow[r, "\det"] \arrow[r, bend left, "\det"] & \mathbb{R}\end{tikzcd}$$

$$\begin{pmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{pmatrix} \simeq (u_1, \dots, u_n) \in \mathbb{R}^{n\times n}$$

Claim: det is n-linear.

Ex: A 1-linear map $\varphi: V \to \mathbb{R}$ linear. This is called a linear functional $\varphi \in V^*$.

Ex. A 0-linear map on V^0 "empty sequence" $\in V^0 \neq \emptyset$

$$w: V^0 = \{()\} \longrightarrow \mathbb{R}$$

$$\frac{22}{7} \overset{?}{=} w(()) \in \mathbb{R}.$$

So, 0-linear is equivalent to a real number.

Def: Define

$$\mathcal{T}^k(V) := \{k\text{-linear maps on } V\}.$$

Warning: this is also almost always called $\mathcal{T}^k(V^*)$.

Ex. $\langle \cdot , \cdot \rangle \in \mathcal{T}^2 V$; $\det \in \mathcal{T}^n V$; $\mathcal{T}^0 V \cong \mathbb{R}$, $\mathcal{T}^1(V) = V^*$.

- If $\dim V = n$, then $\dim \underline{V^*} = n$.

  Identify w/ matrices with n column and 1 row.

- Suppose $(v_1, \dots, v_n)$ is a basis of V. Then, there exists a unique basis $\underline{(\varphi_1, \dots, \varphi_n)}$ of $V^* = \mathcal{T}^1 V$ such that $\varphi_i(v_j) = \delta_{ij} = \begin{cases} 1, & i = j \\ 0, & \text{else} \end{cases}$.

  Dual basis of $(v_1, \dots, v_n)$

  Completely defines (φ_i)

Ex 1: What is the dual basis of $\underline{\{(1,2), (3,4)\}}$?

Basis of $\mathbb{R}^2$

thinking of $\mathbb{R}^2 = \left\{ \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\}$, $(\mathbb{R}^2)^* = \{(\cdot \ \ \cdot)\}$.

So, we want $\varphi_1 = (\text{---})$, and $\varphi_2 = (\text{---})$ s.t.

$$\boxed{\begin{array}{ll} \varphi_1(v_1) = 1 & \varphi_2(v_1) = 0 \\ \varphi_1(v_2) = 0 & \varphi_2(v_2) = 1 \end{array}} \longrightarrow \begin{pmatrix} - \varphi_1 - \\ - \varphi_2 - \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus, basically the matrix for φ_i is the inverse of the matrix v_i: so, compute

$$\begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & 3/2 \\ 1 & -1/2 \end{pmatrix} = \begin{pmatrix} - \varphi_1 - \\ - \varphi_2 - \end{pmatrix}$$

Thus, $\varphi_1 = (-2 \quad 3/2)$ and $\varphi_2 = (1 \quad -1/2)$.

In general, the dual basis of $(v_1, \dots, v_n)$:

$$\begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}^{-1} = \begin{pmatrix} - & \varphi_1 & - \\ & \vdots & \\ - & \varphi_n & - \end{pmatrix}.$$

Ex. In $\mathbb{R}^n$, what is the dual of $(1, 0, 0, \dots, 0)$?

Non-example! This does not make sense!

$(1, 0, \dots, 0)$ is NOT a basis!

We only have a dual to a basis w/ n vectors!

Claim: $\mathcal{T}^k(V)$ is itself a vector space.

Operations

- If $T_1, T_2 \in \mathcal{T}^k$, then $(T_1 + T_2)(u_1, \dots, u_k) = T_1(u_1, \dots, u_k) + T_2(u_1, \dots, u_k)$
- If $T \in \mathcal{T}^k$, then $(\alpha T)(u_1, \dots, u_k) = \alpha\,(T(u_1, \dots, u_k))$.
- $0_{\mathcal{T}^k}(u_1, \dots, u_k) = 0$.

□

Also, there is a map "$\otimes$" ("tensor product / multiplication") $\otimes : \underset{\underset{T_1}{\cup}}{\mathcal{T}^k} \times \underset{\underset{T_2}{\cup}}{\mathcal{T}^l} \to \mathcal{T}^{(k+l)}$ defined as

$$(T_1 \otimes T_2)(u_1, \dots, u_{k+l}) = T_1(u_1, \dots, u_k) \cdot T_2(u_{k+1}, \dots, u_l).$$

Claim: $T_1 \otimes T_2 = T_1 T_2 \in \mathcal{T}^{k+l}$.

Pf: linear in the first k variables b/c T_1 is linear in the first k variables and same for the last l variables.

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LECTURE 43: k-TENSORS (continued)

As seen last time, $\otimes$ is associative, distributive, but *not* commutative.

- **Associative:**

$$\overset{\overset{\mathcal{T}^k}{\cup}}{T_1} \otimes (\overset{\overset{\mathcal{T}^l}{\cup}}{T_2} \otimes \overset{\overset{\mathcal{T}^m}{\cup}}{T_3})(u_1, \dots, u_{k+l+m})$$

$$= T_1(u_1, \dots, u_k) \cdot T_2 \cdot T_3(u_{k+1}, \dots, u_{k+l+m})$$

$$= T_1(u_1, \dots, u_k)\, T_2(u_{k+1}, \dots, u_{k+l})\, T_3(u_{k+l+1}, \dots, u_{k+l+m})$$

$$= (T_1 \otimes T_2) \otimes T_3(u_1, \dots, u_{k+l+m}).$$

- **Distributive:**

Proceed as before

$$(\overset{\overset{\mathcal{T}^k}{\cup}}{T_1} + \overset{\overset{\mathcal{T}^k}{\cup}}{T_2}) \otimes \overset{\overset{\mathcal{T}^l}{\cup}}{T_3} = T_1 \otimes T_3 + T_2 \otimes T_3 \in \mathcal{T}^{k+l}$$

$$T_1 \otimes (T_2 + T_3) = T_1 \otimes T_2 + T_1 \otimes T_3$$

Note that **$\otimes$ is bilinear**: $(\alpha T_1 + \beta T_2)\otimes T_3 = \alpha T_1 \otimes T_3 + \beta T_2 \otimes T_3$.

- **Not Commutative**: $T_1 \otimes T_2 \neq T_2 \otimes T_1$ in general.

Counterexample: $V = \mathbb{R}^2$, $\{e_1, e_2\}$, $\{\varphi_1, \varphi_2\} \in V^* = \mathcal{T}^1(V)$

$$(\varphi_1 \otimes \varphi_2)(e_1, e_2) = \varphi_1(e_1)\,\varphi_2(e_2) \overset{\text{def}}{=} 1.$$

$$(\varphi_2 \otimes \varphi_1)(e_1, e_2) = \varphi_2(e_1)\,\varphi_1(e_2) \overset{\text{def}}{=} 0.$$

NOTATION: $\underline{n} := \{1, \dots, n\}$;

$\underline{n}^k = \{\bar{i} = I = (i_1, \dots, i_k) : i_\alpha \in \underline{n}\ \ \forall \alpha\}$.

Note. $|\underline{n}^k| = n^k$.

Now, $(v_j)_{j=1}^{n} \in V^n$ and $J \in \underline{n}^k$ "multi-index".

$$v_J = (v_{j_1}, v_{j_2}, v_{j_3}, \dots, v_{j_k})$$

If $\varphi_i \in V^*$, $i = 1, \dots, n$, and $I \in \underline{n}^k$

$$\varphi_I = \varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k}.$$

E.g.: $\varphi_1 \otimes \varphi_2 = \varphi_{(1,2)}$ and $\varphi_{(1,2)}(e_{(1,2)}) = 1$.

$\varphi_2 \otimes \varphi_1 = \varphi_{(2,1)}$ and $\varphi_{(2,1)}(e_{(1,2)}) = 0$.

Suppose V is a vector space w/ basis $v_1, \dots, v_n$ and dual basis $\varphi_1, \dots, \varphi_n$ and $I, J \in \underline{n}^k$. Then

$I = (i_1, \dots, i_k)$

$J = (j_1, \dots, j_k)$

$$\mathcal{T}^k \ni \varphi_I(v_J) = (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(v_{j_1}, \dots, v_{j_k})$$

$$= \varphi_{i_1}(v_{j_1}) \cdot \dots \cdot \varphi_{i_k}(v_{j_k})$$

$$= \prod_{\alpha=1}^{k} \varphi_{i_\alpha}(v_{j_\alpha}) = \prod_{\alpha=1}^{k} \delta_{i_\alpha j_\alpha} = \begin{cases} 1, & I = J \\ 0, & I \neq J \end{cases}.$$

$\Rightarrow \varphi_I(v_J) = \delta_{IJ}.$

Thm: Let $(v_1, \dots, v_n)$ be a basis for V and $(\varphi_1, \dots, \varphi_n)$ be the dual basis. Then

$$\{\varphi_I : I \in \underline{n}^k\}$$

is a basis of $\mathcal{T}^k(V)$.

$\Rightarrow$ Hence, $\dim \mathcal{T}^k(V) = n^k = (\dim V)^k.$

Pf: 1) If $T_1, T_2 \in \mathcal{T}^k$, then $T_1 = T_2 \Leftrightarrow \forall I \; T_1(v_I) = T_2(v_I)$, $v_I = (v_{i_1}, \dots, v_{i_k})$.

Pf 1: $(\Rightarrow)$ Obvious.

$(\Leftarrow)$ Assume $\forall I$, $T_1(v_I) = T_2(v_I)$. Let $T = T_1 - T_2$ (need to show $T = 0$).

$$T(u_1, \dots, u_k) = T\left(\sum_{i_1=1}^{n} a_{1i_1} v_{i_1}, \dots, \sum_{i_k=1}^{n} a_{ki_k} v_{i_k}\right)$$

$$= \sum \cdots T(v_{i_1}, \Sigma, \dots, \Sigma)$$

$$= \sum \cdots T(v_{i_1}, v_{i_2}, \Sigma, \dots, \Sigma)$$

$$= \dots = \sum (\text{coeff.})\, T(v_I) = \sum (\text{coeff}) (T_1(v_I) - T_2(v_I))$$

$$= 0.$$

□

2) $\{\varphi_I\}$ spans $\mathcal{T}^k V$. (given $T \in \mathcal{T}^k$, $T \overset{?}{=} \sum a_I \varphi_I$)

$\Rightarrow$ Evaluate on v_J: $T(v_J) = \sum_I a_I \varphi_I(v_J) = \sum_I a_I \delta_{IJ}$

$$= a_J \delta_{JJ} = a_J.$$

Pf 2: given $T \in \mathcal{T}^k$, set $a_I = T(v_I)$, and we claim

$$T = \sum_I a_I \varphi_I .$$

Need to show $T(v_J) = \left(\sum_I a_I \varphi_I\right)(v_J) = a_J$ ✓

$$\begin{array}{c} T(v_J) \\ \| \\ a_J \end{array}$$

□

LECTURE 44: k- TENSORS

Recall: $\underline{n} = \{1, \ldots, n\}$; $\bar{i} = I \in \underline{n}^k$ means $I = \bar{j} = (j_1, \ldots, j_k)$; if $v_j \in V$; $v_I = (v_{j_1}, \ldots, v_{j_k}) \in V^k$. If $\varphi_j \in V^*$;

$$\varphi_I = \varphi_{j_1} \otimes \cdots \otimes \varphi_{j_k} \in \mathcal{T}^k(V).$$

Thm: V w/ basis $v_1, \ldots, v_n$; if $\varphi_1, \ldots, \varphi_n$ is the dual basis then $\{\varphi_I : I \in \underline{n}^k\}$ is a basis of $\mathcal{T}^k(V)$, hence $\dim \mathcal{T}^k(V) = n^k$.

Continuing the proof of the theorem:

3) The φ_I are linearly independent.

Pf: Assume $a_I \in \mathbb{R}$ s.t. $\sum a_I \varphi_I = 0$. So, for every $J \in \underline{n}^k$,

$$\left(\sum_I a_I \varphi_I\right)(v_J) = 0\,(v_J) = 0$$

$$\sum_I a_I \varphi_I(v_J) = 0$$

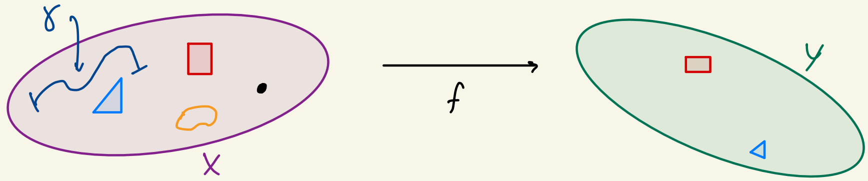
$$a_J = \sum_I a_I \delta_{IJ} = 0 \Rightarrow \forall J,\ a_J = 0.$$

□

FACT: $\dim(\mathcal{T}^k(V)) = n^k$.

Philosophical interlude:

Theme: In math, things either push forward or pull back.

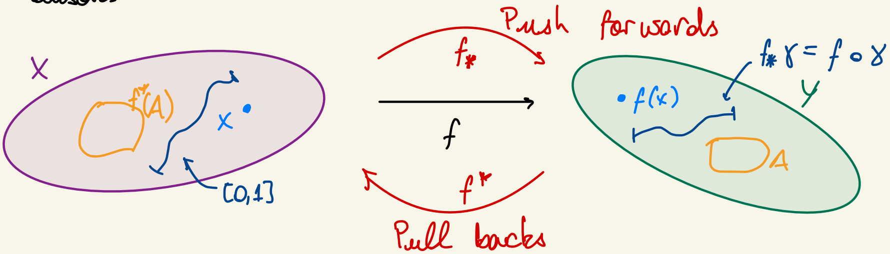


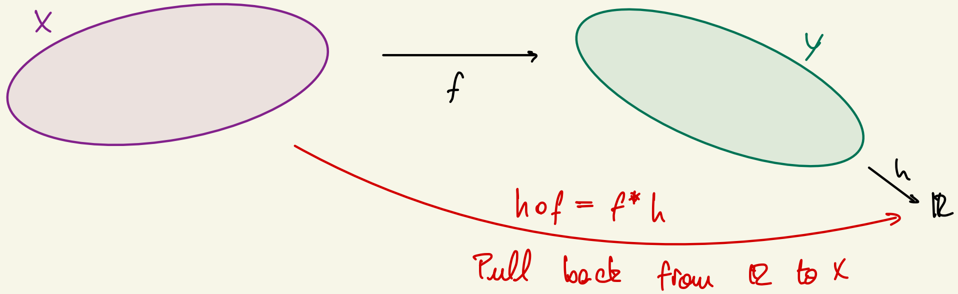
Push $x \mapsto f_* x = f(x)$

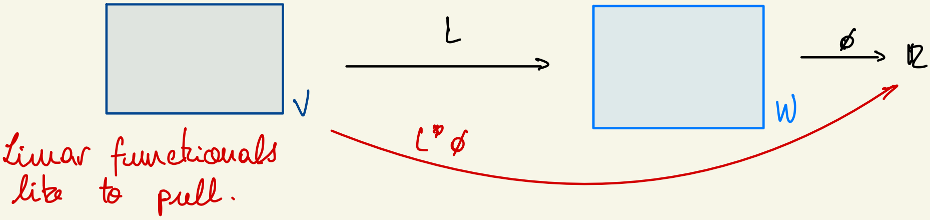
$X \supset A$ $f(A) \subset Y$ $f^{-1}(A)$ well behaved
$\Rightarrow$ $f^* A = f^{-1}(A)$

Push

Points, subsets, paths, functions, linear functionals, k-tensors



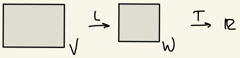




Aside: $L^*: W^* \longrightarrow V^*$ "pull back" or "adjoint of L".

Suppose $V \xrightarrow{L} W$ is linear. Then $\exists L^*$ s.t.

$$\mathcal{T}^k(V) \xleftarrow{L^*} \mathcal{T}^k(W)$$



defined by, for $T \in \mathcal{T}^k(W)$, $(v_1, —, v_k) \in V$,

$$T \mapsto (L^*T)(v_1, —, v_k) := T(Lv_1, —, Lv_k)$$

Claim 1: If $T \in \mathcal{T}^k(W)$, then $L^*T \in \mathcal{T}^k(V)$. (namely, L^*T is k-linear)

Pf: $(L^*T)(v_1, —, \alpha v_i' + \beta v_i'', —, v_k)$

L linear ↷ $= T(Lv_1, —, \alpha Lv_i' + \beta Lv_i'', —, Lv_k)$

$= \alpha T(Lv_1, —, Lv_i', —, L_k) + \beta T(Lv_1, —, Lv_i'', —, Lv_k)$

$= \alpha (L^*T)(v_1, \overset{v_i'}{—}, v_k) + \beta (L^*T)(v_1, \overset{v_i''}{—}, v_k)$. □

Claim 2: $L^*: \mathcal{T}^k(W) \to \mathcal{T}^k(V)$ is linear.

WTS: if $T_1, T_2 \in \mathcal{T}^k(W)$, then $L^*(T_1 + T_2) = L^*T_1 + L^*T_2$ and $L^*(\alpha T) = \alpha L^*T$.

Claim 3: L^* is "compatible" with $\otimes$, i.e., if $T_1 \in \mathcal{T}^k(W)$ and $T_2 \in \mathcal{T}^l(W)$, then

$$\underbrace{L^*(\underbrace{T_1 \otimes T_2}_{\in \mathcal{T}^{k+l}(W)})}_{\in \mathcal{T}^{k+l}(V)} = \underbrace{(\underbrace{L^*T_1}_{\in \mathcal{T}^k(V)}) \otimes (\underbrace{L^*T_2}_{\in \mathcal{T}^l(V)})}_{\in \mathcal{T}^{k+l}(V)}$$

Pf: Trace the definitions and see it works. □

$L^*(T_1 \otimes T_2)(v_1, —, v_k, v_{k+1}, —, v_\ell) = (T_1 \otimes T_2)(Lv_1, —, Lv_k, Lv_{k+1}, —, Lv_\ell)$

$= T_1(Lv_1, —, Lv_k) \cdot T_2(Lv_{k+1}, —, Lv_\ell) = (L^*T_1) \otimes (L^*T_2)$

LECTURE 45: ALTERNATING k- TENSORS

Recall: V w/ basis $(v_i)_{i=1}^{n}$ and dual basis $(\phi_i)_{i=1}^{n}$.

$\Rightarrow \{\phi_I\}_{I\in \underline{n}^k}$ is a basis of $\mathcal{T}^k(V) \Rightarrow \dim \mathcal{T}^k(V) = n^k$

Def: $T\in \mathcal{T}^k(V)$ "kills repetitions": $T(-,u,-,u,-)=0$.

Def: (ALTERNATING TENSOR) A k- tensor $T\in\mathcal{T}^k(V)$ is alternating if

$$T(-,u,-,w,-)=-T(-,w,-,u,-).$$

Also, define $\Lambda^k(V) := \{T\in \mathcal{T}^k(V) : T \text{ is alternating}\}$.

Note: i) $\Lambda^k(V)$ is a subspace of $\mathcal{T}^k(V)$.

ii) Other sources call this as $\Lambda^k(V^*)$.

Prop: If $T\in\mathcal{T}^k(V)$, then T kills repetitions if and only if $T\in\Lambda^k(V)$ (T is alternating).

Pf: ($\Leftarrow$) Suppose $T\in\Lambda^k(V)$. So,

swap "u" and "u"

$$T(-,u,-,u,-) = -T(-,u,-,u,-)$$

$$\Rightarrow T(-,u,-,u,-)=0.$$

($\Rightarrow$) Suppose T kills repetitions. Consider

$$0=T(-,u+w,-,u+w,-) = \cancel{T(-,u,-,u,-)}^{=0}$$
$$+\,T(-,w,-,u,-)$$

$$+\,T(\_,u,\_,w,\_) + \underbrace{\cancel{T(\_,w,\_,w,\_)}}_{} = 0$$

$$\Rightarrow \ T(\_,u,\_,w,\_) = -T(\_,w,\_,u,\_)$$

$\square$

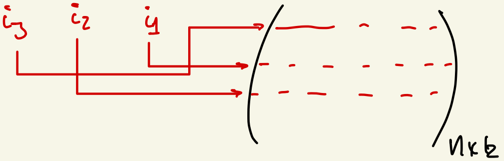
EXAMPLES 1. $\det \in \Lambda^n(\mathbb{R}^n)$

$\cap$

$\mathcal{L}^n(\mathbb{R}^n)$

2. Suppose $k \leq n$ and $\lambda_I \in \Lambda^k(\mathbb{R}^n)$, $\overset{(i_1,\dots,i_k)}{\overset{\|}{I}} \in \underline{n}^k$.

We get a $n \times k$ matrix



$$\xmapsto{\lambda_I} \det\begin{pmatrix} k\times k \text{ matrix highlighted} \\ \text{by rows } i_1,\dots,i_k \end{pmatrix}$$

$$= \det\begin{pmatrix} a_{i_1 1} & \cdots & a_{i_1 k} \\ \vdots & \ddots & \vdots \\ a_{i_k 1} & \cdots & a_{i_k k} \end{pmatrix}$$

i) Clearly λ_I is alternating.

ii) It's pointless to look at I's in which an index is repeating.

iii) Consider

$$\lambda_{(1753)} = -\lambda_{(1735)} = \lambda_{(1375)} = -\lambda_{(1357)}.$$

So, if we want to understand all the λ_I's, it suffices to look at I's such that

$$\underline{n}_a^k := \{ I \in \underline{n}^k : i_1 < i_2 < \cdots < i_k \}$$

↳ "ascending"

Q: What is $|\underline{n}_a^k|$?

E.g.: $n=8$ and $k=3 \Rightarrow 1\ 2\ ③\ 4\ ⑤\ ⑥\ 7\ 8 \leftrightarrow 3,5,6$

So, $|\underline{n}_a^k| = \binom{n}{k} = \frac{n!}{k!(n-k)!}$ = # ways of choosing k out of n objects.

So, rename

$$\underline{n}_a^k \longmapsto \binom{\underline{n}}{\underline{k}} = \text{Collection of ways of choosing } k \text{ from } \underline{n}.$$

$$= \text{Set of ascending sequences of length } k \text{ of integers from } 1 \text{ to } n.$$

Convention: for elements of $\gamma^k(V)$: "T", for elements of $\Lambda^k(V)$, "ω" (omega).

Now, suppose $\omega \in \Lambda^k(V)$ and $(u_1, —, u_k) \in V^k$.

$$\omega(u_{\sigma 1}, —, u_{\sigma k}) = \pm\, \omega(u_1, —, u_k).$$

Def: (PERMUTATIONS) A <u>permutation</u> of order k is a map $\sigma : \underline{k} \to \underline{k}$ which is a bijection. Let

$$S_k = \{\text{permutations } \sigma : \underline{k} \to \underline{k}\}.$$

→ Note $|S_k| = k!$. Moreover, S_k is a group. If $\sigma, \tau \in S_k$, then $\sigma\tau = \sigma \circ \tau \in S_k$. Also, $\iota \in S_k$ is defined by $\iota(i) = i$ for all i (identity).

Properties:

Form a group
1. $(\sigma\tau)\lambda = \sigma(\tau\lambda)$ (associative)
2. $\sigma \cdot \iota = \iota \cdot \sigma = \sigma$ (identity)
3. $\sigma \cdot \sigma^{-1} = \iota$

Obs: Note that $V \to W$ and $\Lambda^k(V) \leftarrow \Lambda^k(W)$.

LECTURE 46: ALTERNATING TENSORS (continued)

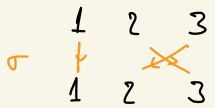
Recall: $\Lambda^k(V)$ is a subspace of $\mathcal{T}^k(V)$. Moreover,

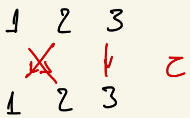
$\underline{n}^k_< := \binom{\underline{n}}{\underline{k}} = \{(i_1, \_\_, i_k) \in \underline{n}^k : i_1 < \dots < i_k\}$. Note that

$\left|\binom{\underline{n}}{\underline{k}}\right| = \binom{n}{k}$ and $S_k = \{\text{bijections } \sigma : \underline{k} \to \underline{k}\}$, "permutations"

which is a group and $|S_k| = k!$

S_k = "Permutation group of k elements". Moreover, S_k is not commutative, i.e., $\exists \sigma, \tau$ s.t. $\sigma\tau \neq \tau\sigma$. For example, take $S_k \ni \sigma = [1 \ 3 \ 2] = [\sigma 1 \ \sigma 2 \ \sigma 3]$, and $\tau = [2 \ 1 \ 3]$



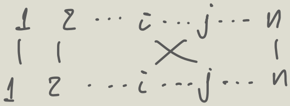


So, $\sigma \cdot \tau = [3 \ 1 \ 2]$ but $\tau \cdot \sigma = [2 \ 3 \ 1]$. S_k is not commutative.

Thm: There exists a unique map $\text{sign} : S_k \to \{0, 1\}$ such that

1. $\text{sign}(\sigma\tau) = \text{sign}(\sigma)\,\text{sing}(\tau)$
2. $\text{sign}(\tau_{ij}) = -1$, where

$$\tau_{ij}(l) = \begin{cases} i, & l = j \\ j, & l = i \end{cases}$$



Notation: $\text{sign}(\sigma) = (-1)^\sigma$ and $\text{sign}(\sigma) = 1 \Rightarrow$ "σ is even" and $\text{sign}(\sigma) = -1 \Rightarrow$ "σ is odd".

Pf. (240/247) □

FORMULA FOR THE SIGN:

$$\boxed{\text{sign}(\sigma) = (-1)^{\left(\begin{smallmatrix}\#\text{ of crossings in the}\\ \text{string diagram}\end{smallmatrix}\right)}}$$

E.g.: $\text{sign}(\tau_{ij}) = -1$

$$\text{sign}(\sigma) = \prod_{i<j} \text{sign}(\sigma_j - \sigma_i) \dots$$

$$\text{sign}(\sigma) = \det(P_\sigma), \quad \text{where } P_\sigma = \begin{array}{c} \begin{array}{ccc} \sigma 1 & \sigma 2 & \end{array} \\ \left(\begin{array}{ccc} 0 & 0 & \\ \vdots & 1 & \\ 1 & \vdots & \cdots \\ \vdots & & \\ 0 & 0 & \end{array}\right) \end{array}$$

Claim: Every σ is a composition of τ_{ij}'s.

$$\text{sign}(\sigma) = (-1)^{\#\text{ of transpositions } i<j}$$

Claim: If $T \in \Lambda^k(V)$ and $\sigma \in S_k$, then

$$T \circ \sigma^* = (-1)^\sigma T,$$

where $T: V^k \to \mathbb{R}$ and $\sigma^*: V^k \to V^k$ such that $\sigma^*(v_1, —, v_k) = (v_{\sigma 1}, —, v_{\sigma k})$. This gives

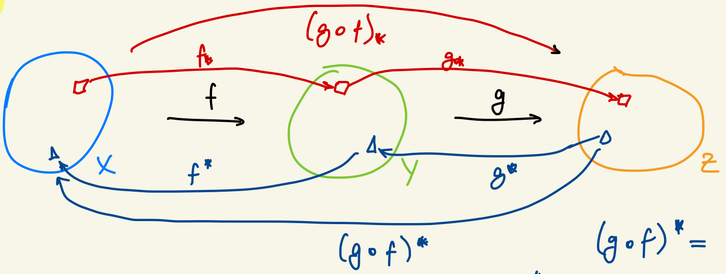
$$(T \circ \sigma^*)(v_1, —, v_k) = T(v_{\sigma 1}, —, v_{\sigma k}).$$

Aside: $V^{\underline{k}} = \{\text{functions } \underline{k} \to V\}$. Now

$$\underline{k} \xrightarrow{\sigma} \underline{k} \longrightarrow V$$

"Pull back of a list of vectors via permutation."

Aside 2:



$(g\circ f)_* = g_* \circ f_*$ "Covariant"

$(g\circ f)^* = f^* \circ g^*$ "Contravariant"

In particular, $(\sigma\circ\tau)^* = \tau^*\circ\sigma^*$ as $V^{\underline{k}} \to V^{\underline{k}}$.

Pf of claim: Write $\sigma = \tau_1 \circ \dots \circ \tau_\ell$, transpositions.

$$T\circ\sigma^* = T\circ(\tau_1\circ\dots\circ\tau_\ell)^* = (T\tau_\ell^*)\tau_{\ell-1}^* \cdots \tau_1^*$$

$$= -T\tau_{\ell-1}^*\cdots\tau_1^* = \dots = (-1)^\ell T$$

$$= (-1)^\sigma T.$$

$\square$

Def: If $I \in \underline{n}^{\underline{k}}$ (especially if $I \in \underline{n}_a^{\underline{k}}$), set

$$\omega_I = \sum_{\sigma\in S_k} (-1)^\sigma (\varphi_I \circ \sigma^*)$$

"anti-symmetrization"

E.g.: $f: \mathbb{R}^2 \to \mathbb{R}$ is anti-symmetric if $f(x,y) = -f(y,x)$. If $g: \mathbb{R}^2 \to \mathbb{R}$ is any function, set $f(x,y) = g(x,y) - g(y,x)$ and $h(x,y) = g(x,y) + g(y,x)$. So, $g = \frac{1}{2}(f+h)$.

Claim: ω_I is alternating.

Pf: Write $\omega_I \circ \tau^* = \left(\sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ \sigma^* \right) \circ \tau^*$

$$= \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ \sigma^* \circ \tau^* = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ (\tau\sigma)^*$$

$$= -\sum_{\sigma \in S_k} (-1)^{\sigma\tau} \varphi_I \circ (\tau\sigma)^* = -\sum_{\lambda \in S_k} (-1)^\lambda \varphi_I \lambda^* = -\omega_I.$$

□

//

LECTURE 47: ALTERNATING TENSORS

Recall: $S_k = \{\text{bijections } \sigma : \underline{k} \to \underline{k}\}$ $\underline{k} = \{1, \ldots, k\}$.

$T \in \Lambda^k(V) \Leftrightarrow T \circ \sigma^* = (-1)^\sigma T$; $\operatorname{sgn}(\sigma) = (-1)^\sigma = (-1)^{\#\text{transpositions to make } \sigma}$.

$\omega_I = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ \sigma^*$ where $\sigma^* : V^k \to V^k$ "pullback"

$\sigma^*(v_1, \ldots, v_k) = (v_{\sigma 1}, \ldots, v_{\sigma k})$; $\underline{n}_a^k = \binom{n}{k}$

$= \left\{ \begin{matrix} a_1 < \cdots < a_k \\ \in \underline{n}^k \end{matrix} \right\}$

thm: The collection $\{\omega_I : I \in \underline{n}_a^k\}$ is a basis of $\Lambda^k(V)$. In particular, $\dim \Lambda^k(V) = \binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Pf: 1. $\omega_I(v_J) = \delta_{IJ}$ I, J are ascending $\underline{n}_a^k$.

Pf 1: $\omega_I(v_J) \overset{\text{def}}{=} \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \varphi_I(\sigma^* v_J)$

$v_J = (v_{j_1}, \ldots, v_{j_k})$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \left(\varphi_{i_1} \bullet \cdots \otimes \varphi_{i_k}\right)\left(v_{j_{\sigma 1}}, \text{—}, v_{j_{\sigma k}}\right)$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma)\, \varphi_{i_1}(v_{j_{\sigma 1}}) \cdot \varphi_{i_2}(v_{j_{\sigma 2}}) \cdots \varphi_{i_k}(v_{j_{\sigma k}})$$

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \begin{cases} 1, & \text{if } i_1 = j_{\sigma 1}, \text{—}, i_k = j_{\sigma k} \\ 0, & \text{else} \end{cases}$$

ascending

$$= \sum_{\sigma \in S_k} \operatorname{sgn}(\sigma) \cdot \begin{cases} 1, & \text{if } (i_1, \text{—}, i_k) = (j_{\sigma 1}, \text{—}, j_{\sigma k}) \\ 0, & \text{otherwise} \end{cases}$$

only way for this to be ascending is if σ is the identity.

$$= (-1)^{\operatorname{sgn}(1)} \delta_{IJ} = \delta_{IJ}$$

2. Suppose $\lambda_1, \lambda_2 \in \Lambda^k(V)$. Then $\lambda_1 = \lambda_2$ iff $\forall I \in \underline{n}_a^k$, $\lambda_1(v_I) = \lambda_2(v_I)$.

($\Rightarrow$) Obvious.

($\Leftarrow$) Assume $\forall I \in \underline{n}_a^k$ $\lambda_1(v_I) = \lambda_2(v_I)$. So, compute

WTS: $(\lambda_1 - \lambda_2)(u_1, \text{—}, u_k) = 0$. By multilinearity, it suffices to show that $(\lambda_1 - \lambda_2)(v_{i_1}, \text{—}, v_{i_k}) = 0$ $\forall i_1, \text{—}, i_k$. So, it's enough to show that $(\lambda_1 - \lambda_2)(v_{i_1}, \text{—}, v_{i_k}) = 0$ $\forall (i_1, \text{—}, i_k) \in \underline{n}_a^k$. This is the same as saying $\lambda_1(v_I) = \lambda_2(v_I)$ $\forall \in \underline{n}_a^k$. □

3. <u>Span</u>: given $\lambda \in \Lambda^k(V)$, find $a_I \in \mathbb{R}$ s.t.

$$\lambda = \sum a_I \omega_I.$$

Take $a_I = \lambda(v_I)$. <u>WTS</u>: $\lambda = \sum a_I \omega_I$. Enough to show

$\forall J \in \underline{n}_a^k$: $\lambda(v_J) = \left(\sum a_I \omega_I\right)(v_J) = \sum a_I \delta_{IJ}$

$\lambda(v_J) =$

$= a_J$.

□

4. Linear independence:

Assume, for $b_I \in \mathbb{R}$,

<u>WTS</u>: $\sum b_I \omega_I = 0$

Evaluate on v_J: $0 = \sum b_I \omega_I(v_J) = \sum b_I \delta_{IJ} = b_J$

□

□

Ex: Let $V = \mathbb{R}^3$ and $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$ and $\underbrace{\varphi_1 = x,\ \varphi_2 = y,\ \varphi_3 = z}_{\text{Dual basis}}$.

Write $\mathbb{R}^3 = \left\{\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right\}$.

Compute basis for

$\Lambda^0(\mathbb{R}^3)$	$\Lambda^1(\mathbb{R}^3)$	$\Lambda^2(\mathbb{R}^3)$	$\Lambda^3(\mathbb{R}^3)$	$\Lambda^4(\mathbb{R}^3)$	$\Lambda^5(\mathbb{R}^3)$
$\omega_{()}$	ω_1	ω_{12}	ω_{123}	$\emptyset$	$\emptyset$
	ω_2	ω_{23}			
	ω_3	ω_{13}			
dim = 1	dim = 3	dim = 3	dim = 1	dim = 0	dim = 0

$\omega_1 = \varphi_1 = dx$
$\omega_2 = \varphi_2 = dy$
$\omega_3 = \varphi_3 = dz$

$\omega_{12} = \varphi_1 \wedge \varphi_2 = dx \wedge dy$
$\omega_{23} = \varphi_2 \wedge \varphi_3 = dy \wedge dz$
$\omega_{13} = \varphi_1 \wedge \varphi_3 = dx \wedge dz$

$\omega_{123} = dx \wedge dy \wedge dz$.

Aside: $\underline{n}_a^k = \left\{(i_1, —, i_k) : \begin{matrix} 1 \le i_\alpha \le n \\ i_1 < \dots < i_k \end{matrix}\right\}$; $|\underline{n}_a^k| = \binom{n}{k}$.

What is

$$\underline{n}_{nd}^k = \left\{(i_1, —, i_k) : \begin{matrix} 1 \le i_\alpha \le n \\ i_1 \le i_2 \le \dots \le i_k \end{matrix}\right\} ?$$

Suppose $n = 5$ and $k = 7$: 1 2 2 3 5 5 5 is valid

$$\{(1223555)\} \longleftrightarrow \{(1 \boxed{* 2 * * 3 * 4 5 * * *})\}$$

a sequence of digits and stars $(=k)$; total length $n+k$ and it has to begin with 1. So, there are

\# of ways of placing k stars within strings of length $n+k-1$. $\longrightarrow \binom{n+k-1}{k}$.

//

LECTURE 48: ALTERNATING TENSORS

Recall: $\{\omega_I^{*} = \sum_{\sigma \in S_k} (\varphi_I \circ \sigma^*)\}_{I \in \underline{n}_a^k}$ make a basis of $\Lambda^k(V)$

so $\dim \Lambda^k(V) = \binom{n}{k}$

* Elementary alternating tensors

We had an operation for k-tensors: $\lambda, \eta \mapsto \lambda \otimes \eta$.

Thm: There exists! an operation $(\underset{\overset{\cap}{\Lambda^k(V)}}{\lambda}, \underset{\overset{\cap}{\Lambda^\ell(V)}}{\eta}) \longmapsto \underset{\overset{\cap}{\Lambda^{k+\ell}(V)}}{\lambda \wedge \eta}$, the **WEDGE PRODUCT** such that

0. **Bilinear:**

$$(\alpha \lambda_1 + \beta \lambda_2) \wedge \eta = \alpha \lambda_1 \wedge \eta + \beta \lambda_2 \wedge \eta$$

$$\lambda \wedge (\alpha \eta_1 + \beta \eta_2) = \alpha \lambda \wedge \eta_1 + \beta \lambda \wedge \eta_2 .$$

1. **Associative**: $(\lambda \wedge \eta) \wedge \phi = \lambda \wedge (\eta \wedge \phi)$

2 **Super - commutative (graded - commutative)**:

$$\lambda \wedge \eta = (-1)^{kl}\, \eta \wedge \lambda, \qquad \lambda \in \Lambda^k(V),\ \eta \in \Lambda^l(V)$$

3. $$\omega_I = \varphi_{i_1} \wedge \varphi_{i_2} \wedge \cdots \wedge \varphi_{i_k} \quad \text{if} \quad I \in \underline{n}_a^k .$$

ω_I is k-tensor $\Leftarrow$ 1-tensors

Pf: (<u>UNIQUENESS</u>: if ω and η are given, we can compute $\omega \wedge \eta$ using only 0 - 3).

1 - tensors

$$\varphi_i \wedge \varphi_j = \begin{cases} \omega_{(i\,j)}, & \text{if } i<j, \text{ then } (i,j) \in \underline{n}_a^k \\ -\varphi_j \wedge \varphi_i = -\omega_{(j\,i)}, & \text{if } j<i, \text{ then } (j,i) \in \underline{n}_a^k \\ \varphi_i \wedge \varphi_j = -\varphi_j \wedge \varphi_i \Rightarrow \varphi_i \wedge \varphi_j = 0, & \text{if } i=j \end{cases}$$

Now, suppose $I \in \underline{n}_a^k$ and $J \in \underline{n}_a^l$,

$$\omega_I \wedge \omega_J \overset{3}{=} (\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}) \wedge (\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l})$$

$$\overset{1}{=} \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l}$$

$$= \begin{cases} 0, & \text{if } I \cap J \neq \phi \\ \omega_{I \cup J}, & \text{if } I \cap J = \phi \end{cases}$$

union and sort and account for (-1)

E.g.

$(\varphi_{i_2} \wedge \varphi_{i_3}) \wedge (\varphi_{i_1} \wedge \varphi_{i_4}) = -\varphi_{i_2} \wedge \varphi_{i_1} \wedge \varphi_{i_3} \wedge \varphi_{i_4} = \varphi_{i_1} \wedge \varphi_{i_2} \wedge \varphi_{i_3} \wedge \varphi_{i_4}$

$\omega_{(2\,3)} \wedge \omega_{(1\,4)} = \omega_{(1\,2\,3\,4)} (-1)^2 = \omega_{(1\,2\,3\,4)}$

$(\varphi_{i_1} \wedge \varphi_{i_3}) \wedge (\varphi_{i_2} \wedge \varphi_{i_4}) = -\varphi_{i_1} \wedge \varphi_{i_2} \wedge \varphi_{i_3} \wedge \varphi_{i_4}$

$\omega_{(1\,3)} \wedge \omega_{(2\,4)} = \omega_{(1\,2\,3\,4)} (-1)^1 = -\omega_{(1\,2\,3\,4)}$

so, for $\omega = \sum a_I \omega_I$ and $\eta = \sum b_J \omega_J$,

$$\omega \wedge \eta = \left(\sum a_I \omega_I\right) \wedge \left(\sum b_J \omega_J\right) = \sum_{I,J} a_I b_J \, \omega_I \wedge \omega_J .$$

But, until now, "$\wedge$" is basis-dependent.... need existence; i.e., basis-independent formulas for $\lambda \wedge \eta$, where $\lambda \in \Lambda^k(V)$ and $\eta \in \Lambda^l(V)$.

so,

$$(\lambda \wedge \eta)(u_1, —, u_{k+l}) := \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)(\lambda \otimes \eta)\,\sigma^*(u_1, —, u_{k+l})$$

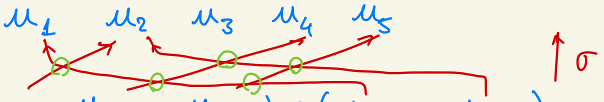
$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} \mathrm{sgn}(\sigma)\, \lambda(u_{\sigma(1)}, —, u_{\sigma(k)})\, \eta(u_{\sigma(k+1)}, —, u_{\sigma(k+l)})$$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma(1) < \cdots < \sigma(k) \\ \sigma(k+1) < \cdots < \sigma(k+l)}} \mathrm{sgn}(\sigma)\, \lambda(u_{\sigma(1)}, —, u_{\sigma(k)})\, \eta(u_{\sigma(k+1)}, —, u_{\sigma(k+l)})$$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma(1) < \cdots < \sigma(k) \\ \sigma(k+1) < \cdots < \sigma(k+l)}} \mathrm{sgn}(\sigma)\, (\lambda \otimes \eta)(u_{\sigma(1)}, —, u_{\sigma(k+l)}).$$

E.g.: $k = 3$; $\lambda \in \Lambda^3(V)$; $l = 2$; $\eta \in \Lambda^2(V)$.

$(\lambda \wedge \eta)(u_1, —, u_5) =$



$$\sum_{\sigma} (-1)^5 \lambda(u_{\sigma(1)}, u_{\sigma(2)}, u_{\sigma(3)}) \cdot \eta(u_{\sigma(4)}, u_{\sigma(5)})$$

$\sigma = [2 \ 4 \ 5 \ 1 \ 3]$

LECTURE 49: WEDGE PRODUCT

Recall: $\exists!\ (\lambda,\eta) \mapsto \lambda\wedge\eta,\ \Lambda^k\times\Lambda^l \to \Lambda^{k+l}$ s.t.

0. Bilinear; 1. Associative; 2. super-commutative $(\lambda\wedge\eta = (-1)^{kl}\eta\wedge\lambda)$

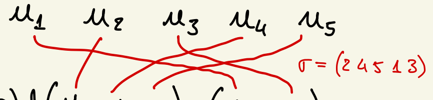
3. $\omega_I = \varphi_{i_1}\wedge\dots\wedge\varphi_{i_k}$.

Existence: Set

$$(\lambda\wedge\eta)(u_1,\text{—},u_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)(\lambda\otimes\eta)(\sigma^* u).$$

$$= \sum_{\substack{\sigma\in S_{k+l}\\ \sigma_1<\dots<\sigma(k)\\ \sigma(k+1)<\dots<\sigma(k+l)}} \mathrm{sgn}(\sigma)\,\lambda(u_{\sigma(1)},\text{—},u_{\sigma(k)})\cdot\eta(u_{\sigma(k+1)},\text{——},u_{\sigma(k+l)})$$

Ex: $k=3$ and $l=2$

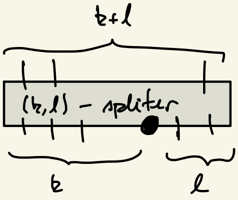


$$(\lambda\wedge\eta)(u_1,u_2,u_3,u_4,u_5) = \frac{1}{3!\,2!}\sum_{\sigma\in S_5}\mathrm{sgn}(\sigma)\,\lambda(u_2,u_4,u_5)\,\eta(u_1,u_3)$$

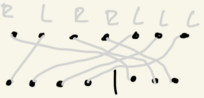
NTS: $\underbrace{\in\Lambda^{k+l}}_{\text{easy}}$; $\underbrace{0,}_{\text{trivial}}$ 1, 2, 3.

(same as "$\omega_I\in\Lambda^k$")

Pictorial sketch: (1-3)



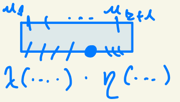
$$= \text{"}\sum_{\substack{\sigma\in S_{k+l}\\ \sigma(1)<\dots<\sigma(k)\\ \sigma(k+1)<\dots<\sigma(k+l)}}(-1)^\sigma\text{"}$$



In particular,

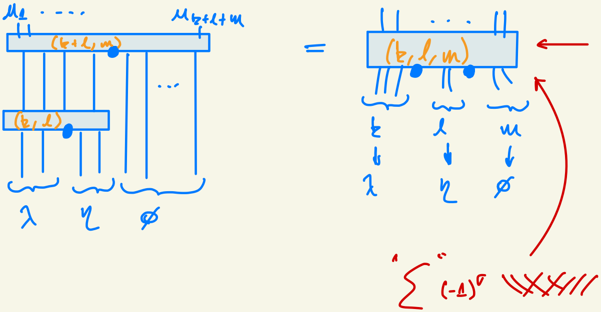
$$(\lambda\wedge\eta)(u_1,\text{—},u_{k+l})$$

$=$



1. **Associativity:** $(\overset{\in\Lambda^k}{\lambda}\wedge\overset{\in\Lambda^l}{\eta})\wedge\overset{\in\Lambda^m}{\phi} = \lambda\wedge(\eta\wedge\phi)$

LHS =



Order of "splittings" doesn't matter... so, associativity holds.

$$\text{“}\sum_{\substack{\sigma\in S_{k+l+m}\\ \sigma(1)<\cdots<\sigma(k)\\ \vdots}}\text{”} (-1)^{\sigma}$$

3. " $\omega_I = \varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k}$ ":

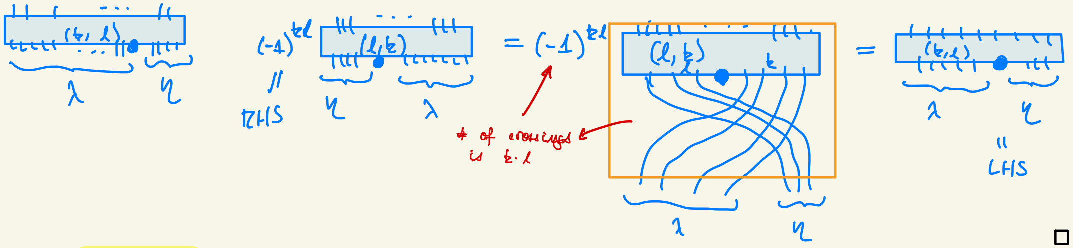
$$(\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k})(u_1,\_\_,u_k) \overset{\text{def}}{=} \frac{1}{k!}\sum_{\sigma\in S_k} \operatorname{sgn}(\sigma)(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k})(u_{\sigma 1},\_\_,u_{\sigma k})$$

$$= \sum_{\substack{\sigma\in S_k\\ \sigma(1)<\cdots<\sigma(k)}} \operatorname{sgn}(\sigma)(\varphi_{i_1}\otimes\cdots\otimes\varphi_{i_k})(u_{\sigma 1},\_\_,u_{\sigma k})$$

← "ascending"

$$\overset{\text{def}}{=} \omega_I \leftarrow I\in\underline{n}_2^k$$

2. " super-commutativity " $\lambda\wedge\eta = (-1)^{kl}\,\eta\wedge\lambda$



□

- **PULLBACKS**:

$$\begin{tikzcd} V \arrow[r, "L", "\text{linear}"'] & W \\ \Lambda^k(V) & \Lambda^k(W) \arrow[l, "L^*"'] \end{tikzcd}$$

Respects all structures

Ex: If $\dim V = n$, then $\underbrace{\Lambda^n(V)} = \Lambda^{\text{top}}(V) = \langle\varphi_1\wedge\cdots\wedge\varphi_n\rangle$

$\dim\Lambda^n(V) = 1$

Thm: If $L: V \to V$, then $L^*: \underbrace{\Lambda^{top}(V)}_{1-\text{dim}} \to \underbrace{\Lambda^{top}(V)}_{1-\text{dim}}$ is multiplication by $\det(L)$. Namely, if $\omega \in \Lambda^{top}$, then

$$(L^*\omega) = \det(L)\ \omega.$$

//

LECTURE 50: ORIENTATIONS

Recall: $\dim V = n$; $\underbrace{\Lambda^n(V)}_{1\text{-dim.}} = \Lambda^{top}(V) = \{\text{volume elements on } V\} \ni \omega$

Moreover, $L: V \to V \Rightarrow L^*\omega = \det(A)\,\omega$. (but could also be called $\det(L)$ b/c det is "basis invariant")

A represents L on a basis

Pf: WLOG, $\omega = \omega_I = \omega_{(1\,2\,\cdots\,n)} = \varphi_1 \wedge \cdots \wedge \varphi_n$, where $\{\varphi_i\}$ are the dual basis for the $\{v_i\}$, basis of V. (b/c all ω's are scalar multiples of the others since $\dim \Lambda^n(V) = 1$). With loss of generality, $n=3$ let $L: V \to V$ be represented by

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}. \rightarrow \text{Relative to } (v_i)$$

So,

$$L^*\omega_{(1\,2\,3)}(v_{(1\,2\,3)}) = L^*\omega_{(1\,2\,3)}(v_1, v_2, v_3)$$

$$\overset{\text{def}}{=} \omega_{(1\,2\,3)}(Lv_1, Lv_2, Lv_3)$$

$$\overset{\text{def}}{=} (\varphi_1 \wedge \varphi_2 \wedge \varphi_3)(Lv_1, Lv_2, Lv_3)$$

$$= (\varphi_1 \wedge \varphi_2 \wedge \varphi_3)(a_{11}v_1 + a_{21}v_2 + a_{31}v_3,\; a_{12}v_1 + a_{22}v_2 + a_{32}v_3,\; a_{13}v_1 + a_{23}v_2 + a_{33}v_3)$$

$$= \sum_{\sigma \in S_3} \operatorname{sgn}(\sigma) \prod_{i=1}^{3} a_{i\sigma(i)} = \det A$$

$$= \det A \; \omega_{(123)}(v_1, v_2, v_3).$$

$\square$

Def: (ORIENTATION) An orientation of a n-dimension vector space V is a choice $(v_1, \text{—}, v_n)$ of an ordered basis. Two such choices are considered the same orientation if the change of basis matrix between them has $\det > 0$.

1. $(v_1, \text{—}, v_n) \overset{A}{\sim} (v_1', \text{—}, v_n') \overset{B}{\sim} (v_1'', \text{—}, v_n'')$, then

   $\det A > 0$ same orientation $\det B > 0$

   $\det(AB) = \det A \det B > 0$

   $(v_1, \text{—}, v_n) \sim (v_1'', \text{—}, v_n'')$.

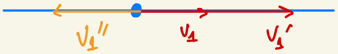
2. If $(v_1, \text{—}, v_n) \sim (v_1', \text{—}, v_n')$, then $(v_1', \text{—}, v_n') \sim (v_1, \text{—}, v_n)$

3. $(v_1, \text{—}, v_n) \overset{Id}{\sim} (v_1, \text{—}, v_n)$

ex: 3. Right (or left) - hands represent well-defined orientation in $\mathbb{R}^3$.

2.

1. Orientation of $\mathbb{R}^1$



$v_1 \sim v_1' \not\sim v_1''$

0. Later.

* Every finite-dimensional vector space have exactly 2 orientations:

$$\begin{tikzcd}(v_1, —, v_n) \arrow[rr, bend right] & \text{and} & (-v_1, v_2, —, v_n)\end{tikzcd}$$

$$\det \begin{pmatrix} -1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{pmatrix} = -1$$

Any other "choice" will be equivalent to one of these

Def 2: (ORIENTATION') An orientation' of V is a choice of $\omega \in \Lambda^n(V)$, $\omega \neq 0$. Moreover, $\omega_1 \sim \omega_2$ if $\omega_1 = \alpha \omega_2$ with $\alpha > 0$.

Thm: The two definitions of orientation are equivalent. Namely,

$$\{\text{orientations}\} \longleftrightarrow \{\text{orientations}'\}.$$

Ordered bases, det (COB) > 0 $\qquad$ $\Lambda^n(V)$; $\alpha > 0$

We say that o' $\omega \in \Lambda^{top}(V)$ agrees with $o = (v_1, —, v_n)$ if $\omega(v_1, —, v_n) > 0$.

- Orientations $V \xrightarrow{L} W$ push or pull?

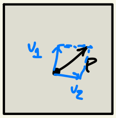
  Neither (in general) and pull (if L is invertible)

LECTURE 51: TANGENT VECTORS

- **TANGENT SPACES:** (purely linguistic)

For $p \in \mathbb{R}^n$,

$$\xi = (\underset{\mathbb{R}^n}{\underset{\cap}{p}}, \underset{\mathbb{R}^n}{\underset{\cap}{v}}) = \text{TANGENT VECTOR TO } \mathbb{R}^n \text{ at } p.$$



So, given $p \in \mathbb{R}^n$, $\{(p, v) : v \in \mathbb{R}^n\} = T_p \mathbb{R}^n = \mathbb{R}^n_p$.

↪ TANGENT SPACE AT p.

1. $T_p\mathbb{R}^n$ is a vector space.

$$\alpha(p, v_1) + \beta(p, v_2) = (p, \alpha v_1 + \beta v_2)$$

2. $T_p\mathbb{R}^n$ has an inner product:

$$\langle (p, v_1), (p, v_2) \rangle = \langle v_1, v_2 \rangle;$$

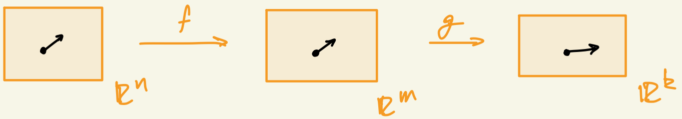
Induces $\|(p, v)\| = \|v\|$.

3. Push forward b/c points push... given $\xi = (p, v)$

$$\mathbb{R}^n \to \mathbb{R}^m, \quad \xi = (p, v) \xmapsto{f_*} f_* \xi = (f(p), f'(p)v)$$

Claims: 1. $f_*: T_p\mathbb{R}^n \to T_{f(p)}\mathbb{R}^n$ is linear

2.



$$(g\circ f)_*: T_p\mathbb{R}^n \longrightarrow T_{g(f(p))}\mathbb{R}^k$$

$\| ?$

$$g_*\circ f_*: T_p\mathbb{R}^n \longrightarrow T_{g(f(p))}\mathbb{R}^k$$

Pf: $\xi = (p, v)$.

$$(g\circ f)_*\xi \overset{\text{def}}{=} \left((g\circ f)(p), (g\circ f)'(p)v\right) = \left(g(f(p)), g'(f(p))f'(p)v\right)$$

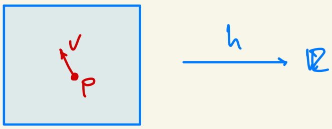
$$g_*\circ f_*\xi = \dots = \left(g(f(p)), g'(f(p))f'(p)v\right).$$

□

4. $T_p\mathbb{R}^n \Leftrightarrow$ "Directional derivatives at p"

$p = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$;

$v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$



$D_{(p,v)}h = D_\xi h =$ "The directional derivative of h in the direction ξ"

$$:= \frac{d}{dt} h(p+tv)\Big|_{t=0}$$

$$= \frac{d}{dt} h\left(\begin{pmatrix} x_1+tv_1 \\ \vdots \\ x_n+tv_n \end{pmatrix}\right)\Bigg|_{t=0}$$

$$= \frac{\partial h}{\partial x_1}(p)v_1 + \dots + \frac{\partial h}{\partial x_n}(p)v_n = \underbrace{h'(p)}_{1\times n \text{ matrix}}(v)$$

Claims: 1. This is "bilinear"

$$D_{\alpha\xi_1+\beta\xi_2}h = \alpha D_{\xi_1}h + \beta D_{\xi_2}h.$$

$$D_\xi(\alpha h_1+\beta h_2) = \alpha D_\xi h_1 + \beta D_\xi h_2.$$

$\xi = (p,v)$

2. Leibniz rule holds. $D_\xi(f\cdot g) = f(p)(D_\xi g) + (D_\xi f)\cdot g(p).$

Finally,

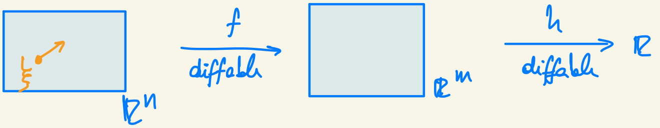
$$D_\xi h = D_{(p,v)} h = \Big(\underbrace{\sum v_i \frac{\partial}{\partial x_i}}_{=\xi} h\Big)\Big|_p$$

e.g. if $(p,(2,3)) \sim 2\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}$

$\sim 2\partial_x + 3\partial_y$

$\sim 2\partial_1 + 3\partial_2$

3. Pushing vectors is compatible with pulling functions.



$$\boxed{D_{f_*\xi} h = D_\xi(f^* h)}$$

$\xi \in T_p\mathbb{R}^n$, $h: \mathbb{R}^m \to \mathbb{R}$, $f: \mathbb{R}^n \to \mathbb{R}^m$.

Pf:

$$D_{f_*\xi} h = D_{(f(p),\, f'(p)v)} h = h'(f(p))(f'(p)v)$$

$//$ Chain Rule

$$D_\xi(f^* h) = D_{(p,v)}(h\circ f) = (h\circ f)'(p)v$$

—————— // ——————

Def: (VECTOR FIELD) A vector field on $\mathbb{R}^n$ is a function $F: \mathbb{R}^n \longrightarrow \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$ such that $F(p) \in T_p\mathbb{R}^n$.

Moreover,

$$F(p) = \left(p, \sum F^i(p) e_i\right),$$

where $F^i: \mathbb{R}^n \to \mathbb{R}$ are the "component functions of F".

Equivalently,

$$F(p) = \sum F^i(p)\frac{\partial}{\partial x_i}.$$

1. F is continuous differentiable $\Leftrightarrow$ $\forall i$ F^i is continuous differentiable

2. $(D_F h)(p) = D_{F(p)} h$.

LECTURE 52: FORMS

Recap: The tangent space at $p \in \mathbb{R}^n$

$$\mathbb{R}^n_p = T_p\mathbb{R}^n = \{(p, \overset{\xi}{v}) : v \in \mathbb{R}^n\} = \{v_p\}$$

tangent vector

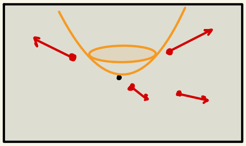
Directional derivatives $D_\xi f$ "bilinear", Leibniz.

- Vector field: $F : \mathbb{R}^n \to \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$ s.t. $\forall p$ $F(p) \in T_p\mathbb{R}^n$ can add, scale, inner multiplicity; can add:

$$F(p) = \sum_i F'(p)(p, e_i) = \sum_i F'(p)\partial_i .$$

Can also scale by real-valued functions $g : \mathbb{R}^n_p \to \mathbb{R}$, $g \cdot F$.

Ex: On $\mathbb{R}^2$



F_1

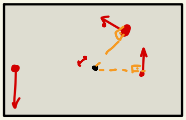
"Radial vector field"

At $p = (x, y)$, $v = (x, y) = x\binom{1}{0} + y\binom{0}{1}$

$$F_1 = x \cdot (p, e_1) + y\,(p, e_2) = x\,\partial_x + y\,\partial_y$$

Now, $D_{F_1}(x^2+y^2) = x\,\partial_x(x^2+y^2) + y\,\partial_y(x^2+y^2) = 2x^2 + 2y^2$.

Ex: On $\mathbb{R}^2$



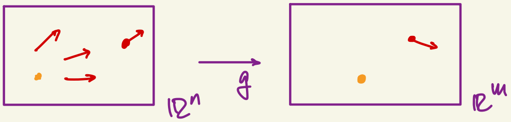
"Rotation by 90° counterclockwise by p"

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

So, $F_2 = -y\,\partial_x + x\,\partial_y$. Compute directional derivative in the direction of F_2 of x^2+y^2:

$$D_{F_2}(x^2+y^2) = -y\cdot 2x + x\cdot 2y = 0.$$

NOTE: Consider:



No natural way of "pushing" or "pulling".

* * *

Def: (k-Form) A <u>k-form</u> on $\mathbb{R}^n$ is a "function"

$$\omega: \mathbb{R}^n \longrightarrow \bigcup_{p\in\mathbb{R}^n} \Lambda^k(T_p\mathbb{R}^n)$$

such that

$$\omega(p) \in \Lambda^k(T_p\mathbb{R}^n).$$

Picks k vectors in $\mathbb{R}^n$ and assign a real number to these k vectors.

- But, in our case, $T_p\mathbb{R}^n \simeq \mathbb{R}^n$, so

$$\omega_I(p)\big((p,v_1),\ldots,(p,v_k)\big) = \omega_I(v_1,\ldots,v_k).$$

Now, given a k-form ω,

form a basis for $\Lambda^k(T_p\mathbb{R}^n)$

$$\omega(p) = \sum_{I\in \underline{n}_a^k} \lambda_I(p)\,\omega_I(p),$$

where for every $I\in\underline{n}_a^k$, $\underbrace{\lambda_I:\mathbb{R}^n\to\mathbb{R}}_{\text{COEFFICIENT FUNCTIONS}}$

Def: ω is continuous, differentiable, C^r, if $\forall I \in \underline{N}_a^k$, λ_I is continuous, differentiable, C^r.

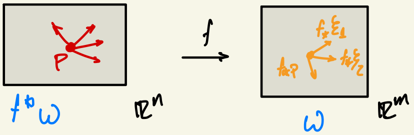
Def:

$$\Omega^k(\mathbb{R}^n) = \{\text{all } C^\infty \text{ } k\text{-forms on } \mathbb{R}^n\} = \text{"differentiable } k\text{-forms"}.$$

TECHNICALITIES: Add; multiply by scalar; multiply by a real-valued function; take

$$\begin{aligned} \wedge : \Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) &\longrightarrow \Omega^{k+l}(\mathbb{R}^n) \\ (\omega, \eta) &\longmapsto \omega \wedge \eta \end{aligned}$$

- Differential forms pull! Namely, $f: \mathbb{R}^n \to \mathbb{R}^n$ get $f^*: \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n)$ such that



$$(f^*\omega)(\overbrace{\xi_1, \text{—}, \xi_k}^{\xi_i \in T_p\mathbb{R}^n}) := \omega(f_*\xi_1, \text{—}, f_*\xi_k)$$

Thm: Pulling differential forms is compatible with everything. Namely,

1. $f^*(\omega_1 + \omega_2) = f^*(\omega_1) + f^*(\omega_2)$;
2. $f^*(7\omega) = 7f^*(\omega)$; $7 \in \mathbb{R}$
3. If $g: \mathbb{R}^m \to \mathbb{R}$, $f^*(g\omega) = f^*(g) \cdot f^*(\omega)$

④ $f^*(\omega\wedge\eta) = f^*(\omega)\wedge f^*(\eta)$.

⑤ Contra-variance. $(h\circ f)^*\omega = f^*(h^*\omega)$.

LECTURE 53: DIFFERENTIAL FORMS

Recap: $\Omega^k(\mathbb{R}^n) = \{\omega : T_p\mathbb{R}^n \to \bigcup_{p\in\mathbb{R}^n} \Lambda^k(T_p\mathbb{R}^n) : \omega(p)\in\Lambda^k(T_p\mathbb{R}^n)\}$
w/ smooth coeffs.

If $\lambda\in\Omega^k(\mathbb{R}^n)$,

$$\lambda(p) = \sum_{I\in\mathbb{N}_a^k} \lambda_I(p)\,\underbrace{\omega_I(p)}_{}. \quad \nearrow (\varphi_{i_1}\wedge\cdots\wedge\varphi_{i_k})(p)$$

Also, $\Omega^0(\mathbb{R}^n) = \{C^\infty$ functions $\mathbb{R}^n\to\mathbb{R}\}$; has $+$, $\cdot$, $\wedge$. If $g:\mathbb{R}^n\to\mathbb{R}^m$, $g^*:\Omega^k(\mathbb{R}^m)\to\Omega^k(\mathbb{R}^n)$ by $(g^*\omega)(\cancel{p})(\xi_1,—,\xi_k) := \omega(\cancel{g*p})(g_*\xi_1,—,g_*\xi_k)$; compatible w/ $+$, $\cdot$, $\wedge$ and contravariant. sometimes omit

Finally, if $f\in\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$ and $\omega\in\Omega^k(\mathbb{R}^n)$, then $f\wedge\omega\in\Omega^k(\mathbb{R}^n)$, so, write $f\wedge\omega = f\cdot\omega$.

Pf: (4) WTS: if $\omega\in\Omega^k(\mathbb{R}^m)$, $\eta\in\Omega^l(\mathbb{R}^m)$ and $g:\mathbb{R}^n\to\mathbb{R}^m$ smooth, then $g^*(\omega\wedge\eta) = (g^*\omega)\wedge(g^*\eta)$.

Pf: $g^*(\omega\wedge\eta)(\xi_1,—,\xi_{k+l})$ $\quad \xi_i\in T_p\mathbb{R}^n$

$$= (\omega\wedge\eta)(g_*\xi_1,—,g_*\xi_{k+l})$$

$$= \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)\,(\omega\otimes\eta)(g_*\xi_{\sigma 1},—,g_*\xi_{\sigma(k+l)})$$

$$= \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)\,\omega(g_*\xi_{\sigma 1},—,\xi_{\sigma(k)})\,\eta(g_*\xi_{\sigma(k+1)},—,g_*\xi_{\sigma(k+l)})$$

On the other hand,

$$(g^*\omega)\wedge(g^*\eta)(\xi_1, \text{—}, \xi_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} \mathrm{sgn}(\sigma)(g^*\omega)(\xi_{\sigma 1}, \text{—}, \xi_{\sigma k})$$
$$(g^*\eta)(\xi_{\sigma(k+1)}, \text{—}, \xi_{\sigma(k+l)}).$$

□

- There exists an operation $d : \underbrace{\Omega^0(\mathbb{R}^n)}_{\text{functions}} \to \underbrace{\Omega^1(\mathbb{R}^n)}_{\text{"machines that eat 1 tangent vector and spit out a number"}}$

Def:

$$(df)(\overset{\overset{(p,v)\in T_p\mathbb{R}^n}{\shortparallel}}{\xi}) := D_\xi f = \sum_{i=1}^n v_i \frac{\partial f}{\partial x_i} \overset{!}{=} \sum_{\boxed{I\in\underline{n}_a^1}} \lambda_I(p)\,\omega_I(v)$$

$\to$ ascending w/ length 1.

$$= \sum_i \lambda_i(p)\,\varphi_i(v) = \sum_i \lambda_i(p)\, v_i$$

$$\Rightarrow \boxed{\lambda_i(p) = \frac{\partial f}{\partial x_i}}.$$

So,

$$df = \sum_i \frac{\partial f}{\partial x_i}\cdot\varphi_i = \sum_i \frac{\partial f}{\partial x_i}\,\omega_{(i)}.$$

In particular, if $f = x_j = \pi_j$, then

$$df = dx_j = \sum_i \frac{\partial x_j}{\partial x_i}\,\omega_{(i)} = \omega_{(j)}.$$

Convention: Never write $\omega_{(j)}$! Instead, write dx_j.

Thus,

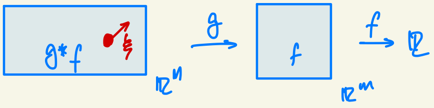
$$\boxed{df = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\, dx_i}.$$

Now,

1. d is linear; i.e., $d(\alpha f+\beta g) = \alpha df + \beta dg$
2. Leibniz Rule: $d(f\cdot g) = (df)\cdot g + f(dg)$
3. Compatible with pullbacks; i.e.,

$$d(g^*f) = g^*(df).$$

Pf 3:

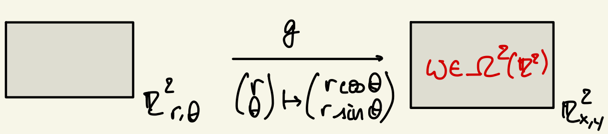


$$d(g^*f)(\xi) = D_\xi(g^*f)$$

$\|$ already shown.

$$g^*(df)(\xi) = (df)(g_*\xi) = D_{g_*\xi} f$$

Ex:



$$\begin{aligned} \omega &= dx\wedge dy \\ &= \varphi_1\wedge\varphi_2 \\ &= \omega_{12} \end{aligned}$$

Compute $g^*(dx\wedge dy)$.

$$\begin{aligned} g^*(dx\wedge dy) &= (g^*dx)\wedge(g^*dy) \\ &= (d(g^*x))\wedge(d(g^*y)) \\ &= d(r\cos\theta)\wedge d(r\sin\theta) \\ &= \left(\frac{\partial}{\partial r}(r\cos\theta)\,dr + \frac{\partial}{\partial\theta}(r\cos\theta)\,d\theta\right)\wedge\left(\frac{\partial}{\partial r}(r\sin\theta)\,dr + \frac{\partial}{\partial\theta}(r\sin\theta)\,d\theta\right) \\ &= (\cos\theta\,dr - r\sin\theta\,d\theta)\wedge(\sin\theta\,dr + r\cos\theta\,d\theta) \\ &= \cancel{\cos\theta\sin\theta\,dr\wedge dr} + r\cos^2\theta\,dr\wedge d\theta - r\sin^2\theta\,d\theta\wedge dr \\ &\qquad \cancel{- r^2\sin\theta\cos\theta\,d\theta\wedge d\theta} \\ &= r\cos^2\theta\,dr\wedge d\theta + r\sin^2\theta\,dr\wedge d\theta \\ &= r(\cos^2\theta+\sin^2\theta)\,dr\wedge d\theta = \boxed{r\,dr\wedge d\theta.} \end{aligned}$$

REMARK:

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, "g"] & \mathbb{R}^n \\
g^* \omega_I = \det g' \cdot \omega_I & \omega_I \in \Lambda^n(\mathbb{R}^n) \arrow[l] \\
r\, dr \wedge d\theta & dx \wedge dy \arrow[l]
\end{tikzcd}$$

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LECTURE 54: EXTERIOR DERIVATIVE

Recall: $d: \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ by $(df)(\xi) := D_\xi f$.

$dx_i = \varphi_i = \omega_{(i)}$, $\quad dx_I := dx_{i_1} \wedge \cdots \wedge dx_{i_k} = \omega_I$. So,

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} \wedge dx_i .$$

Def: (EXTERIOR DERIVATIVE) Define $d: \Omega^k(\mathbb{R}^n) \longrightarrow \Omega^{k+1}(\mathbb{R}^n)$ by

$$d\omega := \sum_{i=1}^n dx_i \wedge \frac{\partial \omega}{\partial x_i} .$$

If $\omega = \sum_{I \in \underline{n}_a^k} \lambda_I \, dx_I$, then $\frac{\partial \omega}{\partial x_i} = \sum_{I \in \underline{n}_a^k} \frac{\partial \lambda_I}{\partial x_i} \cdot dx_I$

Ex: Let $\theta = \frac{y}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy \in \underbrace{\Omega^1(\mathbb{R}^2_{x,y} \setminus \{0\})}$. $\Omega^k(A)$ makes sense for any open $A \subset \mathbb{R}^n$.

Compute $d\theta$.

$dx \wedge dx = 0$

$$d\theta = dx \wedge \frac{\partial \theta}{\partial x} + dy \wedge \frac{\partial \theta}{\partial y} = dx \wedge \left(\cancel{\cdots dx} - \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} dy \right)$$

$dy \wedge dy = 0$

$$+ dy \wedge \left(\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} dx - \cancel{\cdots dy} \right)$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}(dx \wedge dy) + \frac{x^2-y^2}{(x^2+y^2)^2}(dy \wedge dx)$$

$$= \frac{x^2-y^2}{(x^2+y^2)^2}(dx \wedge dy) - \frac{x^2-y^2}{(x^2+y^2)^2}(dx \wedge dy) = 0.$$

Obs: $d(f\,dx + g\,dy) = (-f_y + g_x)\,dx \wedge dy$

Thm: d has the following properties

1. **d is linear:** $d(\alpha\omega + \beta\lambda) = \alpha\, d\omega + \beta\, d\lambda$

   if $\omega \in \Omega^k(\mathbb{R}^n)$

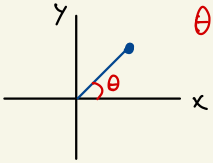
2. **Leibniz Rule:** $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge (d\eta)$

3. **"$d^2 = 0$":** Really $d_k : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ so,

$$\begin{tikzcd} \Omega^k(\mathbb{R}^n) \arrow[r, "d_k"] \arrow[rr, bend right, "d_{k+1} \circ d_k = 0"'] & \Omega^{k+1}(\mathbb{R}^n) \arrow[r, "d_{k+1}"] & \Omega^{k+2}(\mathbb{R}^n) \end{tikzcd}$$

4. **Compatible with pullbacks:** $g^*(d\omega) = d(g^*\omega)$.

Ex: Compute $d\theta$:



$\overset{\text{0-form}}{\theta} = \arctan\left(\frac{y}{x}\right)$ on $\mathbb{R}^2 \setminus [x=0]$

$$\overset{\text{1-form}}{d\theta} = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy$$

$$= \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

$$= -\theta^{old} \quad \text{(previous page)}$$

So, $d^2\theta = d(d\theta) = d(-\theta^{old}) = -d\theta^{old} = 0.$

Remark: "$d^2 = 0$" $\Leftrightarrow$ $\mathrm{im}\, d_k \subset \ker d_{k+1}$

$$\Leftrightarrow \underbrace{\mathrm{im}\, d}_{\text{EXACT FORMS}} \subset \underbrace{\ker d}_{\text{CLOSED FORMS}}$$

EXACT FORMS

ω is exact: $\exists \lambda : \omega = d\lambda$

CLOSED FORMS: ω is closed: $d\omega = 0$

$$\boxed{\text{EXACT FORMS ARE CLOSED}}$$

LECTURE 55: EXTERIOR DERIVATIVE (continued)

$d: \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ by $(df)(\xi) = D_\xi f$.

$d: \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

1. Linear
2. $d^2 = 0$

(exact forms are closed)

3. Leibniz Rule: $\omega \in \Omega^k(\mathbb{R}^n)$

$$d(\omega \wedge \eta) = (d\omega) \wedge \eta + (-1)^k \omega \wedge d\eta$$

4. $d(g^* \omega) = g^*(d\omega)$.

Pf: (3)

$$d(\omega \wedge \eta) = \sum_{i=1}^{n} dx_i \wedge \frac{\partial(\omega \wedge \eta)}{\partial x_i}$$

$$\overset{*}{=} \sum_{i=1}^{n} dx_i \wedge \left(\frac{\partial \omega}{\partial x_i} \wedge \eta + \omega \wedge \frac{\partial \eta}{\partial x_i} \right)$$

$$= \sum_{i=1}^{n} \left(dx_i \wedge \frac{\partial \omega}{\partial x_i} \right) \wedge \eta + (-1)^{\deg \omega} \omega \wedge \sum_{i=1}^{n} \left(dx_i \wedge \frac{\partial \eta}{\partial x_i} \right)$$

$$= (d\omega) \wedge \eta + (-1)^{\deg \omega} \omega \wedge (d\eta).$$

□

Claim:

$$\frac{\partial(\omega \wedge \eta)}{\partial x_i} = \frac{\partial \omega}{\partial x_i} \wedge \eta + \omega \wedge \frac{\partial \eta}{\partial x_i}$$

(2) $d(d\omega) = d\left(\sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}\right) = \sum_{i=1}^{n} d\left(dx_i \wedge \frac{\partial \omega}{\partial x_i}\right)$

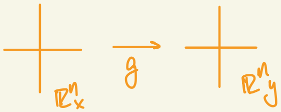
$d(dx_i) = 0$ since dx_i has constant coeffs.

$$= \sum_{i=1}^{n} \cancel{d(dx_i) \wedge \frac{\partial \omega}{\partial x_i}} - dx_i \wedge d\left(\frac{\partial \omega}{\partial x_i}\right)$$

$$= - \sum_{i=1}^{n} dx_i \wedge \left(\sum_{j=1}^{n} dx_j \wedge \frac{\partial^2 \omega}{\partial x_i \partial x_j}\right)$$

$$= - \underbrace{\sum_{i,j} \underbrace{dx_i \wedge dx_j}_{\text{anti-symmetric under } i \leftrightarrow j} \wedge \underbrace{\frac{\partial^2 \omega}{\partial x_i \partial x_j}}_{\text{symmetric under } i \leftrightarrow j}}_{\text{anti-symmetric under } i \leftrightarrow j} = 0.$$

(4) WLOG, $\omega = f \cdot dy_I = f \wedge dy_I$ (by linearity). Now,



$$g^*(d\omega) = g^*(df \wedge dy_I + f \wedge d(dy_I))$$

$$= g^*(df \wedge dy_I)$$

$$= g^*(df) \wedge g^*(dy_I)$$

$$= d(g^*f) \wedge (dg^*y_{i_1}) \wedge \cdots \wedge (dg^*y_{i_k})$$

Recall:

$dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_k}$.

So,

$d(dy_I) = \sum dy_{i_1} \wedge \cdots \wedge \overset{=0}{d(dy_{i_\alpha})} \wedge \cdots \wedge dy_{i_k} = 0$

On the other hand,

$$d(g^*\omega) = d(g^*(f \wedge dy_I)) = d((g^*f) \wedge dg^*y_{i_1} \wedge \cdots \wedge dg^*y_{i_k})$$

$$\overset{\text{Leibniz}}{=} (dg^*f) \wedge dg^*y_{i_1} \wedge \cdots \wedge dg^*y_{i_k}.$$

□

GEOMETRICAL INTUITION: if $\omega \in \Omega^k(\mathbb{R}^n)$, then

$$(d\omega)\underbrace{(\xi_1, \text{—}, \xi_{k+1})}_{\text{define a parallelogram } P(\xi_1,\text{—},\xi_{k+1})} = \sum_{i=1}^{k+1} \pm\, \omega_p(\xi_1, \text{—}, \hat{\xi}_i, \text{—}\, \xi_{k+1}) \pm \omega_{p+v_j}(\xi_1, \text{—}, \hat{\xi}_i, \text{—}, \xi_{+1})$$

$$" \int_{P(\xi_1,\text{—},\xi_{k+1})} d\omega = \int_{\partial P} \omega =$$

LECTURE 56: CHAINS

Thm: With $\xi_i = (p, v_i)$, $\omega \in \Omega^k(\mathbb{R}^n)$,

$$(d\omega)(\xi_1, \text{—}, \xi_{k+1})$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big(\omega(p+\varepsilon v_i)(\varepsilon v_1, \text{—}, \varepsilon \hat{v}_i, \text{—}, \varepsilon v_{k+1})$$

$$- \omega(p)(\varepsilon v_1, \text{—}, \varepsilon \hat{v}_i, \text{—}, \varepsilon v_{k+1}) \Big)$$

Pf: WLOG, $\omega = f \cdot \lambda$ where λ has constant coefficients. So,

$$d\omega = d(f \wedge \lambda) = (df) \wedge \lambda + f \wedge \cancelto{0}{(d\lambda)} = df \wedge \lambda.$$

So,

$$(d\omega)(\xi_1, \text{—}, \xi_{k+1}) = (df \wedge \lambda)(\xi_1, \text{—}, \xi_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1}\, df(\xi_i)\cdot \lambda(\xi_1, \_, \hat{\xi}_i, \_, \xi_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1}\, (D_{\xi_i} f)\cdot \lambda(\xi_1, \_, \hat{\xi}_i, \_, \xi_{k+1})$$

On the other hand,

$$\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big[f(p+\varepsilon v_i)\cdot \lambda(\varepsilon v_1, \_, \widehat{\varepsilon v_i}, \_, \varepsilon v_{k+1}) - f(p)\cdot \lambda(\varepsilon v_1, \_, \widehat{\varepsilon v_i}, \_, \varepsilon v_{k+1}) \Big]$$

λ multilinear
↓

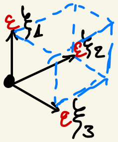
$$= \underline{\lim_{\varepsilon\to 0}} \frac{1}{\varepsilon} \sum_{i=1}^{k+1} (-1)^{i-1} \Big[\underbrace{f(p+\varepsilon v_i) - f(p)}_{D_{(p,v_i)} f} \Big] \lambda(v_1, \_, \hat{v}_i, \_, v_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1} (D_{\xi_i} f)\, \underbrace{\lambda(v_1, \_, \hat{v}_i, \_, v_{k+1})}_{\text{Just omit the base point on notation}} .$$

□

GEOMETRICAL INTERPRETATION:

suppose $k=2$, so



Parallelogram $P(\xi_1, \_, \xi_{k+1})$

Write

$$(d\omega)(\xi_1, \_, \xi_{k+1}) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^{k+1}} (d\omega)(\varepsilon\xi_1, \_, \varepsilon\xi_{k+1}).$$

$\overset{\text{(Thm)}}{\Longrightarrow}$ LHS $\sim$ RHS for small ε.

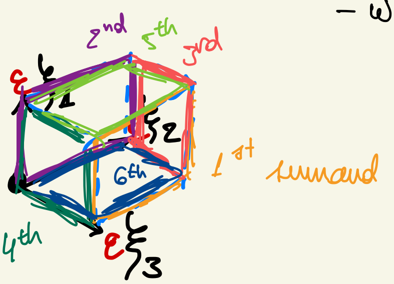
So,

$$\frac{1}{\varepsilon^{k+1}} (d\omega)(\varepsilon \xi_1, \ldots, \varepsilon \xi_{k+1}) = \int_{P(\varepsilon\xi_1, \ldots, \varepsilon\xi_{k+1})} d\omega$$

Also,

$$\frac{1}{\varepsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big(\omega(p+\varepsilon v_i)(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{k+1}) - \omega(p)(\xi_1, \ldots, \hat{\xi}_i, \ldots, \xi_{k+1}) \Big)$$

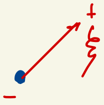
$=$



$$= \int_{\partial P(\varepsilon\xi_1, \ldots, \varepsilon\xi_{k+1})} \omega = \int_{P(\varepsilon\xi_1, \ldots, \varepsilon\xi_{k+1})} d\omega$$

Now,

$(df)(\xi) =$

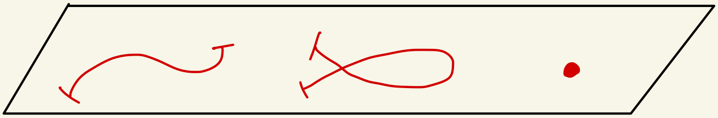


$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \big(f(p+\varepsilon v) - f(p) \big)$$

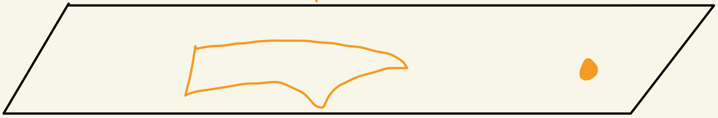
Thus, the external derivative on forms is a generalization of the directional derivative / differential on functions.

Def: (CUBE) A *singular k-cube* in $A_{\text{open}} \subset \mathbb{R}^n$ is a continuous function $c : I^k = [0,1]^k \to \mathbb{R}^n$.

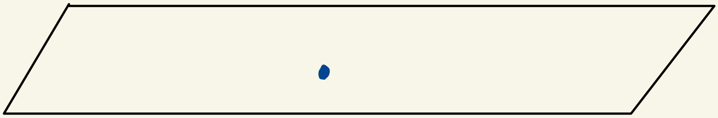
1 - cube (path)



2 - cube ("square")



0 - cube ($I^0 =$ point)



Def: ($C_k(A)$) Define

$$C_k(A) := \begin{pmatrix} \text{space of} \\ k\text{-chains} \\ \text{in } A \subset \mathbb{R}^n \end{pmatrix} = \begin{pmatrix} \text{The free Abelian group} \\ \text{generated by all } k\text{-cubes} \\ \text{in } A \end{pmatrix}.$$

LECTURE 57: CHAINS

Recall:

i) $d\omega(\varepsilon\xi_1, \_\_, \varepsilon\xi_{k+1})$

$$\xrightarrow{\varepsilon \to 0} \sim \sum_{i=1}^{k+1} (-1)^{i-1}\Big[\omega(p+\varepsilon v_i)(\varepsilon v_1, \_\_, \widehat{\varepsilon v_i}, \_\_, \varepsilon v_{k+1}) - \omega(p)(\varepsilon v_1, \_\_, \widehat{\varepsilon v_i}, \_\_, \varepsilon v_{k+1})\Big]$$

ii) A singular k-cube in $A \subset \mathbb{R}^n$ is a continuous $c: I^k \to A$

$I^k = [0,1]^k$

$$C_k(A) := \begin{pmatrix}\text{The space of}\\ k\text{-chains in } A\end{pmatrix} = \begin{pmatrix}\text{Free Abelian group}\\ \text{generated by all } k\text{-cubes}\end{pmatrix}$$

$$= \left\{ \sum_{i=1}^{p} \alpha_i c_i \;:\; \begin{array}{l}\alpha_i \in \mathbb{Z},\\ c_i : I^k \to A\end{array} \right\}$$

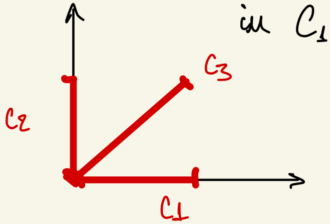
1. Order is immaterial
2. Can drop/ignore $0 \cdot c_i$ from any sum
3. Can merge equal cubes

Remarks:

- Can add: $C_k(A)$ has "+"
- Can multiply by integers.
- It has a zero. (all coeffs. are zero)

A "vector space" over $\mathbb{Z}$

Ex:



in $C_1(\mathbb{R}^2)$. Now, $c_i : [0,1]_t \to \mathbb{R}^2$

$$\begin{cases} c_1 = (t,0) \\ c_2 = (0,t) \\ c_3 = (t,t) \end{cases}$$

Compute $c_1 + c_2 - c_3$.

So, $c_1 + c_2 - c_3 = (t,0) + (0,t) - (t,t)$ NOT $= 0$!

Def: Define for $j \in \underline{k}$ and $\alpha \in \{0,1\}$

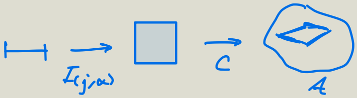
$$I^k_{(j,\alpha)} \in C_{k-1}(I^k),$$

i.e., $I^k_{(j,\alpha)} : I^{k-1}_{y_1,\dots,y_{k-1}} \longrightarrow I^k$ "chooses a specific face of the cube

$$I^k_{(j,\alpha)}(y_1, —, y_{k-1}) := (y_1, —, y_{j-1}, \overbrace{\alpha}^{\text{jth place}}, y_j, y_{j+1}, —, y_{k-1})$$

If $c : I^k \to A$, set

$$c_{(j,\alpha)} := c \circ I^k_{(j,\alpha)}.$$



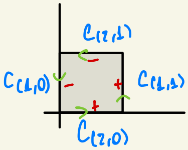
Def:

$$\partial c := \sum_{\substack{j \in \underline{k} \\ \alpha \in \{0,1\}}} (-1)^{j+\alpha} c_{(j,\alpha)} \in C_{k-1}(A).$$

Def: (BOUNDARY OPERATION) Define $\partial : C_k(A) \to C_{k-1}(A)$ by extending by linearity to all of $C_k(A)$:

$$\partial\left(\sum_i \alpha_i c_i\right) = \sum_i \alpha_i \partial c_i$$

Ex: Let $c \in C_2(\mathbb{R}^2)$ given by $c(\overset{\overset{I^2}{\cup}}{x,y}) = (x,y)$



$$\partial c = -c_{(1,0)} + c_{(1,1)} + c_{(2,0)} - c_{(2,1)}$$
$$= -(0,t) + (1,t) + (t,0) - (t,1)$$

Now,

$$C_0(\mathbb{R}^2) \ni \partial(\partial c) = -\partial(0,t) + \partial(1,t) + \partial(t,0) - \partial(t,1)$$
$$= -\left(\cancel{(0,1)} - \cancel{(0,0)}\right) + \left(\cancel{(1,1)} - \cancel{(1,0)}\right)$$
$$+ \left(\cancel{(1,0)} - \cancel{(0,0)}\right) - \left(\cancel{(1,1)} - \cancel{(0,1)}\right)$$
$$= 0.$$

Thm: $(\partial^2 = 0.)$

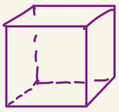
$$\begin{tikzcd} C_k(A) \arrow[r, "\partial"] \arrow[rr, bend right, "0"'] & C_{k-1}(A) \arrow[r, "\partial"] & C_{k-2}(A) \end{tikzcd}$$

LECTURE 58: INTEGRATION ON CHAINS

Recap: $C_k^{1}(A) = \left(\frac{\text{cont. diffable } k\text{-chains}}{\text{in } A \subset \mathbb{R}^n}\right)$

$$= \left\{ \sum \alpha_i c_i : \begin{array}{l} \alpha_i \in \mathbb{Z} \\ c_i : I^k \to A \\ \text{cont. diffable} \end{array} \right\} \Big/ \sim$$

Equivalences between elements of the generating group.



$I^k_{(j,\alpha)} : I^{k-1}_{y_1,\dots,y_{k-1}} \longrightarrow I^k$ by

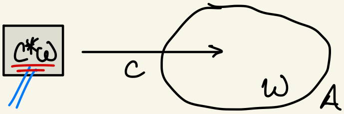
$$(y_1,\dots,y_{k-1}) \mapsto (y_1,\dots,y_{j-1},\overset{\alpha\in\{0,1\}}{\alpha},y_j,y_{j+1},\dots,y_{k-1})$$

$$\partial c = \sum_{j=1}^{k}\sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha}\underbrace{c\circ I^k_{(j,\alpha)}}_{\text{" faces "}}$$ ⇒ Extends to chains?

"$\partial^2 = 0$".

- **INTEGRATION ON CHAINS**: c is a k-chain and $\omega\in\Omega^k(A)$

$$\int_c \omega$$



$f\,(dx_1\wedge\cdots\wedge dx_n)$

Pullback of a k-form to a k-dim. space.
⇒ ω is a multiple of the volume form

Def:

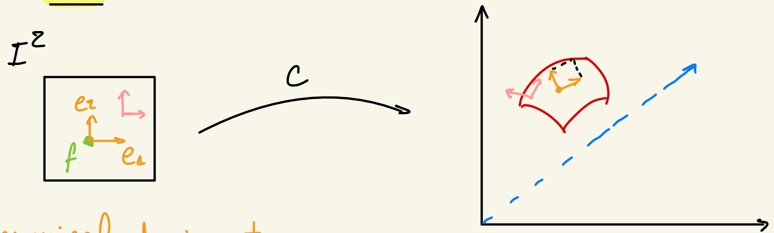
1. $$\int_{I^k_{x_1,\dots,x_k}} f\,dx_1\wedge\cdots\wedge dx_k := \int_{I^k} f$$

2. For a k-chain $c: I^k\to A$ and a k-form $\omega\in\Omega^k(A)$

$$\int_c \omega := \int_{I^k} c^*\omega$$!

3. $$\int_{\sum \alpha_i c_i}\omega := \sum \alpha_i\int_{c_i}\omega$$

Obs:



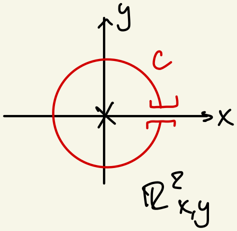
Canonical basis at that tangent space

$\omega \in \Omega^2(\mathbb{R}^3)$

$$\int_c \omega = \int_{I^2} c^* \omega$$

EXAMPLE 0: $\int_{I^k} f \; dx_1 \wedge \cdots \wedge dx_k$ (just old style integration)

EXAMPLE 1: $k = 1$, $n = 2$, $A = \mathbb{R}^2 \setminus \{0\}$



$$\omega := \frac{-y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$$

$c \in C_1^1(A)$, $c : I_t^1 = [0,1]_t \longrightarrow \mathbb{R}^2_{x,y} \setminus \{0\}$

$$c(t) := \begin{pmatrix} \cos 2\pi t \\ \sin 2\pi t \end{pmatrix}$$

Now,

$$\int_c \omega := \int_{I^1 = [0,1]} c^* \omega$$

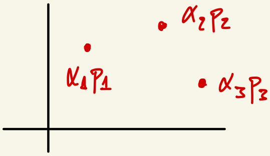
$$= \int_{[0,1]} -\frac{\sin 2\pi t \cdot d(\cos 2\pi t)}{1} + \frac{\cos 2\pi t \cdot d(\sin 2\pi t)}{1}$$

$$= \int_{[0,1]} 2\pi \sin^2(2\pi t)\, dt + 2\pi \cos^2(2\pi t)\, dt$$

$$= 2\pi \int_{[0,1]} dt = 2\pi \int_{[0,1]} 1 = \boxed{2\pi}.$$

EXAMPLE 0: $\underline{k=0}$, $n > 0$

$\hookrightarrow$ 0-chain is a lin. combination of points

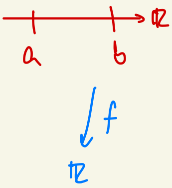


A 0-form is a function $\omega = \underset{\in \Omega^0}{f} \overset{\text{"zero } dx\text{'s"}}{\underbrace{\quad}}$

so,

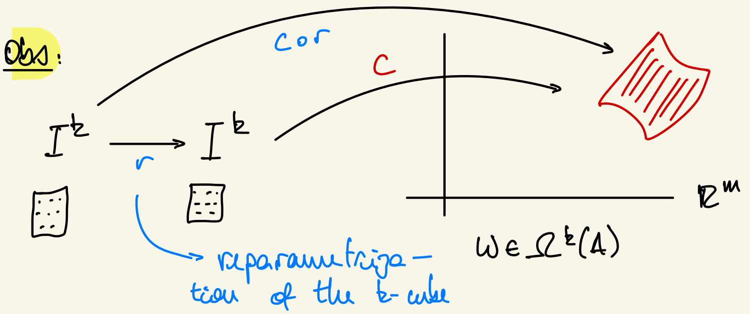
$$\int_{\sum_i \alpha_i P_i} f = \sum_i \alpha_i f(P_i).$$

In particular, in the 157 way...



$$\int_{\underbrace{b+(-1)a}_{\text{as chains}}} f := f(b) - f(a).$$

Obs:



Prop 2: If $r: I^k_{y_1,\dots,y_k} \to I^k_{x_1,\dots,x_k}$ is a reparametrization which is continuously differentiable, 1-1, onto and orientation preserving; i.e., $\det r' > 0$, then

$$\int_c \omega = \int_{c\circ r} \omega$$

Pf 2:

$$\int_{c\circ r} \omega = \int_{I^k} (c\circ r)^* \omega = \int_{I^k} r^*(c^*\omega)$$

Write

$c^*\omega = f\, dx_1 \wedge \cdots \wedge dx_k$

$$= \int_{I^k} r^*(f\, dx_1 \wedge \cdots \wedge dx_k)$$

$$= \int_{I^k} (f\circ r) \cdot r^*(dx_1 \wedge \cdots \wedge dx_k)$$

$$= \int_{I^k} (f\circ r)\, |\det r'|\, dy_1 \wedge \cdots \wedge dy_k$$

$$\overset{\text{cov}}{=} \int_{r(I^k) = I^k} f\, dx_1 \cdots dx_k$$

$$= \int_{I^k} c^* \omega \overset{\text{def}}{=} \int_c \omega .$$

//

LECTURE 59: GEOMETRICAL ASIDE

$$\begin{tikzcd}
\Omega^0(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, leftrightarrow, "Id"'] & \Omega^1(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, leftrightarrow, "\flat"'] & \Omega^2(\mathbb{R}^3) \arrow[r, "d"] \arrow[d, leftrightarrow, "\beta"'] & \Omega^3(\mathbb{R}^3) \arrow[d, leftrightarrow, "*"'] \\
\{f\} \arrow[r, "\text{grad}"] & \left\{\begin{matrix} F_1\, dx_1 \\ + F_2\, dx_2 \\ + F_3\, dx_3 \end{matrix}\right\} \arrow[r, "\text{curl}"] & \left\{\begin{matrix} G_1\, dx_2 \wedge dx_3 \\ + G_2\, dx_3 \wedge dx_1 \\ + G_3\, dx_2 \wedge dx_1 \end{matrix}\right\} \arrow[r, "\text{div}"] & \{ g\, dx \wedge dy \wedge dz \} \\
\text{functions} & \text{vector fields} & \text{vector fields} & \text{functions}
\end{tikzcd}$$

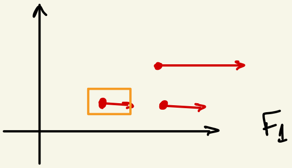
$$\int_{\partial M} \cdot = \int_{3D \cdot \dot{M}}$$

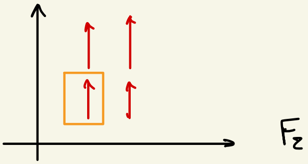
$\text{div}\, G = \partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3$ (any dim.)

$$\text{curl}\, G = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$ (only makes sense in 3-dim).

$\text{grad}\, f = (\partial_1 f, \partial_2 f, \partial_3 f)$ (any dim.)

Consider $\vec{F} = (F_1, F_2)$. Then $\text{div}\, F = \partial_1 F_1 + \partial_2 F_2$.



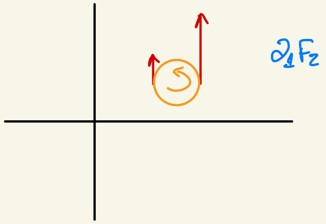


* Overall, if F measures the flow of water, then div measures how much more water flows out of the box than in.

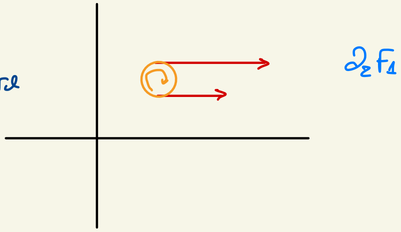
* Now,

$$\operatorname{curl} F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_3 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}.$$

So, $(\operatorname{curl} F)_3 = \underbrace{\partial_1 F_2 - \partial_2 F_1}$ (makes sense in 2-dim)



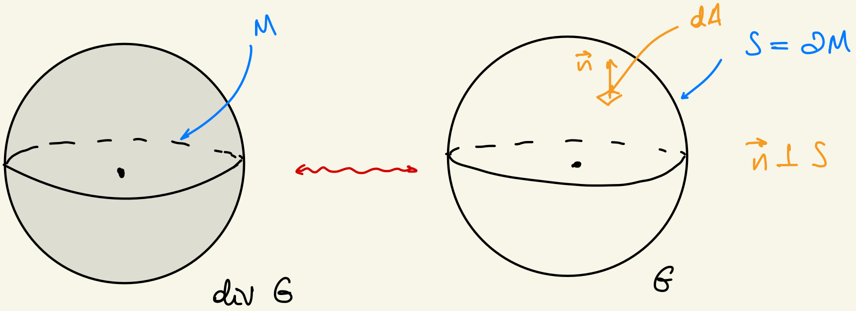
Throw a cork disk and measure how much it wants to spin counterclockwise



So, curl F is a vector specifying the axis of rotation and the speed of rotation around it.

└→ Angular velocity vector induced by a field where the "cork" is.

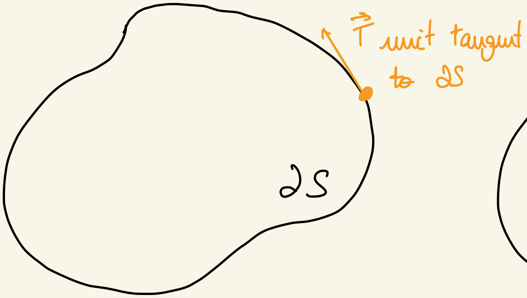
Consider $n = 3$ and $k = 2$. Stokes' Theorem:

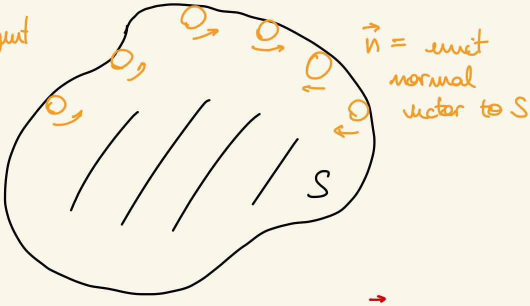


Corollary of Stokes' Thm
$$\left\{ \int_{M\,(3D)} \text{div } G = \int_{\partial M = S\,(2D)} G \cdot \vec{n} \; dA \right. \quad \text{(expectation)}$$

Gauss' Theorem

Now, for a 2-dim. object in $\mathbb{R}^3$:





$$\vec{F} \xrightarrow{\qquad\qquad\qquad} \text{curl } \vec{F}$$

Another corollary of Stokes' Thm, which is also called Stokes' Thm
$$\left. \right\} \int_{\partial S} \vec{F} \cdot \vec{T} \; dl = \int_S (\text{curl } \vec{F}) \cdot \vec{n} \; dA$$

In $\mathbb{R}^2$,

$$\Omega^0(\mathbb{R}^2) \xrightarrow{\quad d \quad} \Omega^1(\mathbb{R}^2) \xrightarrow{\quad d \quad} \Omega^2(\mathbb{R}^2)$$

$$\{f\} \longrightarrow \{\vec{F}\} \longrightarrow \{g\}$$

Also have a 2D Gauss' Theorem and 2D divergence theorem.

$$\int_D (\text{div } \vec{F}) = \int_{\partial D} \vec{F} \cdot \vec{n} \; dA \quad \text{(Green's Theorem)}$$

LECTURE 60: STOKES' THEOREM (on chains)

Recall: $I^k_{(j,\alpha)} : I^{k-1}_{y_1, \dots, y_{k-1}} \to I^k$ by

$$(y_1, \dots, y_{k-1}) \mapsto (y_1, \dots, y_{j-1}, \overbrace{\alpha}^{j}, \dots, y_{k-1}) \qquad \alpha = 0 \text{ or } 1.$$

$$\partial c = \sum_{i=1}^{k} \sum_{\alpha=\{0,1\}} (-1)^{j+\alpha} \, c \circ \overbrace{I_{(j,\alpha)}}^{\text{faces}}$$

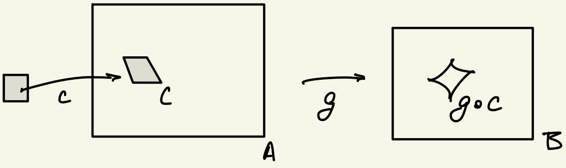
$$\int_{I^k} f \, dx_1 \wedge \cdots \wedge dx_k := \int_{I^k} f \ ; \quad \int_c \omega := \int_{I^k} c^* \omega .$$

- **Cubes: push**

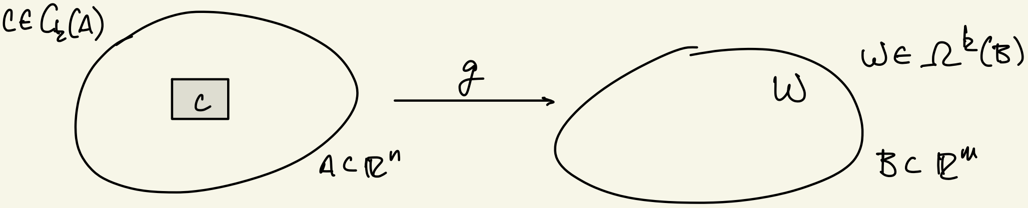
$g : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m \Rightarrow g_* : C_k(A) \to C_k(B)$,

and g_* is compatible with "+" and "·". Moreover, g_* is compatible with "∂", i.e.,

$$\partial(g_* c) = g_* \partial c .$$



Now, push forwards is also compatible w/ integration



Prop:

$$\boxed{\int_c g^* \omega = \int_{g_* c} \omega}$$

Pf:

$$LHS = \int_{I^k} c^*(g^* \omega)$$

$$RHS = \int_{I^k} (g_* c)^* \omega = \int_{I^k} (g \circ c)^* \omega \overset{\substack{\text{Contravariance of}\\ \text{pullbacks.}\\ \swarrow}}{=} \int_{I^k} c^*(g^* \omega).$$

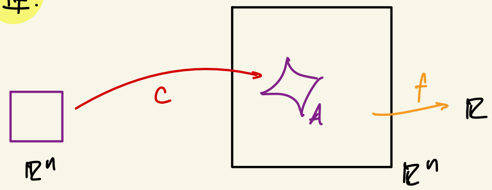
Prop: If $c: I^k \to A$, $\omega \in \Omega^k(A)$, $r: I^k \to I^k$ 1-1, onto and $\det r' > 0$, then

$$\int_c \omega = \int_{c \circ r} \omega .$$

Prop: Suppose $c: I^n \to \mathbb{R}^n_{x_1, \dots, x_n}$ is 1-1 with $\det c' > 0$. Set $A = c(I^n)$. Suppose $f: \mathbb{R}^n \to \mathbb{R}$ is given. Then

$$\int_A f = \int_c f \, dx_1 \wedge \cdots \wedge dx_n .$$

Pf:



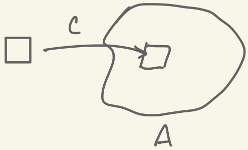
$$\int_A f \overset{?}{=} \underset{\substack{\Uparrow \\ \text{COV}}}{\int_{I^n}} (f \circ c) \cdot \det c'$$

$\square$

Thm: given $c \in C_k^1(A \subset \mathbb{R}^n)$ and $\omega \in \Omega^{k-1}(A)$,

$$\int_c d\omega = \int_{\partial c} \omega .$$

Pf: WLOG, assume $c : I^k \to A$ is a single cube.



Now,

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega)$$

$$\overset{*}{=} \int_{\partial I^k} c^*\omega = \int_{c_*(\partial I^k)} \omega = \int_{\partial(c_* I^k)} \omega$$

$$= \int_{\partial c} \omega .$$

* Assume $\omega \in \Omega^{k-1}(I^k)$. WLOG,

$$\omega = f \cdot dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k =: f \cdot dx_{\text{no } i}$$

Now,

$$\int_{I^k} d\omega = \int_{I^k} dx_i \wedge \frac{\partial f}{\partial x_i} \cdot dx_{\text{no } i} = (-1)^{i-1} \int_{I^k} \frac{\partial f}{\partial x_i} \, dx_1 \wedge \cdots \wedge dx_n$$

$$= (-1)^{i-1} \int_{I^k} \frac{\partial f}{\partial x_i}$$

Fubini, FTC

$$= (-1)^{i-1} \int_{I^{k-1}_{y_1, —, y_{k-1}}} [f(y_1, —, \overset{i\text{-th}}{1}, —, y_{k-1}) - f(y_1, —, \overset{i\text{-th}}{0}, —, y_{k-1})]$$

On the other hand,

$$\int_{\partial I^k} \omega = \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha} \int_{I(j,\alpha)} \omega \qquad \text{(TBC)}$$

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LECTURE 61: PROOF OF STOKES' THEOREM AND MANIFOLDS

Recall. $\partial c = \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha} c \circ I^k_{(j,\alpha)}$; $I^k_{(j,\alpha)}(y_1, —, y_{k-1}) \mapsto (y_1, —, \underbrace{\alpha}_{j}, —, y_{k-1})$

Pf (continued): The RHS: for $\omega = f\, dx_{\text{no } i}$,

$$\int_{\underbrace{\partial I^k}_{= \sum_{j,\alpha} (-1)^{j+\alpha} I^k_{(j,\alpha)}}} \omega = \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha} \int_{I(j,\alpha)} f\, dx_{\text{no } i}$$

$\int$ is linear on cubes

$$= \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha} \int_{I^{k-1}_{y_1, —, y_{k-1}}} (f \circ I_{(j,\alpha)}) \cdot d(x_1 \circ I_{(j,\alpha)}) \wedge \dots \wedge \widehat{d(x \circ I_{(j,\alpha)})} \wedge \dots \wedge d(x_k \circ I_{(j,\alpha)})$$

x_j is constant on the image of $I_{(j,\alpha)}$ (either 0 or 1) $\Rightarrow d(x_j \circ I_{(j,\alpha)}) = 0$

$$= \sum_{\alpha=\{0,1\}} (-1)^{i+\alpha} \int_{I^{k-1}_{y_1,\dots,y_{k-1}}} f(y_1, \_, \overset{i}{\underset{\sim}{\alpha}}, \_, y_{k-1})\, dy_1 \wedge \dots \wedge dy_{k-1}$$

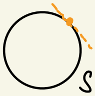
$$= \sum_{\alpha=\{0,1\}} (-1)^{i+\alpha} \int_{I^{k-1}_{y_1,\dots,y_{k-1}}} f(y_1, \_, \overset{i}{\underset{\sim}{\alpha}}, \_, y_{k-1}) = LHS.$$

$\square$

* * *

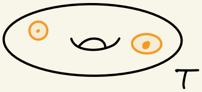
* **MANIFOLDS** (WITH NO BOUNDARY, FOR NOW):

Ex:



$S^1 \subset \mathbb{R}^2$;

$S^2 \subset \mathbb{R}^3$;



$T^2 \subset \mathbb{R}^3$.

- $M^k \subset \mathbb{R}^n$ is a k-manifold if "locally it looks like $\mathbb{R}^k$".

So... what does it mean to "look like"? 3 definition...

Thm: given $k \leq n$, $M \subset \mathbb{R}^n$, $x \in M$, then the following are equivalent:

(M) There exists an open $U \ni p$, open $V \subset \mathbb{R}^n$, and $h: U \to V$ diffeomorphism such that

→ smooth w/ smooth inverse

$$h(U \cap M) = V \cap \left(\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\}\right)$$

(Z) zero set There exists an open $U \ni p$ and a smooth function $g: U \to \mathbb{R}^{n-k}$ such that

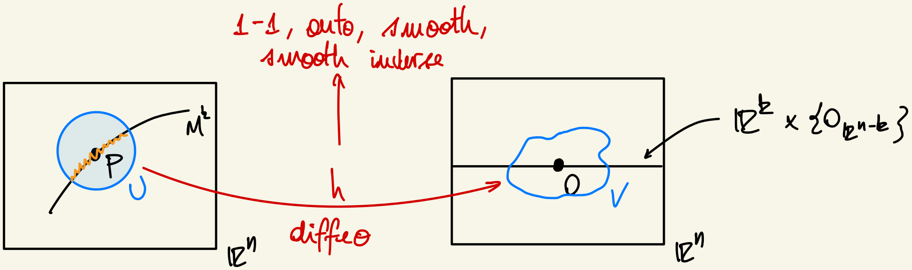
$$U \cap M = U \cap g^{-1}(0)$$

and $\operatorname{rank} g' = n-k$ at p.

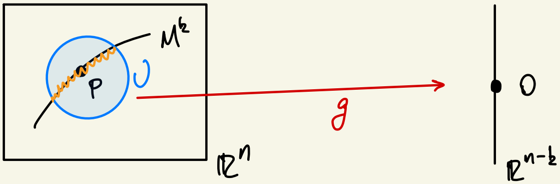
(C) coördinates There exists an open $U \ni p$, open $W \subset \mathbb{R}^k$, and a smooth, 1-1 function $f: W \to \mathbb{R}^n$ such that

1. $f(W) = U \cap M$
2. $f^{-1}: M \cap U \to W$ is continuous
3. $\forall a \in W$ $\operatorname{rank} f'(a) = k$.

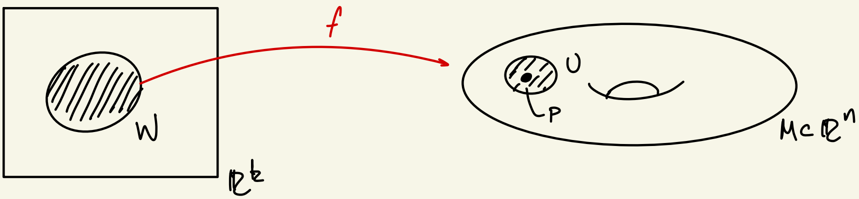
(M)



(Z)



(c)



LECTURE 62: MANIFOLDS

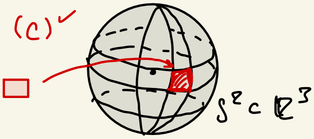
Def: (k-MANIFOLD) $M \subset \mathbb{R}^n$ is a k-manifold if for all $p \in M$, (MZC) holds.

Ex: 1. $S^2 \subset \mathbb{R}^3$, $S^2 = \{(x,y,z): x^2+y^2+z^2-1=0\}$

- $g(x,y,z) = x^2+y^2+z^2-1$

$$S^2 = g^{-1}(0)$$

(c)✓



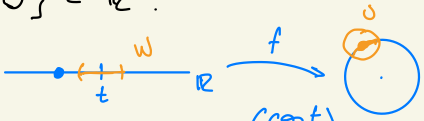
- rank $g' \overset{?}{=} 1$. Well,

$$g'(p) = g'(x,y,z) = \begin{pmatrix} 2x & 2y & 2z \end{pmatrix} \neq 0 \text{ on } S^2$$

$$\Rightarrow \text{rank } g' = 1 \checkmark \ (z)$$

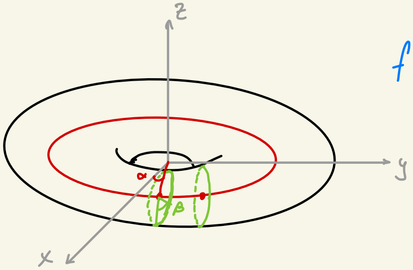
2. $S^1 = \{(x,y): x^2+y^2-1=0\} \subset \mathbb{R}^2$.

(z): $g(x,y) = x^2+y^2-1$; (c):



$$f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

3. $T^2 \subset \mathbb{R}^3$



(c):

$$f(\alpha,\beta) = \begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix} + \frac{1}{3}\left(\cos\beta\begin{pmatrix}\cos\alpha\\ \sin\alpha\\ 0\end{pmatrix} + \sin\beta\begin{pmatrix}0\\ 0\\ 1\end{pmatrix}\right)$$

$$= \begin{pmatrix}\cos\alpha + \frac{1}{3}\cos\beta\cos\alpha\\ \sin\alpha + \frac{1}{3}\cos\beta\sin\alpha\\ \frac{1}{3}\sin\beta\end{pmatrix}.$$

✓ check conditions

4. $\underbrace{SO(3)}_{\text{The set of rotations of } \mathbb{R}^3 \text{ preserving orientation}} \subset M_{3\times3}(\mathbb{R}) = \mathbb{R}^9$ (SO(3) should be 3-dim)

$A \in SO(3) \Leftrightarrow A^TA = Id$ and $\underline{\det A = 1.}$ into a 0-dim set

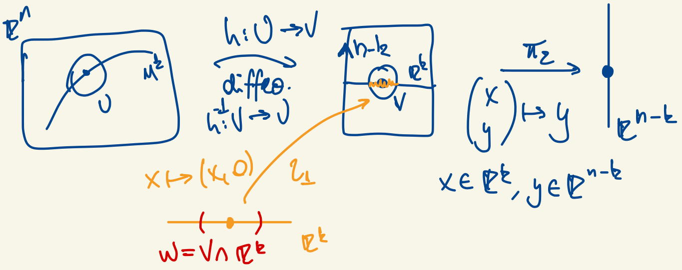
$\Leftrightarrow A^TA - Id = 0 \in$ symmetric matrices

$$\text{symmetric matrices} = \begin{pmatrix} a & d & e\\ d & b & f\\ e & f & c\end{pmatrix} \Big\} \dim = 6$$

Pf: (Equivalent definitions of a k-manifold)

(M ⇒ Z and C) To prove, set $g := \pi_2 \circ h$. ✓

so, M ⇒ Z.

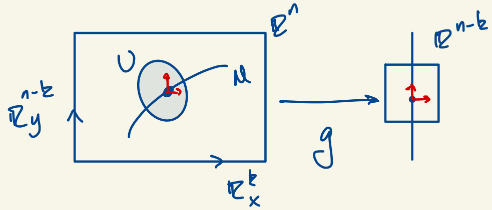


To prove C, set $f := h^{-1} \circ \iota_1$. ✓

LECTURE 63: MANIFOLDS

Pf: (Equivalent definitions of manifolds)

(Z ⇒ M) Name the variables in $\mathbb{R}^n$



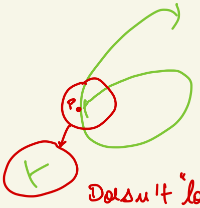
$x_1, \text{—}, x_k$ and $y_1, \text{—}, y_{n-k}$ in an order such that $\underbrace{\frac{\partial g}{\partial y}}_{\in M_{(n-k)\times(n-k)}}$ is non-singular

Define $h(x,y) = (x, g(x,y))$. Now, compute det h' to see if it's invertible locally (and then use IVT)

$$h'(p) = \left(\begin{array}{c|c} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \hline \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{array}\right) = \underbrace{\left(\begin{array}{c|c} Id_{k\times k} & 0 \\ \hline * & \underbrace{\frac{\partial g}{\partial y}}_{\text{non-singular}} \end{array}\right)}_{\text{non-singular}}$$

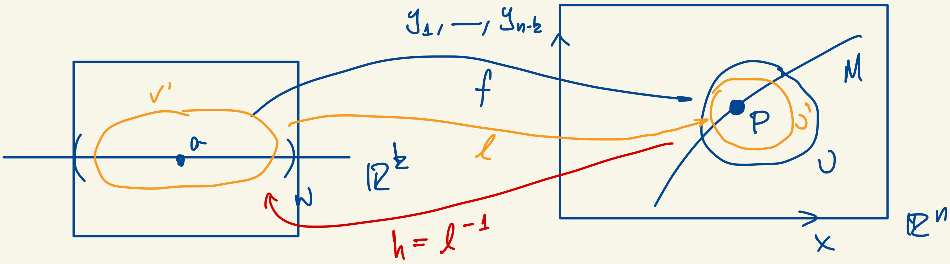
Thus, by the Inverse Function Theorem, $\exists U \ni p$ on which h is invertible and $(h|_U)^{-1}$ is smooth. So, $h: U \to V := h(U)$ is a diffeo.

(C ⇒ M) Remark: if we don't have condition (z)



$= M$ $\quad n = 2$, $k = 1$ This M is not a manifold

Doesn't "look" like $\mathbb{R}^k$.



WLOG,

$$\pi_{x_1, \dots, x_k} \circ f = \pi_1 \circ f$$

is of rank k, hence invertible. Define

$$\ell : V' \subset W_x \times \mathbb{R}_y^{n-k} \longrightarrow U' \subset U \subset$$

$$\ell(x,y) := \underbrace{f(x)}_{\in \mathbb{R}^n} + \underbrace{\begin{pmatrix} 0 \\ y \end{pmatrix}}_{\in \mathbb{R}^n}$$

Compute ℓ' to use IVT again:

$$\ell' = \begin{pmatrix} \frac{\partial \ell_1}{\partial x} & \frac{\partial \ell_1}{\partial y} \\ \frac{\partial \ell_2}{\partial x} & \frac{\partial \ell_2}{\partial y} \end{pmatrix} = \left(\begin{array}{c|c} \frac{\partial f}{\partial x} & 0 \\ \hline * & Id \end{array}\right)$$

$\Rightarrow$ ℓ' is invertible and, hence so is ℓ on some $V' \subset W \times \mathbb{R}^k$. Set $U' = \ell(V')$

NTS: for smaller open $U'' \subset U'$ and $V'' \subset V'$, $\ell(V'' \cap \mathbb{R}^k) = U'' \cap M$.

LECTURE 64: MANIFOLDS (continued)

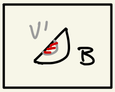
Pf: (Equivalent definitions of a manifold continued)

NTS: for smaller open $U'' \subset U'$ and $V'' \subset V'$, $\ell(V'' \cap \mathbb{R}^k) = U'' \cap M$.

$$\Leftrightarrow f(V'' \cap \mathbb{R}^k) = U'' \cap \mathbb{R}^k.$$

Recall: $g : A \subset \mathbb{R}^n \to B \subset \mathbb{R}^m$ is continuous

$\Leftrightarrow$ $g^{-1}(\tilde{V})$ is $\underset{\text{in } A}{\text{open}}$ whenever $\tilde{V} \subset B$ and $\tilde{V}$ is $\underset{\text{in } B}{\text{open}}$.



"$\tilde{V}$ is open in B" $\Leftrightarrow$ $\exists$ open $V \subset \mathbb{R}^m$ s.t. $\tilde{V} = B \cap V$.

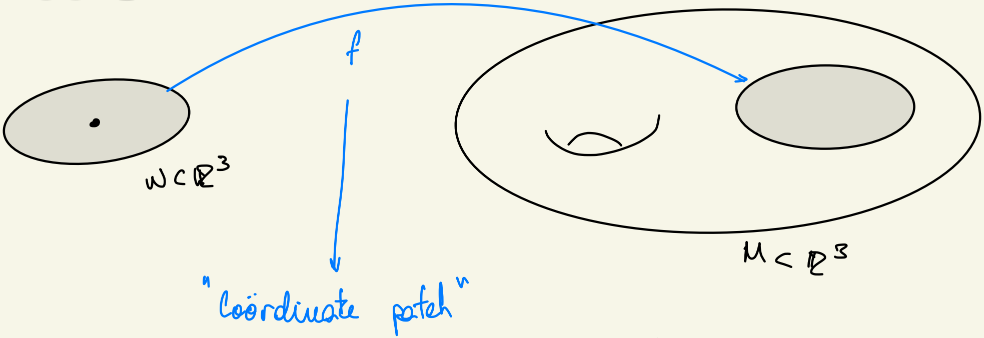
$\Leftrightarrow$ $g^{-1}(V)$ is of the form $A \cap U$ for an open $U \subset \mathbb{R}^n$ whenever $V \subset \mathbb{R}^m$ is open.

So, if f^{-1} is continuous, f is "open", i.e., f carries open sets to open sets. Meaning, in our context, $f(V' \cap \mathbb{R}^k)$ must be open (by continuity of f^{-1}) in $M \cap U_c$. So, there exists open $U'' \subset \mathbb{R}^n$ s.t.

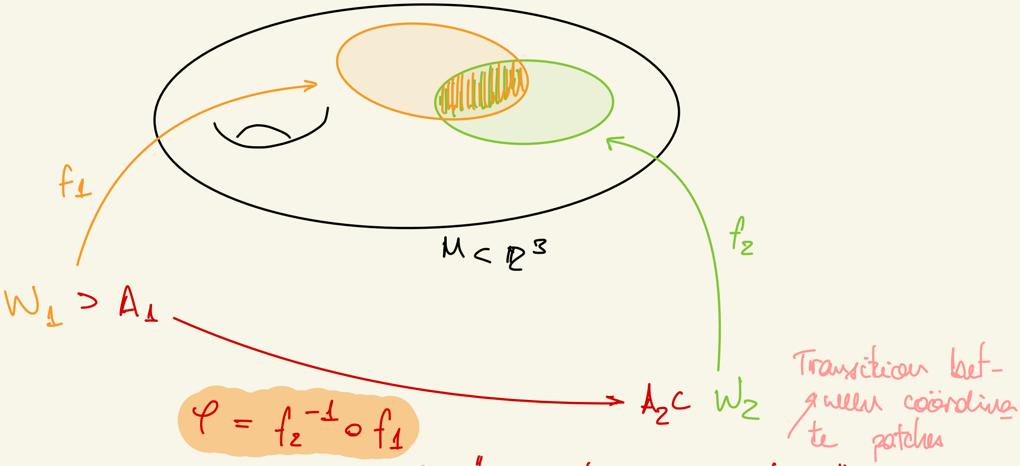
$$\begin{aligned} f(V' \cap \mathbb{R}^k) &= U'' \cap M \cap U_c \\ &= M \cap \underbrace{(U_c \cap U'')}_{=:\, U'''} \\ &= M \cap U'''. \end{aligned}$$

Now, set $V''' := \ell^{-1}(U''')$. □

REMARKS:



If there are two coördinate patches:

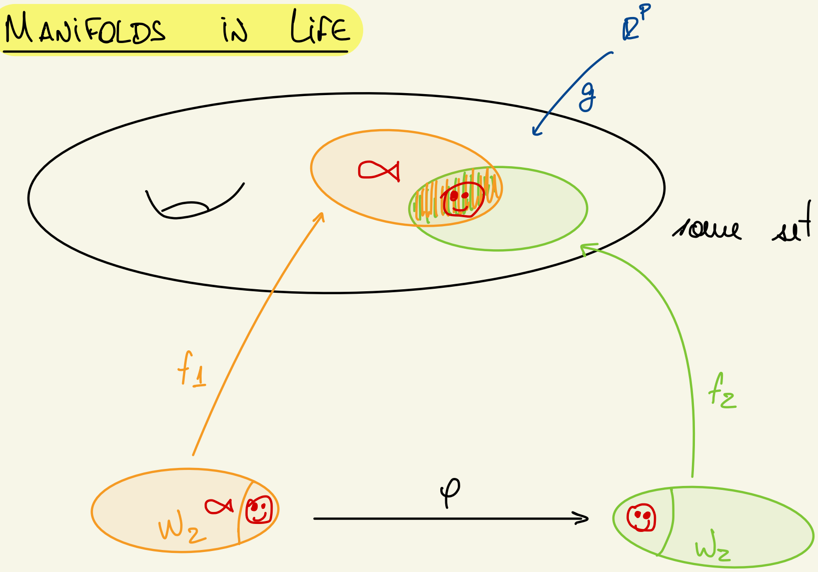


$\varphi = f_2^{-1} \circ f_1$

↪ "Transition function"

φ is a diffeomorphism!

MANIFOLDS IN LIFE



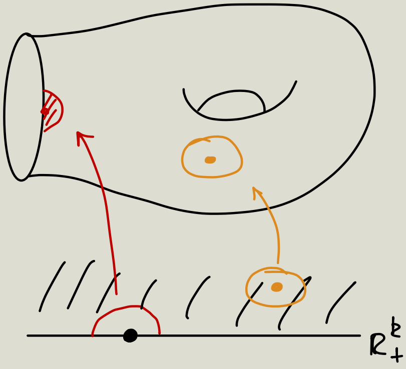
"something is smooth if this something is smooth as viewed by a coördinate patch".

* * *

Def: (MANIFOLD W/ BOUNDARY) A k-manifold with boundary is a subset $M \subset \mathbb{R}^n$ s.t. $\forall p \in M$ $\exists W \subset \mathbb{R}^k_+$ and an open $U \subset \mathbb{R}^n$, $U \ni p$, and a smooth, 1-1 $f: W \to \mathbb{R}^n$ s.t.

1. $f(W) = M \cap U$
2. $f^{-1}: M \cap U \to W$ is cont.
3. f' has maximum rank

at every point in W.



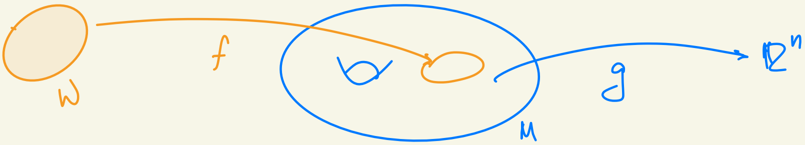
Def: (HALF-SPACE) $\mathbb{R}^k_+ := \{ x \in \mathbb{R}^k : x_k \geq 0 \}$.

* * *

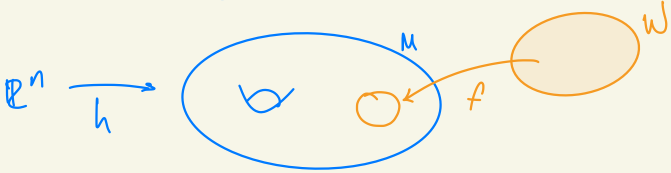
LECTURE 65: BOUNDARIES, TANGENT SPACES, & FORMS ON MANIFOLDS

SMOOTHNESS ON A MANIFOLD: "something" on a manifold is smooth if this "something" is smooth on any coördinate chart

- Function into M or out of M.



g is smooth $:=$ $\forall$ coördinate chart $f : W \subset \mathbb{R}^k \to M$, $g \circ f : W \to \mathbb{R}^n$ is smooth.



h is smooth $:=$ $\forall$ coördinate chart $f : W \to M$, $f^{-1} \circ h|_{h^{-1}(f(W))} : h^{-1}(f(W)) \longrightarrow W$ is smooth.

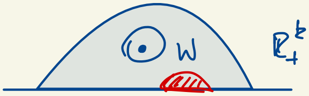
Suppose M^k, N^l are manifolds and

is smooth if

$$\begin{tikzcd} \psi : M \arrow[r] & N \arrow[d, bend left, "f_N^{-1}"] \\ \mathbb{R}^k \arrow[u, "f_M"] \arrow[r] & \mathbb{R}^l \arrow[u, "f_N"] \end{tikzcd}$$

$\forall f_M, f_N$, the partially-defined $f_N^{-1} \circ \psi \circ f_M$ is smooth.

* **MANIFOLD WITH BOUNDARY:** $\forall p \in M$ $\exists$ open $W \subset \mathbb{R}^k_+ = \{x_k \geq 0\}$, open $p \in U \subset \mathbb{R}^n$ and smooth, 1-1, $f: W \to U$ s.t. 1., 2., 3. as before (c) hold.

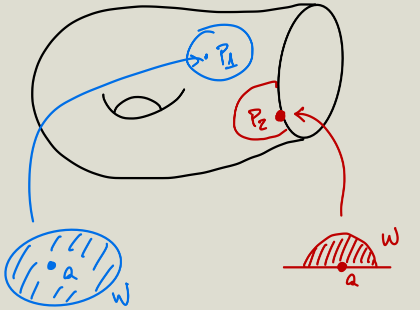


Thm: If M is a manifold with boundary, then for every $p \in M$, either

1. If $f: W \to M$ is a coördinate chart s.t $f(a) = p$, then $a_k > 0$.

or

2. If $f: W \to M$ is a coördinate chart s.t. $f(a) = p$, then $a_k = 0$.



Def: $\partial M = \{ p \in M : \text{ 2. above holds} \}$

WARNING:

1) $\partial \neq \text{bd}$

M = 

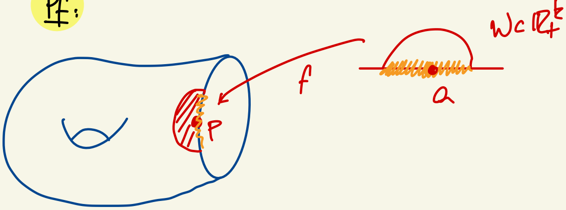
∂M = a circle

$\text{bd}(M) = M$

2) Every manifold is a manifold with boundary, but not vice-versa.

Thm: If M is a k-manifold with boundary, then ∂M is a $(k-1)$-manifold.

Pf:

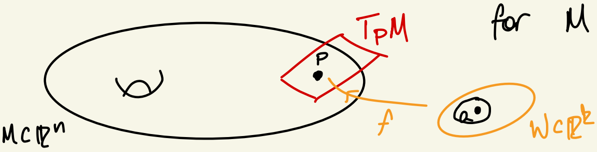


get coörd. charts for the whole boundary.

Thm: $\partial(\partial M) = \emptyset$.

* **TANGENT SPACES**:

Let M be a k-manifold (w/ or w/out boundary) and let $p \in M$. If $f: W \to M$ is a coördinate chart for M with $f(a) = p$, set



$$T_pM = f_*(T_a\mathbb{R}^k) \subset T_p\mathbb{R}^n.$$

Obs: 1. This is well-defined! (independent of choice of coördinate charts); i.e.,

$$f_{1*}(T_{a_1}\mathbb{R}^k) = f_{2*}(T_{a_2}\mathbb{R}^k)$$

$$\| $$

$$(f_1 \circ \phi)_*(T_{a_2}\mathbb{R}^k)$$

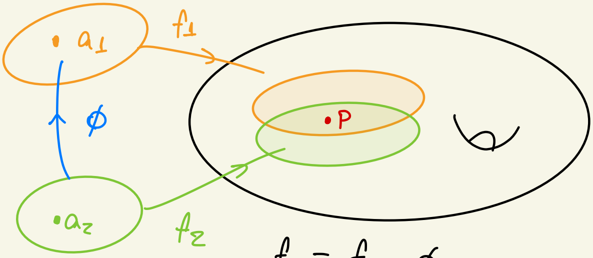
$$\|$$

$$f_{1*}(\phi_* T_{a_2}\mathbb{R}^k)$$

$$\|$$

$$f_{1*}(T_{a_1}\mathbb{R}^k)$$

ϕ_* is invertible as ϕ' is as ϕ is.



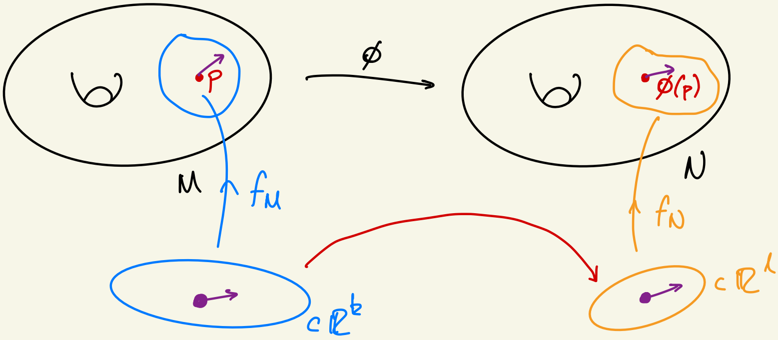
$f_2 = f_1 \circ \phi$

2. $\dim(T_p M) \overset{def}{=} \operatorname{rank} f' = k.$

//

Suppose $\phi : M^k \to N^l$ smooth, $p \in M$, then we can define

$$\phi_* : T_pM \longrightarrow T_{\phi(p)} N$$

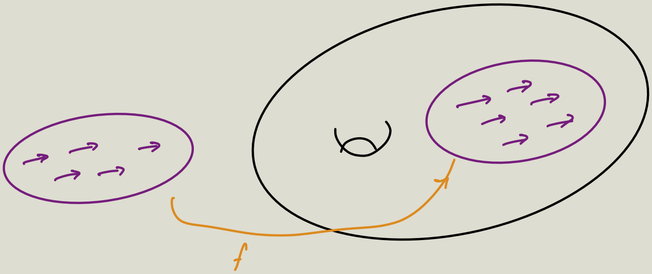


Def: (VECTOR FIELD) A vector field on M is

$$F: M \longrightarrow \bigcup_{p\in M} T_pM$$

such that $F(p) \in T_pM$.

Def: Such F is smooth if it's smooth as viewed under all coördinate charts.



Def: $\omega \in \Omega^l(M^k)$

$$\omega: M \longrightarrow \bigcup_{p\in M} \Lambda^l(T_pM)$$

s.t. $\omega(p) \in \Lambda^l(T_pM)$

LECTURE 66: THINGS ON MANIFOLDS

Recall: $T_pM = f_*(T_a\mathbb{R}^k)$; $\omega \in \Omega^\ell(M)$ if $\omega : M \to \bigcup_{p\in M} \Lambda^\ell(T_pM)$ and $\omega(p) \in \Lambda^\ell(T_pM)$ $\forall p \in M$.

- DIRECTIONAL DERIVATIVE: for $\xi \in T_pM$, $g : M \to \mathbb{R}$ smooth then $\bar{g}$ on W $(f_* g)$ and $\bar{\xi} \in T_a\mathbb{R}^k$, and

$$\mathbb{R} \ni D_\xi g := D_{\bar{\xi}} \bar{g}$$

- DIFFERENTIAL FORMS ON MANIFOLDS: $+$, $\wedge$, $d : \Omega^\ell(M) \to \Omega^{\ell+1}(M)$, ϕ^*, ϕ_* all obey all the rules from $\mathbb{R}^n$, except one: any diff. form

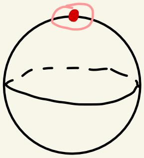
$$\omega = \sum_{I \in \mathbb{N}_0^k} f_I \, dx_I \quad \Big\} \text{ DOESN'T MAKE SENSE ANYMORE}$$

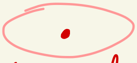
No canonical global coördinate functions on a manifold.

Ex 1: On $S^2 = [x^2+y^2+z^2 = 1] \subset \mathbb{R}^3$

$$\omega_1 = x\, \underbrace{dy}_{} \wedge dz + y\, dz \wedge dx + z\, dx \wedge dy \in \Omega^2(S^2)$$

function on $\mathbb{R}^3 \supset S^2 \Rightarrow$ Makes sense





x, y only

$$= \frac{}{1-x^2-y^2}\, \underbrace{dx \wedge dy}$$

z is almost constant in the pole $\Rightarrow dz = 0$.

THE VOLUME FORM OF S^2

Now $\omega_2 = x\,dx + y\,dy + z\,dz \in \Omega^1(S^2)$

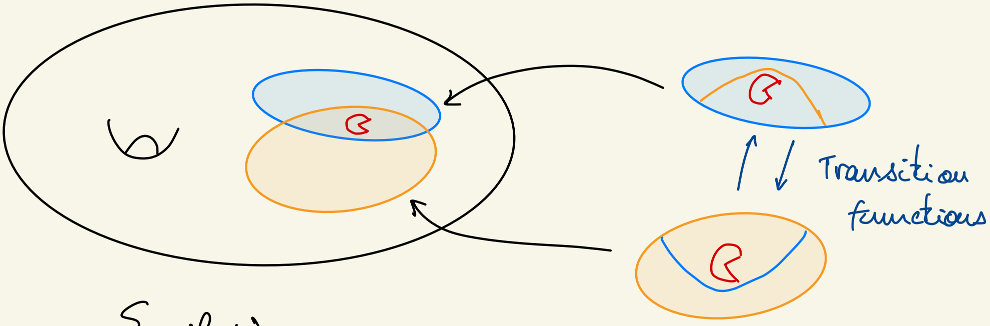
$= 0$ on S^2

↪ Take $g = x^2+y^2+z^2 = 1$ on S^2

$0 = d1 = dg = 2x\,dx + 2y\,dy + 2z\,dz = 2\omega_2$

* **CHAINS**: cubes/chains push and pull; $\phi : M^k \to N^l$, $\phi_* : C_p(M) \to C_p(N)$; compatible w/ ∂. Stokes for chains works for chains on manifolds as well.

* **INTEGRATION**: Want $\int_M d\omega = \int_{\partial M} \omega$



$\sum_i \varphi_i \omega$

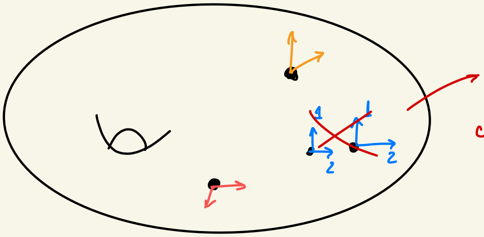
LOOSE DEFINITION: An orientable manifold is a manifold such that an atlas can be chosen such that all transition functions have det > 0.

Reminder: Orientation of V^k

ordered basis / positive det c.o.b $\Leftrightarrow$ $\eta \in \Lambda^k(V)$ / multiply by positive scalar

Def: (ORIENTATION) An *orientation* of a k-manifold M is a continuously varying choice of orientation for T_pM for each $p \in M$.

Namely, it is a choice of $\eta \in \Omega^k(M)$ s.t. $\eta(p) \neq 0$ $\forall p \in M$ modulo $\eta_1 \sim \eta_2$ if $\eta_1 = f\eta_2$, $f > 0$.



Need to be continuously varying

* * *

LECTURE 67: ORIENTATION & INTEGRATION

Recall:

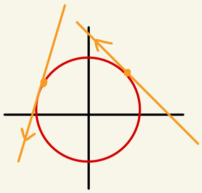
- Orientation of V^k = $\text{ordered basis} \Big/ \overset{*}{\text{pos. det c.o.m.}} \sim \eta \in \Lambda^k(V), \eta \neq 0 \Big/ \text{mult. by pos. scalar}$ $*$

- Orientation of M^k = continuously varying choice of an orientation for T_pM, for each $p \in M$.

$= \left\{ \begin{matrix} \text{Nowhere zero} \\ \text{top form on } M \end{matrix} \right\}^{*} \Big/ \begin{matrix} \eta_1 \sim f\eta_2 \\ \text{if } f: M \to \mathbb{R}, f > 0 \end{matrix}$ $\sim$ A choice $\mathcal{O}_p$ of an ordered basis of T_pM (modulo positive det c.o.b.) for each $p \in M$, s.t. $\exists U \ni p$ and vec. fields $X_1, \dots, X_k$ def. on U, s.t. $\mathcal{O}_{p'} = (X_1(p'), \dots, X_k(p'))$ for each $p' \in U$. $*$

EXAMPLES:

1) $S^1 = \{x^2+y^2=1\}$

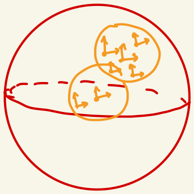


Take

$$\eta = d\theta = \frac{x\,dy - y\,dx}{x^2+y^2} = -y\,dx + x\,dy$$

$X_1\begin{pmatrix}x\\y\end{pmatrix} = \left(\begin{pmatrix}x\\y\end{pmatrix}, \begin{pmatrix}-y\\x\end{pmatrix}\right) \longrightarrow$ Check by evaluating X_1 in η.

2) $S^2 = \{x^2+y^2+z^2=1\}$



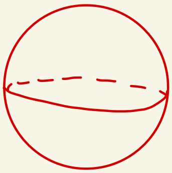
Take

$$\eta = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$$

(volume form)

"You cannot comb a sphere"

= ∄ a nowhere 0 vec. field on S



"Right-hand rule w/ thumb radially outward"

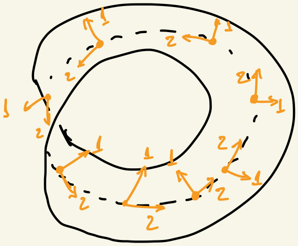
0) Point in $\mathbb{R}^{17}$. (zero-dim. manifold)

In terms of top forms, namely functions (0-forms), namely scalars modulo $\{\eta_1 = a\eta_2,\ a>0\} = \{\text{sign}\}$

⇒ Orientation of a point is a choice of a "+" or "-" sign.

u) MÖBIUS STRIP

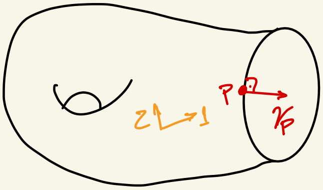
HAS NO ORIENTATION!



Not the only way to write orientation at each point (can rotate)

Def: M is orientable if it is given with a choice of orientation. If M can be oriented, we say it is "orientable".

Given an orientable manifold with boundary M, there is a "canonical" way of orienting ∂M:



The orientation $\mathcal{O}_p^{\partial M}$ of ∂M at $p \in \partial M$ is this such that if you prepend to it the outward pointing normal to ∂M, ν_p, then you get the orientation $\mathcal{O}_p^M$ of M at p; i.e.,

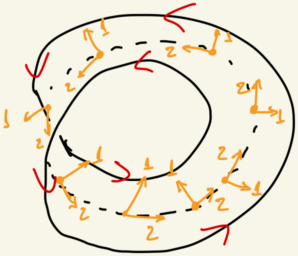
$$\mathcal{O}_p^M = (\nu_p, \mathcal{O}_p^{\partial M})$$

EXAMPLE:

1) $S^2 = \partial D^3$, $D^3 = \{x^2+y^2+z^2 \leq 1\}$

with orientation inherited from $\mathbb{R}^3$: $(\partial_x, \partial_y, \partial_z)$

2) The boundary of Möb is orientable.



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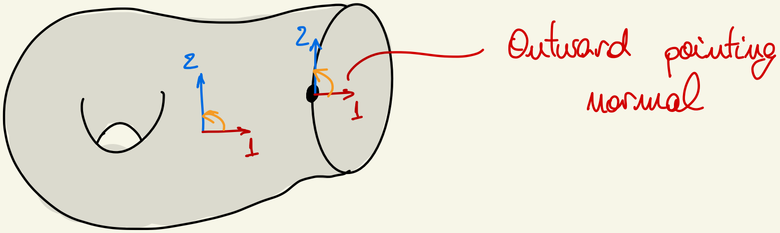
LECTURE 68: ε(ORIENTATIONS) & INTEGRATION

Orienting M^k: nowhere zero $\eta \in \Omega^k(M)$ / multiplication by $f > 0$

More useful

$\sim$ Orienting T_pM for every $p \in M$ in a manner that can be presented locally by k continuous vector fields.

If M is oriented, the induced orientation on ∂M at $p \in \partial M$ is such that if you prepend to it the oriented pointing normal $\nu_p \in T_pM$ to $T_p\partial M$, you get the orientation of M at p.



Alternatively,

INCLUSION MAP

interior product

$$\eta_{\partial M} = \iota^*_{\partial M \to M} \cdot \iota_{\nu} \, \eta_M$$

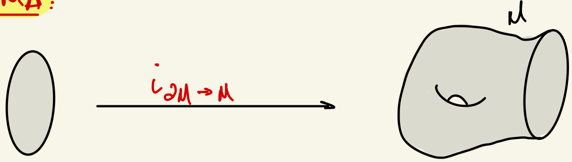
INTERIOR PRODUCT: if X is a vector field on M

$$\iota_X : \Omega^l(M) \longrightarrow \Omega^{l-1}(M)$$

the "inner multiplication by X" is as follows: for $\omega \in \Omega^l(M)$, $\xi_i \in T_pM$

$$(\iota_X \omega)(\xi_1, \text{——}, \xi_{l-1}) = \omega(X(p), \xi_1, \text{——}, \xi_{l-1})$$

INCLUSION MAP:



Claim: The two definitions agree.

Pf: Assume $p \in \partial M$, T_pM is oriented, and say it's given by

$$(v_p, \xi_1, \text{—}, \xi_{k-1}),$$

where $\xi_1, \text{—}, \xi_{k-1} \in T_p \partial M$. Also, assume that this orientation of T_pM is given by some $\eta_M \in \Omega^k(M)$, i.e.,

$$\eta_M(v_p, \xi_1, \text{—}, \xi_{k-1}) > 0. \,^{(*)}$$

Now, the orientation of ∂M at p is given by

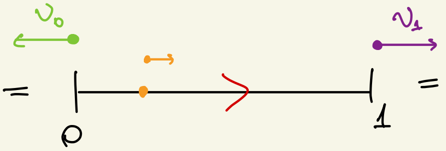
$$\underset{\text{def. 1}}{(\xi_1, \text{—}, \xi_{k-1})} \overset{?}{\sim} \underset{\text{def. 2}}{i^*_{\partial M \to M}\, \iota_{v_p} \eta_M = \eta_{\partial M}}$$

$$0 \overset{?}{<} \eta_{\partial M}(\xi_1, \text{—}, \xi_{k-1})$$

So,

$$\begin{aligned} \eta_{\partial M}(\xi_1, \text{—}, \xi_{k-1}) &= i^*_{\partial M \to M}\, \iota_{v_p} \eta_M (\xi_1, \text{—}, \xi_{k-1}) \\ &= \iota_{v_p} \eta_M (\xi_1, \text{—}, \xi_{k-1}) \\ &= \eta_M (v_p, \xi_1, \text{—}, \xi_{k-1}) \overset{(*)}{>} 0. \end{aligned}$$

$\square$

EXAMPLE: $M =$  $= [0,1]_x$

orientation of a point is a choice of sign

What is ∂M as an oriented manifold?

As a set, $\partial M = \{0, 1\}$

orientation at 0: $-$

orientation at 1: $+$

Consider: $\eta = \frac{dx}{\cancel{1}} = dx$

At $p = 0$,

$$\eta_{\partial M}(0) = i^* \iota_{\nu_0} dx = \iota_{(-\partial_x)} dx = dx(-\partial_x) = -\partial_x x$$
$$= -1$$

Same process at 1.

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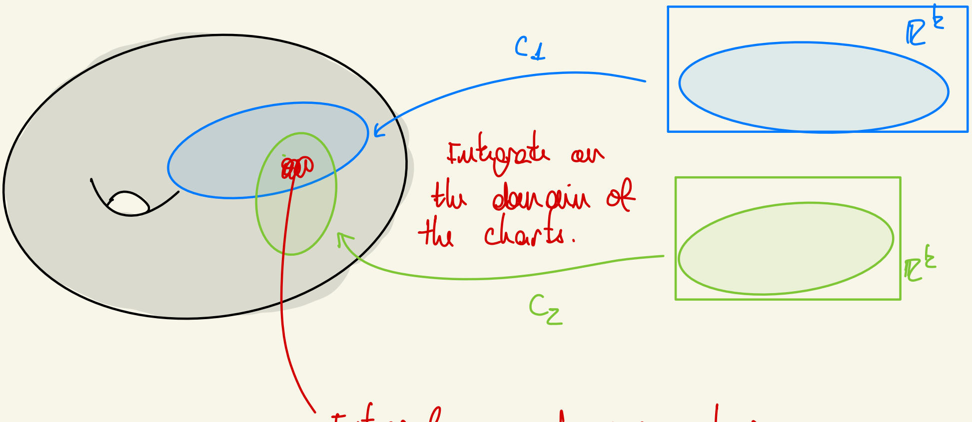
INTEGRATION ON ORIENTED MANIFOLD



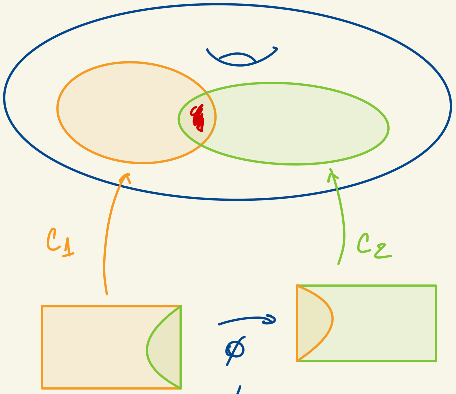
Integral must agree here

Prop: Let c_1, c_2 be smooth injective (with injective c_1', c_2') orientation preserving k-cubes in an oriented M^k, and assume $\omega \in \Omega^k(M)$ is s.t. $\operatorname{supp}\omega \subset \operatorname{im}(c_1) \cap \operatorname{im}(c_2)$. Then

$$\int_{I^k} c_1^*\omega = \int_{c_1} \omega = \int_{c_2} \omega = \int_{I^k} c_2^*\omega$$

$$=: \int_M \omega.$$

Pf:



Orientation preserving, i.e., $\det \phi' > 0$

$$\int_{I^k} c_1^*\omega = \int_{I^k} (c_2 \circ \phi)^*\omega$$

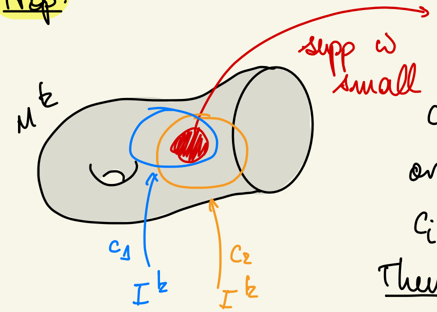
$$= \int_{I^k} \phi^*(c_2^*\omega)$$

$$\overset{\text{cov}}{=} \int_{I^k} c_2^*\omega .$$

$\square$

LECTURE 69: STOKES' THEOREM

Prop:



c_i are smooth, 1-1, orientation preserving, c_i' are also 1-1

Then,
$$\int_{c_1} \omega = \int_{c_2} \omega =: \int_M \omega$$

Suppose $\omega \in \Omega^k(M)$. Choose a POI φ_i subordinate to open sets that can be covered by "good cubes" as on the left. Define

$$\int_M \omega := \sum_i \int_M \varphi_i \omega$$

REMARK:

1) If M is compact, this always makes sense.

2) In general, first define "integrable forms"

Integrable if

$$\sum_i \int_M \varphi_i \underbrace{|\omega|}_{} < \infty$$

$\hookrightarrow c^*(\varphi_i \omega) \in \Omega^k(I^k)$

Then $c^*(\varphi_i \omega) = f \, dx_1 \cdots dx_k$ $\nearrow$ $\int_{I^k} |f| \, dx_1 \cdots dx_k$

IMPORTANT: Always need to show that this is independent of the POI:

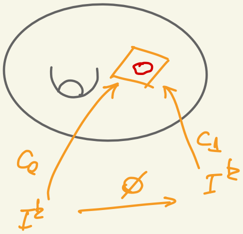
$$\int_M^{(\varphi_i)} \omega \overset{\text{def}}{=} \sum_i \int_M \overset{1=\sum_j \psi_j}{\varphi_i \omega} = \sum_{i,j} \int_M \varphi_i \psi_j \omega$$

$$= \sum_j \int_M \psi_j \omega \overset{\text{def}}{=} \int_M^{(\psi_j)} \omega$$

REMARK:

1. $\int_M \omega$ is linear in ω.

2. $$\int_{\underbrace{-M}_{\text{Orientation reversed}}} \omega = - \int_M \omega$$



$c_1(x_1, \_, x_k) := c_0(1-x_1, \_, x_k)$

$\phi(x_1, \_, x_k) = (1-x_1, x_2 \_, x_k)$

Thm (STOKES THEOREM) If M is a compact and oriented k-manifold and $\omega \in \Omega^{k-1}(M)$, then

$$\int_M d\omega = \int_{\partial M} \omega .$$

Finally... YAY ☺

Pf: Suppose we know the theorem on ω's with "small" supports (i.e., supp $\omega \subset$ in (cube)).

$$\underset{\text{supp }\omega = M}{\int_{\partial M} \omega} = \sum_i \int_{\partial M} \varphi_i \omega = \sum_i \int_M d(\varphi_i \omega)$$

Take POI for M, which is also a POI for ∂M.

By Stokes for "small"

$$= \sum_i \int_M d\varphi_i \wedge \omega + \sum_i \int_M \varphi_i \wedge d\omega$$

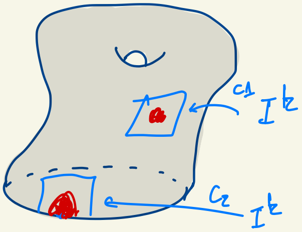
Finite sum b/c M compact

$d(1)=0$

$$= \cancel{\int_M d\left(\sum_i \varphi_i\right)\omega} + \sum_i \int_M \varphi_i \, d\omega .$$

$$= \int_M d\omega .$$

Now, prove Stokes for supp ω "small":



Case 1 (Interior): supp $\omega \subset$ int M and can be covered by a single interior cube.

Case 2 (boundary): supp $\omega \subset \partial M \neq \emptyset$ and can be covered by a single interior cube.

Case 1: $\int_{\partial M} \omega = 0$ since $\omega|_{\partial M} = 0$.

$$\int_M d\omega = \int_{c_1} d\omega = \int_{I^k} c_1^*(d\omega) = \int_{I^k} d(c_1^* \omega)$$

stokes for chains (already proved)

$$= \int_{\partial I^k} c_1^* \omega = \int_{c_{1*}(\partial I^k)} \omega = 0$$

Case 2: Choose c_2 s.t. only $c_2(0, y_1, \dots, y_{k-1})$ intersects ∂M. Then

$$\int_M d\omega = \dots = \int_{\partial I^k} c_2^* \omega = \int_{\partial c_2} \omega$$

Orientation reverses sign relative to the orientation of ∂M

$$= -\int_{(c_2)_{(1,0)}} \omega = -\int_{y_1, \dots, y_{k-1}} \overline{(c_2)_{(1,0)}}^{\,*} \omega$$

$$= \int_{\partial M} \omega .$$

$\square$

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LECTURE 70: APPLICATIONS OF STOKES'

Recall. If M^k is compact and oriented and $\omega \in \Omega^{k-1}(M)$, then

$$\int_M d\omega = \int_{\partial M} \omega .$$

EXAMPLES:

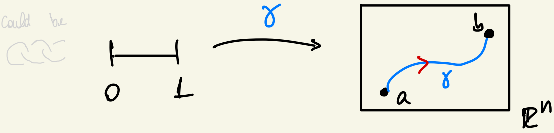
orientation

1) $M^1 = [a,b], \quad \omega = f, \quad \partial M = \{b\} \cup (-\{a\})$

$$d\omega = f' dx$$

$$\int_a^b f' \, dx = \int_M d\omega = \int_{\partial M} \omega = f(b) - f(a).$$

2) $M^1 \subset \mathbb{R}^n$



$$M = \gamma([0,1]); \quad \omega = f, \quad f : \mathbb{R}^n_{x_i} \to \mathbb{R}^n$$

$$f(b) - f(a) = \int_{\partial M} \omega = \int_{\gamma([0,1])} d\omega$$

$$= \int_{\gamma([0,1])} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dx_i$$

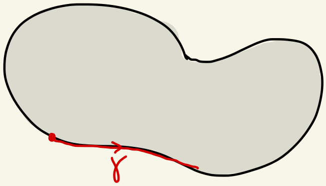
$x_i = \gamma_i(t)$
$dx_i = \gamma_i'(t) dt$

$$= \int_{[0,1]} \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma(t)) \, \gamma_i'(t) \, dt$$

$$f(b) - f(a) = \int_{[0,1]} (\text{grad } f) \cdot \dot{\gamma}(t) \, dt$$

3) $M^2 \subset \mathbb{R}^2$

$\partial D = \gamma$, $\gamma : [0,1] \to \mathbb{R}^2$



Pick $\omega \in \Omega^1(D)$: $\omega = P\,dx + Q\,dy$

$$d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_M d\omega = \int_{\partial M} \omega = \int_\gamma P\,dx + Q\,dy$$

$$\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) = \int_{[0,1]} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \dot{\gamma}\,dt$$

→ GREEN'S THEOREM

Interpretation 1: $F := \begin{pmatrix} P \\ Q \end{pmatrix}$

- $\frac{\partial Q}{\partial x}$ measures how much the y-component grows with x
- $\frac{\partial P}{\partial y}$ measures how much the x-component grows with y

$\Rightarrow$ $\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$ measures how much the field swirls around.

$$\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$$

Interpretation 2.

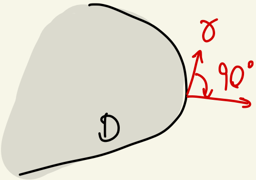
$Q \mapsto P$

$P \mapsto -Q$

$\begin{pmatrix}P\\Q\end{pmatrix}$ rotated 90° clockwise

Then

G.T.:
$$\int_D \underbrace{\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}}_{\operatorname{div}\begin{pmatrix}P\\Q\end{pmatrix}} = \int_{[0,1]} \overbrace{\begin{pmatrix}-Q\\P\end{pmatrix}} \cdot \dot{\gamma}\, dt$$



$$= \int_{[0,1]} \begin{pmatrix}P\\Q\end{pmatrix} \cdot \left(\dot{\gamma} \text{ rotated } 90° \text{ clockwise}\right)$$

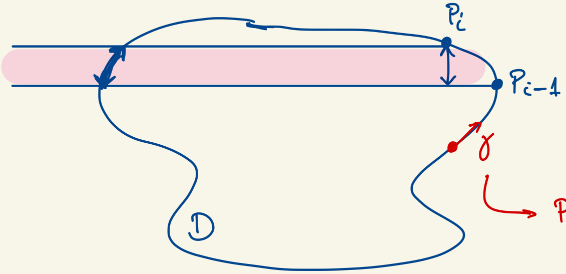
$$= \int_{[0,1]} \begin{pmatrix}P\\Q\end{pmatrix} \cdot \vec{n}$$

outward normal

$$\Rightarrow \int_D \operatorname{div}\begin{pmatrix}P\\Q\end{pmatrix} = \int_{[0,1]} \begin{pmatrix}P\\Q\end{pmatrix} \cdot \vec{n}$$

SUBEXAMPLE: $\omega = x\, dy \Rightarrow d\omega = dx \wedge dy$

$$\text{Area}(D) = \int_D 1 = \int_D d\omega = \int_{\partial D = \gamma} x\, dy$$

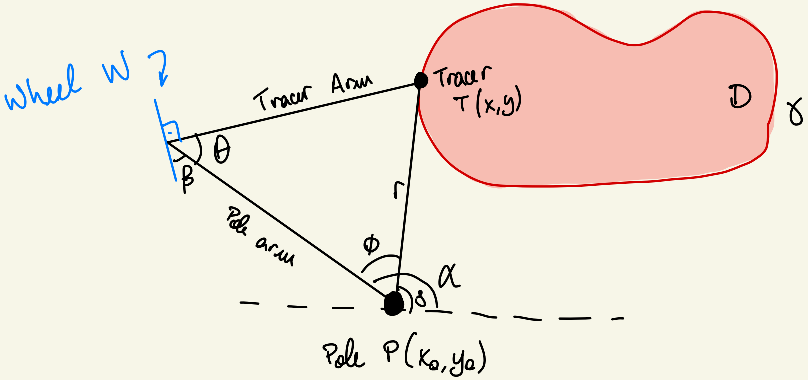


$$\approx \sum_{i=1}^{n} x_i \left(y_i - y_{i-1}\right)$$

Path.gpx = $\begin{pmatrix}x_0 & y_0\\ & \vdots\\ x_n & y_n\end{pmatrix}$

CARTESIAN PLANIMETER.

POLAR PLANIMETER



M = configuration space of the planimeter $\subset \mathbb{R}^{10}_{x_0,y_0,x,y,r,\alpha,\beta,\dots}$

On M there are 10+ functions: $x, y, r, \alpha, \beta, \dots\dots$

$\omega \in \Omega^1(M)$,

ω(tiny motion of the planimeter) = how much W is spinning.

To be continued...

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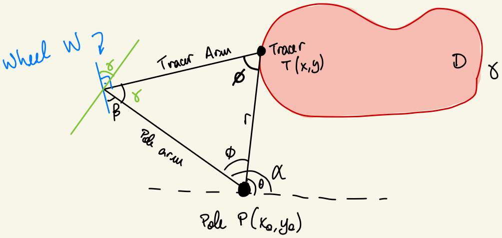
LECTURE 71: VOLUME FORMS IN $\mathbb{R}^3$



ω = how much W turns if the planimeter is pushed a bit.

M = all possible configurations of the planimeter.

$\alpha, \beta, \gamma, \phi, \theta, r, x, y : M \to \mathbb{R}, \quad \omega \in \Omega^1(M)$

so,

$$\int_{\partial D} \omega = \int_D d\omega$$

$\omega \propto d\alpha \curvearrowright$ "change in α", $\quad \omega = (d\alpha) \cos\gamma$

$$\Leftrightarrow \quad \omega = \cos(\pi - 2\phi)\, d(\theta + \phi)$$

$$\omega = -\cos(2\phi)\, d(\theta + \phi)$$

$$d\omega = 2\sin(2\phi)\, d\phi \wedge d\theta$$

$$= \underbrace{2\cos\phi}_{r} \cdot \underbrace{2\sin\phi \;\; d\phi}_{-dr} \wedge d\theta$$

$$= -r\, dr \wedge d\theta = -dx \wedge dy$$

so,

$$\int_{\partial D} \omega = \int_D d\omega = -\int_D dx \wedge dy = -\text{Area}(D).$$

$\square$

CANONICAL THEOREMS IN $\mathbb{R}^3$:

Gauss' Theorem: $\displaystyle \int_{M^3} \operatorname{div} F \; dV = \int_{\partial M^3} F \cdot n\, dA$

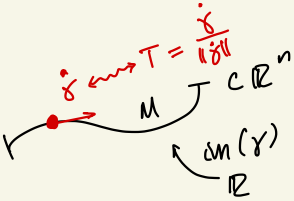
(baby) **Stokes' Theorem:** $\int_{M^2} (\text{curl } F)\, n\, dA = \int_{\partial M^3} F\, T \cdot ds$

Suppose M^k oriented in $\mathbb{R}^n$ w/ orientation given by $\eta \in \Omega^k(M)$, η nowhere zero.

Volume form on M $= dV =$ Is the form for which $dV(\xi_1, \ldots, \xi_k) = 1$ if $\xi_1, \ldots, \xi_k$ make a positive orthonormal basis of $T_p M$.

Agrees w/ orientation, i.e., $\eta(\xi_1, \ldots, \xi_k) > 0$

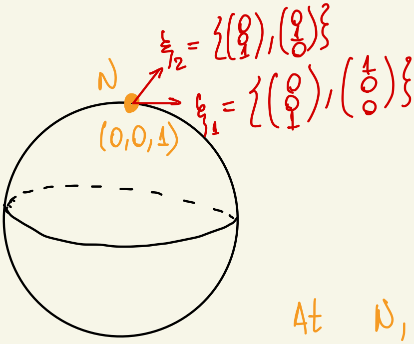
EXAMPLE 1:



$\Omega(M) \ni dV = dl = ds$

$(ds)(T) = 1.$

EXAMPLE 2: $S^2 \subset \mathbb{R}^3_{x,y,z}$



$dV(\xi_1, \xi_2) := dA(\xi_1, \xi_2) \overset{!}{=} 1$

$$(dx \wedge dy)(\xi_1, \xi_2) = dx(\xi_1)\, dy(\xi_2) - dx(\xi_2)\, dy(\xi_1) = 1$$

At N, $dA = dx \wedge dy$.

oriented

Take $M^2 \subset \mathbb{R}^3$. Let $n(x)$, $x \in M$, be the positive unit normal to M:

$$n \cdot M \to \bigcup_{x \in M} T_x \mathbb{R}^3 \quad \text{s.t.}$$

0. $n(x) \in T_x \mathbb{R}^3 \quad \forall x \in M$
1. $n(x) \perp T_x M$
2. $\| n(x) \| = 1$
3. If u, v are tangents to M at x, s.t. (u,v) is positive relative to orientation of M, then (n,u,v) is a positive basis of $\mathbb{R}^3$.

Now,

$$dA\,(u,v) = \begin{vmatrix} \text{———} & u & \text{———} \\ \text{———} & v & \text{———} \\ \text{———} & n & \text{———} \end{vmatrix}$$

Volume form on M

$u, v \in T_x M$

LECTURE 72: 3D THEOREMS

$$\int_D \operatorname{div} G \, dV = \int_{\partial D} G \cdot n \, dA$$ = Gauss' Theorem

$$\int_S (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$$ = Stokes' Theorem

$D \subset \mathbb{R}^3$ compact oriented w/ boundary.
$S \subset \mathbb{R}^3$ surface compact oriented w/ boundary.

For $D \subset \mathbb{R}^3$, $dV = dx \wedge dy \wedge dz$.

For $S \subset \mathbb{R}^3$, n its unit normal,

$$dA(u,v) = \begin{vmatrix} - & u & - \\ - & v & - \\ - & n & - \end{vmatrix} = \overbrace{(u \times v)}\cdot n$$

"Area of the parallelogram defined by u, v."

Note
$(dy \wedge dz)(u,v)$
$= u_2 v_3 - u_3 v_2$

$$= (n_1 \, dy \wedge dz + n_2 \, dz \wedge dx + n_3 \, dx \wedge dy)(u,v)$$

$$\Rightarrow \quad dA = n_1 \, dy \wedge dz + n_2 \, dz \wedge dx + n_3 \, dx \wedge dy$$

Ex: On S^2, $dA = x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy$.

So, $dA(u,v) = (u \times v) \cdot n = \pm |u \times v|$

$$= \pm \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2}$$

So, compute areas as follows:

if S is the image of an orientation preserving 2-cube c:

$$\text{Area}(S) = \int_S dA = \int_{[0,1]^2} c^*(dA)$$

Pullback form = pushforward of tangent vectors

$$= \int_{[0,1]^2} dA(c_* e_1, c_* e_2)$$

$$= \int_{[0,1]^2} dA(\partial_1 c, \partial_2 c)$$

$$= \int_{[0,1]^2} \sqrt{|\partial_1 c|^2 |\partial_2 c|^2 - \langle \partial_1 c, \partial_2 c \rangle^2}$$

Recall: in $\mathbb{R}^3$

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \\
\{\text{functions } f\} \arrow[u, "\omega^0"'] \arrow[r, "\text{grad}"] & \{\text{vec. field } F\} \arrow[u, "\omega^1"'] \arrow[r, "\text{curl}"] & \{\text{vec. field } G\} \arrow[u, "\omega^2"'] \arrow[r, "\text{div}"] & \{\text{functions } g\} \arrow[u, "\omega^3"']
\end{tikzcd}$$

$$\omega_f^0 = f \ ; \quad \omega_F^1 = F_1\, dx + F_2\, dy + F_3\, dz$$

$$\omega_G^2 = G_1\, dy \wedge dz + G_2\, dz \wedge dx + G_3\, dx \wedge dy$$

$$\omega_g^3 = g\, dx \wedge dy \wedge dz$$

Claim 1: On an oriented curve in $\mathbb{R}^3$,

$$\omega_F^1 = (T \cdot F)\, ds$$

Claim 2: On an oriented surface S in $\mathbb{R}^3$,

$$\omega_G^2 = (G \cdot n)\, dA$$

Claim 3: On an oriented domain $D \subset \mathbb{R}^3$,

$$\omega_g^3 = g\, dV.$$

With all that,

$$\boxed{\begin{aligned} &\int_D d\omega_G^2 = \int_{\partial D} \omega_G^2 \Rightarrow \int_D \operatorname{div} G \, dV = \int_{\partial D} G \cdot n \, dA \\ &\int_S d\omega_F^1 = \int_{\partial S} \omega_F^1 \Rightarrow \int_S (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds \end{aligned}}$$

The End