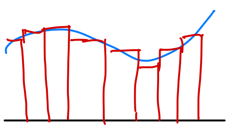
LECTURE 1

Mar 24^{th}, 2026

CHEBYSHEV: $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$, $n \in \mathbb{N}$, $p \in \mathcal{P}_n$ (poly of deg at most n), solve $\max_{x \in [a,b]} |f(x) - p(x)|$

Usual Input Suspects:
- What function are we approximating?
- With what object? (linear fcts, nonlinear, poly...)
- In what sense (distance, etc)
- Is it accurate?
- Numerically stable?
- Can it be carried out in a reasonable time?

EXAMPLE: (QUADRATURE) Goal: Approximate $I(f) = \int_0^1 f(x)\,dx$ using samples.



left-hand sum, right-hand sum

$L_n(f)$, $R_n(f)$,

$$|R_n(f) - I_n(f)| = \frac{C\|f'\|_\infty}{n}$$

More generally,

$$\sum_{k=0}^{n} f(x_k)\, w_k$$

nodes, weight

Trapezoid Rule: $$T_n(f) = \frac{f(0)+f(1)}{2n} + \frac{1}{n}\sum_{k=1}^{n-1} f\left(\frac{k}{n}\right).$$

All error comes from the endpoints!

$$|T_n(f) - I(f)| \le C\|f''\|_\infty n^{-2}$$

Fourier Series: f smooth, $f^{(k)}(0) = f^{(k)}(1) \quad \forall k \geq 0$.

$$f(x) = \sum_{m \in \mathbb{Z}} e^{2\pi i m x} f_m, \quad f_m := \int_0^1 e^{-2\pi i m x} f(x)\, dx.$$

$$\Rightarrow T_n(x) = \sum_{j=0}^{n-1} \left[\sum_{m \in \mathbb{Z}} e^{2\pi i m x_j} f_m \right] \frac{1}{n} \qquad \left(x_j := \frac{j}{n} \right)$$

$$= \sum_{m \in \mathbb{Z}} \frac{f_m}{n} \underbrace{\left(\sum_{j=0}^{n-1} \left(e^{2\pi i m/n}\right)^j \right)}_{= \begin{cases} n, & e^{2\pi i m/n} = 1 \\ \frac{e^{2\pi i m} - 0}{e^{2\pi i m/n} - 1}, & \text{else} \end{cases}}$$

$$= \sum_{m \in n\mathbb{Z}} f_m. \quad \Rightarrow \quad T_n(f) = \sum_{m \in n\mathbb{Z}} f_m$$

$$I(f) = f_0$$

Want f where f_m (Fourier coeffs of f) decay very fast

$$E_n(f) = \left| \sum_{\substack{m \in n\mathbb{Z} \\ m \neq 0}} f_m \right|.$$

Integrate by parts: $$f_m = \frac{-1}{2\pi i m} \int_0^1 \frac{d}{dx}\left(e^{-2\pi i m x}\right) f(x)\, dx$$

by parts $$= \frac{1}{2\pi i m} \int_0^1 e^{-2\pi i m x} f'(x)\, dx$$ Periodicity $\Downarrow$ Boundary terms cancel

$$= \dots$$

$$\rightsquigarrow f_m = \frac{1}{(2\pi i m)^k} \int_0^1 e^{-2\pi i m x} f^{(k)}(x)\, dx$$

$$\Rightarrow \quad |f_m| \le \frac{1}{(2\pi|m|)^k} \|f^{(k)}\|_1$$

$$\Rightarrow \quad \text{Error bound:} \quad E_n \le \frac{\|f^{(k)}\|_1}{(2\pi n)^k} \underbrace{\sum_{\substack{j\in\mathbb{Z}\\ j\neq 0}} \frac{1}{j^k}}_{=C}$$

$$\text{i.e.,} \quad E_n \le \frac{C\|f^{(k)}\|_1}{(2\pi n)^{k}}$$

Trade off between smoothness & integration error

NOTE: If not periodic, can choose p s.t. $g = f - p$ is periodic to make it cancel out the derivatives of f from integration by parts.

* EULER-MACLAURIN: $f : [m,n] \to \mathbb{R}$, $m \le n$,

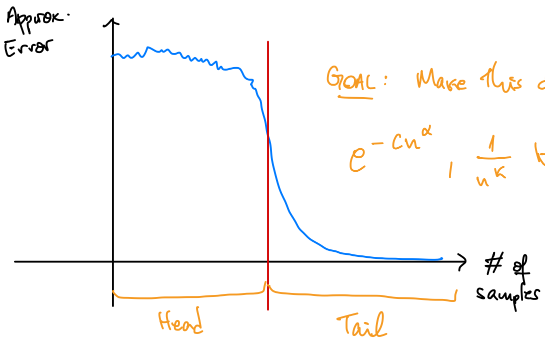
$$\begin{cases} I_{m,n}(f) := \int_n^m f(x)\,dx \\ S_{m,n}(f) := f(m+1) + \cdots + f(n) \end{cases}$$

Then, $\forall k \in \mathbb{N}$, $f \in C^k([m,n])$,

$$S_{m,n} - I_{m,n} = \sum_{j=1}^{k} \frac{B_j}{j!}\left[f^{(j-1)}(n) - f^{(j-1)}(m)\right] + R_k$$

$$|R_k| \le \frac{2\zeta(k)}{(2\pi)^k}\int_m^n |f^{(k)}(x)|\,dx$$ if k even; if k odd, same but w/out the "$\zeta(k)$".

NOTE: Gold-Standard for Approximation



GOAL: Make this decay as fast as possible

$e^{-cn^{\alpha}}$, $\frac{1}{n^{k}}$ $\forall k$, $\frac{1}{n^{\alpha}}$ for α large enough

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BASES OF FUNCTION SPACES

DEF: A subset X of E is a Hamel Basis if every vector $v \in E$ can be expressed as a unique finite linear combination of elements in X; i.e., $\forall v \in E$ $\exists \alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $v_1, \ldots, v_n \in X$ s.t. $v = \alpha_1 v_1 + \cdots + \alpha_n v_n$.

PROPOSITION: Every linear vector space has a Hamel basis.

* BANACH SPACES

DEF: A sequence $\{v_n\}_{n\in\mathbb{N}} \subset X$, X Banach space, is a Schauder Basis if $\forall v \in X$ $\exists! \{\alpha_n\}_{n\in\mathbb{N}}$ s.t. $v = \sum_{j\in\mathbb{N}} \alpha_j v_j$.

← implicitly we need this to converge

WARNING: X needs to be separable for a Schauder basis to exist! Otherwise, $\exists$ counterexamples to existence of Schauder basis.

THM: There exists $M > 0$ s.t. $\forall v \in X$,

$$\left\| \sum_{j=1}^{N} \alpha_j v_j \right\| \leq M \|v\|, \quad \forall N \in \mathbb{N}.$$

— MINIMAL SUCH M IS CALLED THE BASIS CONSTANT: $K = \sup_{N \geq 1} \left\| \sum_{j=1}^{N} \alpha_j v_j \right\|$

Pf: Define $$E := \left\{ \{\alpha_j\}_{j \in \mathbb{N}} : \sum_{j \in \mathbb{N}} \alpha_j v_j \text{ converges in } X \right\}.$$

For $\alpha \in E$, set $\|\alpha\|_F := \sup_{N \in \mathbb{N}} \left\| \sum_{j=1}^{N} \alpha_j v_j \right\|$.

Note $E \overset{\text{bounded bijection}}{\cong} X$. So by Inverse Function Thm, the inverse is also bounded.

Set $S_N(v) := \sum_{j=1}^{N} \alpha_j(v) v_j$. Note: $\alpha_j \in X^*$, so call them v_j^* for each j.

$$\Rightarrow v = \sum_{j \in \mathbb{N}} v_j v_j^*(v).$$

$$\Rightarrow \|v_j\| \, |v_j^*(v)| = \|v_j v_j^*(v)\|$$

$$= \big| \|S_j(v) - S_{j-1}(v)\| \big|$$

$$\leq 2M\|v\| \, ; \text{ i.e., } \|v_j\| \|v_j^*\| \leq 2M \quad \forall j.$$

∎

LECTURE 2

Mar 26, 2026

Functional Anal Qual?

THM: $A \subset X$ precompact iff A bounded & the basis expansion converges uniformly

* PROJECTIONS

DEF: E Banach, $P: E \to E$ is a projection if it is bounded and $P^2 = P$.

THM: $P: E \to E$ is a projection, then $\operatorname{im} P = \ker(\mathbb{1} - P)$ is closed.

THM: If P is a projection, then $\mathbb{1} - P : E \to E$ and $P^* : E^* \to E^*$ are projections.

THM: If $PQP = QP$, then

(1) $\underbrace{\boxed{P + Q - PQ}}_{}$ is a projection onto $(\operatorname{im} P) + (\operatorname{im} Q)$ — $P \oplus Q$ BOOLEAN SUM

(2) QP is a projection onto $(\operatorname{im} P) \cap (\operatorname{im} Q)$

$$(P \oplus Q)^2 = P(P+Q-PQ) + Q(P+Q-PQ) - PQ(P+Q-PQ) = \cdots = P \oplus Q$$

Thm. If U is a finite-dimensional ($\dim U = n < \infty$) subspace of a Banach space E, then $\exists\, P : E \hookrightarrow U$ with norm at most n.

(Baby Auerbach's Thm)

Lemma: If $U \subset E$ has $\dim U = n < \infty$ in E Banach, then $\exists\, u_1, \dots, u_n \in U$ and $\exists\, \ell_1, \dots, \ell_n \in U^*$ s.t.

$$\|u_j\|_U = 1, \quad \|\ell_j\|_{U^*} = 1, \quad \ell_j(u_i) = \delta_{ij}.$$

Pf: (Lemma) Take a basis $v_1, \dots, v_n$ for U. Consider

$$M : (U^*)^{\otimes n} \longrightarrow \mathbb{R}$$

$$M(\omega_1, \dots, \omega_n) := \left| \det(\omega_j(u_i)) \right|.$$

well-defined b/c U finite dimensional

Since U is finite dimensional and M continuous, M attains its max. m_* on $B_{U^*} \times \dots \times B_{U^*}$ at some pt. $(\ell_1, \dots, \ell_n)$. Clearly, $m_* > 0$.

set

$$A_{ji} = \ell_i(v_j) \quad \text{and} \quad C := A^{-1}$$

Set

$$u_j := \sum_{k=1}^{n} c_{j,k} v_k \quad \Rightarrow \quad \ell_i(u_j) = \delta_{ij}.$$

Can normalize things to make $\|\ell_i\|_{U^*} = 1$ $i = 1, \dots, n$.

Take $\psi \in U^*$, $\psi(u_i) = \underbrace{\sum_{j=1}^{n} c_{i,j} \psi(v_j)}_{\text{vector indexed by } i \text{ and solves } Ax = \psi(v)}$

By Cramer's rule,

replace l_i in col. by $\psi(v)$

$$\psi(u_i) = \sum_{j=1}^{n} c_{i,j}\, \psi(v_j) = \frac{\det(A')}{\det(A)}$$

$$|\psi(u_i)| = \left| \frac{M(l_1, \dots, l_{i-1}, \psi, l_{i+1}, \dots, l_n)}{M(l_1, \dots, l_n)} \right| \leq \|\psi\|.$$

max. val. on unit ball

$\square$

NOTE: $P : E \hookrightarrow U$, $u_1, \dots, u_n \in U$

$l_1, \dots, l_n \in U^*$ ← replace U^* by E^* by Hahn–Banach

Pf: (Thm) set $P(v) := \sum_{i=1}^{n} u_i\, l_i(v)$ using Auerbach.

$\operatorname{im} P = U$, $P^2 = P$.

$$\|Pv\| = \Big\| \sum_i u_i\, l_i(v) \Big\| \leq \sum_i \|u_i\|\, \|l_i\|\, \|v\| = n\, \|v\|.$$

$\square$

NOTATION: X is Banach space

$A \subset X$ nonempty subset

$$\operatorname{dist}(v, A) := \inf_{u \in A} \|v - u\|.$$

$$P_A(v) := \{ u_0 \in A : \|v - u_0\| = \operatorname{dist}(v, A) \}.$$

DEF: $A \subset X$ is an <u>existence set</u> if $P_A(v)$ is nonempty $\forall v \in X$

a <u>uniqueness set</u> if $\# P_A(v) \leq 1 \quad \forall v \in X$

a <u>Chebyshev set</u> if $P_A(v)$ existence & uniqueness set $\forall v \in X$.

FACTS:

- Existence $\Rightarrow$ closed

  $\cancel{\Leftarrow}$ Take $A = \{(1 + 1/j)\, e_j\}_{j=1}^{\infty}$ in ℓ^2 (basis: e_j)

- $X = \ell^\infty$, $A = \{v \in \ell^\infty : \|v\|_\infty \leq 1\}$

  is existence but <u>not</u> uniqueness

- X Banach and $\dim A < \infty \Rightarrow A$ is existence.

$$\boxed{\textbf{LEMMA:} \quad A \text{ convex} \Rightarrow P_A(v) \text{ convex}}$$

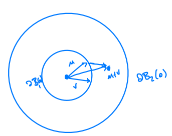
Pf: $u_0, u_1 \in P_A(v)$, $u_\lambda := \lambda u_0 + (1-\lambda) u_1$, $\lambda \in (0,1)$

$$\begin{aligned}
\|v - u_\lambda\| &= \|v - \lambda u_0 - (1-\lambda) u_1\| \\
&= \|\lambda (v - u_0) + (1-\lambda)(v - u_1)\| \\
&\leq \lambda \|v - u_0\| + (1-\lambda)\|v - u_1\| \\
&= \text{dist}.
\end{aligned}$$

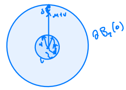
$\square$

DEF: X is <u>strictly convex</u> if $\forall u, v \in \partial B$, $u \neq v$, we have $\|u + v\| < 2$.

$B := B_1(0) \subset X$



DEF: X is <u>uniformly</u> <u>convex</u> if given $\varepsilon > 0$ $\exists\, \delta(\varepsilon) > 0$ s.t. $\forall\, u, v \in \partial B$ with $\|u+v\| > 2-\delta$ we have $\|u-v\| < \varepsilon$.



DEF: X is <u>smooth</u> if for each $v \in \partial B$ $\exists!$ supporting functional (i.e., $\ell \in X^*$, $\|\ell\| = 1$, $\ell(v) = \|v\|$).

LEMMA: If M is convex and the space is strictly convex then M is a uniqueness set

⚠

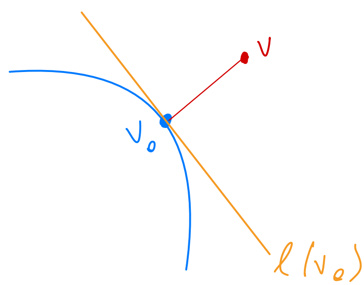
WARNING: $C^0(X)$ is <u>not</u> strictly convex (need to work harder)

THM: (CRITERION FOR BEST APPROXIMATION) If $M \subset X$ is convex in Banach space X, and $v \in X \setminus \overline{M}$, then $v_0 \in M$ is a best approximation to v in M iff $\exists\, \ell \in X^*$ s.t.

(1) $\|\ell\|_{X^*} = 1$

(2) $\ell(v - v_0) = \|v - v_0\|$

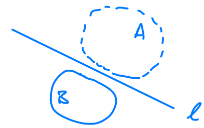
(3) $\mathrm{Re}\left(\ell(u - v_0)\right) \leq 0 \quad \forall\, u \in M$

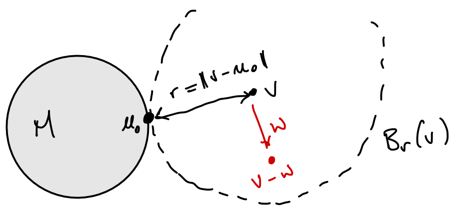


⚠ **KOLMOGOROV CRITERION**

Pf: ($\Rightarrow$) Uses Geometric Hahn-Banach.

THM: (Geometric Hahn-Banach) $A, B \subset X$ disjoint non-empty convex, X Banach, A open, then $\exists\, \ell \in X^*$ bounded and $\alpha \in \mathbb{R}$ s.t. $\operatorname{Re} \ell(u) < \alpha \le \operatorname{Re} \ell(v)$ $\forall u \in A, v \in B$.





By Geometric Hahn-Banach, $\exists\, \tilde{\ell} \in X^*$, $\alpha \in \mathbb{R}$, with $\operatorname{Re} \tilde{\ell}(\operatorname{int} B_r(v)) > \alpha$ and $\operatorname{Re} \tilde{\ell}(\overline{M}) \le \alpha$.

$\operatorname{Re} \tilde{\ell}(B_r(v)) \geq \alpha$

Set $\beta := \operatorname{Re} \tilde{\ell}(v - u_0) \overset{!}{>} 0$

$\ell := \frac{r}{\beta} \tilde{\ell}$

By linearity, $\ell(v - u_0) = \| v - u_0 \|$ and $\operatorname{Re} \ell(v - u_0) \le 0$.

By contradiction, suppose $\|\ell\|_{X^*} > 1$ then $\exists\, w \in B_r(0)$ with $\ell(w) > r$

$z := v - w \in B_r(v)$.

Then
$$\operatorname{Re} \tilde{\ell}(z) = \operatorname{Re} \tilde{\ell}(v - u_0) + \operatorname{Re} \tilde{\ell}(u_0) - \operatorname{Re} \tilde{\ell}(w)$$
$$\le \beta + \alpha - \frac{\beta}{r} \ell(w)$$
$$< \alpha \quad \text{↯}$$

($\Leftarrow$) Conversely, if such ℓ exists then, for any $u \in M$,
$$\| u - v \| \geq \operatorname{Re} \ell(v - u)$$
$$= \operatorname{Re} \ell(v - u_0) - \operatorname{Re} \ell(u_0 - u)$$
$$\geq \| v - u_0 \|.$$

$\square$

EXAMPLE: $X = \mathcal{H}$ Hilbert space

u_0 in M convex is a best approx

$\iff \operatorname{Re}\langle v - u_0, u - u_0 \rangle \leq 0 \quad \forall u \in M$

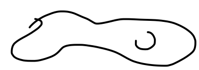
EXAMPLE: $X = L^p(\Omega, d\mu)$, $1 < p < \infty$.

M is a subspace of X then $u_0 \in M$ is a best approx. to $v \in L^p(\Omega, d\mu)$ iff

$$\int_\Omega |v(x) - u_0(x)|^{p-2} \left[\bar{v}(x) - \bar{u}_0(x)\right] u(x) \, d\mu(x) = 0 \quad \forall u \in M.$$

NOTE: Single out v by a subspace as subset of X

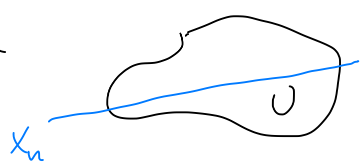
Alternatively, approximate a subspace U by finite dimensional subspaces of X.



LECTURE 3

Mar 31, 2026

GOAL: Approximate a subset $U \subset X$ by a dimension-$n < \infty$ subspace of X



"best n-dim. subspace approx. to U"

DEF: X Banach, $U \subset X$. The Kolmogorov n-width of U in X is given by

$$d_n(U) := d_n(U, X) := \inf_{X_n \subset X} \sup_{u \in U} d(u, X_n).$$

X_n n-dimensional subspaces

PROVE THESE!

PROPERTIES

(1) $d_n(U) = d_n(\bar{U})$

(2) $\text{hull}(U)$, $d_n(\text{hull}(U)) = d_n(U)$

convex hull

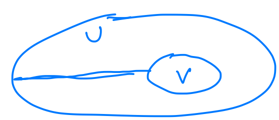
(3) $b(U) := \{\alpha v : v \in U, |\alpha| \leq 1\}$

$$\Rightarrow d_n(U) = d_n(b(U))$$

(4) $d_n(\alpha U) = |\alpha| \, d_n(U) \quad \forall$ scalars α.

(5) $V \subset U$,

$$d_n(U) - \sup_{v \in U} \inf_{w \in V} \|v - w\| \leq d_n(V) \leq d_n(U)$$



(6) $d_n(U) \geq d_{n+1}(U) \quad \forall n$

(approx. gets better by taking larger and larger dimensional spaces)

(7) $Y \supset X$ and Y Banach $\Rightarrow$ $d_n(U, X) \geq d_n(U, Y)$.

(Y bigger space has more options to make the approximation better...)

PROPOSITION: Take $B(X)$ closed unit ball of X, $\dim X > n$, then $d_k(B(X)) = 1$, $k = 0, 1, \dots, n$.

Pf: $d_n(B(X)) \leq 1$ trivially.

On the other hand, given X_n and $v \in B(X)$, $v \notin X_n$.

If w_0 is a best approx. to v from X_n, then 0 is a best approx. to $v - w_0$ from X_n

$$\Rightarrow \text{ for } \tilde{v} := \frac{v - w_0}{\|v - w_0\|}, \quad d(\tilde{v}, X_n) = 1.$$

$\square$

THM: Let $X_{n+1} \subset X$ be a $(n+1)$-dimensional subspace, then

$$d_k(\underbrace{B(X_{n+1})}_{\text{Unit ball of } X_{n+1}}) = 1, \quad k = 0, 1, \dots, n$$

Pf: Clearly $d_k(B(X_{n+1})) \leq 1$.

On the other hand, take Y with $\dim Y \leq n$.

For simplicity, assume X is strictly convex $\rightsquigarrow$ gives uniqueness. (cf. Lecture 2)

Take bases $X_{n+1} \leftrightsquigarrow u_1, \ldots, u_{n+1}$

$Y \leftrightsquigarrow v_1, \ldots, v_k$

Define $\Omega := \left\{ \alpha \in \mathbb{R}^{n+1} : \left\| \sum_{j=1}^{n+1} \alpha_j u_j \right\| \leq 1 \right\}$

For each $\alpha \in \Omega$, set $T(\alpha) \in \mathbb{R}^n$ to be the coeffs. of the best approx. from Y in the $\{v_j\}$ basis.

Can then check that T is odd and continuous from $\partial\Omega \to \mathbb{R}^n$

Also, $\mathrm{int}(\Omega)$ is bdd, open, symmetric nbhd of $0 \in \mathbb{R}^n$.

By Borsuk - Ulam Antipodality, $T(\alpha_*) = 0$ for some α_*.

"$\square$"

<u>COROLLARY</u>: If U contains a ball of radius $\lambda > 0$ in some $(n+1)$-dimensional subspace of X, then

$$d_n(U, X) = \lambda.$$

(Largest such λ is called BERNSTEIN n-WIDTH)

<u>CONTINUOUS FUNCTIONS</u>

- Equipped with $\|\cdot\|_\infty = \sup |\cdot|$

DEF: (Haar subspace) A <u>Haar</u> <u>subspace</u> M is an n-dim. subspace of $C(\Omega)$ such that all elements of M have at most $n-1$ zeros in Ω.

LEMMA: M is an n-dim. Haar subspace

$\Longleftrightarrow$ $\forall$ n pts. $x_1, \dots, x_n \in \Omega$ distinct and all $\beta_1, \dots, \beta_n \in \mathbb{R}$, the interpolation problem Find $u \in M$ s.t. $u(x_i) = \beta_i$ $\forall\, i = 1, \dots, n$ always has a solution.

$\Longleftrightarrow$ For a given basis $\{u_1, \dots, u_n\}$ of M, the map $\phi_{ij} = u_j(x_i)$ is always invertible

IMPORTANT

THM: (HAAR UNIQUENESS) If Ω is locally compact, then a finite dimensional subspace $M \subset C(\Omega)$ is Haar iff every $f \in C(\Omega)$ has a best approx. in M; i.e., M is Chebyshev.

THM: (MAIRHUBER – CURTIS) $\Omega \subset \mathbb{R}^d$, $d \geq 2$, contains an interior point, then $\nexists$ Haar subspace of dim ≥ 2.

LECTURE 4

Apr 2nd, 2026

→ We don't necessarily need constant fcts in the algebra but we can include them w/out problems.

! IMPORTANT !

→ Chair of UChicago Math 1946–1952 ⇝ "Stone Age"

THM: (STONE - WEIERSTRASS) Let X be a compact metric space and $\mathcal{A}$ be a subalgebra of $C(X)$. If $\mathcal{A}$ separates points in X and vanishes nowhere, then $\mathcal{A}$ is dense in $C(X)$.

$\mathcal{A}$ algebra $\Leftrightarrow$ $\vec{\mathcal{A}}$ vector space paired w/ multiplication operation (s.t. $(fg)h = f(gh)$, $f(g+h) = fg + fh$, $\alpha(fg) = (\alpha f)g = f(\alpha g)$)

e.g.: $C(X)$ is an algebra. Typically, $X = [a,b] \subset \mathbb{R}$, $\mathcal{A}$ can be the set of polynomials w/ ptwise multiplication.

$\mathcal{A}$ algebra is normed $\Longleftrightarrow$ $\|\cdot\|$ norm s.t. $\|fg\| \le \|f\|\,\|g\|$.

e.g.: $\|\cdot\|_{C(X)}$ satisfies this since $\|fg\|_\infty = \sup_x |f(x)g(x)| \le \sup_x |f(x)| \cdot \sup_x |g(x)| = \|f\|\|g\|$.

A set $\mathcal{A}$ separates points $\Longleftrightarrow$ $\forall x,y \in X$, $x \neq y$, $\exists f \in \mathcal{A}$ s.t. $f(x) \neq f(y)$.

can't only have bumps

A set $\mathcal{A}$ vanishes nowhere $\Longleftrightarrow$ $\forall x \in X$ $\exists f \in \mathcal{A}$ s.t. $f(x) \neq 0$.

Pf: (Stage 1) Cl: if $f, g \in \mathcal{A}$, then $\max\{f,g\} \in \bar{\mathcal{A}}$

Enough to show that $f \in \mathcal{A} \Rightarrow |f| \in \bar{\mathcal{A}}$ $\quad \max\{0,f\} = \frac{f+|f|}{2}$.

Clearly, if $f \in \mathcal{A} \Rightarrow f \cdot p(f) \in \mathcal{A}$ $\forall$ polynomials p.

WTF: poly. p s.t. $\| f p(f) - |f| \| < \varepsilon$.

Suffices to find p s.t. $|x p(x) - |x|| < \varepsilon \quad \forall x \in [-\|f\|, \|f\|]$

i.e., suffices to approximate $|x|$ on some $[-a, a]$.

WLOG, $a \equiv 1$.

$$|x| = \sqrt{1-(1-x^2)} \overset{\text{Binomial}}{=} \underbrace{\sum_{n=0}^{\infty} (1-x^n)^n (-1)^n \binom{1/2}{n}}_{\text{Converges uniformly on } [-1,1]}$$

Let $q_N(x) := \sum_{n=0}^{N} (1-x^n)^n (-1)^n \binom{1/2}{n} \longleftarrow$ N-th partial sum which is a polynomial

$\forall \varepsilon > 0 \ \exists N$ s.t. $|q_N(x) - |x|| < \varepsilon \quad \forall x \in [-1, 1]$.

Thus, $|q_N(0)| < \varepsilon \implies |[q_N(x) - q_N(0)] - |x|| < 2\varepsilon$.

$q_N(x) - q_N(0) = x P_N(x)$, P_N deg-N poly.

Thus, $f P_N(x)$ is the desired approx. of $|f|$.

(<u>Stage 2</u>) Find $g_x(x) \in \mathcal{A}$ s.t. $g_x(t) = f(t)$ and $g_x(t) > f(t) - \varepsilon$

i.e., $g_x \gtrsim f$.

Since $\mathcal{A}$ separates pts. $\exists h_{x,y} \in \mathcal{A}$ s.t. $h_{x,y}(x) = f(x)$

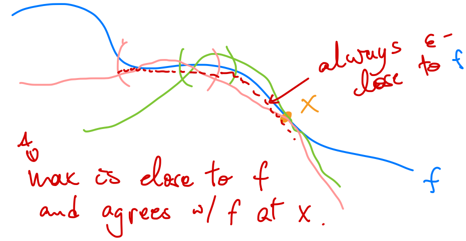
$h_{x,y}(y) = f(y)$.

$\forall y \ \exists B(y)$ ball centered at y s.t. $h_{x,y}(t) = f(t) - \varepsilon$ in $B(y)$

But X is compact $\implies \exists$ finite subset $B_{y_1}, \ldots, B_{y_{n_x}}$ covering X

Let $g_x(t) := \max_{i=1,\dots,n_x} h_{x,y_i}(t)$

$> f(t) - \varepsilon$



This is the desired $g_x \in \bar{A}$

(Stage 3) $\forall x \ \exists \tilde{B}_x$ s.t. $g_x(t) < f(t) + \varepsilon$ in $\tilde{B}_x$.

Then compactness let's us pick a finite covering $\tilde{B}_{x_1}, \dots, \tilde{B}_{x_m}$. Then $\tilde{f}(t) = \min_{i=1,\dots,m} g_{x_i}(t) < f(t) + \varepsilon$

But each $g_{x_i}(t) > f(t) - \varepsilon \quad \forall t \in X$

i.e., $\quad f(t) - \varepsilon < \tilde{f}(t) < f(t) + \varepsilon$

$\Rightarrow \quad \| f - \tilde{f} \|_\infty < \varepsilon$ and $\tilde{f} \in \bar{A}$.

$\square$

REMARK: the $\mathbb{C}$ version of Stone-Weierstrass requires A to be closed under conjugation.

⚠

COROLLARY: $\{ f(x) g(y), \ x \in X, y \in Y : f \in C(X), \ g \in C(Y) \}$ is dense in $C(X \times Y)$.

⚠

COROLLARY: If $K \subset \mathbb{R}^n$ is compact, then the set n-variate polynomials is dense in $C(K)$

THM: (WEIERSTRASS APPROXIMATION) If $f \in C([0,1])$ then $\exists\, p$ polynomial such that $|f(x) - p(x)| < \varepsilon \;\; \forall x \in [0,1]$

⚠ More constructive than Stone-Weierstrass

Pf 1: (BERNSTEIN) Given $f \in C([0,1])$, define

$$B_n(f)(x) := \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^{n-k}.$$

e.g.:

(i) $B_n(1)(x) = 1$

(ii)
$$B_n(t \mapsto t)(x) = \sum_{k=0}^{n} \frac{k}{n}\binom{n}{k} x^k (1-x)^{n-k}$$
$$= \frac{1}{n}\, x \sum_{k=1}^{n} \binom{n}{k} k x^{k-1} (1-x)^{n-k}$$
$$= \frac{x}{n}\, \partial_s (s+t)^n \Big|_{s=x,\, t=1-x} = x.$$

(iii)
$$B_n(t \mapsto t^2)(x) = \cdots\cdots = x^2\underbrace{\left(1 - \frac{1}{n}\right)} + \frac{x}{n}.$$

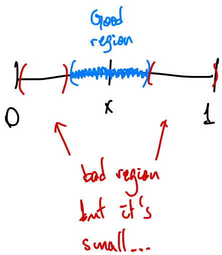
Error of size $O(1/n)$

Given $\delta > 0$ let F_x be the set of k s.t. $\left|\frac{k}{n} - x\right| \geq \delta$

"Far field"

But then

$$\sum_{k \in F_x} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{\delta^2} \sum_{k \in F_x} \binom{n}{k}\left(\frac{k}{n} - x\right)^2 x^k (1-x)^{n-k}$$



$$\leq \frac{1}{\delta^2} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{k^2}{n^2} - 2\frac{k}{n} x + x^2 \right) x^k (1-x)^{n-k}$$

$$\leq \frac{1}{\delta^2} \left[x^2 B_n(1)(x) - 2x B_n(t)(x) + B_n(t^2)(x) \right]$$

$$= \frac{1}{\delta} \left[x^2 - 2x^2 + x^2\left(1 - \frac{1}{n}\right) + \frac{x}{n} \right]$$

$$= \frac{x - x^2}{n \delta^2} \leq \frac{1}{4n\delta^2} .$$

i.e., $\left| f(x) - B_n(f)(x) \right| = \left| \sum_{k=0}^{n} \binom{n}{k} \left(f(x) - f\left(\frac{k}{n}\right) \right) x^k (1-x)^{n-k} \right|$

i.e., given $\varepsilon > 0$ $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ when $|x - y| < \delta$ $\forall x$ by unif. continuity ($[0,1]$ compact and cont. on compact is unif. cont.). With this choice of δ,

$$= \sum_{k \in F_x} \binom{n}{k} x^k (1-x)^{n-k}\, 2\|f\| + \sum_{k \notin F_x} \binom{n}{k} \varepsilon\, x^k (1-x)^{n-k}$$

$$\leq 2\|f\| \frac{1}{4n\delta^2} + \varepsilon \longrightarrow B_n(f) \longrightarrow f \text{ uniformly.}$$

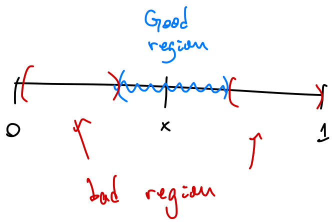
$\square$

LECTURE 5

Apr 7, 2026

THM: (Weierstrass Approximation) $\forall f \in C([0,1])$ $\forall \varepsilon > 0$ $\exists p$ polynomial s.t. $\|f - p\| < \varepsilon$.

Pf 1: Bernstein polynomials $B_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1-x)^k$ $(0 \le x \le 1)$



$$B_n(f)(x) - f(x) = \sum_{k=0}^{n} \overbrace{\left[f\left(\tfrac{k}{n}\right) - f(x)\right]}^{\text{Good region} \Leftrightarrow \text{this is small}} \binom{n}{k} x^k (1-x)^k$$

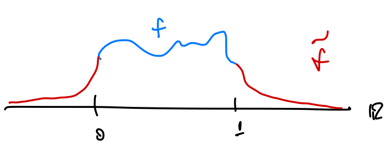
bad region this is small.

REMARK: $B_n(f)(x)$ resembles binomial

$$B_n(f)(x) = \mathbb{E}\left[f\left(\frac{K(n,x)}{n}\right)\right] = \mathbb{E}\left[f\left(\frac{\sum_{i=1}^{n} X_i}{n}\right)\right]$$

$$f(x) = \lim_{n\to\infty} \mathbb{E}[f(X)]$$

Pf 2: (Due to Weierstrass) Don't like $[0,1]$, extend $f \to \tilde{f}$ to a fct $\tilde{f}: \mathbb{R} \to \mathbb{R}$ continuous and compactly supported.



Convolve f with smth compactly supported, or maybe a Gaussian.

$$\begin{cases} \partial_t u = \Delta u \\ u(x,0) = \tilde f \end{cases}$$

PDEs: $\lim_{t \to 0} u(x,t) = \tilde f(x)$ uniformly $\forall x \in [0,1]$

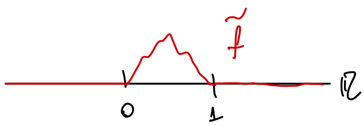
$$u(x,t) = (\Phi * \tilde f)(x,t) = \int_{\mathbb{R}} \underbrace{\frac{1}{(4\pi t)^{1/2}} e^{-|x-y|^2/4t}}_{\text{Approximate this using a poly (e.g. Taylor)}} \tilde f(y)\, dy$$

$\Rightarrow \forall \varepsilon > 0, \quad |P_{t_\varepsilon}(x) - u(x,t)| < \varepsilon.$

$\square$

Pf 3: (Due to Landau) Take $f \in C([0,1])$ and set the extension to $\tilde f \in C_c(\mathbb{R})$

$$\tilde f := f - [f(0) + x(f(1) - f(0))]$$



$$L_n(x) := c_n \int_{-1}^{1} \tilde f(x-t)(1-t^2)^n\, dt$$

w/ c_n s.t. $c_n \int_{-1}^{1} (1-t^2)^n\, dt \overset{!}{=} 1$

Set $s := x - t$ then

$$L_n(x) = c_n \int_{x-1}^{x+1} \tilde f(s) \underbrace{[1-(x-s)^2]^n}_{\text{poly of } x} dt$$

want to get rid of this dependency in x $\leftarrow$ $\tilde f$ vanishes away of $[0,1]$

$$= c_n \int_0^1 \tilde f(s) \underbrace{[1-(x-s)^2]^n}_{\text{polynomial of } x \text{ of deg } 2n} dt \quad \Rightarrow \quad L_n(x) \text{ is a poly. of } x \text{ of deg } 2n.$$

Claim: $L_n(x)$ is close to f:

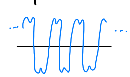
$$\left|L_n(x) - 1\cdot\tilde f(x)\right| = c_n \left|\int_{-1}^{1} \left[\tilde f(x-t) - \tilde f(x)\right](1-t^2)^n \, dt\right|$$

$c_n \int_{-1}^{1} (1-t^2)^n \, dt = 1$

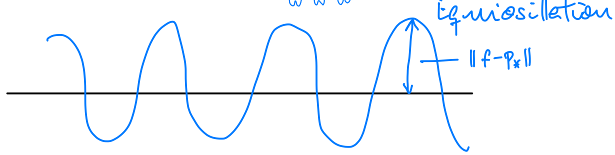
Then: as $n\to\infty$, $(1-t^2)^n$ becomes "peakier" near at fixed x's and the weight is small away from that x ($\forall x\in[0,1]$); so, as $n\to\infty$, it approximates f very well. $\square$

IMPORTANT

EQUIOSCILLATION THEOREM: The space of polynomials of degree at most n on $[-1,1]$ (denoted P_n) is a Chebysev set (i.e., it has a unique best approximation). Moreover, if f is real and P_* is best approximation to f, then P_* is real, and $f-P_*$ equioscillates in at least $n+2$ extreme points. Moreover, if $f-p_*$ equioscillates, then P_* is a best approximation.

 or 

takes max value and min value one right after the other $\longrightarrow$



$\Rightarrow$ 0 is the best approx. to such that equiosillates (i.e., for the error)

$\Rightarrow$ GOAL: Make $f-p_*$ equiosillate in at least $n+2$ pts.

Pf: (Equioscillation $\Rightarrow$ best approximation) Suppose that $f-p$ equioscillates with nodes $x_0 < \cdots < x_{n+1}$ and, by contradiction, that $\|f-q\|_\infty \leq \|f-p\|_\infty$ (i.e., p is not best $\|\cdot\|_\infty$ approx.)

Then

$$|f(x_i) - p(x_i)| = \|f-p\|.$$

So, $q - p = f - p - (f-q)$

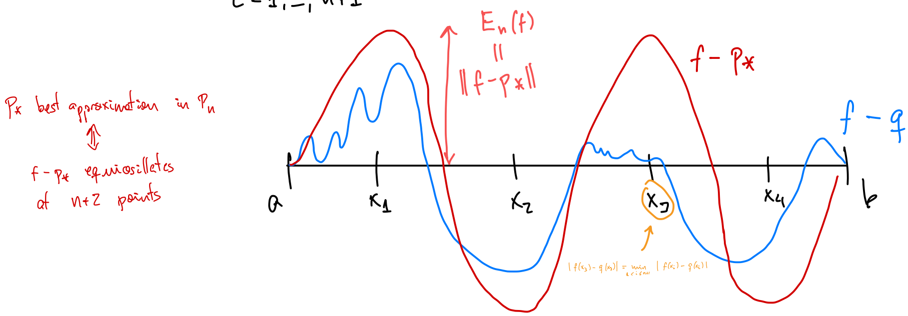
Fencepost { But $f-p$ equioscillates between $n+1$ pts $\Rightarrow$ changes sign $n+2$ times $\Rightarrow$ $n+1$ roots $\Rightarrow$ it's zero.

$\Rightarrow$ Best approx..

$\square$

DE LA VALLÉE POUSSIN THEOREM: Given $f \in C([a,b])$, suppose $q \in P_n$ and $f(x_i) - q(x_i)$ alternates sign at $a \leq x_0 < \cdots < x_{n+1} \leq b$. If $E_n(f)$ is the error of best approximation, then

$$E_n(f) \geq \min_{i=1,\dots,n+1} |f(x_i) - q(x_i)|.$$

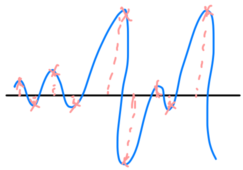


LECTURE 6

Apr 9th 2026

Recall:

oscillation points



← Bad error curve (equioscillation & fact that it doesn't oscillate between min and max enough times).

$$(f-p)-(f-q) = q-p \quad \forall p \in P_n$$ polynomials with at most n roots

Intuition: ∃ element in V that is zero on n pts and has some other value we can choose somewhere else ⇒ {can fix n+1 pts.

THM: Suppose V is an $(n+1)$-dimensional Haar subspace of $C([-1,1])$. Given $f \in C([-1,1])$, let u_* denote the best approximation to f from V. Then, u_* is unique and $f-u_*$ equioscillates in at least $n+2$ extreme points.

Pf: Fencepost → count # of roots: $(n+1)$-dim. Haar → at most n roots

$f-u_*$ equioscillation at $n+2$ pts ⇒ $n+1$ roots at least.

(Optimality ⇒ $f-u_*$ Equioscillates) ⇒ $f-u_* = 0$ ← only fct w/ enough zeros.

u_* is best approx. but $f-u_*$ has an alternant of length $k+1$: $-1 \le x_0 < \dots < x_{k+1} \le 1$, $k<n$ and no longer seq. exists. $E := \|f-u_*\|_\infty$.

GOAL: COMPUTE BEST POLYNOMIAL APPROXIMATION TO x^n IN $P_{n-1}([-1,1])$

$\iff$ To what extent is x^n a polynomial of degree $n-1$?

i.e., $\|x^n - p_*\| < \varepsilon$, what is ε ?

Set $\hat{p} := x^n p_* \in P_n([-1,1])$.

Equioscillation Thm $\Rightarrow \exists$ $n+1$ pts $-1 \le x_0 < x_1 < \cdots < x_n \le 1$

with $|x_i^n - p_*(x_i)| = \|\hat{p}\|$.

Now, $\hat{p}$ is a poly between x_1 and x_{n-1} and it equioscillates
so it attains local max / min at these pts. So, they are extrema,
so

$$\hat{p}'(x_1) = \cdots = \hat{p}(x_{n-1}) = 0 .$$

$\Rightarrow$ $x_0 = -1$ and $x_n = 1$, otherwise, the deg-$(n-1)$ poly $\hat{p}$ would have n or more roots.

$$\Rightarrow \hat{p}(x) = c(x-x_1)\cdots(x-x_{n-1}) .$$

$E^2 = \|\hat{p}\|^2 = \left(\sup_x |\hat{p}(x)|\right)^2$

Set $E := \|\hat{p}\|$ and $q := E^2 - \hat{p}^2 \geq 0$ on $[-1,1]$

and q has same roots as $\hat{p}$. $\Rightarrow$ $x_1, \ldots, x_{n-1}$ are double roots of q for a total of $2n-2$ roots

$\Rightarrow$ q has simple roots at $x_0 = -1$ and $x_n = 1$

$$\Rightarrow q(x) = \tilde{C}(1-x^2)\prod_{i=1}^{n-1}(x-x_i)^2$$

$$\Rightarrow \quad \hat{E}^2 - \hat{p}^2 = \underbrace{\beta}_{\text{leading order coeff}\dots}(1-x^2)(p')^2, \qquad \beta = \frac{1}{n^2}.$$

solve ODE

$$\Rightarrow \quad \arccos\left(\frac{\hat{p}}{\hat{E}}\right) = \pm\, n \arccos x + C$$

$$\hat{p} = \hat{E}\cos(n \arccos x + C)$$

At $x=-1$, $\hat{p} = \pm \hat{E}$,

$$\pm 1 = \cos(n\pi + C), \quad C = k\pi, \quad k \in \mathbb{Z}$$

$$\boxed{\hat{p} = \pm \hat{E}\cos(n \arccos x)}.$$

Note: $$z^n + \frac{1}{z^n} = \left(z^{n-1} + \frac{1}{z^{n-1}}\right)\left(z + \frac{1}{z}\right) - \left(z^{n-2} + \frac{1}{z^{n-2}}\right)$$

$$\Rightarrow \quad \cos(nt) = 2\cos(t)\cos((n-1)t) - \cos((n-2)t)$$

$$\Rightarrow \quad \boxed{T_n(x) := \cos(n \arccos x),} \text{ then}$$

polynomial

i.e., $$\boxed{T_n(x) = 2x\,T_{n-1}(x) - T_{n-2}(x)}$$

$$x\,T_{n-1}(x) = \frac{1}{2}\left(T_n(x) + T_{n-2}(x)\right)$$

COROLLARY: T_n is a polynomial for $n = 0, 1, 2, \dots$ and leading order coefficient is 2^{n-1} ?

i.e., $\hat{p} = \frac{1}{2^{n-1}} T_n(x)$, and so $E = \frac{1}{2^{n-1}}$

$\Downarrow$

really small.

<u>NOTATION:</u> $T_n \equiv$ <u>CHEBYSHEV POLYNOMIALS</u>.

<u>NOTE:</u> T_n has roots that minimize $\max_{x\in[-1,1]} |(x-x_1)\cdots(x-x_n)|$

$$\iff \text{minimize } \max_{x\in[-1,1]} \sum_{j=1}^{n} \log|x-x_j|.$$

$\blacksquare$

THM: (RIVLIN, NEWMAN) For $k<n$, set

$$E_k(x^n) := \max_{p\in P_k} \|x^n - p\|_\infty .$$

Then $E_k(x^n) \le 2e^{-k^2/2n}$ $\left(k = c\sqrt{n} \rightsquigarrow 2e^{-c^2/2}\right)$

$\Rightarrow$ x^n is a polynomial of degree $\sqrt{n}$ for n suff. large and suff. <u>bad</u> glasses.

This bad...

LECTURE 7

Apr 14, 2026

<u>Recall</u>: Monomials as bases are very silly b/c they "all look the same".

<u>THM</u>: If f is Lipschitz on $[-1,1]$, then f has a <u>unique</u> representation as a Chebyshev series

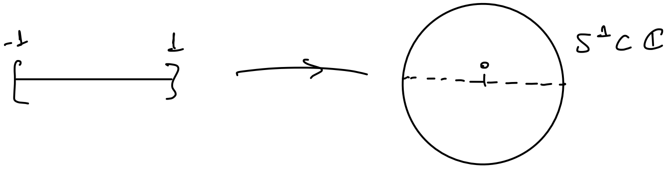
$$f(x) = \sum_{k=0}^{\infty} \alpha_k T_k(x),$$

where

$$\alpha_k = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)\,T_k(x)}{\sqrt{1-x^2}}\,dx, \quad \alpha_0 = \frac{1}{\pi}\int_{-1}^{1}\frac{f(x)}{\sqrt{1-x^2}}\,dx,$$

which is absolutely continuous and unif. convergent

Pf: $T_n(x) = \cos(n \arccos x)$



$z \in \mathbb{C}$, $|z| = 1$, $z = e^{i\theta}$

$$x = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

$$dx = \frac{1}{2}\left(1 - \frac{1}{z^2}\right)dz = \frac{i/z}{2i}\left(z - \frac{1}{z}\right)dz = \pm \frac{i}{z}\sqrt{1-x^2}\,dz$$

"+" if $\operatorname{Im} z \geq 0$

"−" if $\operatorname{Im} z < 0$

$F(x) = f(x(z))$; i.e., f Lipschitz $\Rightarrow$ F Lipschitz

$$F(x) = \frac{1}{2} \sum_{k=0}^{\infty} \underline{\underline{\alpha_k}} \left(z^k + \frac{1}{z^k} \right) = \frac{1}{2} \sum_{k=0}^{\infty} \alpha_k T_k(x(z)) .$$

Laurent series

given by Cauchy integrals:

$$\text{"}z^k\text{"} \left\{ \alpha_k = \frac{1}{i\pi} \int_{|z|=1} z^{-1-k} F(z)\, dz , \quad k > 0 \right. ,$$

$$\text{"}\tfrac{1}{z^n}\text{"} \left\{ \alpha_k = \frac{1}{i\pi} \int_{|z|=1} z^{-1+z} F(z)\, dz , \quad k > 0 \right.$$

$$\Rightarrow \quad \alpha_k = \frac{1}{2\pi i} \int_{|z|=1} \frac{1}{z} \left(z^k + \frac{1}{z^k} \right) F(z)\, dz$$

$$= \frac{1}{\pi i} \int_{|z|=1} \frac{1}{z} T_k(x(z))\, F(z)\, dz$$

Map back to $[-1,1]$ from S^1, get two contributions b/c $x(z)$ is a double cover of S^1.

$$\Rightarrow \quad \alpha_k = \frac{2}{\pi} \int_{-1}^{1} \frac{T_k(x) f(x)}{\sqrt{1-x^2}}\, dx$$

$\square$

<u>NOTE</u>: Nice, but why are Chebyshev polynomials good in particular?

CHEBYSHEV POLYNOMIALS VS. BEST APPROXIMATIONS

REMEMBER FOR TESTS & QUAL:

CHEBYSHEV EXPANSIONS ($\sum_n \alpha_n T_n$) ARE <u>NOT</u> THE BEST APPROXIMATIONS TO <u>ANY</u> FUNCTIONS. They are just "good".

- How far is it from the best?

Ideally,

$$\frac{\|f - p_T\|}{\|f - p_*\|} \le g(n)$$

Chebyshev exp. (p_T)

best approx. (p_*)

IMPORTANT MEMORIZE. (do explicitly for e^x)

THM: If $f \in C([-1,1])$, say p_* is the best polynomial approximation to f in P_n (degree $\le n$) and let p_T be its truncated Chebyshev expansion. Then

$$\frac{\|f - p_T\|}{\|f - p_*\|} \approx \frac{4}{\pi} \log n \quad \text{as } n \nearrow \infty.$$

$\Rightarrow$ If you are approximating a function that <u>should</u> <u>be</u> approximated by polys., then Chebyshev is good. But if your fct shouldn't be approximated by polys. (i.e., if you see the log n error) then Chebyshev is bad b/c probably your fct shouldn't be approximated by polys..

Pf: Let M finite dim. subspace of Banach space E.

$P_M := \mathrm{proj}_M : E \hookrightarrow M$.

Take $f \notin M$ and $p := P_M f$.

$p_* :=$ best approx to f in M.

TRICK: $(f-p)-(f-p_*) \in M$

$$\Rightarrow \quad f-p = f - p_* - \underbrace{(p-p_*)}_{P_M(f-p_*)}$$

$p = P_M f$

$P_M p_* = p_*$

$$\Rightarrow \quad \|f-p\|_E \leq \|f-p_*\|_E + \|P_M(f-p_*)\|_E$$

$$\leq \left[1 + \|P_M\|_{E^*}\right] \|f-p_*\|_E$$

$$\Rightarrow \quad \frac{\|f-p\|_E}{\|f-p_*\|_E} \leq 1 + \|P_M\|_{E^*}$$

Define $E := C([-1,1])$, $\langle u, v\rangle := \int_{-1}^{1} \frac{uv}{\sqrt{1-x^2}}\,dx$, $M := \mathbb{P}_n$

$$\phi_0(x) := \frac{1}{\sqrt{\pi}}, \quad \phi_k(x) = \sqrt{\frac{2}{\pi}}\, T_k(x), \quad k>0$$

$\Rightarrow$ Projection onto M.

$$P(u)(x) = \int_{-1}^{1} \sum_{i=0}^{n} \frac{\phi_i(x)\,\phi_i(y)}{\sqrt{1-y^2}}\, u(y)\, dy$$

$$\|P\| \le \sup_x \int_{-1}^{1} \left| \sum_{i=0}^{n} \frac{\phi_i(x)\,\phi_i(y)}{\sqrt{1-y^2}} \right| dy$$

MIDTERM

MIDTERM $\left[\sum_{i=0}^{n} \phi_i(x)\,\phi_i(y) = \frac{2}{\pi} \sum_{k=0}^{n} \cos(k \arccos x)\cos(k \arccos y) - \frac{1}{\pi} \right]$

and $\left[\cos(ks)\cos(kt) = \frac{1}{2}\left[\cos(k(s-t)) + \cos(k(s+t))\right] \right]$ Good Trick ! MEMORIZE

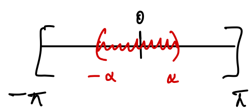
$$\left(\sum_{k=0}^{n} \cos(k\alpha) - \frac{1}{2} = \frac{\sin\left(\alpha\left(n+\frac{1}{2}\right)\right)}{2\sin(\alpha/2)} \right)$$

$$\Rightarrow \int_{-1}^{1} \left| \sum_{i=0}^{n} \frac{\phi_i(x)\,\phi_i(y)}{\sqrt{1-y^2}} \right| dy$$

$$= \frac{1}{2\pi} \int_{-1}^{1} \left| \frac{\sin((n+1/2)(s-t))}{\sin((s-t)/2)} + \frac{\sin((n+1/2)(s+t))}{\sin(s+t/2)} \right| \frac{1}{\sqrt{1-y^2}}\, dy$$

$S = \arccos x, \quad t = \arccos y$

$$\le \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin(t/2)} \right| dt$$



$$\le \underbrace{\frac{1}{2\pi} \int_{|t|<\alpha} \left| \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin(t/2)} \right| dt}_{\left|\sin\left(\left(n+\frac{1}{2}\right)t\right)\right| \le \left(n+\frac{1}{2}\right)|t|} + \underbrace{\frac{1}{2\pi} \int_{\alpha<|t|<\pi} \left| \frac{\sin\left(\left(n+\frac{1}{2}\right)t\right)}{\sin(t/2)} \right| dt}$$

$$|\sin(t/2)| \leq \frac{1}{|t|/2 - \frac{|t|^3}{3!\cdot 8}} = \frac{2}{|t|} + O(\alpha^3)$$

First two terms in Taylor series

$$\Rightarrow \int_{|t|<\alpha} (\dots) \leq \frac{2n+1}{2\pi} \int_{-\alpha}^{\alpha} 1 \, dx + O(n\alpha^3)$$

$$= \frac{2n+1}{\alpha} + O(n\alpha^3)$$

$$\frac{1}{2\pi} \int_{\alpha<|t|<\pi} (\dots) \leq \frac{1}{\pi} \int_{\alpha}^{\pi} \frac{1}{\sin(t/2)} \, dt$$

$$= \frac{2}{\pi} \log\left(\frac{\tan(\pi/4)}{\tan(\alpha/4)}\right)$$

$$= -\frac{2}{\pi} \log(\alpha/4) + O(\alpha^2 \log \alpha)$$

As $\alpha \downarrow 0$, need to balance the log growth from $\int_{\alpha<|t|<\pi}$ by setting $\alpha = \frac{1}{n+\frac{1}{2}}$.

□

$\Rightarrow$ Truncated Chebyshev expansion is "$\log n$" away from being optimal.

- What's the decay rate of the coefficients in the Chebyshev exp.

THM: For $p \in \mathbb{N} \cup \{0\}$, let $f \in C^p([-1,1])$ and

$$V := \int_{-1}^{1} \frac{|f^{(p+1)}(y)|}{\sqrt{1-y^2}}\, dy < \infty,$$

then, for $k \geq p+1$,

$$|\alpha_k| \leq \frac{2V}{\pi (k-p)^{p+1}}$$

Decay of tail coeffs of Chebyshev

Pf: See notes for details... too many indices...

LECTURE 8

Apr 16, 2026

Recall from last time:

THM: If f Lipschitz, then truncated Chebyshev series converges absolutely + uniformly and the coeffs. are contour integrals.

THM: $$\frac{\|f - p_T\|_\infty}{\|f - p_*\|_\infty} \leq \frac{4}{\pi^2} \log n + O(1) \text{ as } n \nearrow \infty.$$

THM: $f, f', \ldots, f^{(p)}$ absolutely continuous, and say

$$V := \int_{-1}^{1} \frac{|f^{(p+1)}(y)|}{\sqrt{1-y^2}}\, dy < \infty$$

then, for $k \geq p+1$, Cheb. coeffs. α_k of f are s.t.

$$|\alpha_k| \leq \frac{2V}{\pi k(k-1)\cdots(k-p)} \leq \frac{2V}{\pi (k-p)^{p+1}}.$$

Cor: $\|f - \underbrace{f_n}_{\uparrow}\|_\infty \leq \frac{2V}{\pi(n-p)^p}$. ← lose 1 order

keep the first $n+1$ terms in Cheb. expansion

Memorize how to do.

Rmk: Can improve this if $f \in C^\omega([-1,1])$.

DEF: JOUKOWSKY MAP $x = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

For $\rho > 1$, the image of the circle of radius ρ via Joukowsky is an ellipse w/ foci at ± 1.

BERNSTEIN ELLIPSES $=: E_\rho$

$$\boxed{\begin{array}{l}\textbf{THM}: f \in C^\omega([-1,1]) \text{ and analytically continuable to } E_\rho, \\ \text{where } |f(x)| \leq M \text{ for } x \in E_\rho. \text{ Then} \\ \qquad |\alpha_0| \leq M, \quad |\alpha_k| \leq 2M\rho^{-k}.\end{array}}$$

Pf: Cauchy's formula: $k = 0$ ← easy from formula.

$k > 0$, $\alpha_k = \frac{1}{\pi i} \int_{|z|=1} z^{-1-k} \underbrace{f(x(z))}_{=: F(z)}\, dz$

Joukowski

$$= \frac{1}{\pi i} \underbrace{\int_{|z|=\rho}}_{\text{analytic..}} z^{-1-k} \underbrace{F(z)}_{\leq M} \, dz$$

i.e., $|\alpha_k| \leq \frac{1}{\pi} \cdot 2\pi\rho \cdot M \cdot \rho^{-1-k}$. $\square$

<u>COROLLARY</u>: $\|f - f_n\|_\infty \leq \dfrac{2M\rho^{-n}}{\rho - 1}$ if f satisfies cond^{ns} of thm above.

<u>EXAMPLE</u>: $f(x) = e^x$ on $[-1, 1]$ $\rightsquigarrow$ $f \in C^\omega([-1,1])$.

Analytic extension: e^z, $z = x + iy$. So, for $\rho \in [1, \infty)$,

$$|e^z| \leq \exp\left[\frac{1}{2}\left(\rho + \frac{1}{\rho}\right)\right] \Rightarrow \|f - f_n\|_\infty \leq \frac{2\exp\left(\frac{\rho + \rho^{-1}}{2}\right)}{\rho^k(\rho - 1)}.$$

$\rightsquigarrow$ Optimize error over ρ.

<u>Q</u>: Can you get better than $\frac{1}{n!}$ from Taylor?

What k should you choose?

<u>A</u>: Depends how you compute these things...

MIDTERM

* INTERPOLATIONS

SETUP: Given pts $x_0, \dots, x_n \in [-1, 1]$ and samples $f_0, \dots, f_n$ from an unknown function f with certain known properties (e.g., Lipschitz, continuous, C^p, etc...).

GOALS:
- Given $x \in [-1, 1]$, can we find $\tilde{f} \approx f(x)$?
- How do we construct f from data ?
- Accuracy of approximation $\tilde{f}$?
- What is "good" ?

1^{st} try: POLYNOMIALS (since we have already so much theory for them)

IDEA 1: VANDERMONDE MATRIX

- Compute V and solve the system

$$\underbrace{\begin{bmatrix} 1 & x_0 & \cdots & x_0^n \\ 1 & x_1 & \cdots & x_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_n & \cdots & x_n^n \end{bmatrix}}_{V} \underbrace{\begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}}_{\vec{\alpha}} = \underbrace{\begin{bmatrix} f_0 \\ f_1 \\ \vdots \\ f_n \end{bmatrix}}_{\vec{f}}$$

Set $\delta_x := (1, x, x^2, \dots, x^n)^T$. Then, if $f \in P_n$,

$$f(x) = \delta_x^T \alpha = \sum_{j=1}^{n} \alpha_j x^j.$$

CAUTION: Usually bad idea b/c V is usually very ill-conditioned.

IDEA 2: LAGRANGE INTERPOLATION

KEY: Choose a basis after receiving data points.
In Vandermonde, we fixed the basis $\{1, x, x^2, -\}$ indep. of the data. Maybe the basis should depend on the data...

$\prod_{i=1}^{n} (x-x_i) = e^{n \sum_{i=1}^{n} \log(x-x_i) \cdot \frac{1}{n}}$ "Riemann" sum

CONSTRUCT: for $i = 1, -, n$, take basis

$$l_i(x) := \frac{\prod_{j \neq i} (x-x_j)}{\prod_{j\neq i} (x_i - x_j)}$$

poly. of deg $= n$

Note: $l_i(x_j) = \delta_{ij}$

$$P_f(x) := f_0 l_0 + \cdots + f_n l_n .$$

⚠

DEF: Given $x_0, -, x_n \in [a,b]$, the LEBESGUE CONSTANT Λ defined as

$P_f(x) := \sum_{j=1}^{n} l_j(x) f(x_j)$

$$\Lambda := \sup_{f \in C([a,b])} \frac{\| P_f \|_{L^\infty}}{\| \underset{\sim}{\vec{f}} \|_{\ell^\infty}} = \sup_{f\in C([a,b])} \frac{\| P_f \|_{L^\infty}}{\| f \|_{L^\infty}}$$

$\vec{f} := (f(x_0), -, f(x_n))$

INTUITION: $P_f \equiv$ projection onto $\mathbb{P}_n$ and $\Lambda =$ operator norm of this projection.

DEF: LEBESGUE FUNCTION $\lambda(x) := \sum_{j=0}^{n} | l_j(x) |$

⚠ $\rightsquigarrow$ $\boxed{\Lambda = \max_x \lambda(x).}$

Proposition: If $x_i \neq x_j \; \forall i \neq j$, then $1 \leq \Lambda < \infty$.

Pf: $\|p_f\|$ continuous fct. of $f_0, \dots, f_n$.

Scale so that, WLOG, $\|\vec{f}\|_{\ell^\infty} \leq 1$.

By compactness of $[a,b]$, $\Lambda \in [1, \infty)$.

Thm: Let $x_0, \dots, x_n \in [a,b]$ and Λ be the Lebesgue constant. Given $f \in C([a,b])$, let p_f be its Lagrange approximation and p_* its best poly. approximation in P_n. Then,

$$\|f - p_f\|_\infty \leq (\Lambda + 1) \|f - p_*\|_\infty .$$

$$\sum_{j=0}^{n} [(f-p_f)]\big|_{x_j} \ell_j(x) = \sum_{j=0}^{n} \Big(f(x_j)\,\ell_j(x) - \sum_{k=0}^{n} f(x_k) \underbrace{\ell_k(x_j)}_{=\delta_{kj}} \ell_j(x) \Big)$$
$$= \sum_{j=0}^{n} \big(f(x_j)\,\ell_j(x) - f(x_j)\,\ell_j(x) \big)$$
$$= 0$$

Pf: $p_f - p_* \in P_n$. So,

$$f - p_f - (f - p_*) = \sum_{j=0}^{n} \big[(f - p_f) - (f - p_*) \big]\Big|_{x_j} \ell_j(x)$$

$$= - \sum_{j=0}^{n} (f - p_*)\big|_{x_j} \ell_j(x) ,$$

hence

$$\|f - p_f\|_\infty \leq \|f - p_*\|_\infty \Big(1 + \max_x \sum_{j=0}^{n} |\ell_j(x)| \Big)$$

$$= \|f - p_*\|_\infty (1 + \Lambda) .$$

$\square$

$\Rightarrow$ Λ is closely related to how well Lagrange interpolation works at approximating cont. fcts. (at least w.r.t. its best poly. approx.).

THM: On $[-1,1]$, if Λ_n is the Lebesgue constant for <u>any</u> set of $n+1$ distinct points $x_0, \dots, x_n \in [-1,1]$.

Then,

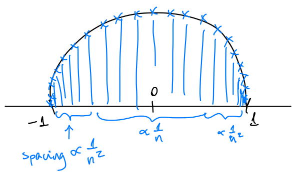
Euler-Mascheroni

$$(1) \quad \Lambda_n \geq \frac{2}{\pi}\log(n+1) + \frac{2}{\pi}\left[\gamma + \log\left(\frac{4}{\pi}\right)\right].$$

cannot be the utmost best... but... quite good still

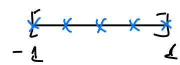
(2) For $x_0, \dots, x_n \in [-1,1]$ Chebyshev nodes (i.e., roots of T_{n+1}),

$$\Lambda_n \leq \frac{2}{\pi}\log(n+1) + 1.$$



$\Rightarrow$ CHEBYSHEV NODES ARE "OPTIMAL SAMPLING LOCATIONS"

NOTE: For equispaced points $x_0, \dots, x_n \in [-1,1]$, the Lebesgue constant Λ_n is s.t.



$$\underbrace{\Lambda_n > \frac{2^{n-2}}{n^2}, \quad \Lambda_n \approx \frac{2^{n+1}}{e\, n \log n}}$$

$\Rightarrow$ Equispaced is bad for Lagrange interpolation.

PROVE THIS AS EXERCISE !!!

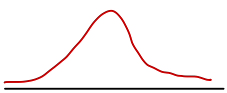
The badness of this is called Runge phenomena.

LECTURE 9

Apr 21^{st}, 2026

Recall: Equispaced pts are bad for Lagrange interpolation.

Ex: (WITCH OF AGNESI)



$$f(x) = \frac{1}{1+a^2x^2} \quad \text{with equispaced pts } x_0, \dots, x_n.$$

$$\ell(x) := \prod_{j=0}^{n} (x - x_j)$$

$$P_f \equiv \text{Lagrange interpolation of } f.$$

For $x \neq x_j$,

$$\phi_x(t) = f(t) - P_f(t) - \left(\frac{f(x) - P_f(x)}{\ell(x)}\right)\ell(t).$$

$$\phi_x(x_j) = 0 \quad \text{for } j = 0, \dots, n.$$

$$\phi_x(x) = 0.$$

$\Rightarrow$ ϕ_x has $n+2$ roots in $[-1,1]$

ϕ_x' has $n+1$ roots in $[-1,1]$ ↓ Rolle's Thm

$\vdots$

$\phi^{(n+1)}$ has $\underbrace{1 \text{ root}}_{=:\,\xi_x}$ in $[-1,1]$ $\longrightarrow$ $\phi_x^{(n+1)}(\xi_x) = 0$

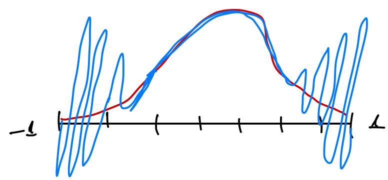
$$\Rightarrow \quad \phi_x^{(n+1)}(\xi_x) = f^{(n+1)}(\xi_x) - 0 - \frac{f(x) - p_f(x)}{\ell(x)}(n+1)! = 0$$

$$|f(x) - p_f(x)| = \frac{1}{(n+1)!}\,|\ell(x)|\,|f^{(n+1)}(\xi_x)|$$

$\Rightarrow$ For the witch fct.:

$$\|f^{(n+1)}\|_\infty \leq a^n\, n!$$

$$\|\ell\|_\infty \leq h^{n+1}\, n!$$



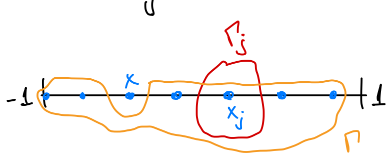
$$|h - p_f| \leq \frac{h^{n+1}\, a^n\, n!}{(n+1)!} \approx \frac{1}{n^{3/2}}\sqrt{8\pi}\left(\frac{2n}{e}\right)^n$$

HERMITE INTERPOLANTS & INTEGRAL FORMULA

Given the pts $x_0, \ldots, x_n$,

$$\ell(x) := \prod_{k=0}^{n}(x - x_k), \quad \ell_j(x) = \frac{\ell(x)}{\ell'(x_j)(x-x_j)} = \frac{\prod_{j\neq i}(x - x_i)}{\prod_{j\neq i}(x_j - x_i)}.$$

$$p_f(x) = \sum_{j=0}^{n} f(x_j)\,\ell_j(x).$$



Cauchy Integral Formula:

$$\ell_j(x) = \frac{1}{2\pi i}\int_{\Gamma_j}\frac{\ell(x)}{\ell(t)(x-t)}\,dt$$

Residue Thm $\Rightarrow$ Residue encircled by Γ_j is $\dfrac{1}{\ell'(x_j)(x-x_j)}$

$$\Rightarrow \ell_j(x) f(x_j) = \frac{1}{2\pi i}\int_{\Gamma_j} \frac{\ell(x)\, f(t)}{\ell(t)(x-t)}\, dt$$

$$\Rightarrow p_f(x) = \frac{1}{2\pi i}\int_{\Gamma} \frac{\ell(x)\, f(t)}{\ell(t)\,(x-t)}\, dt$$

$\leftarrow$ provided f is suff. analytic

Γ encloses all x_j's but not x

So, near x, the integrand has a pole at $t=x$ with residue $= -f(x)$. Thus, expand the contour to include x and call it $\tilde{\Gamma}$. Then

HERMITE INTEGRAL FORMULA

$$\boxed{p_f(x) - f(x) = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{\ell(x)\, f(t)}{\ell(t)\,(x-t)}\, dt.}$$

$$\Rightarrow \boxed{p_f(x) = \frac{1}{2\pi i}\int_{\tilde{\Gamma}} \frac{[\ell(x)-\ell(t)]\, f(t)}{\ell(t)\,(x-t)}\, dt.}$$

Errors:

$$|p(x) - f(x)| \le \underbrace{\max_{t\in\tilde{\Gamma}} \left|\frac{\ell(x)}{\ell(t)}\right|}_{\text{decays fast as } n\uparrow\infty \;\to\; T_1} \cdot \underbrace{\frac{1}{2\pi}\int_{\tilde{\Gamma}} \frac{|f(t)|}{|t-x|}\, dt}_{T_2 \;\leftarrow\; \text{indep. of } n.}$$

Set $\gamma_n(x,t) := \left| \dfrac{\ell(t)}{\ell(x)} \right|^{1/(n+1)}$

$$\alpha_n := \min_{x \in X,\, t \in \Gamma} \gamma_n(x,t).$$

Then, if $\alpha_n \geq \alpha > 1$, we have $\boxed{\|p_f - f\|_\infty = O(\alpha^{-n})}$.

BARICENTRIC INTERPOLATION FORMULA (same as Lagrange but more stable numerically – cf. MAT1800 HW2 Q1)

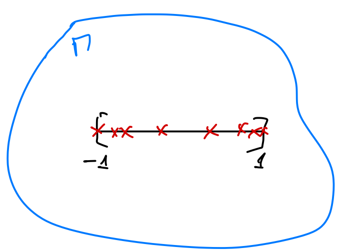
$$p(x) = \frac{\sum_{j=1}^n \frac{w_j}{x - x_j} f(x_j)}{\sum_{j=1}^n \frac{w_j}{x - x_j}}.$$

LECTURE 10

Apr 28, 2026

Recall: Hermite Integral Formula

$$p(x) = \frac{1}{2\pi i} \int_\Gamma \frac{f(t)\,[\ell(t) - \ell(x)]}{\ell(t)\,(x - t)}\, dt$$



$$\ell(x) := \prod_{j=0}^{n} (x - x_j), \quad p(x) = \sum_{j=0}^{n} f(x_j)\, \ell_j(x).$$

* INTEGRATION

- Suppose we have a basis we know $\{\phi_j\}$ and that we have a good interpolant $\sum_j a_j \phi_j$ to a fct f

$$\| f - \sum_j a_j \phi_j \|_\infty < \varepsilon \implies \int f \simeq \sum_j a_j \int \phi_j .$$

- Quadrature scheme: $\displaystyle \int f \, dx \approx \sum_{j=1}^{N} f(x_j)\, w_j$

  for choices of nodes x_j
  and weights w_j.

- Take $n = 2m$ and consider $P_n([-1,1])$

  GOAL: Construct a $p < n+1$ point quadrature rule and exactly integrate elements of $P_n([-1,1])$.

- Note: $q_1, q_2 \in P_m \implies q_1 \cdot q_2 \in P_n$.

  If we can integrate q_1 and q_2, we can integrate inner products

- Suppose we have a scheme with $p = m+1$. Then we get a map

$$T : P_m \to \mathbb{R}^{m+1}, \quad T(q) = \begin{pmatrix} q(x_0)\sqrt{w_0} \\ \vdots \\ q(x_{m+1})\sqrt{w_{m+1}} \end{pmatrix}$$

Isometry from $P_m \subset L^2([-1,1])$ to $\ell^2(\mathbb{R}^{m+1})$

- Need a way to get a good (i.e., orthonormal) basis to $L^2([-1,1])$.

- Given $p_0, \ldots, p_k \in \mathcal{P}_k$ s.t. $\langle p_\ell, p_j \rangle = 0 \quad \forall j \neq \ell$
- $i < k-1$,

$$\langle p_k x, p_j \rangle = \langle p_k, \underbrace{p_i x}_{\in \mathcal{P}_{k-1}} \rangle = 0$$

$$a_k := \langle p_k x, p_{k+1} \rangle$$

$$b_k := \langle p_k x, p_{k-1} \rangle \overset{?}{=} \int_{-1}^{1} x p_k^2 \, dx \overset{\text{by symmetry}}{=} 0$$

$$x p_k - a_{k-1} p_{k-1} \perp p_0, \ldots, p_k$$

TYPOS...

LOOK AT NOTES.

<u>Upshot</u>: Try to make

$$q_{k+1} := x p_k + c_k p_k + d_k p_{k-1} \overset{!}{\perp} p_0, \ldots, p_k .$$

$$\Rightarrow \quad x p_k = \alpha_k p_k + \beta_k p_{k-1} .$$

Set $p_k(1) = 1$

$\Rightarrow$ LEGENDRE (or GAUSS - LEGENDRE) POLYNOMIALS

$$(k+1)\, p_{k+1}(x) = (2k+1)\, x p_k(x) - k p_{k-1}(x)$$

$$p_0(x) = 1, \quad p_1(x) = x$$

REMARK: 1) $\int_a^b p_n p_m \, d\mu(x) \longrightarrow$ Need another term in the recurrence:

$$x p_k = \alpha_k p_{k+1} + \beta_k p_{k-1} + \underline{c_k p_k}$$

2) $$\int_{-1}^{1} p_n p_m \, dx = \frac{2}{2n+1} \delta_{n,m}$$

3) $$\| p_n \|_{L^2([-1,1])} = \sqrt{\frac{2}{2n+1}}$$

4) $q \in P_n$, $n = 2m$, then by the Remainder Thm,

$$q(x) = p(x) \underline{P_{m+1}(x)} + r(x), \quad p \in P_{m-1}, \; r \in P_m$$

P_{m+1} is $(m+1)$-Legendre polynomial

Let x_i, $i = 0, \ldots, m$, be the roots of P_{m+1}. Then

$$q(x_i) = p(x_i) P_{m+1}(x_i) + r(x_i)$$

$$\int_{-1}^{1} q(x) \, dx = \int_{-1}^{1} \underbrace{p(x) P_{m+1}(x) \, dx}_{= 0 \text{ b/c } \perp} + \int_{-1}^{1} r(x) \, dx$$

$$= \int_{-1}^{1} r(x)\, dx .$$

If $r(x) = \sum_{j=0}^{m} c_j P_j(x)$, then $q(x_i) = \sum_{j=1}^{m} c_j P_j(x_i)$

Write: $V_{ji} := P_j(x_i) \rightsquigarrow \underline{\vec{w}} = V^{-1} \begin{pmatrix} 2 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$

$W = \mathrm{diag}(w_1, -, w_m)$

$\int_{-1}^{1} 1\, dx = 2$

$$\int_{-1}^{1} P_j(x)\, dx = 2\, \delta_{j,0} = \sum_{i=0}^{m} P_j(x_i)\, w_i$$

$$\Rightarrow \boxed{\int_{-1}^{1} f(x)\, dx = \sum_{i=0}^{m} q(x_i)\, w_i}$$

Roots of $P_{m+1}(x)$ are the optimal sampling locations for polynomials in P_{2m}

* PROJECTIONS

- Can project functions f onto polynomials of degree at most n

- After a lot of computations, you get the following formula for the projection:

CHRISTOFFEL – DARBOUX FORMULA

$$S_n[f](x) = \int_{-1}^{1} \frac{(n+1)}{2(x-t)} \left[P_{n+1}(x)\, P_n(t) - P_{n+1}(t)\, P_n(x) \right] f(t)\, dt$$

THM: If $f, f', f'', \ldots, f^{(j-1)}$ are all absolutely continuous on $[-1,1]$. Also, suppose

$$V := \int_{-1}^{1} \frac{|f^{(j)}(x)|}{(1-x^2)^{1/4}}\, dx < \infty .$$

Then, $\forall\, n > j+1$,

$$|f_n| \le \sqrt{\frac{2}{\pi(n-j+1/2)}}\; \frac{V}{(n-\frac{1}{2})\cdots(n-j+3/2)}$$

coefficients in Legendre basis

$$\boxed{\text{n-th LEGENDRE COEFFICIENT OF } f: \quad f_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)\, P_n(x)\, dx}$$

LECTURE 11

Apr 30, 2026

RECALL : – Gauss-Legendre polynomials to form quadrature
– Get 3-term recurrence

$$P_{n+1} = c_n P_n + d_n x P_n + f_n P_{n-1}$$

INTEGRATION : P_{n+1} has $n+1$ roots. Using these roots $x_0, \dots, x_n$, we can build a quadrature rule that integrates P_{2n}

$$\begin{pmatrix} f(x_0)\sqrt{w_0} \\ \vdots \\ f(x_n)\sqrt{w_n} \end{pmatrix}$$

CHRISTOFFEL – DARBOUX FORMULA

$$S_n(f)(x) = \int_{-1}^{1} K_n(x,t)\, f(t)\, dt,$$

$$K_n(x,t) = \frac{(n+1)}{2(x-t)} \left[P_{n+1}(x) P_n(t) - P_{n+1}(t) P_n(x) \right]$$

THM : If f smooth, then the coefficients in a Legendre expansion decay fast (analogous to the Chebyshev expansion).

WEIGHT FUNCTIONS	ORTHOGONAL POLYNOMIALS
$w(x) = (1-x^2)^{-1/2}$	Chebyshev 1st kind
$w(x) = (1-x^2)^{1/2}$	Chebyshev 2nd kind

⋮ ← See more in notes.

LINEAR ALGEBRA CONSEQUENCES

$Ax = b, \quad A \in \mathbb{R}^{n \times n}$

$x_0, x_1, x_2, \ldots$

$r_0 := b - Ax_0, \; r_1 := b - Ax_1, \ldots$

$\frac{\|r_n\|}{\|r_0\|}$?

KRYLOV SUBSPACES

$K_\ell(A,b) := \operatorname{span}\{b, Ab, A^2 b, \ldots, A^{\ell-1} b\}$

If $\dim K_\ell < \ell$, then

$$b + \frac{c_1}{c_0} Ab + \cdots + \frac{c_{\ell-1}}{c_0} A^{\ell-1} b = 0$$

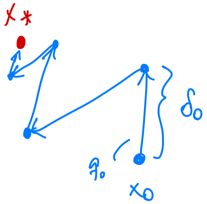
$$\rightsquigarrow \quad b = A\left(\hat{c}_0 b + \hat{c}_1 Ab + \cdots + \hat{c}_{\ell-2} A^{\ell-2} b\right)$$

$$\Rightarrow \quad K_\ell(A,b) = \{ p(A) b : p \in P_{\ell-1})$$

$$\begin{cases} p(A) = p(0) = 1 \\ p(A) b = 0 \end{cases}$$

* CONJUGATE GRADIENT

GOAL: Solve $Ax = b$, for $A^T = A \succ 0$.



$$x_{k+1} = x_k + \delta_k q_k$$

In L^2 sense: $\langle x_* - (x_k + \delta_k q_k), q_k \rangle = 0$

! WARNING

But this requires knowing x_* in the first place...

$\Rightarrow$ USE INNER PRODUCT WITH A: $\leftarrow$ instead of the usual L^2 one.

$$\langle v, w \rangle_A := \langle v, Aw \rangle = \langle Av, w \rangle$$

and measure "optimality" using the Mahalanobis norm.

$$\Rightarrow \quad \langle x_* - x_{k+1}, q_k \rangle_A \overset{!}{=} 0$$

$$x_{k+1} = x_k + \delta_k q_k \quad \rightsquigarrow \quad \delta_k = \frac{\langle r_k, q_k \rangle}{\langle q_k, A q_k \rangle}$$

- Take $v_0, \ldots, v_{n-1}$ basis for $\mathbb{R}^n$ and construct $q_0, \ldots, q_{n-1}$ via Gram-Schmidt

$$\langle q_j, q_i \rangle_A \overset{!}{=} 0 \quad \text{if } i \neq j.$$

$q_j \in \operatorname{span}\{v_0, \ldots, v_j\}$.

- Set $e_i := x_* - x_i$

$$e_i = e_0 + \sum_{j=0}^{i-1} \alpha_j q_j = x_* - x_0 - \delta_0 q_0 - \dots - \delta_{i-1} q_{i-1}$$

Also, since $\{q_j\}_{j=0}^{n-1}$ is a basis of $\mathbb{R}^n$, then $\exists \{\sigma_j\}_{j=0}^{n-1}$ s.t.

$$e_0 = \sum_{j=0}^{n-1} \sigma_j q_j . \quad \underset{e_i \perp_A q_0, \dots, q_{i-1}}{\xRightarrow{A\text{-orthogonality}}} \quad \sigma_j = -\alpha_j$$

$(*)$

$$\rightsquigarrow \quad e_i = \sum_{j=i}^{n-1} \alpha_j q_j .$$

- Say we have an update vector $w \in \operatorname{span}\{q_0, \dots, q_{i-1}\}$

$$\|w - e_0\|_A^2 = \langle A(w - e_0), w - e_0 \rangle$$

expand w in the $\{q_j\}$'s

$$= \sum_{j=0}^{i-1} (w_j - \sigma_j)^2 \langle q_j, q_j \rangle_A + \sum_{j=i}^{n-1} \sigma_j^2 \langle q_j, q_j \rangle_A$$

$$\overset{(*)}{\geqslant} \|e_i\|_A^2 \quad \Longrightarrow \quad \text{This search direction is } A\text{-optimal indeed.}$$

Still need to choose the basis.

$$CG \longrightarrow \boxed{v_i = r_i} \quad \rightsquigarrow \quad \text{With this choice, } \langle r_i, r_j \rangle = 0$$

$$\langle q_j, r_i \rangle = \langle q_j, A e_i \rangle \overset{i \neq j}{=} 0, \quad j < i$$

Strategy: Choose $\{v_i\}$ so that you don't need to run Gram-Schmidt on the matrix each time (too expensive...)

$\rightsquigarrow$ $r_j \in \text{span}\{q_0, \ldots, q_j\}$

$\rightsquigarrow$ $x_{i+1} = x_i + \delta_i q_i$

$\rightsquigarrow$
$$\begin{aligned} r_{i+1} &= A(x_* - x_{i+1}) \\ &= A(x_* - x_i - \delta_i q_i) \\ &= r_i - \delta_i A q_i . \end{aligned}$$

Need to find a recurrence that automatically makes them orthogonal. (just like above...)

RECALL:
$$q_i = v_i + \sum_{j=0}^{i-1} \beta_{i,j} q_j$$

$$\rightsquigarrow \quad \beta_{i,j} = - \frac{\langle r_i, q_j \rangle_A}{\langle q_j, q_j \rangle_A}$$

$$\rightsquigarrow \quad 0 = \langle r_j, r_{i+1} \rangle = \langle r_j, r_i \rangle - \delta_i \langle r_j, q_i \rangle_A$$

$$\rightsquigarrow \quad \langle r_j, q_i \rangle_A = \frac{1}{\delta_i} \langle r_j, r_i \rangle_A$$

$$\rightsquigarrow \quad \beta_{i,j} = \begin{cases} \dfrac{1}{\delta_{i-1}} \dfrac{\langle r_i, r_j \rangle}{\langle q_{i-1}, q_{i-1} \rangle_A}, & i = j+1 \\ 0, & \text{otherwise} \end{cases} .$$

Upshot: (CG) $r_0 = b - Ax_0$

$$\delta_i = \frac{(r_i, r_i)}{(q_i, q_i)_A}$$

$$x_{i+1} = x_i + \delta_i q_i$$

$$r_{i+1} = r_i - \delta_i A q_i$$

$$\beta_{i,j} = \frac{(r_{i+1}, r_{i+1})}{(r_i, r_i)}$$

$$q_{i+1} = r_{i+1} + \beta_{i+1,i} q_i$$

SPECTRAL THEOREM (similar in func anal)

NOTE: Since $A^T = A > 0$, look at spectrum & eigenbasis.

$\rightsquigarrow$ $p_i(A) r_0$ might have a simple formula in the eigenbasis of A (b/c it's diagonal there).

$A \rightsquigarrow (\lambda_j, w_j)$ eigenpairs of $A > 0$,

$$e_0 = \sum_{j=1}^{n} \xi_j w_j, \quad e_i = \sum_{j=1}^{n} p_i(\lambda_j) \xi_j w_j$$

$$A e_i = \sum_{i=1}^{n} \lambda_j p_i(\lambda_j) \xi_j w_j$$

$$\| e_j \|_A^2 = \sum_{j=1}^{n} \xi_i^2 |p_i(\lambda_j)|^2 \lambda_j$$

$\rightsquigarrow$ optimality in the Krylov space gives:

$$\frac{\|e_i\|_A^2}{\|e_0\|_A^2} \leq \min_{\substack{p \in P_i \\ p(0)=1}} \max_{\lambda \in \mathrm{Spec}(A)} |p(\lambda)|^2 .$$

- Replace Spec(A) by $[\lambda_{min}, \lambda_{max}]$ and set:

$$S(\lambda) = 1 - 2\left(\frac{\lambda - \lambda_{min}}{\lambda_{max} - \lambda_{min}}\right), \qquad S(0) = \frac{\lambda_{max} + \lambda_{min}}{\lambda_{max} - \lambda_{min}} .$$

$$S(0) =: S_*$$

$$\Rightarrow \frac{\|e_i\|_A^2}{\|e_0\|_A^2} \leq \min_{p \in P_i} \max_{s \in [-1,1]} \frac{|p(s)|^2}{|p(S_*)|^2}$$

might not be tight

$\mapsto$ Need p to have roots in $[-1,1]$ as well as min and max evenly spread on $[-1,1]$ ($\approx$ equioscillate)

$$\leq \min_{p \in P_i} \max_{\lambda \in [\lambda_{min}, \lambda_{max}]} \frac{|p(A)|}{|p(0)|}$$

we know Chebyshev polynomials are of this form.

Chebyshev polynomial $\rightsquigarrow$ $\boxed{\text{Chebyshev polys only come up in the bounds for CG (not really related to CG algorithm per se).}}$

p optimal

$$\hookrightarrow q(s) = \frac{p(s)}{|q(S_*)|} - \frac{T_i(s)}{|T_i(S_*)|},$$

$s_j \equiv$ max / min pts of T_i

$$\left|\frac{p(s_j)}{p(S_*)}\right| \leq \left|\frac{T_i(s_j)}{T_i(S_*)}\right|, \quad j = 0, \ldots, i$$

$\Rightarrow$ q alternates sign at $i+1$ pts in $[-1,1]$

$\Rightarrow$ q has i roots in $[-1,1]$

However: $s_* \in [-1,1]$ and $q(s_*) = 0$

$\Rightarrow$ $q \in P_i$ and has $i+1$ roots $\Rightarrow$ $q \equiv 0$.

Upshot: $$\| e_i \|_A \leq \frac{1}{|T_i(s_*)|} \| e_0 \|_A$$

$$\kappa := \frac{\lambda_{max}}{\lambda_{min}} \qquad \Rightarrow \boxed{\| e_i \|_A \leq 2 \left(\frac{\sqrt{\kappa}+1}{\sqrt{\kappa}-1} \right)^{-i} \| e_0 \|_A .}$$

$$s_* = \frac{\kappa+1}{\kappa-1}$$

NOTE: CG is viable b/c T_i's are the polynomials of largest growth outside of $[-1,1]$.

LECTURE 12

May 5, 2026

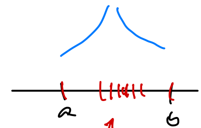
CHANGING GEARS: If function to be approximated is not smooth enough it's not enough to only add more nodes at the endpoints...

$\Downarrow$

ADAPTIVE APPROXIMATION (quadrature)

$\rightsquigarrow$

SPLINES (CAD geometry, etc)



subdivide interval further and further until the intervals are so small that they don't matter.

* SPLINES

- Suppose we have $\begin{cases} \text{nodes} \to x_0, \ldots, x_n \\ \text{samples} \to f_0, \ldots, f_n \end{cases}$

WANT: local polynomial approximation of degree p to f.

- Adaptive interpolation stats:

  - TOTAL DEGREES OF FREEDOM: $n(p+1) = np + n$
  - CONSTRAINTS: $n-1 \longrightarrow$ continuity

    $n+1 \longrightarrow$ consistency (i.e., matching f_j's at x_j's)

    $(n-1)q \longrightarrow$ continuity of the q derivatives we want our curve to have $\Rightarrow$ q can be at most $p-1$!!

• If $q = p-1 \longrightarrow (n-1)(p-1) + (n+1) + (n-1)$
$= np \cancel{-n} - p + 1 + n \cancel{+1}$
$\cancel{+n} \cancel{-1}$

$= np - p + 1 + n$

$= n(p+1) - \underline{\underline{(p-1)}}$

left-over degrees of freedom

$$\Longrightarrow \boxed{S_p := \left\{ s \in C^{p-1} : s\big|_{[x_{i-1}, x_i]} \in P_p \; , \; 1 \leq i \leq n-1 \right\}}$$

and $\dim S_p = n + p$ (SPACE OF SPLINES WITH THE DESIRED PROPERTIES WE IMPOSED.)

• CUBIC SPLINES : $p = 3$

- Natural to impose the condition that $s''(x_0) = s''(x_n) \stackrel{!}{=} 0$.

(SPACE OF NATURAL CUBIC SPLINES)

$$\boxed{S_3^N := S_3^{\text{natural}} := \left\{ s \in S_3 : s''(x_0) = s''(x_n) = 0 \right\}}$$

DEF: If s passes through all data points at the nodes, it is called an INTERPOLATION.

THM: The natural cubic interpolating spline is unique.

- Why is the 2^{nd} derivative condition above "NATURAL"?

Answer:

THM: Set $H^2([a,b])$ to be completion of $C^2([a,b])$ w.r.t.

$$\|f\|_{H^2} := \left(\|f\|_{L^2}^2 + \|f'\|_{L^2}^2 + \|f''\|_{L^2}^2 \right)^{1/2}$$

Given $f \in H^2([x_0, x_n])$ set $f_i := f(x_i)$. If $s \in S_3^N$ is the natural cubic spline interpolant of f, then

$$\langle f'' - s'', s'' \rangle = 0 .$$

Pf: Since $s \in S_3$, s'' is linear on each subinterval $[x_i, x_{i+1}]$.

So,

$$\int_{x_i}^{x_{i+1}} (f'' - s'') s'' \, dx \overset{\text{by parts}}{=} \left[(f' - s') s'' \right] \Big|_{x_i}^{x_{i+1}}$$

$$- \int_{x_i}^{x_{i+1}} (f' - s') s''' \, dx$$

$$= \int_{x_i}^{x_{i+1}} \cancel{(f - s)} \underbrace{s''''}_{=0} \, dx + \left[(f' - s') s'' \right] \Big|_{x_i}^{x_{i+1}}$$

$$- \left[\cancelto{=0}{(f - s)} s''' \right] \Big|_{x_i}^{x_{i+1}}$$

$\Rightarrow$ By telescoping,

$s \in S_3^N$ i.e., $s''(x_0) = s''(x_n) = 0$

$$\langle f'' - s'', s'' \rangle = \left[(f' - s') s'' \right] \Big|_{x_0}^{x_n} = 0$$

$\square$

$\Rightarrow$ Natural to force $s''(x_0) = s''(x_n) = 0$ b/c that's precisely the condition that makes it so that

$$\langle f'' - s'', s'' \rangle = 0.$$

COROLLARY: Let $s \in S_3^N$ be the natural cubic interpolating spline passing through (x_i, v_i), $i = 0, \dots, n$. For any $g \in H^2([x_0, x_n])$ with $g(x_i) = v_i$, we have

ENERGY SEMI-NORM

$$\hookrightarrow [g]_{L^2}^2 := \| g'' \|_{L^2}^2 = \| g'' - s'' \|_{L^2}^2 + \| s'' \|_{L^2}^2, \quad i = 0, \dots, n,$$

$\leftarrow$ completion of C^2...

NOTE: s as above is the unique H^2 function passing on the data points and minimizing the semi-norm $[\cdot]_{L^2}$, $[u]_{L^2} := \| u'' \|_{L^2}$

Pf: $\langle g'', g'' \rangle = \underline{\langle g'' - s'', g'' \rangle} + \underline{\langle s'', g'' \rangle}$

$\langle g'' - s'', g'' - s'' \rangle + \cancel{\langle g'' - s'', s'' \rangle}_{=0}$ $\quad$ $\cancel{\langle s'', g'' - s'' \rangle}_{=0} + \langle s'', s'' \rangle$ $\square$

by theorem above

THM: $\exists C$ s.t. $\forall f \in H^2$, $\|f - s\|_{L^\infty} \leq C\, h_x^{3/2} \|f''\|_{L^2}$,

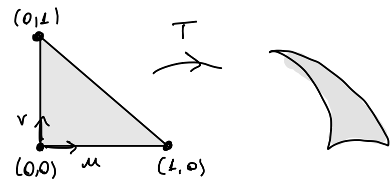
where s is the natural interpolating cubic spline and h_x is the maximum distance between adjacent nodes.

LECTURE 13

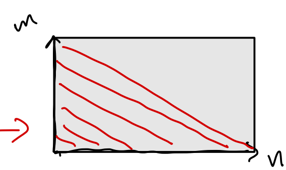
May 7^{th}, 2026

* <u>HIGHER DIMENSIONAL SPLINES</u> ← see notes for details / formulas

- <u>POLYNOMIALS</u>: Natural to take triangulations and look at polynomials on "the edges" of the triangles.



- Nice polynomial basis?
- Monomials? ← Ill-conditioned as in 1-dim.
- Ordering of basis elements?



Partial ordering: fix max degree to be d

$\{1, u, v, u^2, uv, v^2, \ldots\}$

Possible basis for the square

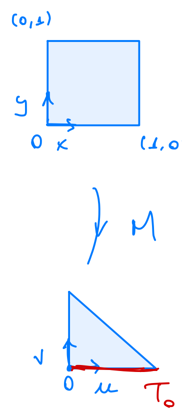
$P_n(x)\, P_m(y)$, $0 \leq n, m \leq d$

- Can orthonormalize this set $\{1, u, v, u^2, uv, v^2, \ldots\}$ to get an orthonormal basis of polynomials

- RESULT: what you get after orthonormalizing $\rightarrow$ w.r.t. $\langle \cdot , \cdot \rangle_{L^2}$ standard this list w/ this specific ordering:

$$\underbrace{K_{i,j}}_{\text{Koornwinder Polynomials}}(u,v) = \underbrace{P_j^{(0,2i+1)}}_{\text{Jacobi Polynomials}}(1-2u)(1-u)^i \; \underbrace{P_i}_{\text{Legendre Polynomials}}\left(\frac{2v+u-1}{1-u}\right)$$

DERIVATION: (of Koornwinder polys.)



- Consider the map $M(x,y) := (x, (1-x)y)$

- Suppose p, q are polys. orthogonal on T_0, then

$$\int_0^1\int_0^1 p(M(x,y))\, q(M(x,y))\,(1-x)\,dx\,dy = 0$$

- Say $\hat{p} \in P_d$ (one-variable poly. of deg. $\leq d$)

$$\hat{p}(u,v) = (1-x)^d\,\hat{p}(y) = (1-u)^d\,\hat{p}\left(\frac{v}{1-u}\right)$$

$\Rightarrow$ this new poly. has degree $= d$ on (u,v).

$$\Rightarrow \quad \hat{p} = \sum_{j,k} c_{jk}\, u^j v^k \quad , \qquad \begin{array}{l} 0 \le j+k \le d \\ 0 \le j,k \end{array}$$

Hence $j+k=d$ for ≥ 1 of the terms

Set $\quad q_{i,j}(x,y) := \underbrace{q_j^{(i)}}_{\deg = j}(x)\,(1-x)^i\, \underbrace{p_i}_{\deg = i}(y)$

$$\int_0^1 \int_0^1 q_{i,j}(x,y)\, q_{i',j'}(x,y)\,(1-x)\, dx\, dy = c_{ij}\, \delta_{i,i'}\, \delta_{j,j'}$$

$$\|$$

$$\int_0^1 \int_0^1 q_j^{(i)}(x)\,(1-x)^i\, p_i(y) \cdot q_{j'}^{(i')}(x)\,(1-x)^{i'}\, p_{i'}(y)\, dx\, dy$$

Change order of integration and note that $p =$ Legendre gives orthogonality

$\Longrightarrow \quad p_i(y) \overset{!}{=} \underbrace{P_i}_{i\text{th Legendre}}(\overbrace{1-2y}^{})$

P_i defined in $(-1,1)$ and here integral is on $(0,1)$

$$\Rightarrow \int_0^1 \int_0^1 q_j^{(i)}(x)\, q_{j'}^{(i)}\,(1-x)^{2i+1}\, dx = c_{ij}\, \delta_{j,j'}$$

$\Rightarrow$ Polynomials q satisfying this are called JACOBI POLY.
$P_j^{(0,2i+1)}(2x-1)$.

$\Rightarrow$ Koornwinder $K_{i,j}$ formula from before ! $\square$

* Kimura Operators : Note

$$\tilde{\mathcal{L}}_1^{\text{kim}}[\psi](u,v) := u(1-u)\,\partial_u^2 \psi - 2uv\,\partial_u \partial_v \psi$$
$$+ v(1-v)\,\partial_v^2 \psi + (1-3u)\,\partial_u \psi$$
$$+ (1-3v)^2\,\partial_v \psi$$

Claim: Koornwinder polynomials $K_{i,j}$ are the eigenfunctions of this Kimura operator $\tilde{\mathcal{L}}_1^{\text{kim}}$.

Facts:

1. $\tilde{\mathcal{L}}_1^{\text{kim}}$ is self-adjoint

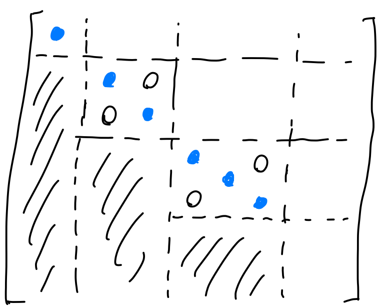
2. It leaves $P_m := \text{span}\{u^j v^k : k+j \leq m\}$ invariant.

3. $\tilde{\mathcal{L}}_1^{\text{kim}}\big|_{P_m} \rightsquigarrow$ eigenfunctions
   - with orthogonality
   - will form a basis for P_m.

4. $\tilde{\mathcal{L}}_1^{\text{kim}}[u^j v^k] = -\left[(j+k)^2 + 2(j+k)\right] u^j v^k + \underbrace{r_{j+k-1}}_{\text{lower degree}}$

only depends on the total order of the monomial

$$\Rightarrow \tilde{\mathcal{L}}_1^{\text{kim}} \begin{bmatrix} 1 \\ u \\ v \\ u^2 \\ uv \\ v^2 \\ \vdots \end{bmatrix} =$$



$$\begin{bmatrix} 1 \\ u \\ v \\ u^2 \\ uv \\ v^2 \\ \vdots \end{bmatrix}$$

$\Rightarrow$ $K_{i,j}$ are eigenfunctions with eigenvalues $-(i+j)^2 - 2(i+j)$

THEN: $\langle \tilde{\mathcal{L}}_1^{kin} f, g \rangle = \langle f, \tilde{\mathcal{L}}_1^{kin} g \rangle$

$\Rightarrow$ Coeffs of f in the Koornwinder basis are:

$$\alpha_{i,j} = \frac{\langle K_{i,j}, f \rangle}{\langle K_{i,j}, K_{i,j} \rangle^{1/2}} = \frac{1}{\lambda_{i,j}^{P}} \frac{\langle K_{i,j}, (\tilde{\mathcal{L}}_1^{kin})^P f \rangle}{\| K_{i,j} \|_{L^2(T_0)}}$$

$\Rightarrow$ DECAY OF THE COEFFS. of f IN KOORNWINDER BASIS:

(in the triangle ! :))

$$\boxed{|\alpha_{i,j}| \leq \frac{1}{m^{2k}} \frac{\| K_{i,j} \|_{L^\infty(T_0)} \| (\tilde{\mathcal{L}}_1^{kin})^k [f] \|_{L^1(T_0)}}{\| K_{i,j} \|_{L^2(T_0)}}}$$

BRAMBLE – HILBERT LEMMA: Consider $\Omega \subset \mathbb{R}^n$

$$W^{k,p}(\Omega) = \{ u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \quad \forall |\alpha| \leq k \}$$

with

$$\|u\|_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| \leq k} \| D^\alpha u \|^p_{L^p(\Omega)} \right)^{1/p} .$$

Define the semi-norm

$$[u]_{W^{k,p}(\Omega)} := \left(\sum_{|\alpha| = k} \int_\Omega |D^\alpha v(x)|^p dx \right)^{1/p} .$$

Let $F : W^{k,p}(\Omega) \to \mathbb{R}$ be such that

- F is bounded (i.e., $|F(v)| \leq C_1 \|v\|_{W^{k,p}(\Omega)}$)
- F is sub-linear (i.e., $|F(u+v)| \leq C_2 (|F(v)| + |F(u)|)$)
- F annihilates $P_{k-1}(\Omega)$ (i.e., $F(P_{k-1}(\Omega)) = \{0\}$)

Then, $\exists C > 0$ s.t.

$$|F(v)| \leq C [v]_{W^{k,p}(\Omega)} , \quad \forall v \in W^{k,p}(\Omega)$$

LECTURE 14

May 12^{th}, 2026

* CONVERGENCE OF FINITE ELEMENT METHODS (ctd)

- On each patch, we have a projection operator P_i typically designed so that $P_i u = u|_{T_i}$ if u is a poly. of deg. at most k.

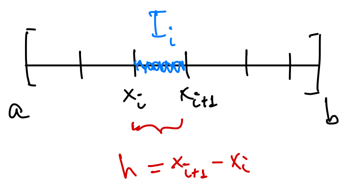
- Typically, we want that

$$\sum_i \| P_i u - u|_{T_i} \|_{H^{k+1}(T_i)} \longrightarrow 0 \quad \text{as } \underset{\substack{\uparrow \\ \text{diameter of patch}}}{\underline{\underline{h}}} \to 0$$

$$\|u\|^2_{H^k(T_i)} := \int_{T_i} |u|^2 + |Du|^2 + \dots + |D^k u|^2 .$$

— PROBLEM: This doesn't demonstrate higher order / optimal order convergence (typically $O(h)$).

- Start in $\mathbb{R}$: Want a bound for $\| P_i u - u|_{I_i} \|_{L^2}$ in terms of $\| P_i u - u \|_{H^1}$



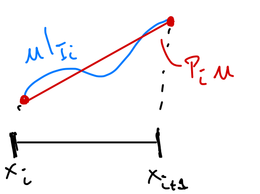
i.e., $\underline{P_i}\, \ell := \ell|_{I_i}$ for ℓ linear

$\Rightarrow P_i \equiv$ build linear approx. for u in I_i

Set $\tilde{u} : [0,1] \to \mathbb{R}$ to be

$$\tilde{u}(s) := u\big(\underline{(x_{i+1} - x_i)}\, s + x_i \big)$$

$$= u(\underline{h}s + x_i)$$



$$\Rightarrow \quad \| P_i u - u|_{I_i} \|_{L^2}^2 = h \int_0^1 | P_i \tilde{u} - \tilde{u} |^2 \, ds$$

$$\leq h \, \| P_i \tilde{u} - \tilde{u} |_{I_i} \|^2_{H^1((0,1))}$$

- To proceed, we need POINCARÉ INEQUALITY: $\forall v \in H_0^1(I)$, then $\exists C > 0$ s.t.

$$\| v \|_{H^1(I)} \leq C \underbrace{[v]_{H^1(I)}}_{} := C \underbrace{\sqrt{\int_I |v'|^2}}_{}$$

$H^1(I)$ seminorm

b/c it's zero for nonzero constants (so not a "full norm")

- There is another form of POINCARÉ'S INEQUALITY: for any I $\exists C > 0$ s.t.

$$\underbrace{\| v - \textstyle\fint_I v \|_{H^1(I)}}_{} \leq C \, [v']_{H^1(I)}$$

Excursion from the mean is bounded by the size of the gradient

$\Rightarrow$ Any choice of P_i that enforces agreement somewhere allows to conclude that $\exists C$ s.t.

Any interpolation rule will automatically make this work.

$$h \| P_i \tilde{u} - \tilde{u} \|^2_{H^1(0,1)} \leq C h \, [P_i \tilde{u} - \tilde{u}]^2_{H^1(0,1)}$$

$$\Rightarrow \quad \| P_i u - u|_{I_i} \|^2_{L^2(x_i, x_{i+1})} \leq Ch \int_0^1 |(P_i \tilde{u})' - \tilde{u}'|^2 \, ds$$

$$= C \int_{x_i}^{x_{i+1}} h^2 |(P_i u)' - u'|^2 \, dx$$

$$(\tilde{u} = h u')$$

$$= Ch^2 \left[P_i u - u|_{I_i} \right]^2_{H^1((x_i, x_{i+1}))}$$

$$\leq Ch^2 \| P_i u - u|_{I_i} \|^2_{H^1(x_i, x_{i+1})}$$

Summing over i:

$$\| Pu - u \|^2_{L^2(0,1)} = \sum_i \| P_i u - u|_{I_i} \|^2_{L^2(0,1)}$$

$$\leq Ch^2 \sum_i \| P_i u - u|_{I_i} \|^2_{H^1(x_i, x_{i+1})}$$

$$= Ch^2 \| Pu - u \|^2_{H^1(0,1)}$$

$$\Rightarrow \quad \boxed{\| Pu - u \|_{L^2(0,1)} \leq h \sqrt{C} \| Pu - u \|_{H^1(0,1)}}$$

i.e., convergence in $H^1(0,1) \Rightarrow$ convergence in $L^2(0,1)$ with an order of h better.

- This generalizes to $\mathbb{R}^n$ and H^{k+1} with the appropriate "general Poincaré inequality". (aka Bramble - Hilbert)

BRAMBLE – HILBERT THEOREM: Let $F: H^{k+1} \to \mathbb{R}$ be a functional on H^{k+1} s.t.

1. F is bounded i.e. $|F(v)| \leq C_1 \|v\|_{H^{k+1}}$
2. F is sublinear i.e. $|F(v+w)| \leq C_2(|F(v)| + |F(w)|)$
3. F annihilates polynomials of deg $\leq k$

Then, $\exists C > 0$ s.t. $|F(v)| \leq C\,[v]_{H^{k+1}(\Omega)}$

In particular: for us, $F_i(u) := \|P_i \tilde{u} - \tilde{u}_i\|_{H^\ell(T_0)}$

- Applying Bramble - Hilbert on each patch

$$\|P_i u - u|_{T_i}\|^2_{H^\ell(T_i)} \leq \underset{\text{Bramble - Hilbert}}{C_\ell} \underbrace{\left(\frac{\text{diam}(T_i)}{\text{diam}(T_0)}\right)^{-2\ell}}_{\text{change of variables in } D^\ell u} \underbrace{\frac{\text{area}(T_i)}{\text{area}(T_0)}}_{\text{change of variables}} [\tilde{u}]^2_{H^{k+1}(T_0)}$$

- Now, we know $[\tilde{v}]^2_{H^{k+1}(T_0)} \longrightarrow [u|_{T_i}]_{H^{k+1}(T_i)}$

$$\| P_i u - u|_{T_0} \|^2_{H^\ell(T_i)} \leq C_\ell \left(\frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_0)} \right)^{-2\ell} \left(\frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_0)} \right)^{2(k+1)} [u|_{T_i}]^2_{H^{k+1}(T_i)}$$

$$= C_\ell \left(\frac{\mathrm{diam}(T_i)}{\mathrm{diam}(T_0)} \right)^{2(k+1-\ell)} [u|_{T_i}]^2_{H^{k+1}(T_i)}$$

- If all diameters are $\leq h$, then

$$\| P_i u - u \|_{H^\ell} \leq C_\ell \, h^{k+1-\ell} \, \| u|_{T_i} \|_{H^{k+1}}$$

- Summing over i:

$$\| P u - u \|_{H^\ell(\Omega)} \leq C_\ell \, h^{k+1-\ell} \, \| u \|_{H^{k+1}}$$

- If P_i is exact for polynomials of degree $\leq k$, we get convergence of order $O(h^{k+1-\ell})$ in H^ℓ

$$\Rightarrow O(h^{k+1}) \text{ in } L^2$$

$$\Rightarrow O(h^1) \text{ in } H^k$$

– Can show convergence of finite element methods

* TRIGONOMETRIC POLYNOMIALS

GOAL: Build approximations using trig. polys. of the form

$$f(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

Claim 1: $\cos(kx) + \cos((k-2)x) = 2\cos((k-1)x)\cos x$] Cheb. recursion in disguise

↳ Recurse : $\cos(kx) = \underline{P_k}(\cos x)$

↑ polynomial of degree k with leading coeff $= 2^k$

Claim 2: Similarly,

$$\sin((k+1)x) - \sin((k-1)x) = 2\cos(kx)\sin(x)$$

↳ Recurse: $\sin((k+1)x) = \underline{Q_k}(\cos x) \cdot \sin x$

↑ degree k

Upshot: Substituting back into the general form of trig. poly.

$$f(x) = \underbrace{P_N}_{\text{poly of deg}=N}(\cos x) + \underbrace{Q_{N-1}}_{\text{poly of deg}=N-1}(\cos x) \cdot \sin x$$

→ GOAL: Show trig. polys. are an algebra to use Stone-Weierstrass.

NOTE: $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x) = \tilde{P}(\cos x)$

$\Rightarrow$ If f, g are trig. polys., then so is fg and $f+g$.

$\Rightarrow$ Trig polys form an algebra

• GOAL: Separates pts? Vanishes nowhere?

chur

→ If $x, y \in [0, 2\pi)$, $x \neq y$, then

$\cos x \neq \cos y$ or $\sin x \neq \sin y$

$\Rightarrow$ Separates pts. $\Longrightarrow$ can apply Stone-Weierstrass

IMPORTANT !

THM: If $f \in C^{2\pi} := C([0, 2\pi))$, $\exists$ trig poly T_ε such that $\| f - T_\varepsilon \| < \varepsilon$.

THM: Trigonometric polynomials of degree n form a Haar subspace of dimension $\underbrace{2n+1}_{n \text{ cosines, } n \text{ sines, } 1 \text{ constants}}$.

Pf (Haar): Suppose

$$f(x) = a_0 + \sum_{n=1}^{N} a_n \cos(nx) + b_n \sin(nx)$$

has $f(x_i) = 0$ at $x_0, \ldots, x_{2n}$ disjoint.

Using

$$\cos(nx) = \frac{e^{inx} + e^{-inx}}{2}, \quad \sin(nx) = \frac{e^{inx} - e^{-inx}}{2i}$$

$\Rightarrow \exists$ constants $\{c_n\}$ s.t. $f(x) = \sum_{n=-N}^{N} c_n e^{inx}$

In matrix form of $f(x_i) \overset{!}{=} 0$, $i = 0, \ldots, 2n$, we have

$$\underbrace{\begin{bmatrix} e^{-iNx_0} & \cdots & e^{iNx_0} \\ e^{-iNx_1} & \cdots & e^{iNx_1} \\ \vdots & \ddots & \vdots \\ e^{-inx_{2n}} & \cdots & e^{-iNx_{2n}} \end{bmatrix}}_{U} \begin{bmatrix} c_{-N} \\ c_{-N+1} \\ \vdots \\ c_N \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

From HW, know how to compute this det in terms of entries

$\leftarrow$ scalar multiple of Vandermonde matrix

$$\Rightarrow \det(U) = e^{-iN(x_0 + \cdots + x_{2N})} \prod_{0 \le j < l \le 2N} (e^{ix_l} - e^{ix_j})$$

$$\neq 0$$

$\Rightarrow U$ invertible $\Rightarrow c_n \equiv 0 \;\forall n \Rightarrow f \equiv 0$

$\Rightarrow$ Haar. $\square$

REMARK: The general Equioscillation Thm, for all $f \in C^{2\pi}$,

$\exists!$ best approximation $f_* \in \mathcal{T}_n$ $:=$ trig. polys. of deg $\leq n$

(optimality $\iff$ equioscillation)

$\implies$ All results we had about best approxs. apply !

LECTURE 15

May 14, 2026

* FOURIER SERIES

- Suppose f is 2π-periodic, bounded, integrable.

Fourier Series for f: $\frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$,

$$\begin{cases} a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} \cos(kt)\, f(t)\, dt \\ b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(kt)\, f(t)\, dt \end{cases}$$

- Can also write it as: $\sum_{k \in \mathbb{Z}} c_k e^{-ikx}$, $c_k := \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} f(x)\, dx$

NOTE: $\overline{c_k} = c_{-k}$.

- N-th Partial Sum:

$$S_n(f)(x) = \frac{a_0}{2} + \sum_{k=1}^{N} \left[a_k \cos(kx) + b_k \sin(kx) \right]$$

PROPOSITION: $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{1}{\sqrt{\pi}} \cos(kx), \frac{1}{\sqrt{\pi}} \sin(kx) \right\}_{k\in\mathbb{N}}$

are orthonormal on $[0, 2\pi)$ w.r.t. $\langle \cdot, \cdot \rangle_{L^2([0,2\pi))}$.

FACT: $S_n^2 = S_n \implies S_n$ polynomial from $C^{2\pi}$ to T_n.

trig. poly. of degree at most n

COROLLARY: $f \in C^{2\pi}$, and $a_k = b_k = 0 \ \forall k$, then $f \equiv 0$.

Pf: $\int_{-\pi}^{\pi} T f \, dx = 0 \quad \forall T$ trig. poly.

By Weierstrass: $\exists \tilde{T}$ best approx to f s.t. $\| f - \tilde{T} \|_\infty < \varepsilon$

$$\implies \int_{-\pi}^{\pi} f^2 \, dx \leq \int_{-\pi}^{\pi} f \tilde{T} \, dx + \varepsilon \int_{-\pi}^{\pi} |f| \, dx$$

$$= \varepsilon \int_{-\pi}^{\pi} |f| \, dx$$

$$\varepsilon \text{ arbitrary} \implies \int_{-\pi}^{\pi} f^2 \, dx \xrightarrow{f \text{ continuous}} f \equiv 0.$$

Corollary:

$$\frac{1}{\pi}\|S_n(f)\|_{L^2(-\pi,\pi)} = \frac{1}{\pi}\int_{-\pi}^{\pi} |S_n(f)(x)|^2\,dx = \frac{a_0^2}{2} + \sum_{k=1}^{\infty}(a_k^2+b_k^2)$$

$$\le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx$$

$$= \frac{1}{\pi}\|f\|^2_{L^2(-\pi,\pi)}.$$

$\Rightarrow$ coeffs are square integrable

Corollary: $f \in C^{2\pi} = C([0,2\pi))$, then

$$\lim_{n\to\infty} S_n(f) = f \quad \text{in } L^2.$$

Detour: $f(x) = \operatorname{sgn}(x)$ on $[-\pi,\pi)$. Compute Fourier Series.

$\int (\text{even})\cdot(\text{odd})\,dx = 0 \Rightarrow a_k = 0 \quad \forall k$

Abramowitz & Stegun: $b_k = \frac{2}{\pi k}\left[1-(-1)^k\right].$

$$S_{2n}(f) = \sum_{k=1}^{n} \frac{4}{(2k-1)\pi}\sin((2k-1)x)$$

Know: $S_{2n}(f) \xrightarrow{L^2} f$. What about in $C^{2\pi}$, $\|\cdot\|_\infty$?

Set: $x_n = \frac{\pi}{2n}$, $n \in \mathbb{N}$.

$$\Rightarrow \quad S_{2n}(f)\left(\frac{\pi}{2n}\right) = \frac{4}{\pi}\Big[\underbrace{\sin\left(\frac{\pi}{2n}\right) + \frac{1}{3}\sin\left(\frac{3\pi}{2n}\right) + \dots + \frac{1}{2n-1}\sin\left(\frac{(2n-1)\pi}{2n}\right)}_{\text{looks like Riemann Sum for } \int.}\Big]$$

$$= \frac{4}{\pi}\frac{\pi}{2n}\left[\frac{1}{\frac{\pi}{2n}}\sin\left(\frac{\pi}{2n}\right) + \frac{1}{\frac{3\pi}{2n}}\sin\left(\frac{3\pi}{2n}\right) + \dots + \frac{1}{\frac{(2n-1)\pi}{2n}}\sin\left(\frac{(2n-1)\pi}{2n}\right)\right]$$

$$\approx \frac{2}{\pi}\int_0^{\pi}\frac{\sin t}{t}\,dt$$

$\Rightarrow$ Jump discontinuity at zero

$\Downarrow$

<u>GIBBS PHENOMENON</u>.

$$\gtrsim 1 + 2\,(\text{Wilbraham - Gibbs constant})$$

$\Downarrow$

Fourier series is a "bad" approximation to <u>dis</u>continuous functions.

(also happens when using general polys. to represent discontinuous functions)

- **PROPERTIES OF PROJECTIONS S_n :**

- Take $S_n(f)(x) := \sum_{k=-n}^{n} c_k e^{ikx}$, $c_k := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iky} f(y)\, dy$.

$$\Rightarrow S_n(f) = \sum_{k=-n}^{n} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-iky} f(y)\, dy \cdot e^{ikx}$$

Parkour $\longrightarrow$ $$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \sum_{k=-n}^{n} \left[e^{i(x-y)}\right]^k f(y)\, dy$$

k=0 counted twice

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left(\mathrm{Re}\left(\sum_{k=0}^{n} e^{ik(x-y)}\right) - \frac{1}{2}\right) f(y)\, dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \left[\mathrm{Re}\left(\frac{1 - e^{i(n+1)(x-y)}}{1 - e^{i(x-y)}}\right) - \frac{1}{2}\right] f(y)\, dy$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} \underbrace{\frac{\sin\left[(x-y)(n+1/2)\right]}{\sin\left[(x-y)\cdot 1/2\right]}}_{= D_n(x-y)} f(y)\, dy$$

Much easier to compute than the sum !!

$$D_n(t) := \frac{\sin\left[t(n+1/2)\right]}{\sin(t/2)}$$ (DIRICHLET KERNEL)

THM: (About Dirichlet Kernel D_n)

1. D_n continuous, $\lim_{t \to 0} D_n(t) = 2n+1$, D_n is even.

2. $\frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(t) \, dt = 1$

3. $\frac{\left| \sin\left(t(n+1/2)\right) \right|}{|t/2|} \leq |D_n(t)| \leq \frac{\pi}{|t|}$, $t \in (-\pi, \pi)$

4. $\frac{8}{\pi} \log n \leq \| D_n \|_1 \leq 6\pi + \pi \log n$.

COROLLARY: $\exists f_{bad} \in C^{2\pi}$ for which ← ⊙⊙

$\| S_n(f_{bad}) \|_{C^{2\pi}}$ is unbounded in n.

Pf: $S_n : C^{2\pi} \to C^{2\pi}$ are continuous, hence bounded $\forall n$.

If $\sup_n \| S_n(f) \| < \infty$ $\forall f \in C^{2\pi}$ $\xRightarrow{}$ $\sup_n \| S_n \| < \infty$

Uniform Boundedness Principle

But $\| S_n(f) \| \leq \frac{1}{2\pi} \| f \|_\infty \int_{-\pi}^{\pi} | D_n(t) | \, dt \longrightarrow \infty$

Now, $\quad S_n(f)(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} D_n(y)\, f(y)\, dy$

1st attempt: $\quad f(y) = \operatorname{sgn}(D_n(y))$ (not continuous)

2nd attempt: smooth this out a bit.

$\square$

THM: Let $f \in C^{2\pi}$ and T_* be its best approximation from $\mathcal{T}_n$ (trig. poly. of deg $\leq n$). Let $S_n(f)$ be f's truncated Fourier series at order n. Then

$$\| f - S_n(f) \|_\infty \leq \Big(1 + \underbrace{\| D_n \|_1}_{\sim \log n}\Big) \| f - T_* \|_\infty .$$

$\Rightarrow$ Fourier series is very good
except when you shouldn't be using polys. to approximate your fct

REMARK: (CONVOLUTIONS)

$$(f * g)(x) := \int_{-\pi}^{\pi} f(x-y)\, g(y)\, dy = \int_{-\pi}^{\pi} f(y)\, g(x-y)\, dy .$$

$$\Rightarrow \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{-ikx} (f*g)(x)\, dx = 2\pi\, \underset{\hat{f}_k}{\overset{||}{c_k}}(f)\, \underset{\hat{g}_k}{\overset{||}{c_k}}(g)$$

$$S_n(f) = \frac{1}{2\pi} (D_n * f) \quad \Rightarrow \quad \widehat{S_n(f)}_k = \underbrace{\widehat{(D_n)}_k}\, \hat{f}_k$$

$$\widehat{(D_n)}_k = \begin{cases} 1, & |k| \leq n \\ 0, & \text{else} \end{cases} .$$

LECTURE 16

May 19th, 2026

FOURIER TRANSFORM:

$$(\mathcal{F}f)(\xi) := \hat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi\cdot x} f(x)\, dx$$

$$(\mathcal{F}^{-1}\hat{f})(x) := f(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi\cdot x} \hat{f}(\xi)\, d\xi$$

$\mathcal{F}$ is unitary then...

FACTS:

1. $f \in L^1(\mathbb{R}) \Rightarrow \hat{f} \in C(\mathbb{R})$

2. RIEMANN-LEBESGUE LEMMA:

   $f \in L^1 \Rightarrow \hat{f}$ continuous and $|\hat{f}(\xi)| \xrightarrow{|\xi|\nearrow\infty} 0$

   Pf: Approximation by C_c^∞ fcts.

3. PROPOSITION: Let $j \in \mathbb{N}$ and f has j integrable derivatives. Then $\exists C > 0$ s.t.

   $$|\hat{f}(\xi)| \le \frac{C}{(1+|\xi|^2)^{1/2}}$$

   Conversely, if $\hat{f}(\xi)$ decays like $|\xi|^{-j+1+\varepsilon}$ $(\varepsilon > 0)$, then f is continuous with j continuous derivatives

Upshot: f "smooth" (C^j) $\iff$ $\hat{f}$ decays "fast" ($O(\xi^j)$)

4. f compactly supported $\rightsquigarrow$ $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} f(x)\, e^{i\xi \cdot x}\, dx$

$f \in L^2$.

$\Rightarrow$ For $\xi \in \mathbb{C}$ then this still well-defined $\rightsquigarrow$ $\xi = \tau + i\eta$.

$$\hat{f}^{(\ell)}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (ix)^{\ell} f(x)\, e^{-i\xi \cdot x}\, dx$$

$\Rightarrow$ $\hat{f}$ is entire (i.e., analytic everywhere in $\mathbb{C}$)

5. PARSEVAL: $\|f\|_{L^2} = \|\hat{f}\|_{L^2}$.

SIGNAL PROCESSING

- Aliasing
- Nyquist Frequency $\rightsquigarrow$ sample 1 data pt. per trough and 1 per peak.

Avoids confusing high freq. w/ low freq. $\rightarrow$

- Assume f is bandlimited; i.e., $\operatorname{supp} \hat{f} \subset [-A, A]$.

Goal: Evaluate f at $t_j = j\tau$, $j \in \mathbb{Z}$

Shannon–Nyquist Theorem: If $f \in L^1(\mathbb{R})$ and $\operatorname{supp} \hat{f} \subset [-\Omega, \Omega]$

then

$$f(t) = \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{A}\right) \operatorname{sinc}(At - n\pi)$$

with equality in L^2 sense $\left(\tau = \frac{\pi}{A}\right)$.

Poisson Summation: Suppose $\sum_{n \in \mathbb{Z}} \hat{f}(\xi + 2An) \in L^2([0, 2A])$.

with $f, \hat{f} \in L^1$. Then

$$\sum_{n \in \mathbb{Z}} \hat{f}(\xi + 2nA) = \frac{\sqrt{2\pi}}{2A} \sum_{n \in \mathbb{Z}} f\left(\frac{n\pi}{A}\right) e^{-2\pi i \xi n/2A}$$

Pf: Set $\phi(\xi) := \sum_n \hat{f}(\xi + 2nA)$, then ϕ is periodic and, by assumption, $\in L^2[0, 2A]$. Take Fourier Series: $\phi(\xi) = \sum_n c_n e^{-2\pi i \xi n/2A}$

$$c_n = \frac{1}{2A} \int_0^{2A} e^{2\pi i \xi n/2A} \phi(\xi)\, d\xi = \frac{1}{2A} \int_0^{2A} e^{2\pi i \xi n} \left(\sum_n \hat{f}(\xi + 2nA)\right) d\xi$$

$$\overset{\text{Parseval}}{=} \sum \leftrightarrow \int \underset{\text{variables}}{\overset{\text{change}}{=}} \frac{1}{2A} \int_{\mathbb{R}} e^{2\pi \xi n/2A} \hat{f}(\xi)\, d\xi = \frac{\sqrt{2\pi}}{2A} f\left(\frac{n\pi}{A}\right) \quad \square$$

Pf: (SHANNON – NYQUIST) $\operatorname{supp} \hat{f} \subset [-A, A]$, hence

$$\sum_{n \in \mathbb{Z}} \hat{f}(\xi + 2nA) = \hat{f}(\xi) \quad \text{as long as} \quad \xi \in [-A, A].$$

Now, can write $\hat{f}(\xi) = \chi_{[-A,A]}(\xi) \sum_{n \in \mathbb{Z}} \hat{f}(\xi + 2An)$

So, $f(x) = \mathcal{F}^{-1} \hat{f} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i \xi \cdot x} \hat{f}(\xi)\, d\xi$

$$= \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{i\xi \cdot x} \sum_{n \in \mathbb{N}} \hat{f}(\xi + 2An)\, d\xi$$

Poisson Summation

$$= \frac{1}{\sqrt{2\pi}} \int_{-A}^{A} e^{i\xi x} \frac{\sqrt{2\pi}}{2A} \sum_{n \in \mathbb{N}} f\left(\frac{n\pi}{A}\right) e^{-2\pi i \xi n / 2A}\, d\xi$$

$\mathcal{F}^{-1}[\chi_{[\dots]}] = \operatorname{sinc}(\dots)$

$$= \sum_{n \in \mathbb{N}} f\left(\frac{n\pi}{A}\right) \operatorname{sinc}(Ax - \pi n).$$

$\square$

Upshot: To sample signal using equispaced sampling points, need to sample at Nyquist Frequency: $f_N = \frac{\pi}{A}$

* TIME – LIMITED & BANDLIMITED :

→ Can only have a fct that is both cptly supp. on $\mathcal{F}$ – domain and space up to an ε !

(not fully, because the only analytic cptly supp fct on $\mathbb{C}$ is $\equiv 0$ by Liouville)

$$(\Delta x)^2 := \int_{\mathbb{R}} x^2 |f(x)|^2 dx - \left(\int_{\mathbb{R}} x |f(x)|^2 dx \right)^2$$

$$(\Delta p)^2 := \int_{\mathbb{R}} \xi^2 |\hat{f}(\xi)|^2 d\xi - \left(\int_{\mathbb{R}} \xi |\hat{f}(\xi)|^2 d\xi \right)^2$$

$$f_c(x) := e^{ip_0 x} f(x + x_0)$$

(so we can shift in space / freq. domains)

$$\int_{\mathbb{R}} x |f_c|^2 dx = \int_{\mathbb{R}} \xi |\hat{f}_c|^2 d\xi = 0$$

Set $u = xf$, $v = f'$ $\quad (\hat{v} = (i\xi)\hat{f})$

Replace $f \leftrightarrow f_c$ below :

$$(\Delta x)^2 = \int_{\mathbb{R}} |u|^2 dx , \quad (\Delta p)^2 = \int_{\mathbb{R}} |v|^2 dx$$

$$\mathrm{Re}\left(\int_{\mathbb{R}} \bar{u} v \, dx \right) = \frac{1}{2} \int_{\mathbb{R}} x \bar{f} f' + (\bar{f})' x f \, dx$$

$$= \frac{1}{2} \int_{\mathbb{R}} \underbrace{x \bar{f} f' - \bar{f} (xf)'}_{= \bar{f} \cdot [x, \frac{d}{dx}] f} dx$$

$$= -\frac{1}{2}\int_{\mathbb{R}} |f|^2\,dx$$

$$= -\frac{1}{2}.$$

Cauchy-Schwarz

$$\left|\operatorname{Re}\left(\int_{\mathbb{R}} \bar{u}\, v\, dx\right)\right| \leq \|u\|_{L^2}\|v\|_{L^2} = (\Delta x)(\Delta p)$$

i.e., $\boxed{\frac{1}{2} \leq (\Delta x)(\Delta p)}$ (UNCERTAINTY PRINCIPLE)

$\iff$ The more you want to localize in space, the more you "leak out" in frequency space, and vice-versa

OBS: Prolate Spheroidal Wavefunctions are maximally band limited and time-limited.

LECTURE 17

May 21, 2026

→ <u>Again</u>: Can we have band-limited (space) and time-limited (Fourier) functions? $\mathbb{C}$-analysis: No

Life: YES, up to an ε, choose Gaussian

uncertainty principle quantifies this ε...

→ Is there an optimally time & band-limited fct? PROLATE SPHEROIDALS

$$(Q_T)f(x) = \begin{cases} f(x), & |x| < T \\ 0, & \text{else} \end{cases} \qquad P_\Omega := \mathcal{F}^{-1} Q_\Omega \mathcal{F}, \quad \Omega \in \mathbb{R}.$$

$$\frac{\|P_\Omega Q_T f\|^2}{\|f\|^2} < 1$$

$$\frac{\langle P_\Omega Q_T f, P_\Omega Q_T f\rangle}{\|f\|^2} = \frac{\langle f, Q_T P_\Omega Q_T f\rangle}{\|f\|^2}$$

Q_T, P_Ω self-adjoint & P_Ω proj.

Eigenval. Problem

$$\underline{(Q_T P_\Omega Q_T)}(f)(x) = \lambda \int_{-\pi}^{\pi} \operatorname{sinc}(\Omega\lambda - \lambda t)\, f\left(\frac{tT}{\pi}\right) dt$$

Eigenfcts of ↑ are called PROLATE SPHEROIDAL FUNCTIONS

$$\boxed{\lambda = \frac{\Omega T}{\pi} \quad (\approx \text{area in phase space})}$$

* **BAND-LIMITED FUNCTIONS:**

$$B_\Omega := \{ f \in L^2(\mathbb{R}) : \operatorname{supp} \hat{f} \subset [-\Omega, \Omega] \}$$ (space of bandlimited fcts)

FACT: $B_\Omega \subset L^2(\mathbb{R})$ is a closed subspace, hence $(B_\Omega, \langle \cdot, \cdot \rangle_{L^2(\mathbb{R})})$ is a Hilbert space.

$$|f(z)| \leq \frac{1}{\sqrt{2\pi}} \|\hat{f}\|_{L^1} e^{\Omega (\operatorname{Im} z)}, \quad z \in \mathbb{C}$$

NOTE: $f \in B_\Omega$, $f \in L^1(\mathbb{R})$.

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \hat{f}(\xi)\, d\xi$$

$$= \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} \int_{-\Omega}^{\Omega} e^{-i\xi x} f(x)\, dx\, d\xi$$

$$= \frac{1}{2\pi} \int_{-\Omega}^{\Omega} \int_{\mathbb{R}} e^{i\xi(x-y)} f(y)\, dy\, dx$$

$$= \underbrace{\frac{1}{\pi}} \int_{\mathbb{R}} \underbrace{\frac{\sin(\Omega(x-y))}{(x-y)}} f(y)\, dy$$

Set $e_x(y) := \dfrac{\sin(\Omega(x-y))}{\pi(x-y)} \in L^2(\mathbb{R})$, actually, $e_x \in B_\Omega$

Then $$f(x) = \int_{\mathbb{R}} e_x(y)\, f(y)\, dy$$

and $$f(x) = \langle \underline{\underline{e_x}}, f \rangle_{L^2(\mathbb{R})}$$

"δ function" but, unusually, $e_x \in B_\Omega$:)

Set $K(x,y) = e_x(y)$,

$$f(x) = \int_{-\infty}^{\infty} K(x,y)\, f(y)\, dy, \quad \underline{K(x,\cdot) \in B_\Omega}$$

"REPRODUCING KERNEL"

<u>REPRODUCING KERNEL HILBERT SPACES</u> (RKHS)

fcts on X

<u>DEF</u>: $\mathcal{H}$ is an RKHS if $\forall x \in X \; \exists\, l_x \in \mathcal{H}^* \cong \mathcal{H}$ $(\|l_x\|_{\mathcal{H}^*} < \infty)$
s.t. $\underline{\underline{l_x}}(f) = f(x) \quad \forall f \in \mathcal{H}$

e.g., $e_x(\cdot)$ from above
$\langle e_x, f \rangle = l_x(f)$

<u>NON-EXAMPLE</u>: $L^2([0,1]) \rightsquigarrow$ ptwise evaluation is meaningless in L^2...

!

<u>EXAMPLE</u>: HARDY SPACES — Analytic fcts → force L^2 on bdry

SZEGŐ KERNEL
STUDY FOR FINAL / QUAL

<u>EXAMPLE</u>: $H_0^1([-1,1])$ w.r.t. $\langle f, g \rangle_{H^1} = \langle f, g \rangle_{L^2} + \langle f', g' \rangle_{L^2}$

* REPRESENTER THEOREMS:

- $$\min_{f \in \mathcal{H}} \Big\{ \overbrace{\sum_{i=1}^{n} [y_i - f(x_i)]^2 + \lambda \, \| (Id - \underline{P}) f \|^2}^{=: F(f)} \Big\} \quad (*)$$

orthogonal projection

- Assume $\dim \underbrace{\ker(Id - P)}_{=: \mathcal{H}_0} < \infty$ $\longrightarrow$ basis $\phi_1, \ldots, \phi_m$ for $\mathcal{H}_0$

- Since projection: $\mathcal{H} = \mathcal{H}_0 \oplus \underline{\mathcal{H}_1}$

Let K_1 be reproducing kernel for $\mathcal{H}_1$

THM: Solution to $(*)$ is of the form

$$f_* = \sum_{j=1}^{m} \alpha_j \phi_j + \sum_{k=1}^{n} \beta_k e_k, \quad e_k := K_1(\cdot, x_k).$$

$\Rightarrow$ Philosophically:

in very high-dimensional problems, it's silly to find a basis for 1 billion dims to only study 1,000 data pts...

But in low dimensions (e.g. $\mathbb{R}$), it's much better to take a basis (e.g. Chebyshev polys)

Q: HOW TO FIND RKHS ?

A: Many ways... one way is Radial Basis fcts,

* RADIAL BASIS FUNCTIONS :

Approximate f by cubic splines $\sum_{j=1}^{n} a_j \underbrace{|x - x_j|^3}_{=: \phi(x-x_j)} + P_2$

$\phi(\cdot) := |\cdot|^3$

PHILOSOPHY : Choose the basis after data is given (data driven)
Opposite to what we were doing before with
Chebyshev nodes and poly. bases.

DEF: $\phi \in C(\mathbb{R}^d, \mathbb{C})$ is positive semi-definite if $\forall N \in \mathbb{N}$, and pairwise distinct points $X := \{x_1, \dots, x_N\} \subset \mathbb{R}^d$, and $\alpha \in \mathbb{C}^N$, we have

$$\sum_{j,k=1}^{N} \alpha_j \overline{\alpha_k} \phi(x_j - x_k) \geq 0$$

ϕ positive definite if "$>$"

Set $F_\phi := \operatorname{span}\{\phi(\cdot - y) : y \in \Omega\}$, $\Omega \subset \mathbb{R}^d$.

$$\sum_{j=1}^{N} \sum_{k=1}^{M} \alpha_j \overline{\beta_k} \phi(x_j - y_k).$$

THM: If ϕ is symmetric (i.e., $\phi(\cdot,\cdot)$) and positive definite then $\langle \cdot,\cdot \rangle_\phi$ defines an inner product on the pre-Hilbert space F_ϕ). Moreover, ϕ is the reproducing kernel

THM: (MOORE - ARONSZAJN) If ϕ symmetric & positive definite on X, then $\exists!$ Hilbert space of functions on X for which ϕ is the reproducing kernel.

Converse is also true.