Lecture 1 | Introduction

1) Complex Numbers and Complex Functions:

$\mathbb{C}$ is the field of complex numbers: $a = \alpha + i\beta$.

In particular, that $\frac{1}{\alpha + i\beta} \in \mathbb{C}$

Identify with (α, β)

$\Downarrow$

$i = (0, 1)$

$$\frac{1}{\alpha + i\beta} \cdot \frac{\alpha - i\beta}{\alpha - i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2} \in \mathbb{C}$$

i is the root of $z^2 + 1 = 0$.

For $a = \alpha + i\beta$, $\alpha = \mathrm{Re}(a)$ and $\beta = \mathrm{Im}(a)$ and its conjugate is $\bar{a} = \alpha - i\beta$.

$$\overline{a+b} = \bar{a} + \bar{b} \quad \text{and} \quad \overline{ab} = \bar{a} \cdot \bar{b}$$

The modulus of $a = \alpha + i\beta$ is

$$|a| = \sqrt{\alpha^2 + \beta^2} = \sqrt{\bar{a} \cdot a}$$

Note that

$$\mathrm{Re}(a) = \frac{1}{2}(a + \bar{a})$$

$$\mathrm{Im}(a) = \frac{1}{2i}(a - \bar{a})$$

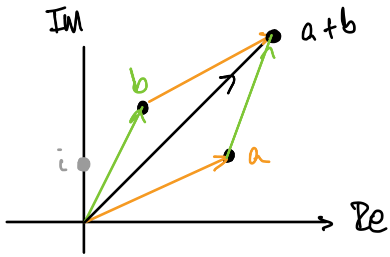
Can write this b/c the map $a \mapsto \bar{a}$ is invertible.

Triangle Inequality: $|a+b| \leq |a| + |b|$

Cauchy-Schwarz:

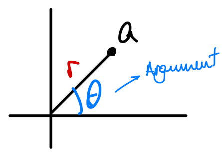
$$\left| \sum_{j=1}^{n} a_j b_j \right|^2 \leq \left(\sum_{j=1}^{n} |a_j|^2 \right) \left(\sum_{j=1}^{n} |b_j|^2 \right)$$

Addition:



Polar Coördinates: For $a = \alpha + i\beta$,

$$\alpha = r\cos\theta \quad , \quad \beta = r\cos\theta$$



$$\Longrightarrow \quad \boxed{a = r(\cos\theta + i\sin\theta)}.$$

Multiplication: For

$$a = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad b = r_2(\cos\theta_2 + i\sin\theta_2),$$

we have

$$a \cdot b = r_1 r_2 \left(\cos\underbrace{(\theta_1+\theta_2)} + i\sin\underbrace{(\theta_1+\theta_2)} \right)$$

Addition of the arguments: $\arg(ab) = \arg(a) + \arg(b) \pmod{2\pi}$

Powers:

$$a^n = r^n\left(\cos n\theta + i\sin n\theta\right), \quad n \in \mathbb{Z}.$$

Note: $$a^{-1} = \frac{1}{a} = \frac{1}{r}\cdot\frac{1}{\cos\theta + i\sin\theta}$$
$$= \frac{1}{r}(\cos\theta - i\sin\theta)$$
$$= r^{-1}(\cos(-\theta) + i\sin(-\theta)).$$

De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$$

n-th roots of a: solutions to $z^n = a$.

For

$$a = r(\cos\theta + i\sin\theta); \qquad z = \rho(\cos\varphi + i\sin\varphi)$$

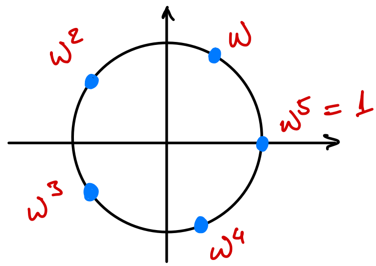
$$\rightsquigarrow z = r^{1/n}\left[\cos\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right)\right],$$
$$k = 0, 1, \ldots, n-1$$

n-th roots of the unit: $z^n = 1$

e.g., $z = \omega$, where $\omega = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$.

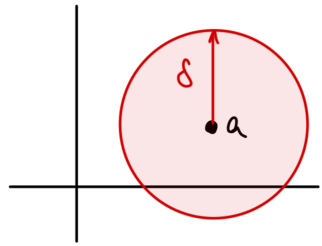
In general, $z = \underbrace{1}_{=\,\omega^n}, \omega, \omega^2, \ldots, \omega^{n-1}$

e.g., $n = 5$



$\varepsilon - \delta$ proofs

$$|z - a| < \delta \xrightarrow[\text{geometrically}]{}$$



We could think this in terms of rectangles as well since "for every point contained in an open ball, there is an open rectangle that contains the point and is contained in the ball."

"z lies in the open disks with centre a and radius δ."

LECTURE 2 | RIEMANN SPHERE

RIEMANN SPHERE: extend $\mathbb{C}$ to include ∞.

Properties:

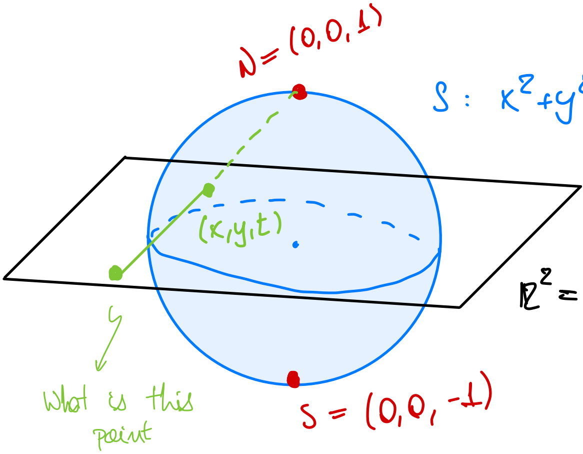
$$a + \infty = \infty + a = \infty$$

$$a \cdot \infty = \infty \cdot a, \quad a \neq 0$$

$$a/\infty = 0, \quad a \neq \infty$$

$$a/0 = \infty, \quad a \neq 0$$

So,



$\mathbb{R}^2 = \mathbb{C}$, $(x,y) = x+iy = z$

STEREOGRAPHIC PROJECTION FROM THE NORTH POLE N:

$$\boxed{z = \frac{x+iy}{1-t}}.$$

The 3 points

$$(0,0,1), \quad (x,y,t), \quad \text{and} \quad \left(\frac{x}{1-t}, \frac{y}{1-t}, 0\right)$$

are colinear.

This defines a homeomorphism of $S^2 \setminus \{N\}$ onto $\mathbb{C}$ (N is the point at ∞). Equivalently, this defines a coördinate chart for S^2.

We also consider the stereographic projection from the south pole S:

complex conjugate

$$z' = \frac{x-iy}{1+t},$$

With stereographic proj., we can cover S^n with two coördinate charts which is a homeomorphism from $S^2 \setminus \{S\}$ onto $\mathbb{C}$.

Note that, for any point of $S^2 \setminus \{N, S\}$,

$$z'z = \left(\frac{x-iy}{1+t}\right)\left(\frac{x+iy}{1-t}\right) = \frac{x^2+y^2}{1-t^2} = 1$$

$\rightsquigarrow x^2+y^2+t^2=1$

$$\Rightarrow \boxed{z'z = 1 \iff z' = \frac{1}{z}}$$

z' coördinate at infinity.

Under the stereographic projection from N, any straight line on the plane corresponds to a circle through N.

Any circle in S^2 corresponds to a circle or straight line in $\mathbb{C}$.

Any circle in S^2 lies in a plane

$$ax + by + ct = d.$$

$$(x, y, t) \mapsto \frac{x+iy}{1-t}$$

Note that

$$|z|^2 = \bar{z}z = \frac{x-iy}{1-t} \cdot \frac{x+iy}{1-t} = \frac{1+t^2}{(1+t)^2} = \frac{1+t}{1-t},$$

i.e., $t = \dfrac{|z|^2-1}{|z|^2+1}$ or $1-t = \dfrac{2}{|z|^2+1}$.

So,

$$\frac{1}{2}(z+\bar{z}) = \frac{x}{1-t}$$

$$\Rightarrow x = \frac{1}{2}(z+\bar{z})(1-t) = \frac{z+\bar{z}}{|z|^2+1}.$$

$$\frac{1}{2}(z-\bar{z}) = \frac{y}{1-t}$$

$$\Rightarrow y = \frac{z-\bar{z}}{i(|z|^2+1)}$$

Now, substitute in the plane equation

$$a(z+\bar{z}) + \frac{b}{i}(z-\bar{z}) + c(|z|^2-1) = d(|z|^2+1).$$

Write $z = u+iv$

(*) $(d-c)(u^2+v^2) - 2au - 2bv + (d+c) = 0$

circle if $d \neq c$, line if $d = c$.

Note that this is 1-1 since any circle or line can be written as (*)

COMPLEX FUNCTIONS:

1) **LINEAR FUNCTIONS**: $T: \mathbb{C} \to \mathbb{C}$, $T(z) = az$ is a homothetic linear transformation. It is a composition of $\underbrace{\text{rotations}}_{|a|=1}$ and $\underbrace{\text{dilations}}_{a>0}$ (scaling) and

preserves angles and orientation.

For example, $z \mapsto \bar{z}$ is $\mathbb{R}$-linear but <u>not</u> $\mathbb{C}$-linear

$\mathbb{R}$-linear transformation of $\mathbb{R}^2$: $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\mathbb{C}$-linear transformation of $\mathbb{C} = \mathbb{R}^2$.

$$az = (\alpha + i\beta)(x + iy) = (\alpha x - \beta y) + i(\beta x + \alpha y).$$

As $\mathbb{R}$-linear transformation:

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

Exercise: find all angle preserving $\mathbb{R}$-linear transformations T of $\mathbb{C}$.

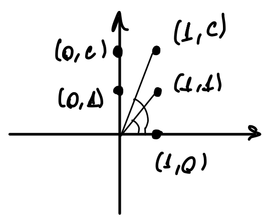
Let S be a homothetic linear transformation such that $S^{-1}T$ fixes $(1, 0)$.

$$S^{-1}T : (0,1) \mapsto (0, c), \quad c \neq 0.$$

preserves angles

So,

$$S^{-1}T : (1,1) \mapsto (1, c)$$



$$\Rightarrow c = \pm 1$$

If $c = 1$, then $S^{-1}T = \mathrm{id}$ and if $c = -1$, then $S^{-1}T : z \mapsto \bar{z}$, so $T(z) = a\bar{z}$.

2) POLYNOMIAL FUNCTIONS.

$$f(z) = a_n z^n + \cdots + a_1 z + a_0$$
$$= a_n (z - c_1) \cdots (z - c_n)$$

3) RATIONAL FUNCTIONS

$R(z) = \dfrac{P(z)}{Q(z)}$, P, Q poly. with no common factors.

* Zeros of $Q(z)$ are called the poles of $R(z)$.
the points z_0 s.t. $R(z_0) = \infty$, $\lim_{z \to z_0} R(z) = \infty$.

* The order of a pole is the order of the correspon-ding zero dimension.

* Poles of $R'(z)$ are the same as the poles of $R(z)$. If z_0 is a pole of order k of $R(z)$, then z_0 is a pole of order $k+1$ of $R'(z)$.

$$R'(z) = \frac{P'(z) Q(z) - P(z) Q'(z)}{[Q(z)]^2}$$

* Define $R(\infty)$ as $\lim_{z \to \infty} R(z)$.

* Order of a pole at ∞: think of both z and

$R(z)$ as points on the Riemann Sphere. Consider $R(z)$ in coördinates at ∞. So, consider

$$R_1(z) = R(1/z).$$

So, saying that $R(z)$ has a pole at ∞ is the same as saying that R_1 has a 0 pole at 0.

$\Rightarrow$ Order of a $\begin{smallmatrix}\text{zero}\\ \text{or}\\ \text{pole}\end{smallmatrix}$ of $R(z)$ at ∞ is the order of the $\begin{smallmatrix}\text{zero}\\ \text{or}\\ \text{pole}\end{smallmatrix}$ of $R_1(z)$ at 0.

Ex: Let

$$R(z) = \frac{a_m z^m + \cdots + a_0}{b_n z^n + \cdots + b_0} \quad (a_m, b_n \neq 0).$$

Then, setting $R_1(z) = R(1/z)$, we get

$$R_1(z) = z^{n-m} \cdot \frac{a_0 z^m + a_1 z^{m-1} + \cdots + a_m}{b_0 z^n + b_1 z^{n-1} + \cdots + b_n}.$$

If $n > m$, then $R(z)$ has a zero at ∞ of order $n-m$

If $m > n$, then $R(z)$ has a pole at ∞ of order $m-n$

If $m = n$, then $R(\infty) = \dfrac{a_m}{b_n} \neq 0$. (not a zero or a pole)

MORAL: the total number of zeros of $R(z)$ in the Riemann Sphere (including at ∞) is $\max(m,n)$ which is the same number of poles of $R(z)$ in the Riemann Sphere.

counted w/ multiplicity

"Order of $R(z)$"

A rational function $R(z)$ of order k zeros and k poles and every equation $R(z) = a$ has k roots.

LECTURE 3

Rational function: $R(z) = \dfrac{P(z)}{Q(z)}$, P, Q have no common factors.

$$\boxed{\operatorname{order}(R) = \max\{\deg P, \deg Q\}},$$

if $\operatorname{order}(R) = k$ and R is not constant, then R has exactly k poles and k zeros in S^2, and every equation $R(z) = a$ k roots.

EXAMPLE: Rational function of order 1

$$S(z) = \frac{az+b}{cz+d}, \quad ad-bc \neq 0$$

FRACTIONAL LINEAR TRANSFORMATION (or Möbius linear transf.)

Any equation $S(z) = w$ has exactly 1 root; i.e.,

$$w = S^{-1}(z) = \frac{dz-b}{-cz+a}$$

e.g., $S(z) = z+a$ (translation) $\rightarrow$ 1 fixed point at ∞

e.g., $S(z) = \frac{1}{z}$ (inversion) $\rightarrow$ ± 1 are fixed points and 0 and ∞ are interchanged.

* REPRESENTATION OF RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

For $R(z) = \frac{P(z)}{Q(z)}$, first do long division until the degree of the numerator is $\leq$ the degree of denominator:

$$P(z) = G(z)Q(z) + \tilde{H}(z), \quad \deg \tilde{H} \leq \deg Q$$

$$R(z) = \underbrace{G(z)}_{\text{poly. without constant term}} + \underbrace{H(z)}_{\text{finite at } \infty}, \qquad H(z) = \frac{\tilde{H}(z)}{Q(z)}$$

$\deg G(z) =$ order of pole of $R(z)$ at ∞.

$\hookrightarrow$ $G(z)$ is the <u>singular part</u> of $R(z)$ at ∞

Now, let $\beta_1, \dots, \beta_\ell$ be distinct finite poles of $R(z)$

$$z = \beta_j + \frac{1}{\zeta}, \qquad \zeta = \frac{1}{z - \beta_j}$$

$R\left(\beta_j + \frac{1}{\zeta}\right)$ rational function of ζ with pole at ∞. So, we can write

$$R\left(\beta_j + 1/\zeta\right) = \underbrace{G_j(\zeta)}_{\text{poly. without constant term}} + \underbrace{H_j(\zeta)}_{\text{finite at } \infty}$$

like before.

$$\Rightarrow \quad R(z) = \underbrace{G_j\left(\frac{1}{z-\beta_j}\right)}_{\text{poly. in } \frac{1}{z-\beta_j} \text{ without constant term}} + \underbrace{H_j\left(\frac{1}{z-\beta_j}\right)}_{\text{finite at } z = \beta_j}$$

$\hookrightarrow$ singular part of R at β_j

Consider

$$(\star) \quad R(z) - G(z) - \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right),$$

a rational function whose poles are at most β_j or ∞.

- $\underline{z = \beta_j}$: only 2 terms that can become ∞, namely, $R(z)$ and $G_j(1/z-\beta_j)$. But

$$R(z) - G_j\left(\frac{1}{z-\beta_j}\right) = \underbrace{H_j\left(\frac{1}{z-\beta_j}\right)}_{\text{finite when } z=\beta_j}$$

- $\underline{z = \infty}$: same argument (only R, G can become ∞)

$\Rightarrow$ $(\star)$ is a rational function with no poles

$\Rightarrow$ $(\star)$ is constant.

So, we can absorb the constant term into $G(z)$ to obtain

$$R(z) = G(z) + \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right)$$

What about the real case?

Roots of denominator occur in conjugate pairs.

$$\frac{1}{z-\beta} + \frac{1}{z-\bar{\beta}} = \frac{*}{(z-\beta)(z-\bar{\beta})} = \frac{*}{z^2 - \underbrace{(\beta+\bar{\beta})}_{\text{Real}}z + \underbrace{\bar{\beta}\beta}_{\text{Real}}}$$

LECTURE 4 | RATIONAL FUNCTIONS

* Classification of rational functions of order 2 → Up to coordinate changes; i.e., fractional linear transformations of the source and the target.

A rational function of order 2 has 2 poles.

(1) Either 1 double pole or

(2) 2 distinct poles.

(1) Make a fractional linear transformation to move β to ∞:

$$z = \beta + \frac{1}{\zeta}.$$

So, we get a rational function with a double pole at ∞; i.e.,

$$w = az^2 + bz + c.$$

$$= a\left(z + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$\Leftrightarrow \underbrace{w + \frac{b^2}{4a} - c}_{=:w_1} = \underbrace{a\left(z + \frac{b}{2a}\right)^2}_{=:z_1^2}$$

Thus, we can represent any such rational function of order 2 w/ 1 double pole as

$$\boxed{w = z^2.}$$

(2) Now, consider the case with 2 distinct poles. Make a fractional linear transformation to move a, b to ∞, 0:

$$\frac{z-b}{z-a}.$$

So, the rational function of order 2 with poles at 0, ∞ can be written as $w = \underbrace{Az + B}_{\text{at } \infty} + \underbrace{\frac{C}{z}}_{\text{at } 0}$

Make the coefficients of z and $1/z$ equal via

$$z' = \sqrt{\frac{A}{C}}\, z .$$

Then

$$w = A\left(z + 1/z\right) + B$$

$$\Rightarrow \frac{1}{A}(w - B) = z + \frac{1}{z}$$

Thus, we can represent any such rational function of order 2 w/ distinct poles as

$$\boxed{w = z + \frac{1}{z}}$$

* Classification of rational functions of order 1.

$$w = S(z) = \frac{az+b}{cz+d}, \quad ad - bc \neq 0$$

Note that

$$S(\infty) = \frac{a}{c}, \qquad S\left(-\frac{d}{c}\right) = \infty$$

- If $c = 0$, then $w = az + b$ (composition of translation and homothety)

- If $c \neq 0$, we have

$$\frac{az+b}{cz+d} = \frac{a/c\,(z+d/c) + (bc-ad)/c^2}{(z+d/c)}$$

$$= \frac{a}{c} + \frac{(bc-ad)/c^2}{z+d/c},$$

which is a composition of a translation

$$z_1 = z + d/c$$

with an inversion

$$z_2 = 1/z_1$$

and a homothety

$$z_3 = z_2\left(\frac{bc-ad}{c^2}\right)$$

and a translation

$$z_4 = z_3 + \frac{a}{c}.$$

<u>Upshot</u>: can write any fractional linear transformation as a composition of translations, inversions and homotheties.

Thm: Fractional linear transformations takes the family of circles & lines to itself. Moreover, given any pair of circles or lines, there exists a fractional linear transformation taking one to the other.

Lemma: Given any 3 distinct points $z_2, z_3, z_4 \in S^2$, there is a unique fractional linear transformation S:

$$z_2, z_3, z_4 \mapsto 1, 0, \infty.$$

Pf:

$$S(z) = \frac{z - z_3}{z - z_4} \bigg/ \frac{z_2 - z_3}{z_2 - z_4}.$$

- If $z_2 = \infty$, then

$$S(z) = \frac{z - z_3}{z - z_4}$$

- If $z_3 = \infty$, then

$$S(z) = \frac{z_2 - z_4}{z - z_4}$$

- If $z_4 = \infty$, then

$$S(z) = \frac{z - z_3}{z_2 - z_3} .$$

Uniqueness: suppose that T also takes z_2, z_3, z_4 to $1, 0, \infty$. Then $\underbrace{ST^{-1}}_{\uparrow}: 1, 0, \infty \mapsto 1, 0, \infty$

$$\frac{az+b}{cz+d}, \quad c \neq 0 \ldots .$$

$\square$

Def: (CROSS-RATIO)

$$(z_1 : z_2 : z_3 : z_4) = S z_1 .$$

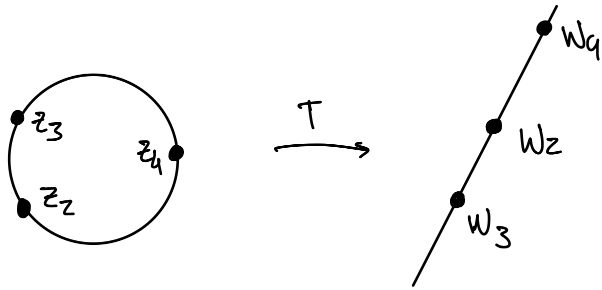
Thm 1: If $z_1, \ldots, z_4 \in S^2$ are distinct and T is a fractional linear transformation, then

$$(Tz_1 : Tz_2 : Tz_3 : Tz_4) = (z_1 : z_2 : z_3 : z_4)$$

Thm 2: $(z_1 : z_2 : z_3 : z_4)$ is real if and only if all four points lie on a circle or a line.

Pf of Thm: 1st statement follows from Thm 1 and Thm 2.

2nd statement follows from



$w = Tz$:

$$(w : w_2 : w_3 : w_4) = (z : z_2 : z_3 : z_4).$$

Pf of Thm 1: Let $Sz = (z : z_2 : z_3 : z_4)$. Then

ST^{-1} : $Tz_2, Tz_3, Tz_4 \mapsto 1, 0, \infty$. But

$$(Tz_1 : Tz_2 : Tz_3 : Tz_4) \overset{\text{def}}{=} ST^{-1}(z_1)$$
$$= Sz_1$$
$$= (z_1 : z_2 : z_3 : z_4).$$

$\square$

Claim: The image of the real axis under a fractional linear transformation T^{-1} is a circle or line.

Pf: Look at $w = T^{-1}z$, $z \in \mathbb{R}$. Want to see that w satisfies the equation of the circle or line. Now

$$w = T^{-1}z \Rightarrow Tw = z$$

$$Tw = \frac{aw+b}{cw+d}$$

So,

$$z \in \mathbb{R},\ z = Tw \Rightarrow Tw = \overline{Tw}$$

$$\frac{aw+b}{cw+d} = \frac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$$

$$(aw+b)(\bar{c}\bar{w}+\bar{d}) = (\bar{a}\bar{w}+\bar{b})(cw+d)$$

$$\underbrace{(a\bar{c}-\bar{a}c)}_{\text{Imaginary}}|w|^2 + \underbrace{(a\bar{d}-\bar{b}c)}w + \underbrace{(b\bar{c}-\bar{a}d)}\bar{w} + \underbrace{b\bar{d}-\bar{b}d} = 0$$

$w = x+iy$. Write with $x+iy$, and, since everything is imaginary, divide by i.

Circle if $a\bar{c} - \bar{a}c \neq 0$.

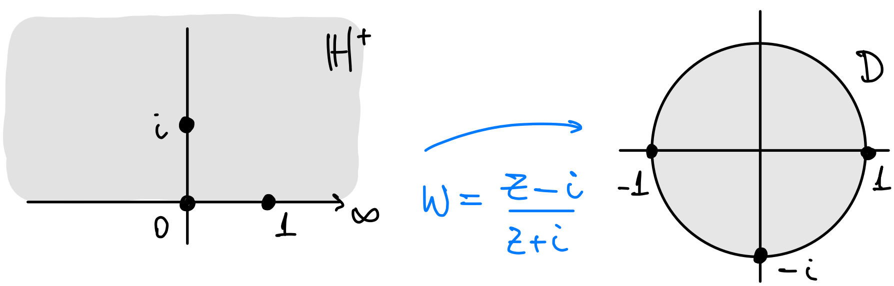
Line if $a\bar{c} - \bar{a}c = 0$.

Claim: $Sz = (z : z_2 : z_3 : z_4)$ is real on the image of the real axis under S^{-1} and nowhere else.

$$\begin{tikzcd} z_2, z_3, z_4 \arrow[r, bend left, "S"] & 1, 0, \infty \arrow[l, bend left, "S^{-1}"] \end{tikzcd}$$

$\square$

EXAMPLE: Fractional linear transformation that takes the upper-half plane $\mathbb{H}^+$ to the unit disk D.



$0 \mapsto -1$; $\infty \mapsto 1$; $1 \mapsto -i$

* **HOLOMORPHIC FUNCTIONS:** $f(z)$ complex-valued function on an open set $\Omega \subset \mathbb{C}$.

Say that **f is HOLOMORPHIC at $z \in \Omega$** if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = c$$

exists; i.e.,

$$f(z+h) - f(z) = ch + \varphi(h)h,$$

where $\lim_{h\to 0} \varphi(h) = 0$.

Note that, writing $z = x+iy$ $\quad c = a+ib$

$f(z) = u+iv$ $\quad h = \xi + i\eta$

The derivative of f at z

$$h \mapsto ch$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \underbrace{\begin{matrix} a \\ b \end{matrix}}_{\frac{\partial f}{\partial x}} & \underbrace{\begin{matrix} -b \\ a \end{matrix}}_{\frac{\partial f}{\partial y}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

i.e.,

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} \longmapsto \underbrace{\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}}_{\frac{\partial f}{\partial z}} \begin{pmatrix}\xi\\ \eta\end{pmatrix}$$

So, f holomorphic means differentiable at z (as a function (u,v) of (x,y)) and the partial derivatives at z satisfy the equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad ; \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

<u>or</u>

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

CAUCHY–RIEMANN EQUATIONS

LECTURE 5 | HOLOMORPHIC FUNCTIONS

- f is holomorphic if it is a complex-valued function on an open $\Omega \in \mathbb{C}$.

- f is <u>holomorphic at $z \in \mathbb{C}$</u> if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = c \in \mathbb{C}$$

i.e., limit exists

i.e.,

$$f(z+h) - f(z) = ch + \varphi(h)h, \quad \lim_{h \to 0} \varphi(h) = 0$$

Now, consider

$$f(z) = u + iv; \quad z = x + iy$$

$$c = a + ib; \quad h = \xi + i\eta$$

So, the derivative of f at z is a linear transformation

$$h \mapsto ch$$

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \longmapsto \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$\Rightarrow$ f is holomorphic at z if and only if f is differentiable at z (as a function of x and y) and the partial derivatives satisfy the Cauchy-Riemann equations:

$$\frac{\partial f}{\partial x} + i\,\frac{\partial f}{\partial y} = 0$$

$$\Leftrightarrow \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

* Jacobian determinant:

$$\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = a^2 + b^2 = |c|^2 = |f'(z)|^2.$$

Consider $f(x,y)$ differentiable and complex-valued.

Differential: $\quad df = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$

Ex: Take $z = x+iy$, $\bar{z} = x-iy$. Then

$$dz = dx + i\,dy, \qquad d\bar{z} = dx - i\,dy.$$

So that

$$dx = \frac{1}{2}(dz + d\bar{z}), \qquad dy = \frac{1}{2i}(dz - d\bar{z}).$$

Going back to the differential of f:

$$df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z}$$

Define

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

Thus,

$$df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}.$$

So, we can write the Cauchy-Riemann equation as

$$\boxed{\frac{\partial f}{\partial \bar{z}} = 0.}$$

$\Rightarrow$ Holomorphic functions only depend on z, not $\bar{z}$.

Def: (HARMONIC FUNCTIONS) A function $f(x,y)$ is harmonic if $f \in C^2$ and f satisfies the Laplace equation:

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \qquad (*)$$

$(*)$ Can be rewritten as $\dfrac{\partial^2 f}{\partial z \, \partial \bar{z}} = 0$

* We will see that holomorphic functions are harmonic.

$\rightarrow$ Real and imaginary parts of holomorphic functions are harmonic.

Remark: $$\frac{\partial f}{\partial \bar z} = 0 \iff \frac{\partial \bar f}{\partial z} = 0$$

(Holomorphic)

Pf. $$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

If $z = x+iy$ and $f(z) = u+iv$, then $\overline{f(z)} = u-iv$.

$\Big\downarrow$ Conjugate

$$\overline{\left(\frac{\partial f}{\partial \bar z}\right)} = \frac{1}{2}\left(\frac{\partial \bar f}{\partial x} - i\frac{\partial \bar f}{\partial y}\right) = \frac{\partial \bar f}{\partial z}.$$

Lemma: If $f(z)$ is holomorphic in a connected open set and $f'(z) \equiv 0$, then f is constant.

Pf:

$$df = \cancel{\frac{\partial f}{\partial z}}\,dz + \cancelto{0}{\frac{\partial f}{\partial \bar z}\,d\bar z} = 0$$

$f'(z) \equiv 0$

$\Rightarrow f$ is constant. $\square$

Lecture 6

* **Properties of Holomorphic Functions**

- If $f(z)$ is holomorphic in a connected open set Ω and $f'(z) \equiv 0$ in Ω, then f is constant.

Prop: Given $f(z)$ holomorphic in a connected open set Ω,

(1) if $|f(z)|$ is constant, then $f(z)$ is constant.

(2) if $\mathrm{Re}(f)$ is constant, then $f(z)$ is constant.

Pf: (1) $|f(z)|^2 = f(z)\overline{f(z)}$ is constant, so the derivative w.r.t. z is zero:

$$0 = \frac{\partial f}{\partial z} \cdot \overline{f(z)} + \cancel{f(z)\frac{\partial \overline{f(z)}}{\partial z}}$$

$= 0$ b/c f is holomorphic: $\dfrac{\partial \overline{f}}{\partial z} = \overline{\dfrac{\partial f}{\partial \bar{z}}} = 0$

f holo.

$$= \frac{\partial f}{\partial z} \cdot \overline{f(z)} \Rightarrow \text{either } \frac{\partial f}{\partial z} = 0$$

or $\overline{f(z)} = 0$ in a neighborhood of every z. $\Rightarrow f$ is constant.

(2) $\operatorname{Re}(f) = \frac{1}{2}(f + \bar{f})$ is constant

$\Rightarrow 0 = d(f + \bar{f})$

f holomorphic

$$\Rightarrow \frac{\partial f}{\partial z} dz + \cancel{\frac{\partial f}{\partial \bar{z}}} d\bar{z} + \cancel{\frac{\partial \bar{f}}{\partial z}} dz + \frac{\partial \bar{f}}{\partial \bar{z}} d\bar{z} = 0$$

$$\Rightarrow \underbrace{\frac{\partial f}{\partial z}} dz + \underbrace{\frac{\partial \bar{f}}{\partial \bar{z}}} d\bar{z} = 0$$

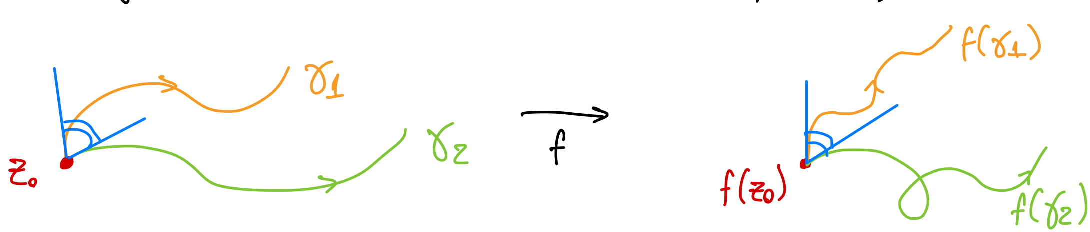
$\Rightarrow \frac{\partial f}{\partial z}$ and $\frac{\partial \bar{f}}{\partial z} = 0 \Rightarrow f$ constant.

$\square$

* **MAPPING PROPERTIES**

Consider $f(z)$ holomorphic at some point $z_0 \in \mathbb{C}$. Then, the <u>tangent mapping</u> of f at z_0 is a $\mathbb{C}$-linear map $w(z) = cz$, where $c = f'(z_0)$.

* If $c \neq 0$, then the tangent mapping preserves angles and their orientation; i.e.,



Def: We say that a holomorphic function f is <u>conformal</u> at any point z_0 if $f'(z_0) \neq 0$.

Lemma: A $\mathbb{R}$-linear transformation from $\mathbb{C} \to \mathbb{C}$ which preserves angles is of the form: either

$$w(z) = cz \quad \text{or} \quad w(z) = c\bar{z}$$

Pf: HW exercise

* Consider $w = f(z)$ in a connected open set Ω. Assume $f \in C^1$ (as a $\mathbb{C}$-valued function of (x,y)) and that the Jacobian determinant is nonzero everywhere.

If f preserves angles at every point of Ω, then either

$$\frac{\partial f}{\partial z} = 0 \quad \text{or} \quad \frac{\partial f}{\partial \bar{z}} = 0$$

at every point of Ω.

Remark: they can't be both zero at a given point, b/c, if they were, the Jacobian det would be zero at that point $\hookrightarrow\!\!/\!\!/\!\!\hookleftarrow$

Since $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar{z}}$ are continuous, the sets

$\left\{ \frac{\partial f}{\partial z} = 0 \right\}$ and $\left\{ \frac{\partial f}{\partial \bar{z}} = 0 \right\}$ are disjoint closed sets whose union is equal to Ω.

But, since Ω is connected, one of those sets is empty; i.e.,

- either $\frac{\partial f}{\partial \bar{z}} = 0$ on $\Omega \Rightarrow f$ holomorphic
- or $\frac{\partial f}{\partial z} = 0$ on $\Omega \Rightarrow f$ antiholomorphic

<u>Prop</u>: Consider $w = f(z)$ in a connected open set Ω. Assume $f \in C^1$ (as a $\mathbb{C}$-valued function of (x,y)) and that the Jacobian determinant is nonzero everywhere. Then, f preserves angles at every point of Ω if and only if f is either holomorphic or antiholomorphic.

Thm: (INVERSE FUNCTION THEOREM) Suppose f is holomorphic in a neighborhood of z_0 and the derivative $f'(z_0) \neq 0$. Then, there are neighborhoods U of z_0 and V of $w_0 = f(z_0)$ such that f maps U onto V, with an inverse $g(w_0) = z_0$ which is holomorphic. It follows that, if $w = f(z)$,

$$g'(w) = \frac{1}{f'(z)}.$$

Pf: To be completed later: use the fact that the partial derivatives of holomorphic functions are continuous.

Then the existence of C^1 inverse follows from the real Inverse Function Thm.

Since f is holomorphic,

$$f'(z) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}.$$

$f(z) = w$

So,
$$g'(w) = \frac{1}{a^2+b^2}\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

$\Rightarrow$ so g satisfies the Cauchy-Riemann eqs.

$\Rightarrow$ g is holomorphic.

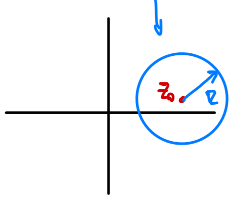
$\square$

LECTURE 7

* COMPLEX POWER SERIES

Def: A complex power series is of the form
$$\sum_{n=0}^{\infty} a_n (z-z_0)^n, \quad a_n \in \mathbb{C}, \quad z_0 \in \mathbb{C}$$

We'll see that such power series defines a function at



(R could be 0, ∞ and everything in between)

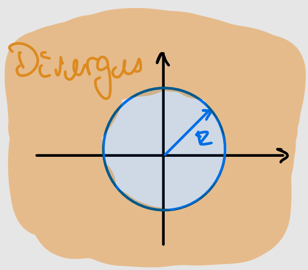
Thm: (from Spivak's Calculus) For a power series $\sum_{n=0}^{\infty} a_n z^n$, there exists R, $0 \leq R \leq \infty$, s.t.

(1) For all $r < R$, the series $\sum_n a_n z^n$ converges uniformly and absolutely in the disk $|z| \leq r$.

(2) If $|z| > R$, then the series diverges (in fact, even the terms of the series are unbounded).

R = radius of convergence

$\frac{1}{R} := \limsup_{n \to \infty} |a_n|^{1/n}$



Converges absolutely and uniformly

(3) Derived series: $\sum_{n=1}^{\infty} n a_n z^{n-1}$ has the same radius of convergence R. Moreover, by setting

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad |z| < R,$$

then $f(z)$ is holomorphic and its derivative is the derived series.

EXAMPLES:

1) $\sum_{n=0}^{\infty} n!\, z^n$. Radius of convergence $= 0$.

Terms of the series are unbounded.

2) $\sum_{n=0}^{\infty} \frac{1}{n!} z^n$, $R = \infty$ by the ratio test.

3) $\sum_{n=0}^{\infty} z^n$, $R = 1$ by ratio test.

4) $\sum_{n=0}^{\infty} \frac{1}{n} z^n$, $R = 1$.

5) $\sum_{n=0}^{\infty} \frac{1}{n^2} z^n$, $R = 1$.

* EXPONENTIAL AND LOGARITHMIC FUNCTIONS:

Def: (exp)

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n$$

(Radius of convergence $= 1$)

Properties:

- $$\frac{d}{dz} e^z = e^z$$

- $$e^{z+w} = e^z e^w$$ (Cauchy product) or from some solutions to an ODE: let $g(z) = e^z e^{c-z}$. Then

$$g'(z) = e^z e^{c-z} - e^z e^{c-z} = 0 \Rightarrow g \text{ constant}$$

For $z = 0$, $g(0) = e^c \Rightarrow$ constant $= e^c$. Thus,

$$e^z e^{c-z} = e^c.$$

Take $c = z + w$.

So, if $z = x + iy$, then $e^{x+iy} = e^x e^{iy}$, where

$$e^{iy} = \cos y + i \sin y$$

Plug iy in the series from the definition of the exp.

$$e^{i\pi} = -1$$ ☺

Now, $|e^{iy}| = 1$; i.e., $e^{iy} \in S^1 = \{|z| = 1\} \subset \mathbb{C}$.

So, $|e^{x+iy}| = e^x$.

Now, consider the mapping

group under $+$ $\quad \mathbb{R} \longrightarrow S^1 \subset \mathbb{C}$ $\quad$ group under mult.

$$y \longmapsto e^{iy} \quad \text{(Group homomorphism)}$$

Onto since $e^{iy} = \cos y + i \sin y$ (i.e., every point in S^1 can be written as $(\cos y, \sin y)$).

Kernel $= 2\pi k$, $k \in \mathbb{Z}$.

$\Rightarrow$ We get an isomorphism $\mathbb{R}/2\pi k \to S^1$.

This is a homeomorphism, where the LHS has the quotient topology.

Continuous bijective mapping of compact Hausdorff spaces.

$$\mathbb{R}/2\pi\mathbb{Z} \xrightarrow{\simeq} S^1$$

Inverse is called arg

$$\underbrace{\arg(z)}_{\text{Defined up to an integer multiple of } 2\pi.} \longleftarrow\!\!| \; z$$

For any nonzero $z \in \mathbb{C}$, we define

$$\arg(z) = \arg\left(\frac{z}{|z|}\right).$$

Thus, we can write

$$z = |z| e^{i \arg(z)}.$$

Now, we can define trig. functions from the definition of exp:

$$\cos z := \frac{e^{iz} + e^{-iz}}{2}.$$

$$\sin z := \frac{e^{iz} - e^{-iz}}{2i}.$$

This means we can define trig. functions on all of $\mathbb{C}$: $e^{iz} = \cos z + i \sin z$. For example, for any $z \in \mathbb{C}$,

$$\cos^2 z + \sin^2 z = 1.$$

Moreover,

$$(\cos z)' = -\sin z$$

$$(\sin z)' = \cos z$$

Similarly,

$$\begin{cases} \cos(z+w) = \cos z \cos w - \sin z \sin w \\ \sin(z+w) = \cos z \sin w + \cos w \sin z \end{cases}$$

* COMPLEX LOGARITHM

$\log z$ is the solution to $e^w \overset{(*)}{=} z$, $z \neq 0$.

Recall

$$z = |z| e^{i \arg z}$$

$$= e^{\log|z|} e^{i \arg z}$$

$$= e^{(\log|z| + i \arg z)}$$

This shows that the solution of (*) is

$$w = \log|z| + i \arg z.$$

Def:
$$\log z = \log|z| + i \arg z.$$
Only defined up to integer multiples of $2\pi i$.

Not good...

but $e^{\log z} = z$

* If $z = x$ positive real number, then we get the "classical" $\log x$ if we allow only the value 0 for $\arg(z)$.

* If $z, z' \neq 0$, then
$$\log(zz') = \log z + \log z' \pmod{2\pi i}$$

→ BRANCHES OF $\log z$: Let $f(z)$ be a continuous function in a connected open set Ω. We say $f(z)$ is a branch of $\log z$ if, for all $z \in \Omega$,
$$e^{f(z)} = z.$$
Not always can solve this for cont. f.

We'll study: what conditions must Ω satisfy for a branch of log to exist? simply connected?

Lemma: Suppose there is a branch $f(z)$ of $\log z$ in a connected open set Ω. Then any other branch has form $f(z) + 2k\pi i$, for some $k \in \mathbb{Z}$.

Conversely, for all $k \in \mathbb{Z}$, $f(z) + 2k\pi i$ is a branch of $\log z$.

Pf: Suppose $f(z)$ and $g(z)$ are branches of $\log z$. Let $h(z) := \dfrac{f(z) - g(z)}{2\pi i}$. Then $h(z)$ is continuous with values in $\mathbb{Z}$. Since Ω is connected, h is constant.

Likewise, we can define a branch of $\arg z$ in a connected open set not containing the origin.

↳ Any branch of $\arg z$ defines a branch of $\log z$, and vice-versa.

Prop: If $f(z)$ is a branch of $\log z$ in a connected open set Ω, then $f(z)$ is holomorphic and

$$f'(z) = 1/z.$$

Pf:

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{h\to 0} \frac{f(z+h)-f(z)}{z+h-z}$$

f is branch of log

$$= \lim_{h\to 0} \frac{f(z+h)-f(z)}{e^{f(z+h)}-e^{f(z)}}$$

$w = f(z+h)$

$$= \lim_{w\to f(z)} \frac{w-f(z)}{e^{w}-e^{f(z)}}$$

$$= 1/(\text{derivative of } e^w \text{ at } f(z))$$

$$= (1/e^w)\Big|_{w=f(z)}$$

$$= 1/e^{f(z)} \overset{\text{branch of log}}{=} 1/z.$$

$\square$

LECTURE 8 POWER SERIES OPERATIONS

Complex Power Series

$$f(w) = \sum_{n=0}^{\infty} a_n w^n.$$

coefficients are complex

w is just a variable

$$g(z) = \sum_{p=0}^{\infty} b_p z^p.$$

The composition:

$$f(g(z)) = a_0 + a_1(b_0 + b_1 z + \cdots) + a_2(b_0 + b_1 z + \cdots)^2 + \cdots$$

Coeff of z^n: infinite series

This makes sense if $b_0 = 0$.

E.g., infinite Taylor series of $f(g(z))$ at $z = z_0$:
$w_0 = g(z_0)$.

$\Rightarrow$ Taylor series of f at w_0:

$$\sum_{n=0}^{\infty} a_n (w - w_0)^n$$

b_0

substitute

w = Taylor series of g at z_0:

$$\sum_{p=0}^{\infty} b_p (z - z_0)^p$$

Thm (FORMAL INVERSE FUNCTION THEOREM) given a formal power series $f(w) = \sum_{n=0}^{\infty} a_n w^n$ such that $f'(0) \neq 0$ and $f(0) = 0$, there is a unique power series $g(z) = \sum_{p=0}^{\infty} b_p z^p$ such that $b_0 = 0$ and $f \circ g = id$.

In this case, g is uniquely defined by f and $g(f(w)) = w$.

Pf: (Method of Indeterminate Coefficients)

$$a_0 + a_1(b_1 z + b_2 z^2 + \cdots) + a_2(b_1 z + b_2 z^2 + \cdots)^2 + \cdots = z$$

By comparing both sides, $a_0 = 0$ and $a_1 b_1 = 1$ (so $a_1 = f'(0) \neq 0$ and $a_0 = f(0) = 0$ are necessary conditions).

Conversely, they are sufficient to solve uniquely for the coefficients of g.

Note that the coeff. of z^n in LHS is the same as the coeff. of z^n in $\cancel{a_0}^{0} + a_1 g(z) + \cdots + a_n g(z)^n$.

$$\Rightarrow \quad a_1 \cdot b_n + P(a_2, \dots, a_n, b_1, \dots, b_{n-1})$$

poly. w/ nonnegative integer coeff. and linear in $a_1, \dots, a_n$.

$b_1 = 1/a_1$

b_n = calculated recursively.

Now, since $g(0) = 0$, $g'(0) \neq 0$, there is a unique $f_1(w)$ s.t. $g \circ f_1 = id$

$$f_1 = id \circ f_1 = (f \circ g) \circ f_1 = f \circ (g \circ f_1)$$
$$= f \circ id = f.$$

$\square$

<u>Prop</u>: if $f(w) = \sum_{n=0}^{\infty} a_n w^n$ and $g(z) = \sum_{p=1}^{\infty} b_p z^p$ are convergent, then so is $f \circ g$. In fact, take $r > 0$ s.t. $\sum_{p=1}^{\infty} |b_p| r^p < R(f)$, $R(f)$ = radius of convergence of f. Then $R(f \circ g) \geq^{(1)} r$, and, if

$|z| < r$, then $|g(z)| \overset{(z)}{<} R(f)$. (clear since $|g(z)|$ is majorized by $\sum_{p=1}^{\infty} |b_p| r^p < R(f)$)

Finally, $f(g(z)) = (f \circ g)(z)$ (rearrangement of an absolutely convergent power series).

Pf: (1)

$$\underbrace{\sum_{n=0}^{\infty} |a_n| \left(\sum_{p=1}^{\infty} |b_p| r^p \right)^n}_{\sum_{k=0}^{\infty} \gamma_k r^k} < \infty$$

Say

$$(f \circ g)(z) = \sum_{k=0}^{\infty} c_k z^k .$$

Then $|c_k| \leq \gamma_k$. So, $\sum |c_k| z^k < \infty$, i.e, $R(f \circ g) \geq r$.

$\square$

* Reciprocal of a Power Series

Prop: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $a_0 \neq 0$, then there is a unique power series $g(z)$ s.t. $f(z)g(z) = 1$ and, if f has positive radius of convergence, then so does g.

Pf: Assume $a_0 = 1$. Write

$$f(z) = 1 - h(z) \qquad (\text{i.e., } h(z) = 1 - f(z))$$

$$\Rightarrow h(0) = 0.$$

So,

$$(1 - h(z))^{-1} = \text{composite of } \underbrace{1 + \sum_{n=1}^{\infty} w^n}_{(1-w)^{-1}}$$

with $w = h(z)$

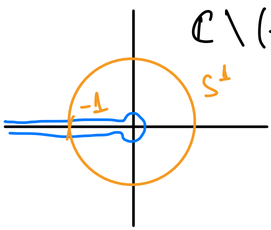
$\square$

* Inverse Function Theorem for Convergent Power Series

In the previous statement, if $f(w)$ has positive radius of convergence then so does $g(z)$.

↳ Can be done by direct estimate <u>or</u> from the Inverse Function Thm for holomorphic functions and once we know that holomorphic functions have infinite Taylor series, that converges and represents the function.

* **PRINCIPAL BRANCH OF $\log z$**



$\mathbb{C} \setminus (-\infty, 0]$ In here, there is a unique value of $\arg z \in (-\pi, \pi)$. Call it

$\mathrm{Arg}(z)$ = Principal branch of $\mathrm{Arg}(z)$

Note that $\mathrm{Arg}(z)$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$.

It is enough to show continuity on $S^1 \setminus \{-1\}$.

$$\begin{tikzcd} S^1 \setminus \{-1\} \arrow[r, bend right, "\mathrm{Arg}(z)"'] & (-\pi, \pi) \arrow[l, bend right, "z = e^{i\theta}"'] \end{tikzcd}$$

$S^1 \setminus$ [neighborhood of -1]

↳ Restrict $[-(\pi - \varepsilon), \pi - \varepsilon]$

$\underline{\log|z| + i\,\mathrm{Arg}(z)}$ is continuous on $\mathbb{C} \setminus (-\infty, 0]$

PRINCIPAL BRANCH OF LOG (coincides w/ the real log on $(0, \infty)$)

Prop: Power series: $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n} =: f(z)$

Real sum = principal branch of $\log(1+z)$

Pf: The power series $f(z)$ and $g(w) = \underbrace{\displaystyle\sum_{n=1}^{\infty} \frac{w^n}{n!}}_{e^w - 1}$ are inverses

$$g(f(z)) = z, \qquad |z| < 1$$

$$e^{f(z)} = 1 + z \quad \text{i.e., } f(z) \text{ is a branch of } \log(z)$$

To determine it's the principal branch, just check at 1 point; e.g., $z = 0$

$\square$

Def: For $z \neq 0$ and $\alpha \in \mathbb{C}$,

$$z^\alpha = e^{\alpha \log z}.$$

For fixed $\alpha \in \mathbb{C}$, this is a many-valued function of z.

↳ This has a branch in any domain Ω where log has a branch.

connected open subset of $\mathbb{C}$.

Moreover, any branch of $\log z$ in Ω defines a branch of z^α.

* **Binomial Series**

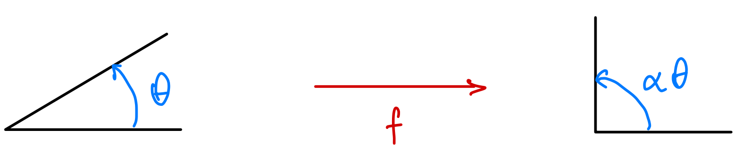
$$(1+z)^\alpha = e^{\alpha \log(1+z)}$$

Power series expansion in $|z| < 1$ is:

$$\sum_{n=0}^{\infty} \binom{\alpha}{n} z^n .$$

Lecture 9 — Mapping Properties of Holomorphic Functions

Take $f(z) = z^\alpha$, $\alpha \in \mathbb{R}$ and $\alpha > 0$.



Conformal outside 0 $\quad (f'(z) \neq 0 \quad \forall z \neq 0)$

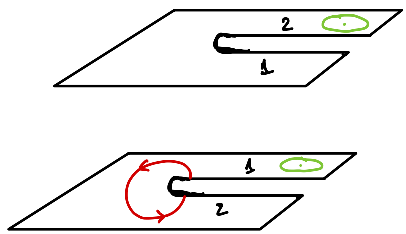
z^α | is <u>not</u> 1-1 if $\alpha \neq 1$
| is multi-valued if α is fractional

Holomorphic in wedge

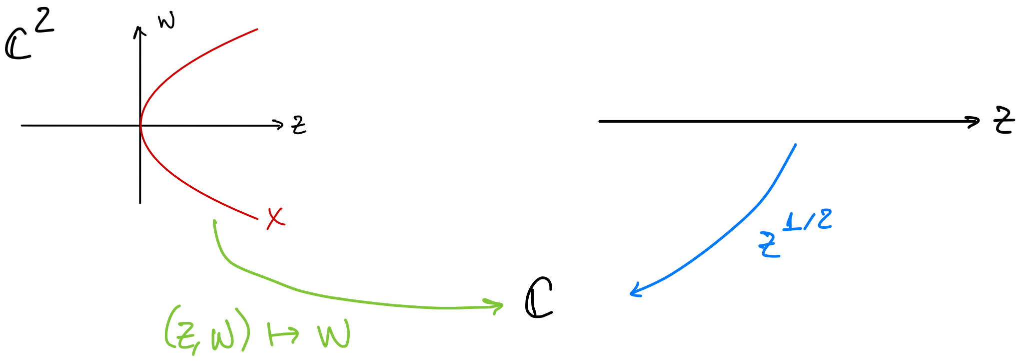
$$\frac{dw}{dz} = \alpha \frac{w}{z} \qquad \left(z^\alpha = e^{\alpha \log(z)} \right)$$

$\boxed{\text{Ex}}$: $\quad w = z^{1/2}$

$$\begin{tikzcd}
X \arrow[d, "\text{branched covering}"'] \arrow[dr, "\text{single valued function}"] & \\
\mathbb{C} \arrow[r, "z^{1/2}"', "\text{multivalued}"] & \mathbb{C}
\end{tikzcd}$$



$z = w^2$. Let $X := \{ z = w^2 \} \subset \mathbb{C}^2$

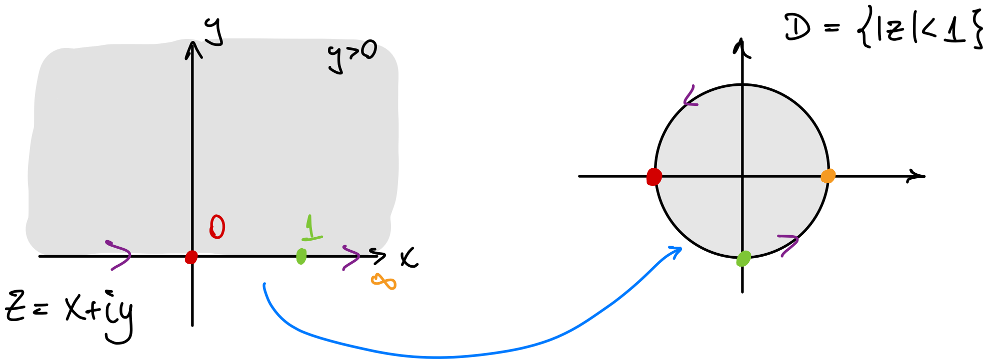


* A multivalued function $w = z^{1/2}$ lift to single-valued function $(z, w) \mapsto w$ on a covering surface X (RIEMANN SURFACE)

* w/ ∞:

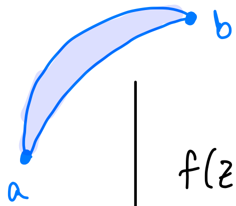
When you glue the two together in this way, you get another sphere ($\simeq \mathbb{C} \cup \{\infty\}$).

Consider $\mathbb{H}_+ = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$



$$w = \frac{z-i}{z+i}$$

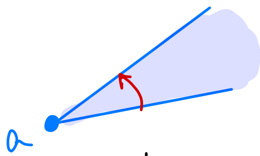
* Conformal Mapping of Circular Wedge onto $\mathbb{D}$ or $\mathbb{H}_+$



$$\downarrow$$

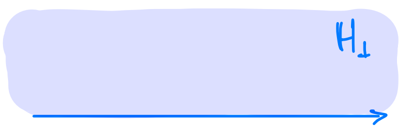
$f(z) = \dfrac{z-a}{z-b}$ maps $a \mapsto 0$ and $b \mapsto \infty$

so (arcs) $\mapsto$ (rays)



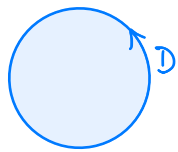
$$\downarrow$$

$g(z) = z^{\alpha}$ to "straighten the angle"
(after rotation)

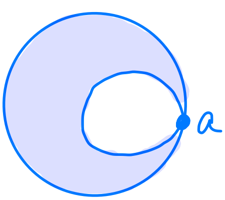


$$\xrightarrow{\quad\quad\quad}$$

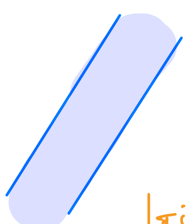
$h(z) = \dfrac{z-i}{z+i}$



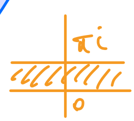
Now,

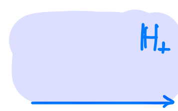


$$\xrightarrow[a \mapsto \infty]{f(z) = \frac{1}{z-a}}$$

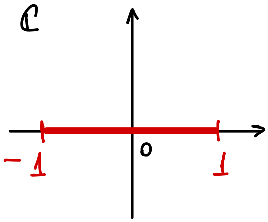


$$\xrightarrow[\exp]{}$$

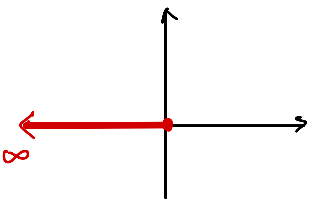




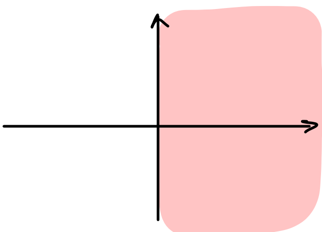
EXERCISE: Conformal Mapping of Complement of line segment to the interior (or exterior) of the unit disk.



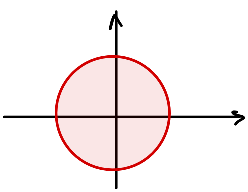
$$f(z) = \frac{z+1}{z-1} \qquad \begin{pmatrix} -1 \mapsto 0 \\ 1 \mapsto \infty \\ 0 \mapsto -1 \end{pmatrix}$$



$$g(z) = \left(f(z)\right)^{1/2}$$



$$h(z) = \frac{g(z)-1}{g(z)+1}$$



Compose everything and we get

$$\boxed{h(z) = z - \sqrt{z^2-1}}.$$

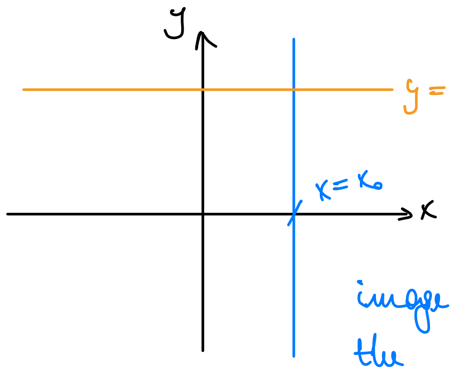
Which square root? $\underbrace{\left(z-\sqrt{z^2-1}\right)}_{\text{b/c } |w|<1}\left(z+\sqrt{z^2-1}\right) = 1$

Lecture 10

* **Mapping Properties of Exp, Log**

$w = e^z$ is $2\pi i$-periodic

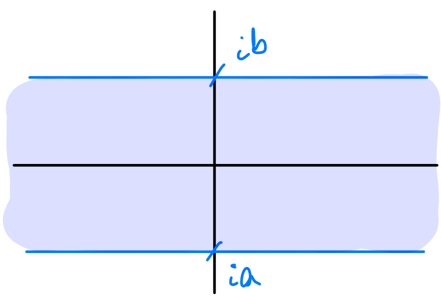
$$e^z = e^x e^{iy} = e^x(\cos y + i \sin y)$$



$y = y_0$ maps to a ray through 0.

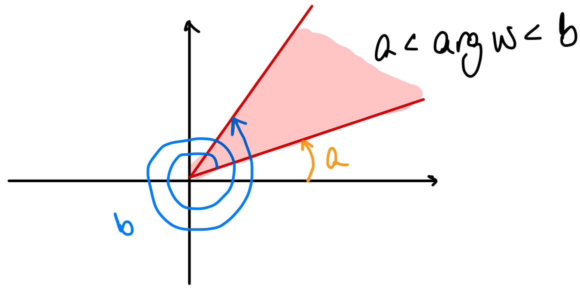
image is a circle around the origin

* e^z maps any other line to a log spiral.

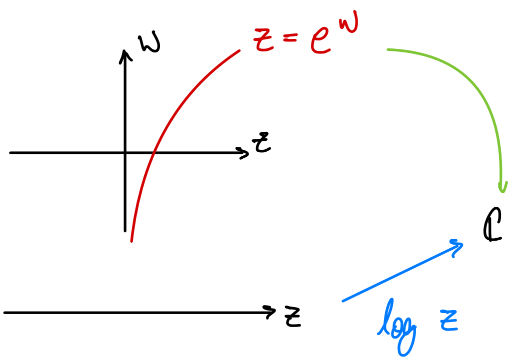


e^z is injective on open strip $a < \operatorname{Im} z < b$ provided that $b - a < 2\pi$

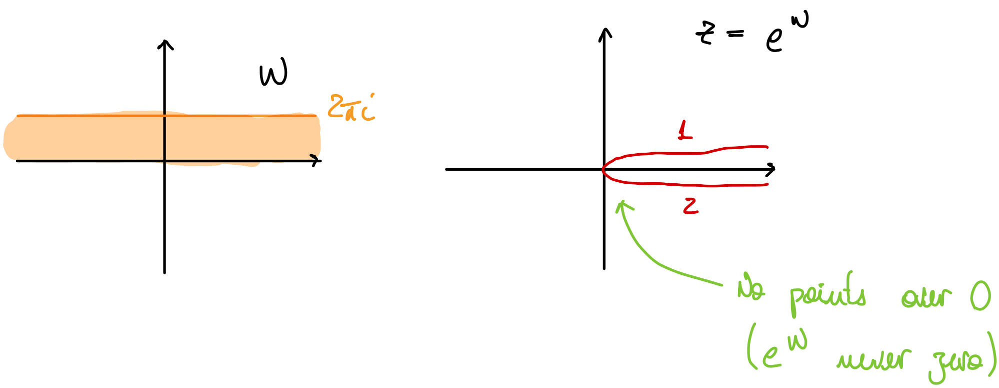
image under exp



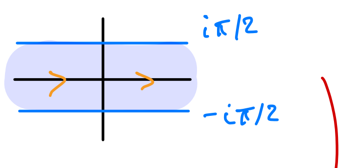
* RIEMANN SURFACE OF $w = \log z$



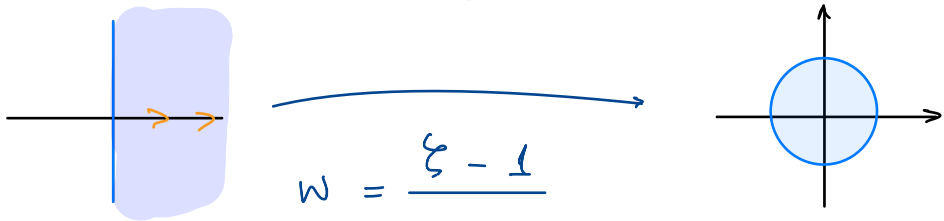
$$\begin{tikzcd} X \arrow[r, "\text{single-valued}"] \arrow[d, "\text{“covering space”}"'] & \mathbb{C} \\ \mathbb{C} \arrow[ur, "\log z"'] & \end{tikzcd}$$



* **EXAMPLE** Map the open strip conformally onto the open unit disk



$$\zeta = e^z$$



$$w = \frac{\zeta - 1}{\zeta + 1}$$

$$\Rightarrow \quad w(z) = \frac{e^z - 1}{e^z + 1}.$$

---

* ANALYTIC FUNCTIONS

Def: (ANALYTIC FUNCTIONS) f is <u>analytic</u> in an open set Ω if f has a convergent power series representation at every point $z_0 \in \Omega$.

i.e, for any $z_0 \in \Omega$, there is a convergent power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ s.t.

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

in $|z - z_0| < r$ for some $r \le$ radius of convergence.

$\hookrightarrow$ If f has a convergent power series representation at z_0 then there is a convergent power series g at z_0 such that $g'(z) = f(z)$ in some disk $|z - z_0| < r$.

In this sense:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

Integrate term-by-term $\downarrow$

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

Uniquely determined up to a constant

Same radius of convergence

Does convergent power series define an analytic function?

Prop: If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is a convergent power series with radius of convergence R, then f is analytic in $|z| < R$.

i.e., for any $|z_0| < R$, f has convergent power series representation at z_0.

Remark: we will show that there is a power series which converges absolutely and uniformly in $|z-z_0|<r$, for any $r<R-|z_0|$

Pf: Write

$$f(z)=\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n\big(z_0+(z-z_0)\big)^n$$

Binomial Thm

$$=\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n}\binom{n}{k} z_0^{\,n-k}(z-z_0)^k .$$

Now, check abs. convergence:

$$\sum_{n=0}^{\infty}|a_n|\big(|z_0|+|z-z_0|\big)^n$$

$$=\sum_{n=0}^{\infty}|a_n|\sum_{k=0}^{n}\binom{n}{k}|z_0|^{n-k}|z-z_0|^k$$

Re-arrangement of abs. convergent $\Rightarrow$ we can change the order of summation $\Longrightarrow$

$$f(z)=\sum_{k=0}^{\infty}\underbrace{\left(\sum_{n=k}^{\infty} a_n\binom{n}{k} z_0^{\,n-k}\right)}(z-z_0)^k$$

$\hookrightarrow \frac{1}{k!} f^{(k)}(z_0)$

$\square$

* **PRINCIPLE OF ANALYTIC CONTINUATION**

Thm: If f is analytic in a domain Ω, and $z_0 \in \Omega$, then the following are equivalent:

(1) $f^{(n)}(z_0) = 0, \quad n = 0, 1, \ldots$

(2) f is identically zero in a neighborhood of z_0

(3) f is identically zero in Ω

Pf: $(3) \Rightarrow (1)$: tautology

$(1) \Rightarrow (2)$: coeffs. of convergent power series representation of z_0.

$(2) \Rightarrow (3)$. Let

$$\Omega' := \{ z \in \Omega : \text{f identically 0 in a neighborhood of z in Ω} \}$$

Not empty b/c $f(z_0) = 0 \Rightarrow z_0 \in \Omega'$

Ω' is open by definition.

Now, take $z \in \overline{\Omega'}$ (closure). Then $f^{(n)}(z) \overset{(3)}{=} 0$

for all n. So, f is identically zero in a neighborhood of z b/c $(1) \Rightarrow (2)$.

$$\Rightarrow z \in \Omega'.$$

$\square$

Corollary: If f, g are analytic in a domain Ω and $f = g$ in a neighborhood of the same point, then $f = g$ in Ω.

Corollary: The ring $\mathcal{A}(\Omega)$ of analytic functions on a domain Ω is an integral domain.

Pf: $fg = 0$, $f, g \in \mathcal{A}(\Omega)$. Suppose f is not identically 0. Then $\exists$ z_0 s.t. f is nonvanishing in a neighborhood U of z_0. So, $g = 0$ in U, and therefore $g \equiv 0$ in Ω.

$\square$

* ## Zeros and Poles

Consider f analytic in a neighborhood of z_0.
Then $f(z) = \sum a_n (z-z_0)^n$, $|z-z_0|$ small enough.
Suppose $f(z_0) = 0$, but f is not identically zero near z_0.

Let k be the smallest integer such that

$$f^{(k)}(z_0) \neq 0.$$

(i.e., $a_k \neq 0$). Then $\underbrace{f(z) = (z-z_0)^k g(z)}$ where g is analytic $g(z_0) \neq 0$:

$f(z) \neq 0$ in $0 < |z-z_0| < \varepsilon$, some $\varepsilon > 0$.

$$g(z) = \sum_{n=k}^{\infty} a_n (z-z_0)^{n-k}$$

k = order or multiplicity of the zero at z_0 of f.

Characterized by $f^{(j)}(z_0) = 0$, $j < k$.
$f^{(k)}(z_0) \neq 0$

* So, if f is analytic and not identically zero, then its zeros are isolated.

Make a coordinate change near z_0

$$\zeta = (z - z_0)\, g(z)^{1/k}.$$

Then

$$f(z(\zeta)) = \zeta^k$$

i.e., $w = f(z)$ becomes $w = \zeta^k$ in the new coordinates.

* Quotient of analytic functions $\frac{f(z)}{g(z)}$ where $g(z)$ not identically zero.

Then $\frac{f(z)}{g(z)}$ is well-defined, and analytic in a neighborhood of any z_0 such that $g(z_0) \neq 0$.

If $g(z_0) = 0$,

$$\begin{aligned} f(z) &= (z - z_0)^k f_1(z) \\ g(z) &= (z - z_0)^l g_1(z) \end{aligned}, \quad \text{where} \quad \begin{aligned} f_1(z_0) &\neq 0 \\ g_2(z_0) &\neq 0 \end{aligned}$$

$$\Rightarrow \quad \frac{f(z)}{g(z)} = (z-z_0)^{k-l} \boxed{\frac{f_1(z)}{f_2(z)}}$$

analytic, nowhere vanishing in a neighborhood of z_0

- $k \geq l$: f/g extends to be analytic at z_0
- $k < l$: z_0 is a pole of $f(z)/g(z)$ of order $l-k$. Note that

$$\left| f(z)/g(z) \right| \xrightarrow{z \to z_0} \infty$$

$\Rightarrow$ f/g makes sense as a function with values in the Riemann sphere.

Def: (MEROMORPHIC FUNCTION) A meromorphic function in an open set Ω is a function which is well-defined and analytic in the complement of discrete sets and expressible in the neighborhood of any point of Ω as a quotient f/g of analytic functions (with $g \neq 0$).

Remark: if f is merom. in Ω, then f' is also merom. in Ω with same poles of f.

If z_0 is a pole of order k of $f(z)$ then z_0 is a pole of order $k+1$ of $f'(z)$.

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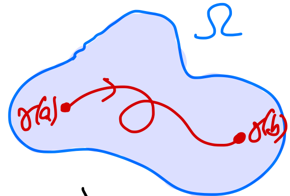
LECTURE 11: CAUCHY'S INTEGRAL FORMULA

* INTEGRATION OVER CHAINS

Consider $\Omega \subset \mathbb{R}^2$ open.



- Curve in Ω: $\gamma : [a,b] \to \Omega$ usually C^1. Write $\gamma(t) = (x(t), y(t))$

- Differential (1-) form $\omega = P\,dx + Q\,dy$, where P, Q are $\mathbb{R}$- or $\mathbb{C}$-valued cont. functions.

Def:

$$\int_\gamma \omega = \int_a^b F(t)\,dt,$$

where $F(t) = P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)$.

Remark : $\int_\gamma \omega = \int_a^b \gamma^* \omega$,

$$\gamma^* P = P \circ \gamma$$

$$\gamma^*(dx) = d(\gamma^* x) = d(x \circ \gamma) = d(x(\gamma(t)))$$

$$= d(x(t)) = x'(t)\, dt$$

* CHANGE OF PARAMETERS

$\delta(u) = \gamma(t(u))$, where $t : [c,d] \to [a,b]$

$t(c) = a$, $t(d) = d$, $t'(u) > 0$

Now,

$$\delta^* \omega = F(t(u))\, t'(u)\, du$$

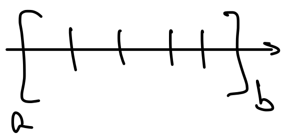
$\Rightarrow \displaystyle\int_\delta \omega = \int_\gamma \omega$ by integration by substitution

So, $\int_\gamma \omega$ depends on ω, γ and <u>not</u> in the choice of parameters.

oriented curve

If the change of parameters δ is orientation reversing, then

$$\int_{\delta} \omega = -\int_{\gamma} \omega .$$



t_0 t_1 ...
(Partition)

Then $\displaystyle\int_{\gamma} \omega = \sum_{i=1}^{n} \int_{\gamma_i} \omega$,

$\gamma_i = \gamma|_{[t_{i-1}, t_i]}$, is independent of choice of parameters. So, $\int_{\gamma}$ makes sense for <u>piecewise</u> C^1 curves.

Closed curve: $\gamma(a) = \gamma(b)$.

$\int_{\gamma}$ is independent of choice of initial/final points.

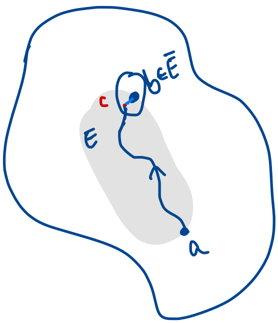
<u>LEMMA</u>: Any 2 points of a connected open $\Omega \subset \mathbb{R}^2$ can be joined by a piecewise C^1 curve.

Pf: Fix $a \in \Omega$. Let

$$E := \{ b \in \Omega : \text{a, b can be joined by a piecewise } C^1 \text{ curve} \}.$$

$E \neq \emptyset$, open, closed

take $b \in \bar{E}$, $B_\delta(b)$ will contain some portion of E, i.e., we can find $c \in E$ and join a to c by a curve and then c to b by another curve.



$\square$

* PRIMITIVE OF ω: C^1 function F on Ω s.t.

ω needs to have cont. coeff. functions

$$\omega = dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$$

$\gamma(t) = (x(t), y(t))$

$$\int_\gamma dF = \int_a^b (F \circ \gamma)'(t)\, dt$$

$$= F(\gamma(b)) - F(\gamma(a)).$$

e.g., if Ω is connected and $dF = 0$, then F is constant.

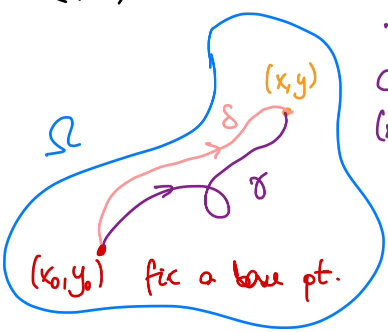
Prop: ω has a primitive if and only if $\int_\gamma \omega = 0$ for every piecewise C^1 <u>closed</u> curve γ.

Pf: $(\Rightarrow)$ then $\exists F$ s.t. $\omega = dF$ so

$$\int_\gamma \omega = \int_\gamma dF = F(\gamma(b)) - F(\gamma(a)) = 0$$

γ closed

$(\Leftarrow)$

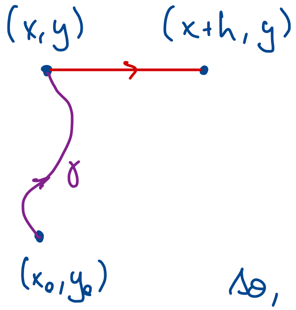


γ piecewise C^1 curve from (x_0,y_0) to (x,y)

Define

$$F(x,y) := \int_\gamma \omega.$$

Check it's well defined: it should <u>not</u> depend on the choice of γ. Take another curve δ, but then $\int_{\delta \circ \gamma} \omega = 0 \Rightarrow$ independent of the choice of δ.



Now,

$$F(x+h,y) - F(x,y) = \int_x^{x+h} P(t,y)\,dt$$

so,

$$\lim_{h\to 0} \frac{1}{h}\left[F(x+h,y) - F(x,y)\right] = \lim_{h \to 0} \int_x^{x+h} P(t,y)\,dt$$

$$\overset{\text{FTC}}{=} P(x,t)$$

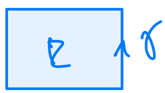
$\square$

Lecture 12: Cauchy's Theorem

Consider $\omega = P\,dx + Q\,dy$ differential form in an open $\Omega \subset \mathbb{R}^2$, with P, Q continuous.

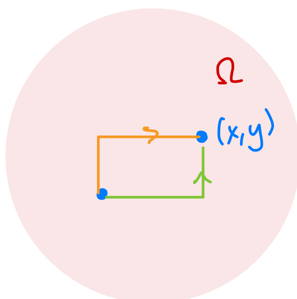
As seen last time, ω has a primitive in Ω iff $\int_\gamma \omega = 0$ for every piecewise C^1 closed curve γ in Ω.

Can we replace the condition by "$\int_\gamma \omega = 0$ whenever γ is the boundary of a rectangle in Ω?"



In general, no.

But YES, if Ω is AN OPEN DISK!



Def: (CLOSED FORM) ω is closed if

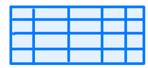
$$\int_\gamma \omega = 0$$

whenever γ is the boundary of a sufficiently small rectangle R in Ω.

↳ Not "$d\omega = 0$" b/c we're assuming $P, Q \in C^0$, not necessarily C^1 (so "d" might not make sense)

The above definition is equivalent to saying

$\int_\gamma \omega = 0$ whenever γ is the boundary of any rectangle in Ω. $\longrightarrow$ Can write a "big rectangle" as the "union" of several "small rectangles"



$\Rightarrow$ Closed differential forms ω in a disk have a primitive !

Def: (CLOSED FORM) A form ω is closed if ω locally has a primitive.

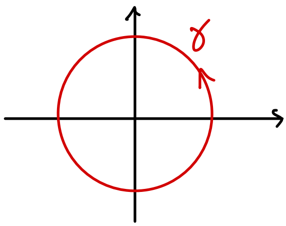
* Closed differential form in an arbitrary domain Ω need <u>not</u> have a primitive:

EXAMPLE $\Omega = \mathbb{C} \setminus \{0\}$, $\omega = \dfrac{dz}{z}$

ω is closed because locally at every point of Ω there is a branch of log $\longrightarrow$ primitive.

But no global primitive (just find a closed curve over which $\int_\gamma \omega \neq 0$)

Take $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$.



$$z = e^{it}$$

$$dz = i e^{it}\, dt$$

$$\int_\gamma \omega = \int_\gamma \frac{dz}{z} = \int_0^{2\pi} i\, dt = 2\pi i \neq 0.$$

Obs: example needn't be complex:

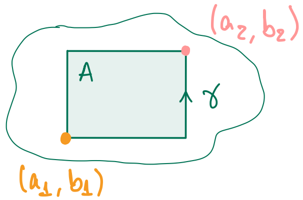
$$\frac{dz}{z} = \frac{d(x+iy)}{x+iy}$$

$$= \underbrace{\frac{x\,dx + y\,dy}{x^2+y^2}}_{\text{integral over } \gamma = 0} + i \boxed{\frac{x\,dy - y\,dx}{x^2+y^2}} =: \eta$$

The integral of this over γ is equal to 2π

$$\eta = dt, \quad t = \arctan\left(\frac{y}{x}\right)$$

Aside: Green's Theorem: assume P, Q are continuous w/ continuous partial $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ in a neighborhood of a closed rectangle A. Then



$$\int_\gamma \underbrace{P\,dx + Q\,dy}_{\omega} = \iint_A \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{d\omega}$$

Pf: $$\iint_A \frac{\partial Q}{\partial x}\,dx\,dy = \int_{b_1}^{b_2}\left(\int_{a_1}^{a_2} \frac{\partial Q}{\partial x}\,dx\right) dy$$

$$= \int_{b_1}^{b_2} Q(a_2, y) - Q(a_1, y)\,dy$$

$$= \int_\gamma Q\,dy.$$ Similar for P.

So, Green's Thm says that, if $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ exist and are continuous, then $\int_\gamma \omega = 0$ when γ

is the boundary of a small rectangle is equivalent to saying that $\iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx dy = 0$ for a sufficiently small rect. A.

$$\Longleftrightarrow \quad \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0 \quad \Longleftrightarrow \quad d\omega = 0.$$

!!

Thm: (CAUCHY'S THEOREM) If $f(z)$ is holomorphic in an open $\Omega \subset \mathbb{C}$, then $f(z)\,dz$ is closed; i.e., $\exists F$ s.t. $F' = f$, i.e., $f(z)$ has a primitive.

Assume further that $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ exist and are continuous.

Pf: (Cauchy's Thm)

$$f(z)\,dz = f(z)\,dx + i f(z)\,dy$$

By Green's Thm, it's enough to show that "$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$", $P = f(z)$ and $Q = i f(z)$; i.e., WTS: $\dfrac{\partial f}{\partial y} = i \dfrac{\partial f}{\partial x}$] Cauchy - Riemann equations.

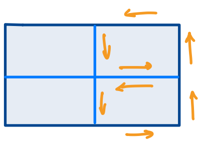
Since f is holom., f satisfies these eqs. $\square$

Pf: (CAUCHY'S THEOREM) Enough to show that

$\int_\gamma f(z)\,dz = 0$ whenever γ is the boundary

of a rectangle R.

R



Divide R into 4 equal parts R_i, each with an oriented boundary γ_i. Then,

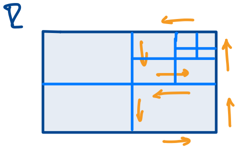
$$\mu(R) := \underbrace{\int_\gamma f(z)\,dz} = \sum_{i=1}^{4} \int_{\gamma_i} f(z)\,dz .$$

So, for at least one i,

$$\left| \int_{\gamma_i} f(z)\,dz \right| \geq \frac{1}{4} \left| \int_\gamma f(z)\,dz \right|$$

Say $R_i = R^{(1)}$, $\gamma_i = \gamma^{(1)}$. So $|\mu(R^{(1)})| \geq \frac{1}{4} |\mu(R)|$.

Continue to subdivide: $R \supset R^{(1)} \supset R^{(2)} \supset \dots$



So,

$$\left|\mu(R^{(k)})\right| = \left|\int_{\gamma^{(k)}} f(z)\,dz\right| \geq \frac{1}{4^k}\left|\int_{\gamma} f(z)\,dz\right| = \frac{1}{4^k}\left|\mu(R)\right|$$

Now, $\exists!\, z_0 \in \bigcap_k R^{(k)}$ (Cauchy's convergence criterion)

Since f is holomorphic,

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \underbrace{\varphi(z)}\,|z - z_0|$$

$\lim_{z \to z_0} \varphi(z) = 0$, i.e., $\forall \varepsilon > 0\ \exists \delta > 0$
$|z - z_0| < \delta \Rightarrow |\varphi(z)| < \varepsilon$

So,

$$\int_{\gamma^{(k)}} f(z)\,dz = \overbrace{\int_{\gamma^{(k)}} f(z_0)\,dz}^{=0 \text{ b/c } z \text{ has primitive}} + \overbrace{\int_{\gamma^{(k)}} f'(z_0)(z - z_0)\,dz}^{=0 \text{ b/c has primitive}}$$

$$+ \int_{\gamma^{(k)}} \varphi(z)\,|z - z_0|\,dz$$

Now, given $\varepsilon > 0$, if $|z - z_0| < \delta$, then

$$\left| \int_{\gamma^{(k)}} \varphi(z) |z - z_0| \, dz \right| \leq \varepsilon \left| \int_{\gamma^{(k)}} |z - z_0| \, dz \right|$$

$$\leq \varepsilon \underbrace{\operatorname{diam} R^{(k)}}_{\frac{1}{2^k} \operatorname{diam} R} \cdot \underbrace{\operatorname{perim} R^{(k)}}_{\frac{1}{2^k} \operatorname{perim} R}$$

$$= \frac{1}{4^k} \varepsilon \operatorname{diam} R \operatorname{perim} R$$

$$\Rightarrow |\mu(R)| \leq 4^k \left| \int_{\gamma^{(k)}} f(z) \, dz \right|$$

$$\leq \varepsilon \operatorname{diam} R \operatorname{perim} R$$

for any $\varepsilon > 0$

$\Rightarrow \mu(R) = 0$ so $f(z)\,dz$ is closed.

$\square$

Corollary 1: A holomorphic function $f(z)$ in an open $\Omega \subset \mathbb{C}$ locally has a primitive which is holomorphic.

Pf: Consider local primitive $F(z)$. Then

$$\underbrace{f(z)\,dz}_{} = dF = \underbrace{\frac{\partial F}{\partial z}\,dz}_{} + \underbrace{\frac{\partial F}{\partial \bar z}\,d\bar z}_{}$$

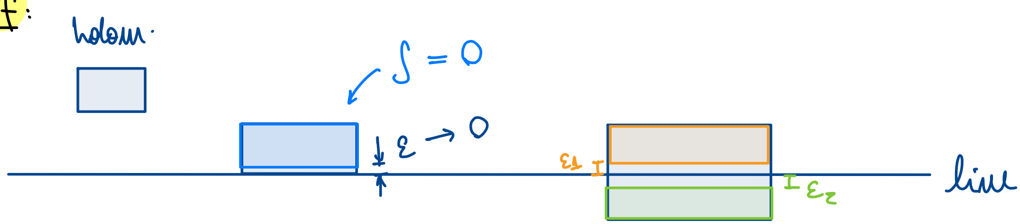
No $d\bar z$ term

$$\Rightarrow \frac{\partial F}{\partial \bar z} = 0 \Rightarrow F \text{ holomorphic}$$

(Cauchy-Riemann eqs.) □

Corollary 2: (GENERALIZED CAUCHY'S THM) In Cauchy's Theorem, it is enough to assume f is continuous in Ω except on, e.g., a line parallel to the x-axis

Pf:



□

Thm: Closed differential forms in a simply-connected open subset Ω of $\mathbb{R}^2$ has a global primitive.

Next time: a closed diff. form $\omega = P\,dx + Q\,dy$ in open $\Omega \subset \mathbb{R}^2$ always has a primitive along a curve $\gamma(t)$, $t \in [a,b]$ i.e., a cont. $f(t)$ s.t. $\forall t_0 \in [a,b]$ $\exists$ primitive F of ω in a neighborhood of $\gamma(t_0)$ s.t. $f(t) = F(\gamma(t))$, t suff. close to t_0.

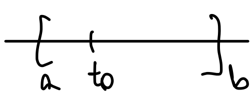
LECTURE 13: HOMOTOPY

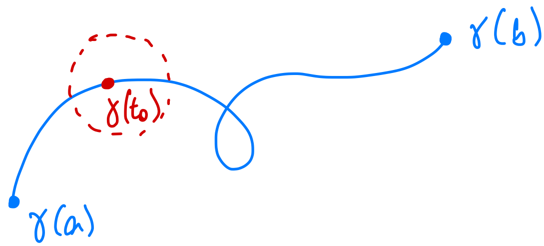
* PRIMITIVE OF A CLOSED DIFFERENTIAL FORM

Closed diff. forms ω in $\Omega \subset \mathbb{R}^2$ doesn't have necessarily have a "global primitive".

But it always have primitives along a curve

Prop: Consider a closed differential form ω in $\Omega \subset \mathbb{R}^2$ and a continuous curve $\gamma : [a,b] \to \Omega$. Then there is a continuous function $f(t)$ on $[a,b]$ s.t. for every $t_0 \in [a,b]$ there is a primitive F of ω in a neighborhood of $\gamma(t_0)$ s.t. $f(t) = F(\gamma(t))$ for t in a neighborhood of t_0. F is uniquely determined up to a constant.





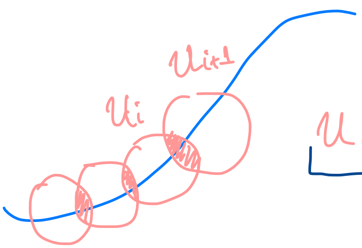
Pf: <u>Uniqueness</u>: suppose f_1, f_2 are primitives of ω along γ. Then in a neighborhood of t_0,

$$\underbrace{f_1(t_0) - f_2(t)} = F_1(\gamma(t_0)) - F_2(\gamma(t_0))$$

Locally constant, hence $f_1 - f_2 = \text{const.}$ b/c continuous $\Longleftarrow$ Differ by a constant (two local primitives in the same set)

<u>Existence</u>: There is a partition $a = t_0 < t_1 < \cdots < t_n = b$ (finite by compactness) of $[a,b]$ s.t. every $\gamma([t_{i-1}, t_i])$ lies in an open disk U_i in which $\underbrace{\omega \text{ has a primitive } F_i}$

ω is closed $\Rightarrow$ locally has a primitive



$\underbrace{U_i \cap U_{i+1}}$ is connected

$\to$ here, $F_{i+1} - F_i$ is constant

So, choose constants step by step to make

$$F_{i+1} = F_i \quad \text{in} \quad U_{i+1} \cap U_i .$$

Define $f(t) := F_i(\gamma(t))$ for $t \in [t_{i-1}, t_i]$. $\square$

C^1 on each interval of the partition

If γ is piecewise C^1 and f is the primitive of ω along γ, then

$$\int_\gamma \omega = f(b) - f(a).$$

This is because: consider the partition

$$a = t_0 < t_1 < \cdots < t_n = b.$$

Then $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$. So

$$\int_\gamma \omega = \sum_i \int_{\gamma_i} \omega$$

$$= \sum_i \left[F_i(\gamma(t_i)) - F_i(\gamma(t_{i-1}))\right]$$

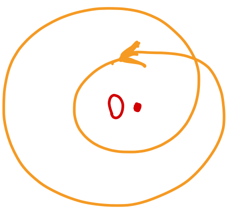
$$= \sum_i \left[f(t_i) - f(t_{i-1})\right]$$

Telescopic sum

$$= f(b) - f(a).$$

So, we can define $\int_\gamma \omega$ for any cont. curve as $f(b) - f(a)$

EXAMPLE: $$\omega = \frac{dz}{z}$$



γ closed (i.e., $\gamma(a) = \gamma(b)$)

Then, for f primitive along γ,

$$\int_\gamma \omega = \int_\gamma \frac{dz}{z}$$

Local primitive is log

$$= f(b) - f(a)$$

= difference between 2 branches of log at $\gamma(a) = \gamma(b)$

$$= 2\pi i n, \quad n \in \mathbb{Z}$$ (in this picture, $n=2$)

Now, if you take $\omega = \dfrac{x\,dy - y\,dx}{x^2+y^2}$, then

$$\int_\gamma \omega = 2\pi n, \quad n \in \mathbb{Z}$$

<u>Aside</u>: $b : [0,1]_t \to \mathbb{R}^2 \setminus \{0\}$ $\qquad d\theta \overset{\text{def}}{=} \dfrac{-y\,dx + x\,dy}{x^2+y^2}$

$$b(t) = (\cos 2\pi t, \sin 2\pi t)$$

$$\int_b d\theta = \int_0^1 b^* d\theta = \int_0^1 -\sin 2\pi t \cdot 2\pi(-\sin 2\pi t) + \cos 2\pi t \cdot 2\pi (\cos 2\pi t)\, dt$$

$$= 2\pi .$$

<u>Def</u>: (HOMOTOPY) Two continuous curves

$$\gamma_0, \gamma_1 : [0,1] \to \Omega$$

with same endpoints (i.e., $\gamma_0(0) = \gamma_1(0)$ and $\gamma_0(1) = \gamma_1(1)$) are <u>homotopic with fixed endpoints</u> <u>in</u> Ω if there is a continuous function

$$\gamma : \underset{s}{I} \times \underset{t}{I} \to \Omega \quad \text{s.t.}$$

$$\gamma(0,t) = \gamma_0(t), \quad \gamma(1,t) = \gamma_1(t)$$

$$\gamma(s,0) = \gamma_0(0) = \gamma_1(0)$$

$$\gamma(s,1) = \gamma_0(1) = \gamma_1(1).$$

Now, two continuous curves are <u>homotopic</u> <u>as closed curves in Ω</u> if there is a cont. function

$$\gamma : \underset{s}{I} \times \underset{t}{I} \to \Omega \quad \text{s.t.}$$

$$\gamma(0,t) = \gamma_0(t), \quad \gamma(1,t) = \gamma_1(t)$$

$$\gamma(s,0) = \gamma(s,1).$$

<u>Thm</u>: If ω is a closed differential form in Ω and $\gamma_0, \gamma_1 : [0,1]$ cont. curves in Ω, homotopic either

1. w/ fixed endpoints or
2. as closed curves

then $\displaystyle\int_{\gamma_0} \omega = \int_{\gamma_1} \omega.$

Lecture 14: Homotopy

Thm: If ω is a closed form in an open $\Omega \subset \mathbb{C}$ and $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are continuous curves either

1) homotopic w/ fixed endpoints
2) homotopic as closed curves

then

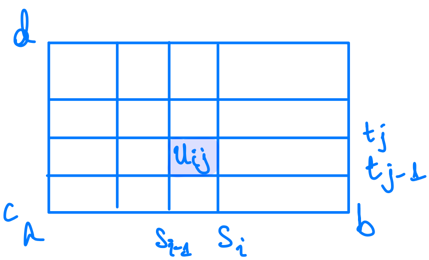
$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega .$$

Lemma: take a closed form ω in Ω and $\gamma : [a,b] \times [c,d] \to \Omega$ a continuous curve, then there is a continuous function $f : [a,b] \times [c,d] \to \mathbb{C}$ s.t., for every $(s_0, t_0) \in [a,b] \times [c,d]$, there is a primitive F of ω defined in a neighborhood of $\gamma(s_0, t_0)$ s.t. $f(s,t) = F(\gamma(s,t))$ in some neighborhood of (s_0, t_0). Moreover, f unique up to a constant. Call f "a primitive along γ".

Pf: Choose partitions $\{s_i\}$ of $[a,b]$
$\{t_i\}$ of $[c,d]$

s.t. γ maps every $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$ into an open set U_{ij} where ω has primitive F_{ij}.

For fixed j, there is a primitive f_j along $\gamma|_{[a,b]\times[t_{j-1},t_j]}$ <u>like</u> <u>before</u> (p. 85). So,



F_{ij} and $F_{i+1,j}$ differ by a constant in $U_{ij} \cap U_{i+1,j}$ (b/c both are local primitives of ω).

So we can adjust the constants one at a time to make $F_{ij} = F_{i+1,j}$ in the overlaps. Then, we can define f_j as $F_{ij} \circ \gamma$ in $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$.

For each j, f_j and f_{j+1} differ by a constant on $[a,b] \times \{t_j\}$. We can again adjust these constants so the functions coincide in overlaps.

$\square$

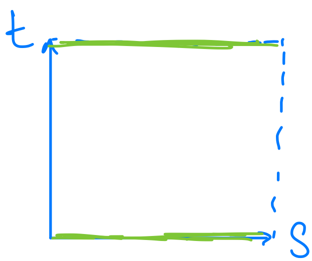
Pf: (Thm)

(1) Suppose we have a homotopy

$$\gamma : [0,1] \times [0,1] \longrightarrow \Omega$$

fixed endpoints
$$\begin{cases} \gamma(0,t) = \gamma_0(t), \quad \gamma(1,t) = \gamma_1(t) \\ \gamma(s,0) = \gamma_0(0) = \gamma_1(0), \\ \gamma(s,1) = \gamma_0(1) = \gamma_1(1) \end{cases}$$

Let f be the primitive of ω along γ



f is constant on the two horizontal sides, i.e.,

$$f(0,0) = f(1,0)$$
$$f(0,1) = f(1,1)$$

$$\int_{\gamma_0} \omega = f(0,1) - f(0,0)$$

)) equal

$$\int_{\gamma_1} \omega = f(1,1) - f(1,0)$$

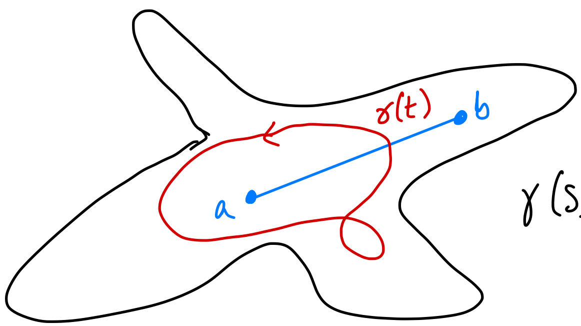
□

Def: (SIMPLY-CONNECTED) Ω is <u>simply-connected</u> if Ω is connected and any closed curve in Ω is <u>null-homotopic</u>

Cor: In simply-connected open sets, any closed form has a global primitive.

EXAMPLES OF SIMPLY-CONNECTED OPEN SETS

1) star shaped open sets are simply-connected.



$$\gamma(s,t) = a + s(\gamma(t) - a)$$

2) Closed form $\omega = \frac{dz}{z}$ has a primitive in any simply-connected open set Ω not containing zero; i.e., $\log z$ has a branch in any simply connected

open set not containing zero:

let
$$\log z := w_0 + \int_{z_0}^{z} \frac{dz}{z},$$
where $e^{w_0} = z_0$

3) $\mathbb{C} \setminus \{0\}$ is <u>not</u> simply-connected b/c S^1 is <u>not</u> homotopic to a point in $\mathbb{C}\setminus\{0\}$ (b/c $\int_{S^1} \frac{1}{z}\, dz = 2\pi i \neq 0$)

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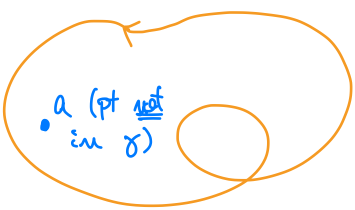
* CAUCHY'S INTEGRAL FORMULA



γ closed curve in $\mathbb{C}$

Define the <u>winding number</u> of γ with respect to a as
$$W(\gamma, a) = \frac{1}{2\pi i} \int_{\gamma} \frac{dz}{z-a} \in \mathbb{Z}$$

Properties:

(1) Fix a. Then $w(\gamma, a)$ is invariant under homotopies of γ not passing through a. (by 1st thm of this lecture)

(2) In particular, if γ lies in a simply-connected open set not containing a, then $w(\gamma, a) = 0$.

(3) Fix γ. Then $w(\gamma, a)$ is constant on connected components of the complement of a.

Enough to show locally. A small shift of a is equiv. to a small shift of γ in the opposite direction (this preserves w by (1)).

(4) If γ is a circle described in the positive sense (i.e., $w(\gamma,$ centre of circle$) = +1$), then

$$w(\gamma, a) = \begin{cases} 0, & \text{if } a \text{ outside } \gamma \\ 1, & \text{if } a \text{ inside } \gamma \end{cases}$$

Constant in connected components of the complement of γ (3)

Thm: (Cauchy Integral Formula) If $\Omega \subset \mathbb{C}$ is open, $a \in \Omega$, $f: \Omega \to \mathbb{C}$ is holomorphic in Ω, γ is a closed curve in Ω, not containing a, homotopic to a point in Ω, then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz = f(a) \cdot w(\gamma, a).$$

Cor: If f is holomorphic in a neighborhood of a closed disk D whose boundary is γ (in the positive sense), then

$$\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a}\, dz = \begin{cases} f(a), & a \text{ inside disk} \\ 0, & a \text{ outside disk} \end{cases}$$

Pf: (Cauchy's Integral Formula)

Let

$$g(z) = \begin{cases} \dfrac{f(z)-f(a)}{z-a}, & z \neq a \\ f'(a), & z = a \end{cases}.$$

Note that $g(z)$ is continuous and holomorphic when $z \neq a$. Then $g(z)\,dz$ is closed (by Cauchy's Thm).

So,

$$0 = \int_\gamma g(z)\,dz = \int_\gamma \frac{f(z)-f(a)}{z-a}\,dz$$

$$\Leftrightarrow \int_\gamma \frac{f(z)}{z-a}\,dz = \int_\gamma \frac{f(a)}{z-a}\,dz$$

$$= f(a) \underbrace{\int_\gamma \frac{1}{z-a}\,dz}_{2\pi i\, W(\gamma, a)}$$

□

Prop: Consider a continuous function $f : \Omega \to \mathbb{C}$. Then, the following are equivalent:

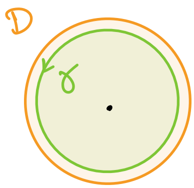
(1) f is holomorphic in Ω

(2) $f(z)\,dz$ is closed

(3) $f(z) = \dfrac{1}{2\pi i} \displaystyle\int_\gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta$!

when z is in the interior of a closed disk D in Ω with (positively) oriented boundary γ.

Remark: A holomorphic function $f(z)$ in an open disk D is infinitely differentiable in D.



γ boundary of a smaller disk inside D

For z inside γ, by Cauchy's Integrable Formula,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z} \, dz . \quad \Big] \text{ inside } \gamma$$

So,

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2} \, d\zeta$$

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta$$

Pf: (1) $\Rightarrow$ (2) "Original" Cauchy's thm

(1) $\Rightarrow$ (3) Cauchy Integral Formula

(3) $\Rightarrow$ (1) Remark above

$(2) \Rightarrow (1)$ Assume $f(z)\,dz$ is closed. So, (Morera's Thm) $f(z)$ locally has a primitive $g(z)$ which is holomorphic. This implies $f(z) = g'(z)$ is holomorphic (we can differentiate g ∞-many times, so we can do the same for f ...). $\square$

A continuous function which is holomorphic except on a line is holomorphic everywhere.

Except on a finite union of lines/circles also works.

Apply Möbius transf. to make them lines.

//

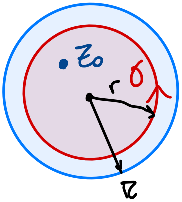
Lecture 15: Applications of Cauchy's Formula

Recall: Consider $f(z)$ holomorphic in $|z| < R$.

For z s.t. $|z| < r < R$,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

and, for all $n \in \mathbb{N}$,



$0 < r < R$.

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta,$$

Taylor Expansion of f at 0:

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta^{n+1}} \, d\zeta$$

Thm: $f(z)$ has a convergent power series expansion in $|z| < R$.

Pf.

$$\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1 - \frac{z}{\zeta}\right)^{-1}$$

$$= \frac{1}{\zeta}\left(1 + \frac{z}{\zeta} + \frac{z^2}{\zeta^2} + \cdots\right),$$

which is convergent for $|z| < r$, $|\zeta| = r$. Thus, by Cauchy's Formula,

$$f(z) = \frac{1}{2\pi i} \int_\gamma \sum_{n=0}^{\infty} \frac{z^n \overbrace{f(\zeta)}^{\text{bdd}}}{\zeta^{n+1}} \, d\zeta$$

For fixed z, $|z| < r$, this is uniformly convergent on $|\zeta| = r$.

By comparison w/ geom. series $\Leftarrow \left|\frac{z}{\xi}\right| < 1$
and Weierstrass M- test

So, the $\int_\gamma$ commutes w/ $\sum_n$.

So, we can write

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where}$$

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi^{n+1}} \, d\xi$$

$\square$

Cor: Every holomorphic function is analytic.

* Could use Fourier series

$$f(re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{im\theta},$$

so,

$$a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} \, d\theta$$

if $n = m$, then $\int = 2\pi$, if $n \neq m$, $\int = 0$.

* Could also do this by setting $\xi := re^{i\theta}$
$d\xi = ire^{i\theta}$

and letting γ = circle of radius r centred at 0 and integrating.

* The integral formula gives an upper-bound for the Taylor coefficients a_n:

Let $\boxed{M(r) = \sup_{\theta} |f(re^{i\theta})|}$. Then

Biggest Abs of f on the circle of radius r

$$|a_n r^n| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} \, d\theta \right|$$

Cauchy's Integral Formula

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \underbrace{|e^{-in\theta}|}_{=1} \, d\theta$$

$$\leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| \, d\theta$$

$$\leq M(r)$$

! $\Rightarrow$ **Cauchy's Inequalities**:

$$|a_n r^n| \leq M(r),$$
$$|a_n| \leq \frac{M(r)}{r^n}.$$

Thm: (LIOUVILLE'S THEOREM) A bounded holomorphic function in $\mathbb{C}$ is constant.

Pf: $M(r) \leq M$ for some $M > 0$ since f is bounded.

Now,

Cauchy's Inequalities

$$|a_n| \leq \frac{M}{r^n} \quad \text{for all } r > 0$$

But, as $r \nearrow \infty$, $1/r^n \to 0$; so, $a_n = 0$ if $n \geq 1$. Thus, $f(z) = a_0$ (constant).

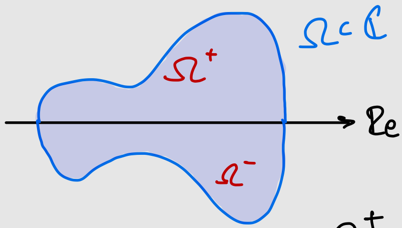
Indep. coeff. of power series expansion.

$\square$

Cor: (Fundamental Theorem of Algebra) Every non-constant polynomial has a root.

Pf: (Contradiction) Assume poly. $P(z)$ has no roots. Then $1/P(z)$ is a holomorphic function on $\mathbb{C}$. Note that $1/P(z)$ is bounded. So, by Liouville's Thm, $P(z)$ is constant $\Leftarrow\!\!\!/\!\!\Rightarrow$ $\Rightarrow$ Non-constant has at least one root. $\square$

Schwarz Reflection Principle:



$\Omega\subset\mathbb{C}$ is a domain symmetric to the Re-axis. Take $f(z)$ continuous on Ω^+, real on $\Omega\cap\mathbb{R}$, holomorphic on $\Omega^+\cap\{\operatorname{Im}(z)>0\}$. Then $f(z)$ can be extended (uniquely) to a holomorphic function in Ω by reflection.

Analytic continuation

Pf: Define

$$g(z)=\begin{cases} f(z), & z\in\Omega^+ \\ \overline{f(\bar z)}, & z\in\Omega^- \end{cases}$$

Then $g(z)$ is holomorphic on $\Omega\cap\{\operatorname{Im}(z)>0\}$ and on $\Omega\cap\{\operatorname{Im}(z)<0\}$, continuous on Ω. Therefore, holomorphic.

$\square$

LECTURE 16:

Recall: Consider $f(z) = \sum_{n=0}^{\infty} a_n z^n$. If $r > 0$ is small enough and we write $z = re^{i\theta}$, then

$$a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta.$$

So that

$$f(0) = a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta})\, d\theta$$

is the mean value of $f(z)$ on $|z| = r$.

Prop: (MEAN VALUE PROPERTY) If $f(z)$ is holomorphic on an open $\Omega \subset \mathbb{C}$, then $f(z)$ has the mean value property; i.e., for any closed disk in Ω,
f (centre) = mean value on boundary.

Re & Im parts of a function w/ MVP also have MVP.

Prop: (MAXIMUM MODULUS PRINCIPLE) If f is a continuous complex-valued function w/ MVP on an open $\Omega \subset \mathbb{C}$, and $|f|$ has a local max. at $a \in \Omega$, then f is constant on a neighborhood of a.

Pf: (If f is holomorphic, proof is way simpler; trivial if Ω is connected)

- If $f(a) = 0$, trivial
- If $f(a) \neq 0$, then we can assume $0 < f(a) \in \mathbb{R}$.

For $r > 0$ small enough,

local max at a

$$M(r) := \sup_{\theta \in [0, 2\pi]} \left| f(a + re^{i\theta}) \right| \leq f(a) \leq M(r)$$

$$\Rightarrow f(a) = M(r).$$

Let $g(z) := \mathrm{Re}\left(f(a) - f(z) \right)$

$$= \underbrace{f(a)}_{\sup_{|z-a|<r} |f(z)| \geq \mathrm{Re}(f(z))} - \mathrm{Re}(f(z)) \geq 0 \Rightarrow g(z) \geq 0 \text{ on } |z-a| = r.$$

Note that $g(z) = 0 \Leftrightarrow f(z) = f(a)$. So, the mean value of $g(z)$ on $|z-a| = r$ is $g(a) = 0$. Since $g(z)$ is continuous, and nonnegative on $|z-a| = r$, then $g \equiv 0$ on $|z-a| = r$. Thus, $f(z) = f(a)$ on $|z-a| = r$.

$\square$

Cor: Suppose $\Omega \subset \mathbb{C}$ is bounded. Let $f(z)$ be continuous and complex-valued on $\overline{\Omega}$ w/ MVP in Ω. Set

$$M := \sup |f(z)| \text{ on the } \underbrace{\text{frontier}}_{\partial\Omega = \overline{\Omega}\setminus\Omega.} \text{ of } \overline{\Omega}.$$

Then $|f(z)| \leq M$ for $z \in \Omega$ and if $|f(z_0)| = M$ for some $z_0 \in \Omega$, then f is constant.

Pf: Let $M' := \max_{z\in\overline{\Omega}} |f(z)|$. M' is attained at a point $a \in \overline{\Omega}$.

→ If $a \in \{\text{frontier}\}$, ✓

→ If $a \in \Omega$, $\{z \in \Omega : f(z) = f(a)\}$ is open by max modulus principle but closed (since a is one point). □

! **Thm:** (SCHWARZ'S LEMMA) Let $f(z)$ be holomorphic in $|z| < 1$ and such that $|f(z)| < 1$ if $|z| < 1$ and $f(0) = 0$. Then

1) $|f(z)| \leq |z|$ if $|z| \leq 1$

2) If $|f(z_0)| = |z_0|$ for some $z_0 \neq 0$, $|z_0| < 1$, then $f(z) = \lambda z$, $|\lambda| = 1$.

Pf: Since $f(0) = 0$, $\frac{f(z)}{z}$ is holomorphic on $|z|<1$.

So, we can write $f(z) = \sum_{n=0}^{\infty} a_n z^n$.

Moreover,

$$\left| \frac{f(z)}{z} \right| < \frac{1}{r} \quad \text{on } |z| = r,$$

where $r < 1 \Rightarrow$ on $|z| \leq r$

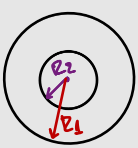
So, for fixed z,

$$\left| \frac{f(z)}{z} \right| < \frac{1}{r} \quad \forall r : |z| \leq r < 1.$$

Now, $\lim_{r \to 1} |f(z)| \leq |z|$. If $|f(z_0)| = |z_0|$, $z_0 \neq 0$, then $f(z)/z$ attains max. mod. at some λ. $\square$

! <u>LAURENT</u> <u>EXPANSION</u>: A holomorphic function $f(z)$ on the annulus $0 \leq R_2 < |z| < R_1 \leq \infty$ has convergent Laurent expansion on the annulus; i.e.,

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n = \underbrace{\sum_{n<0} a_n z^n}_{\text{converges if } |z| > R_2} + \underbrace{\sum_{n \geq 0} a_n z^n}_{\text{converges if } |z| < R_1}$$

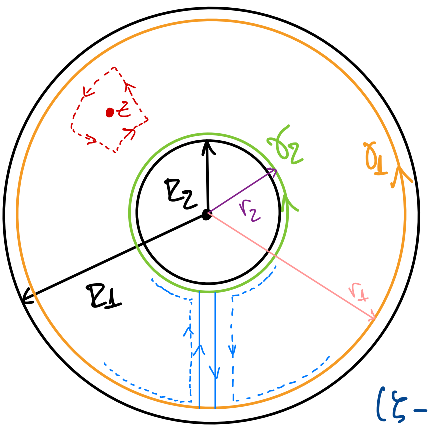


Write $z = 1/\zeta$. Then

$$\sum_{n<0} a_n z^n = \sum_{n<0} a_n \zeta^{-n} = \underbrace{\sum_{n>0} a_{-n} \zeta^n}_{\substack{\text{converges if} \\ |\zeta| < 1/R_2}}$$

Pf: (By Cauchy's Theorem) Write

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i} \int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$



1) $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ (as before)

where $\displaystyle a_n = \frac{1}{2\pi i} \int_{\gamma_1} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta$

(Recall how we got the formula for $(\zeta - z)^{-1} = \frac{1}{\zeta}\left(1 - z/\zeta\right)^{-1}$).

2) If $z > \zeta$,

$$\frac{1}{\zeta - z} = -\frac{1}{z}\left(\frac{1}{1 - \zeta/z}\right) = -\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} = -\underbrace{\sum_{n<0} \frac{z^n}{\zeta^{n+1}}}_{\substack{\text{Uniformly and absolutely} \\ \text{convergent on } |\zeta| = r_2.}}$$

$\square$

! **LAURENT EXPANSION:**

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad a_n = \begin{cases} \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_1} \dfrac{f(\zeta)}{\zeta^{n+1}}\, d\zeta, & n \geq 0 \\ \dfrac{1}{2\pi i} \displaystyle\int_{\gamma_2} \dfrac{f(\zeta)}{\zeta^{n+1}}\, d\zeta, & n < 0 \end{cases}.$$

Uniformly and absolutely convergent on $r_2 \leq |z| \leq r_1$.

Def. (ISOLATED SINGULARITY) A holomorphic function $f(z)$ in the punctured disk $0 < |z| < r$ has an <u>isolated singularity</u> at $z = 0$ if it <u>cannot</u> be extended to a holomorphic function in $|z| < r$.

Note that the Laurent expansion of $f(z)$ on the punctured disk $0 < |z| < r$ still exists:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n.$$

This isolated singularity can be either:

(i) <u>Pole</u>: finitely many negative exponents in the Laurent expansion;

(ii) <u>Essential singularity</u>: infinitely many negative exponents in the Laurent expansion.

LECTURE 17:

* Isolated singularities and residues:

Holomorphic function $f(z)$ in a punctured disk $0<|z|<R$ has an ISOLATED SINGULARITY at 0 if f cannot be extended to be holomorphic in $|z|<R$:

Pole or Essential singularity

Only finitely many negative exponents in Laurent series

Infinitely many...

* Extension to holomorphic function in $|z|<R$ is possible if and only if f is bounded in a neighborhood of zero.

Why? Write $f(z)=\sum_{n=-\infty}^{\infty} a_n z^n$

$$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta} \quad (0<r<R)$$

where

$$a_n r^n = \frac{1}{2\pi} \int_0^{-\pi} e^{-in\theta} f(re^{i\theta})\, d\theta .$$

Let $M(r) := \sup_{|z|=r} |f(z)|$. Then

$$|a_n| r^n < M(r) \text{ ; i.e., } |a_n| \leq \frac{M(r)}{r^n}, \quad n \in \mathbb{Z}.$$

f bounded in punctured disk $\iff$ $M(r) \leq M$ (const.)

- If $n < 0$, then $|a_n| \leq M r^{-n}$. $\to 0$ as $r \to 0$ b/c $-n$ positive

So, $a_n = 0$, $n < 0$, which implies f is holomorphic.

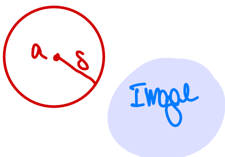
- If zero is a pole, then $\lim_{z \to 0} f(z) = \infty$.

A meromorphic function is a holomorphic function w/ values in the Riemann sphere S^2.

Thm: (WEIERSTRASS THEOREM) If zero is an essential singularity of f, then, for any $\varepsilon > 0$, $f(0<|z|<\varepsilon)$ is dense in $\mathbb{C}$.

Pf: Otherwise, there is some $\delta > 0$ such that

$$|f(z) - a| > \delta \quad \text{if} \quad 0 < |z| < \varepsilon.$$



Let

$$g(z) = \frac{1}{f(z) - a}.$$

Holomorphic in $0<|z|<\varepsilon$, bounded by $1/\delta$. Therefore holomorphic in the whole disk $|z|<\varepsilon$.

$\Longleftrightarrow$ $f(z) = a + \dfrac{1}{g(z)}$ is meromorphic (quotient of holomorphic function). So, zero is a pole. $\lightning$

$\square$

In fact, the image of any puncture disk (i.e, $f(0<|z|<\varepsilon)$) omits at most 1 value.

Picard's big theorem

At infinity: Suppose f is holomorphic in $|z| > R$.

- f is holomorphic at ∞ if $f(1/z)$ is holomorphic in $|z| < 1/R$.

f has a pole at ∞ if $f(1/z)$ has a pole at 0.

f has an essential singularity at ∞ if $f(1/z)$ has an essential singularity at 0.

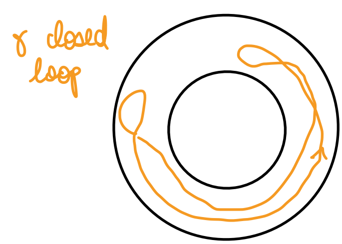
Consider the Laurent expansion in $|z| > R$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n .$$

- **Pole at ∞** if there are finitely many a_n, $n > 0$, are nonzero.

EXERCISE: Let $f(z)$ be holomorphic in the annulus $R_2 < |z| < R_1$. Compute

$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz .$$



We know $f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$

$$= \frac{a_{-1}}{z} + \sum_{n \neq -1} a_n z^n$$

has a primitive

So,

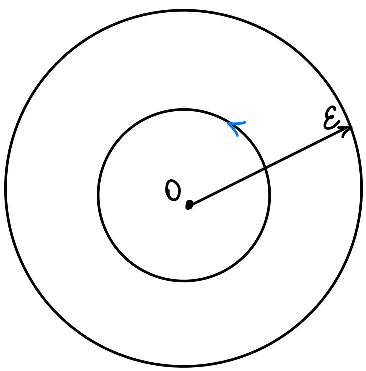
$$\frac{1}{2\pi i} \int_\gamma f(z)\, dz = \frac{1}{2\pi i} \int_\gamma \frac{a_{-1}}{z}\, dz$$

$$= a_{-1}\, w(\gamma, 0).$$

In the example, $w(\gamma, 0) = 0 \ldots$

$\square$

In particular, if f is holomorphic in the punctured disk $0 < |z| < \varepsilon$.



γ circle around 0 in the positive sense.

!

Residue of f at 0; or residue of form $\omega = f(z)\, dz$ at 0.

$$\boxed{\frac{1}{2\pi i} \int_\gamma f(z)\, dz} = a_{-1}, \text{ where}$$

f has Laurent series $\sum_{n \in \mathbb{Z}} a_n z^n$.

!

Residue at ∞: of $f(z)$ holomorphic in $|z|<R$ is defined as

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz,$$

where γ is a circle around ∞ negatively oriented w.r.t. ∞.

In coördinates at ∞: $z = 1/z'$,

$$f(z)\,dz = -\frac{1}{z'^2} f(1/z')\,dz'.$$

So,

$$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = -\frac{1}{2\pi i}\int_{\tilde{\gamma}} \frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'.$$

$\tilde{\gamma}$ in the positive sense.

LECTURE 18: RESIDUE THEOREM

Recall: $f(z)$ is holomorphic in a punctured neighborhood of point a.

RESIDUE: of $f(z)\,dz$ at a is defined as

$$\operatorname{res}(f(z)dz, a) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz.$$

$$\| $$

$$\operatorname{res}(f, a)$$

γ: circle around a oriented in the positive sense. (small enough so it lies inside the punctured disk where f is holomorphic)

Compute the residue: a_{-1} in the Laurent expansion of f at a $\sum_{n\in\mathbb{Z}} a_n (z-a)^n$.

RESIDUE AT ∞: of $f(z)\,dz$ at ∞ is defined

$$\operatorname{res}(f(z)dz, \infty) = \frac{1}{2\pi i}\int_\gamma f(z)\,dz$$

γ: small positive circle around ∞ (i.e., a big negative circle on the plane $\mathbb{C}$).

Compute this by $z' := 1/z \Rightarrow f(z)\,dz = -\frac{1}{z'^2} f(1/z')\,dz'$

$$\operatorname{res}(f, \infty) = -\frac{1}{2\pi i}\int_\gamma \frac{1}{z'^2} f\left(\frac{1}{z'}\right) dz'$$

γ: small circle around $z' = 0$.

So, $\operatorname{res}(f,\infty) = -a_{-1}$

using the Laurent expansion of $f(z)$ at ∞:

$$\sum_{n\in\mathbb{Z}} a_n z^n = \dots + a_{-2}\underbrace{\frac{1}{z^2}}_{z'^2} + a_{-1}\underbrace{\frac{1}{z}}_{z'} + a_0 + a_1 \underbrace{z}_{\frac{1}{z'}} + a_2 z^2 + \dots$$

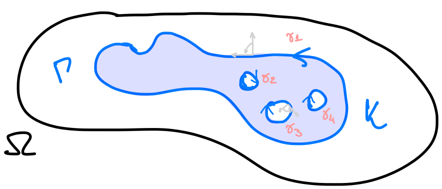
divide by z'^2

$$\Rightarrow \operatorname{Res}(f,\infty) = -\operatorname{Res}\left(\frac{1}{z^2} f\left(\frac{1}{z}\right), 0\right)$$

!!

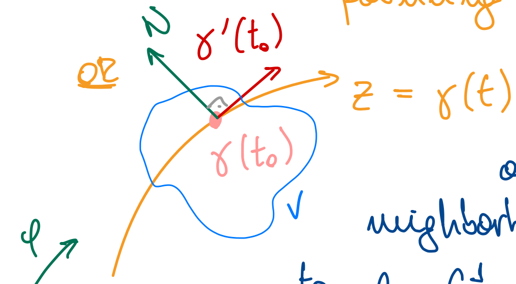
! **Thm:** (RESIDUE THEOREM) Let Ω be an open set in $\mathbb{C}\cup\{\infty\}$ and $f(z)$ be holomorphic in Ω except maybe at isolated points. Let K be a compact set with piecewise C^1 oriented boundary Γ in Ω, where Γ contains no singularities or ∞. Then,

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{\substack{\text{singularities}\\ z_k \text{ of } f \text{ in } K}} \operatorname{res}(f, z_k).$$



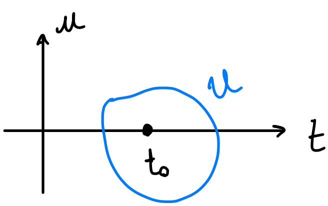
Γ is a disjoint union of closed piecewise C^1 curves γ (1-1 except for initial and final points) positively oriented with respect to K.

$\hookrightarrow$ orientation induced by the outward pointing normal.



<u>Claim</u>: there exists an open disk $\mathcal{U}$ at $(t_0, 0)$ and open neighborhood V of $\gamma(t_0)$ s.t. $\gamma(t)$ extends to a C^1 map $\varphi : \mathcal{U} \to V$ with C^1 inverse s.t. $\varphi(t, 0) = \gamma(t)$ and $\det \varphi' > 0$ (invertible and orientation preserving). $\hookrightarrow$ Why?

Pick a normal vector N at $\gamma(t_0)$ s.t. $N \perp \gamma'(t_0)$ and let $\varphi(t, u) = \gamma(t) + uN$. If this gives $\det \varphi' < 0$, then just use $-N$.



- γ is positively oriented with respect to K if the upper half of the disk gets mapped to the inside of K (i.e., N points "towards the hole" of K).

<u>Recall</u>: (Green's Theorem) If P, Q are C^1 in a neighborhood of K, then

$$\int_\Gamma P\,dx + Q\,dy = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy .$$

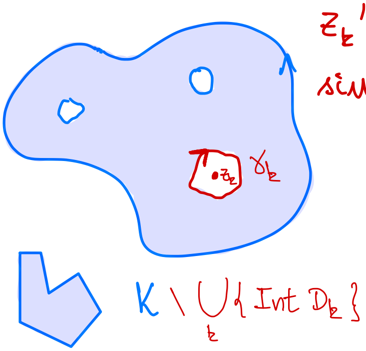
In particular, if $P\,dx + Q\,dy$ is closed, then

$$\int_\Gamma P\,dx + Q\,dy = 0$$

(integral of closed form is zero)

Pf: (RESIDUE THEOREM) **Case 1**: $\infty \notin K$

z_k' are the singular points



Let γ_k be the boundary of closed disk D_k in $\text{Int}(K)$ around z_k oriented in the positive sense with respect to z_k.

Let $K' = K \setminus \bigcup_k \{\text{Int } D_k\}$.

But, note that $f(z)$ is holomorphic in a neighborhood of K'. By Green's Thm & Cauchy's Thm:

$$\int_{\partial K'} f(z)\,dz = 0 .$$

$\partial K' \longrightarrow \partial K - \bigcup_k \gamma_k$

Negative because of the "relative orientation" between Γ and the γ_k

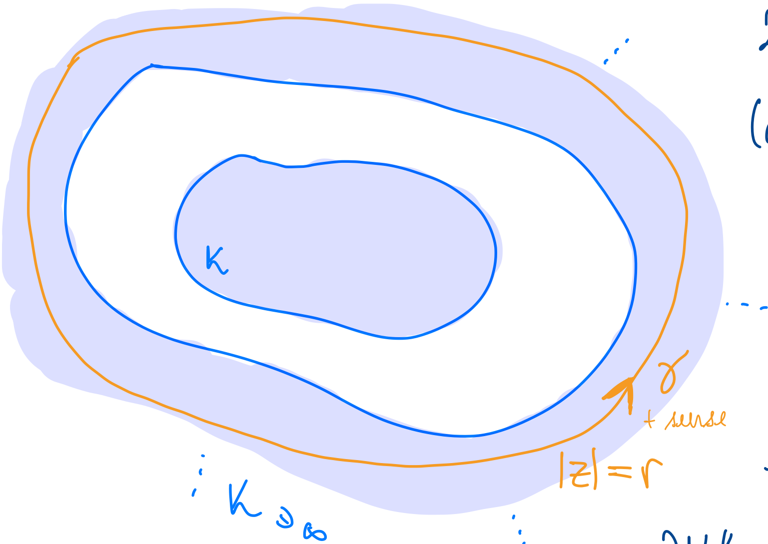
So,

$$\int_\Gamma f(z)\,dz - \sum_k \int_{\gamma_k} f(z)\,dz = 0$$

$$\Rightarrow \int_\Gamma f(z)\,dz = \sum_k \int_{\gamma_k} f(z)\,dz$$

$$\Rightarrow \int_\Gamma f(z)\,dz = 2\pi i \sum_k \operatorname{res}(f, z_k).$$

Case 2: $\infty \in K$:



Choose r big enough s.t. $\{|z| \leq r\} \subset \operatorname{Int} K$ (and $f(z)$ is holomorphic in $|z| > r$ except at ∞). Let

$$K'' := K \setminus \{|z| > r\}$$

so that

$$\partial K'' = \Gamma \cup \gamma.$$

Now, as before in Case 1:

$$\int_\Gamma f(z)\,dz + \underbrace{\int_\gamma f(z)\,dz}_{-2\pi i \cdot \operatorname{res}(f,\infty)} = 2\pi i \sum_k \operatorname{res}(f, z_k).$$

$\square$

CONSEQUENCES OF THE RESIDUE THM:

1) If $K = S^2$, then $\sum_k \operatorname{res}(f, z_k) = 0$.

$\Rightarrow$ sum of residues of any rational function is zero.

2) CALCULATE RESIDUES !!

n = order of pole of f at a.

$$\operatorname{Res}(f, a) = \frac{1}{(n-1)!} \lim_{z \to a} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

(i) If $f(z) = \dfrac{g(z)}{h(z)}$ and a is a point s.t.

$g(a) \neq 0$ and h has a $\underbrace{\text{simple zero}}$ at a.

$h(a) = 0$

$h'(a) \neq 0$

The Taylor exp. of g at a:

$$g(z) = g(a) + g'(a)(z-a) + \dots$$

The Taylor exp. of g at a:

$$h(z) = h'(a)(z-a)(1 + \dots).$$

!!

So,

$$\operatorname{res}(f, a) = \frac{g(a)}{h'(a)}$$

$\rightarrow f(z) = \underbrace{\frac{g(a)}{h'(a)} \frac{1}{z-a}}_{a_{-1} \text{ term}} + \dots$

(ii) $$f(z) = \frac{e^{iz}}{z(z^2+1)^2}.$$ Singular points: $0, \pm i$

Let $\xi = z - i$

$$f(i+\xi) = \frac{e^{i(i+\xi)}}{\xi^2 (i+\xi)(2i+\xi)^2}$$

Taylor exp

(a) $e^{i(i+\xi)} = e^{i^2} e^{i\xi} = \frac{1}{e}(1 + i\xi + \cdots)$

(b) $(i+\xi)^{-1} = -i(1 - i\xi)^{-1} = -i(1 + i\xi + \cdots)$

(c) $(2i+\xi)^{-2} = -\frac{1}{4}\left(1 - \frac{i}{2}\xi\right)^{-2} = -\frac{1}{4}(1 + i\xi + \cdots)$

$(a)\cdot(b)\cdot(c) = \frac{i}{4e}(1 + 3i\xi + \cdots)$

$$\Rightarrow \operatorname{res}(f, i) = \frac{i}{4e}(3i) = -\frac{3}{4e}$$

(iii) Meromorphic function $f(z)$ in a neighborhood of $z = a$.

$$f(z) = (z-a)^k g(k),$$

where g is holomorphic in a neighborhood of a, $g(a) \neq 0$.

$$\frac{f'}{f} = \frac{k}{z-a} + \frac{g'}{g}.$$

Product rule

So, $\dfrac{f'}{f}$ has a simple pole at a with

$$\operatorname{res}\left(\frac{f'}{f}, a\right) = k.$$

* **ARGUMENT PRINCIPLE:**

Let $f(z)$ be a nonconstant meromorphic function in open Ω. Let K be a compact set with oriented boundary Γ in Ω. Suppose $f(z)$ doesn't take value a on Γ and has no poles on Γ. Then

Residue Thm

$$\frac{1}{2\pi i}\boxed{\int_\Gamma \frac{f'(z)}{f(z)-a}}\,dz = \underbrace{Z}_{} - \underbrace{P}_{}$$

zeros of $f(z) = a$

* poles of $f(z)$ b/c (the poles of f' are the same as f).

Both w/ multiplicity

Variation of the arg. of $f(z)-a$ as z describes Γ (even if Γ is not closed)

LECTURE 19: CONSEQUENCES OF RESIDUE THM

Recall: Variation of argument

Let $f(z)$ be meromorphic in a neighborhood of $a \in \mathbb{C}$, then we have that

$$f(z) = (z-a)^k g(z),$$

g holomorphic, $g(a) \neq 0$.

$<0 \to$ pole

$>0 \to$ zero.

So,

Product rule

$$\frac{f'}{f} = \frac{k}{z-a} + \underbrace{\frac{g'}{g}}_{\text{holomorphic} \Rightarrow \text{res} = 0}$$

$$\Rightarrow \text{res}\left(\frac{f'}{f}, a\right) = k.$$

* **ARGUMENT PRINCIPLE**: consider $f(z)$ nonconstant and meromorphic in an open Ω. Let K be a compact set with oriented boundary Γ.

Given a, assume $\left\{\begin{array}{l}\text{no zeroes of } f(z)-a \\ \text{no poles of } f(z)\end{array}\right\}$ on Γ.

Thm,

$$\frac{1}{2\pi i}\int_{\Gamma} \frac{f'(z)}{f(z)-a}\,dz = \underbrace{Z}_{} - \underbrace{P}_{}$$

Z = # of zeroes of $f(z)-a$

P = # of poles of $f(z)$ in K counted with multiplicity.

Pf: (of the above) By Residue Thm and the logarithmic derivative example. □

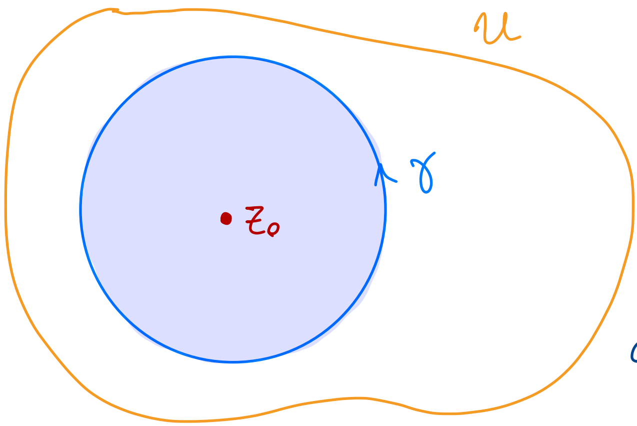
Thm: Consider $f(z)$ nonconstant holomorphic function on a neighborhood of $z = z_0$, where z_0 is a root of order k of $f(z)-a$, $a \in \mathbb{C}$.

For every sufficiently small neighborhood U of z_0 and $b \in \mathbb{C}$ sufficiently close to a, $f(z) - \underline{b}$ has k simple roots in U.

$b \neq a$

k roots of multiplicity 1

Pf: Take a neighborhood small enough so that $f(z)-a$ has no other zeros in the



closed disk and $f'(z) \neq 0$ in the closed disk except at z_0 (so the roots are simple).

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}\,dz$$ constant for b in a connected component of $f \circ \gamma$ (by integration by substitution and the winding number). Thus,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}\,dz = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\,dz$$

$w(\gamma, a) = k$ $= k$

$\Rightarrow$ By argument principle, $f(z)-b$ has k roots inside γ. All simple b/c $f'(z) \neq 0$ from the beginning.

$\square$

! Compare roots of holomorphic functions

Thm: (ROUCHÉ'S THEOREM) Let $f(z)$ and $g(z)$ be holomorphic functions in open Ω and K a compact set w/ oriented boundary Γ in Ω. If

$$|f(z) - g(z)| < |f(z)| \quad \text{on } \Gamma$$

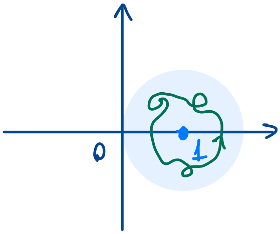
then $f(z)$ and $g(z)$ have the same number of zeros in K counted w/ multiplicity.

Remark: If $|f(z) - g(z)| < |f(z)|$, then f and g don't have zeros on Γ (would violate the inequality).

Pf: Divide the inequality by $|f(z)|$

$$\left| 1 - \frac{g(z)}{f(z)} \right| < 1 \quad \text{on } \Gamma$$

$\Rightarrow$ Values of $F(z) = \dfrac{g(z)}{f(z)}$ on Γ lie in the open disk of radius 1 centred at 1:



Thm

meromorphic

\# of zeroes of F

$$\underbrace{\frac{1}{2\pi i}\int_\Gamma \frac{F'(z)}{F(z)-0}\,dz}_{\text{Sum of winding numbers of } F\circ\gamma \text{ at zero } = 0} = (\#\text{ of zeroes of } g(z)) - (\#\text{ of zeroes of } f(z))$$

w/ multiplicity

\# poles of F

* **EVALUATION OF DEFINITE INTEGRALS** by Residue Calculus

$$\int : A \longrightarrow B$$

elementary functions

as sums of residues of suitable holom. functions

LECTURE 20:

* EVALUATING DEFINITE INTEGRALS w/ RESIDUE CALC.

1) $$\int_0^{2\pi} \underbrace{R(\sin\theta, \cos\theta)}\, d\theta$$

Rational function of sin and cos w/ no poles in the unit circle

- set $z := e^{i\theta}$

$$\Rightarrow \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right) \; ; \; \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

$$dz = i e^{i\theta}\, d\theta \Rightarrow d\theta = -i\frac{dz}{z}.$$

Then

$$\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

$$= -i \int_{|z|=1} \underbrace{R\left(\frac{1}{2}(z+1/z), \frac{1}{2}(z-1/z)\right)}\, dz$$

$$= 2\pi i \cdot (-i) \sum_{|z|<1} \operatorname{res}(\,\cdot\,)$$

Ex: $\displaystyle\int_0^{\pi} \frac{d\theta}{a+\cos\theta}, \quad a > 1.$

cos even

$$\overset{\cos\text{ even}}{=} \frac{1}{2}\int_0^{2\pi} \frac{d\theta}{a+\cos\theta} \qquad \rightsquigarrow \frac{d\theta}{a+\cos\theta} \overset{z:=e^{i\theta}}{=} -i\,\frac{dz}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$= -i \int_0^{2\pi} \frac{dz}{z^2+2az+1}$$

Want to compute residue inside the unit circle $\Rightarrow$ look at roots of $z^2+2az+1$:

$$:= z^2+2az+1 = (z-\alpha_1)(z-\alpha_2)$$

$$\Rightarrow \alpha_{\pm} = -a \pm \sqrt{a^2-1}$$

Since $a>1$, only α_+ will be inside the unit circle $\Rightarrow$ pole in $|z|<1$: $-a+\sqrt{a^2-1}$.

$$\operatorname{res}\left(f, -a+\sqrt{a^2-1}\right) = \frac{1}{2\alpha_+ + 2a} = \frac{1}{-2a+2\sqrt{a^2-1}+2a}$$

see page 122

$$= \frac{1}{2\sqrt{a^2-1}}.$$

Residue Thm

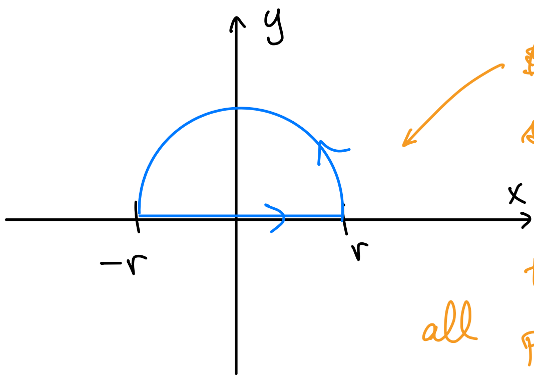
$$\overset{\downarrow}{=} 2\pi i(-i)\left(\frac{1}{2\sqrt{a^2-1}}\right) = \boxed{\frac{\pi}{\sqrt{a^2-1}}}.$$

2) $\boxed{\displaystyle\int_{-\infty}^{\infty} \underbrace{R(x)}\, dx}$ → Rational function w/ no poles in the real line

→ Converges when $\int_{-\infty}^{0}$ and $\int_{0}^{\infty}$ both converge separately. This converges iff R vanishes with order ≥ 2 at $\pm\infty$; i.e.,

$$\deg(\text{denominator}) \geq \deg(\text{numerator}) + 2.$$

Equivalently, $\lim\limits_{x\to\pm\infty} xR(x) = 0$.



Integrate here and then send $r \to \infty$.

r big enough so that the curve in blue encircles all poles in the upper-half-plane $y > 0$.

So, by the Residue Theorem,

$$\int_{-r}^{r} R(x)\,dx + \underbrace{\int_{\text{half-circle}} R(z)\,dz}_{\le M(r)\cdot \pi r \xrightarrow{r\to\infty} 0} = 2\pi i \sum_{y>0} \operatorname{res}(R(z))$$

$$\le M(r)\cdot \underbrace{\pi r}_{\text{length of the half-circle}} \xrightarrow{r\to\infty} 0, \quad M(r) = \sup_{|z|=r} |R(z)|$$

b/c of convergence conditions

Could use lower half-plane, but then $\int_{-\infty}^{\infty} R(x)\,dx = -2\pi i \sum_{y<0} \operatorname{res} R(z)$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i \sum_{y>0} \operatorname{res} R(z)}$$

Ex: $\displaystyle\int_0^\infty \frac{dx}{1+x^6}$. Note that $\dfrac{1}{1+z^6}$ has 6 poles all in the unit circle.

In the upper half-plane $y>0$, the poles are at

$$e^{i\pi/6},\ e^{i\pi/2},\ e^{5i\pi/6}.$$

At each pole α, the residue (as in pg. 122) is

$$\frac{1}{6\alpha^5} = -\frac{\alpha}{6}$$

root of unity

$$\Rightarrow \int_0^\infty \frac{1}{1+x^6}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{dx}{1+x^6}$$

Residue Thm
beginning of previous page

$$= \frac{1}{2}\,2\pi i \cdot \frac{-1}{6}\left(e^{i\pi/6} + e^{i\pi/2} + e^{5i\pi/6}\right)$$

$$= -\frac{\pi i}{6}\left(e^{i\pi/6} + e^{i\pi/2} + e^{5i\pi/6}\right)$$

$$= \frac{\pi}{6}\left(2\sin\frac{\pi}{6} + 1\right)$$

$$= \boxed{\frac{\pi}{3}}.$$

3) $\boxed{\int_{-\infty}^\infty R(x)\,e^{ix}\,dx}$ → Real and imaginary parts give

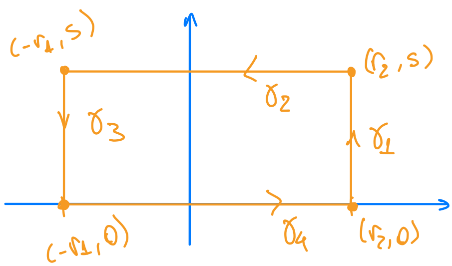
$$\int_{-\infty}^\infty R(x)\cos x\,dx \ ; \ \int_{-\infty}^\infty R(x)\sin x\,dx.$$

* If $R(z)$ has zero of order ≥ 2 at ∞, then

$$\int_{-\infty}^\infty R(x)\,e^{ix}\,dx = 2\pi i \sum_{y>0} \operatorname{res}\left(R(z)\,e^{iz}\right).$$

because, $|e^{iz}| = e^{-y}$ $\rightsquigarrow$ bounded in the upper half-plane $y>0$.

• In fact, it's enough here that $R(z)$ has a zero of order ≥ 1 at ∞ (i.e., $|z\,R(z)|$ bounded). For this, consider the contributions of each side of the rectangle: $|R(z)| \leq \frac{(\text{constant})}{|z|}$.



$$\left|\int_{\gamma_1}\right| \leq (\text{constant})\cdot\int_0^s e^{-y}\,\frac{ds}{|z|}$$

$$\leq \frac{(\text{constant})}{r_2}\int_0^\infty e^{-y}\,dy \leq 1;$$

$$\left|\int_{\gamma_3}\right| \leq \frac{(\text{constant})}{r_1}\;; \qquad \left|\int_{\gamma_2}\right| \leq (\text{constant})\,\frac{e^{-s}}{s}\,(r_1+r_2)$$

$$\xrightarrow{s\nearrow\infty} 0 \quad \text{for fixed } r_1, r_2.$$

So,

$$\left|\int_{-r_1}^{r_2} R(x)\,e^{-ix}\,dx - 2\pi i\sum_{y>0}\operatorname{res}\left(R(z)\,e^{iz}\right)\right|$$

$$\leq (\text{constant}) \left(\frac{1}{r_1} + \frac{1}{r_2} \right)$$

so now let $r_1, r_2 \to \infty$ separately.

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} R(x)\, e^{ix}\, dx = 2\pi i \sum_{y>0} \operatorname{res}\left(R(z)\, e^{iz}\right)}$$

* If we had $\int_{-\infty}^{\infty} R(x)\, e^{-ix}\, dx$, we can re-do this process but substituting "upper half-plane" for "lower half-plane" since e^{-ix} is bounded for $y<0$.

* Likewise, for integrals involving e^{imx}, $\cos(mx)$, $\sin(nx)$, $\cos^m x$, $\sin^n x$. (Express these as linear combinations of what we did above).

Ex:
$$\int_0^{\infty} \frac{\cos mx}{x^2+1}\, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\cos mx}{x^2+1}\, dx$$

$$= \frac{1}{2} \operatorname{Re}\Big(\underbrace{\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1}\, dx}_{*} \Big).$$

$*$: $\displaystyle\int_{-\infty}^{\infty} \frac{e^{imx}}{x^2+1}\,dx = \frac{2\pi i}{m} \sum_{y>0} \operatorname{res}\left(\frac{e^{iz}}{1+z^2/m^2}\right)$

$= \dfrac{m^2 e^{iz}}{z^2+m^2}$. Only pole in $y>0$ is $z = im$

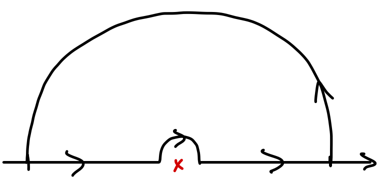
$\Rightarrow \operatorname{res} = \dfrac{m^2 e^{-m}}{2im}$ (p 122)

$$= \pi e^{-m}.$$

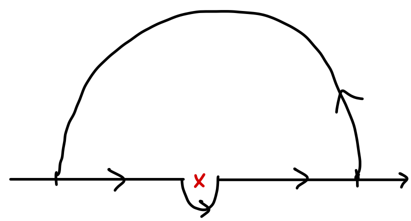
$$\Rightarrow \int_0^{\infty} \frac{\cos mx}{x^2+1}\,dx = \boxed{\frac{\pi e^{-m}}{2}}.$$

$*$ What if $P(z)$ has poles in the real axis?

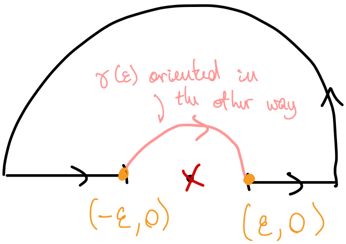
If it's a simple pole: choose contour to by-pass or enclose:



or



If zero is a simple pole of $f(z)$:



$$\boxed{\lim_{\varepsilon \to 0} \int_{\underline{\gamma(\varepsilon)}} f(z)\, dz = \pi i \operatorname{res}(f, 0)}$$

positively oriented semi-circle

semi-circle $\Rightarrow \frac{1}{2} \cdot$ contribution.

Why? $\quad f(z) = \frac{a}{z} + \underline{g(z)}$ ← Holomorphic

$$\Rightarrow \int_{\gamma(\varepsilon)} f(z)\, dz = \underbrace{\int_{\gamma(\varepsilon)} \frac{a}{z}\, dz}_{\pi i a} + \underbrace{\int_{\gamma(\varepsilon)} g(z)\, dz}_{\to 0 \text{ as } \varepsilon \to 0.}$$

Ex:

$$\boxed{\int_0^{\infty} \frac{\sin x}{x}\, dx} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx$$

$$= \frac{1}{2} \operatorname{Im} \left[\lim_{\varepsilon \to 0} \left(\int_{-\infty}^{-\varepsilon} \frac{e^{ix}}{x}\, dx + \int_{\varepsilon}^{\infty} \frac{e^{ix}}{x}\, dx \right) \right]$$

→ No residues in the upper half-plane. Only contribution is due to the simple pole at 0.

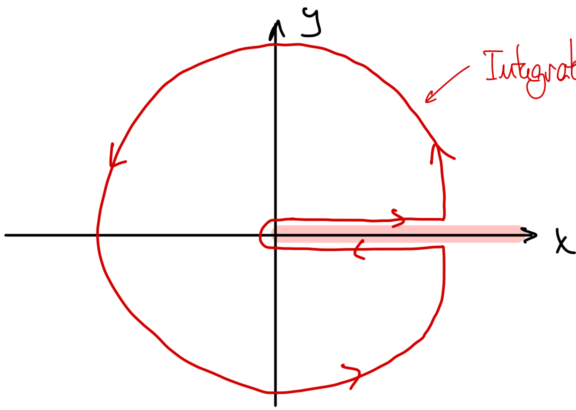
$$= \frac{1}{2}\, \mathrm{Im}\Big[\underbrace{\lim_{\varepsilon\to 0} \int_{\gamma(\varepsilon)} \frac{e^{iz}}{z}\, dz}_{\pi i \, \underbrace{\mathrm{res}\left(\frac{e^{iz}}{z},\, 0\right)}_{=1 \ (\text{pg } 122)} = \pi i} \Big]$$

$$= \frac{1}{2}\, \mathrm{Im}(\pi i) = \boxed{\frac{\pi}{2}}.$$

multivalued .....

4) $\displaystyle\int_0^\infty \frac{R(x)}{x^\alpha}\, dx$

5) $\displaystyle\int_0^\infty R(x) \log x \, dx$



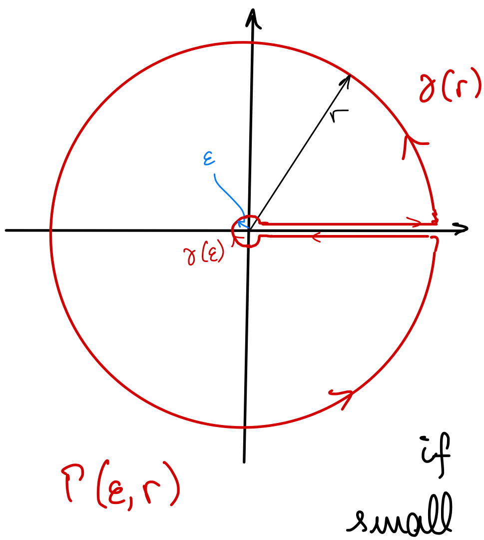
Integrate over this contour

LECTURE 21:

4) $\displaystyle\int_0^\infty \frac{R(x)}{x^\alpha}\,dx$ → $R(x)$ rational, no poles on $[0,\infty)$ and $0<\alpha<1$.

→ Improper integral converges at zero. It converges at ∞ iff $R(x)$ has a zero of order ≥ 1 at ∞; i.e., $x\,R(x)$ is bounded.

Consider the following contour:



Take $\arg z$ in $[0, 2\pi)$.

Then

$$\int_{\Gamma(\varepsilon,r)} \frac{R(z)}{z^\alpha}\,dz = 2\pi i \sum_{\mathbb{C}\setminus[0,\infty]} \operatorname{res}\left(\frac{R(z)}{z^\alpha}\right)$$

if r is large enough and ε small enough.

• if $\arg z = 2\pi$, then $z^\alpha = e^{2\pi i\alpha}|z|^\alpha$.

$$\text{LHS} = \underbrace{\int_{\gamma(r)} \frac{R(z)}{z^\alpha}\,dz}_{\to 0 \text{ as } r\nearrow\infty} + \underbrace{\int_{\gamma(\varepsilon)} \frac{R(z)}{z^\alpha}\,dz}_{\to 0 \text{ as } \varepsilon\to 0} + \int_\varepsilon^r \frac{R(x)}{x^\alpha}\,dx$$

b/c $\frac{zR(z)}{z^\alpha} \to 0$ as $z \to 0$ or $z \to \infty$

$$- e^{-2\pi i \alpha} \int_\varepsilon^r \frac{R(x)}{x^\alpha}\, dx$$

$$= \left(1 - e^{-2\pi i \alpha}\right) \int_0^\infty \frac{R(x)}{x^\alpha}\, dx$$

$$= 2\pi i \sum_{\mathbb{C}\setminus[0,\infty]} \operatorname{res}\left(\frac{R(z)}{z^\alpha}\right)$$

Ex: $\displaystyle\int_0^\infty \frac{dx}{x^\alpha(1+x)}$, $0 < \alpha < 1$

We have

Only this b/c we exclude $[0,\infty]$.

$$\left(1 - e^{-2\pi i \alpha}\right) \int_0^\infty \frac{dx}{x^\alpha(1+x)} = 2\pi i \operatorname{res}\left(\frac{1}{z^\alpha(1+z)}, -1\right)$$

$$= 2\pi i \frac{1}{e^{i\pi\alpha}}.$$

Thus,

$$\int_0^\infty \frac{dx}{x^\alpha(1+x)} = \frac{2\pi i}{e^{i\pi\alpha}\left(1 - e^{-2\pi i \alpha}\right)} = \boxed{\frac{\pi}{\sin(\pi\alpha)}}$$

5)

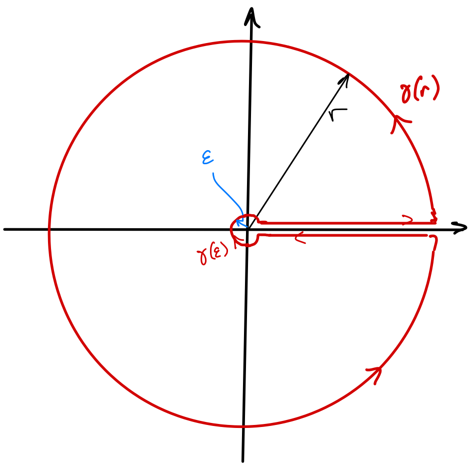
$$\boxed{\int_0^\infty R(x)\log x \, dx}$$

No poles on $[0,\infty)$

→ Converges at zero (log well behaved there).

At ∞, need $\lim_{x\to\infty} xR(x) = 0$ for the integral to converge.

Consider the same contour. As before, choose $\arg z$ on $[0, 2\pi)$, but now integrate $R(z)(\log z)^2$.



When $\arg z = 2\pi i$, we'll have

$$\log z = \log|z| + 2\pi i.$$

- As before, $\int_{\gamma(r)}, \int_{\gamma(\varepsilon)} \to 0$ as $\begin{matrix} r \nearrow \infty \\ \varepsilon \searrow 0 \end{matrix}$.

So,

$$\int_0^\infty R(x)(\log x)^2 \, dx - \int_0^\infty R(x)(\log x + 2\pi i)^2 \, dx$$

$$= 2\pi i \sum_{\mathbb{C}\setminus[0,\infty)} \operatorname{res}\left(R(z)(\log z)^2\right)$$

$\Leftrightarrow$

$$-2\int_0^\infty R(x)\log x\,dx - 2\pi i\int_0^\infty R(x)\,dx = \sum_{\mathbb{C}\setminus[0,\infty)} \operatorname{res}\left(R(z)(\log z)^2\right)$$

• If $R(x)$ is real, then we can equate real and imaginary parts.

Ex:

$$\int_0^\infty \frac{\log x}{(1+x)^3}\,dx$$

1 pole on $\mathbb{C}\setminus[0,\infty)$

Residue of $\dfrac{(\log z)^2}{(1+z)^3}$ at $z=-1$. Make change of variables to focus on 0 instead of -1: $\xi := z+1$. Then

$$\log(-1+\xi) = \log(-1) + \log(1-\xi)$$
$$= \pi i + \log(1-\xi).$$

So,

$$(\log z)^2 = \left[\pi i + \left(-\xi - \frac{\xi^2}{2} + \dots\right)\right]^2$$

$$= (1-\pi i)\xi^2 + \cdots$$

$$\Rightarrow \operatorname{res}\left(\frac{(\log z)^2}{(1+z)^3}, -1\right) = 1 - \pi i$$

$$\Rightarrow \int_0^\infty \frac{\log x}{(1+x)^3}\,dx = -\frac{1}{2}\operatorname{Re}(1-\pi i) = -\frac{1}{2}$$

LECTURE 22: HARMONIC FUNCTIONS

* HARMONIC FUNCTIONS: Dirichlet and Mean Value Property.

Def. $f(x,y)$ of two real variables in an open subset of $\mathbb{R}^2$ (or $\mathbb{C}$: $z = x+iy$) ($\mathbb{R}$- or $\mathbb{C}$-valued) is <u>harmonic</u> if $f \in C^2$ and $\Delta f = 0$, $\Delta = \partial_x^2 + \partial_y^2$.

Recall: $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$; $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

duals of

$$dz = dx + i\,dy \quad ; \quad d\bar{z} = dx - i\,dy$$

So,

$$\boxed{\Delta = 4 \frac{\partial^2}{\partial z \partial \bar{z}}}.$$

Consequence:
- holomorphic functions are harmonic
- real and imaginary parts of harmonic functions are harmonic
- real and imaginary parts of holomorphic functions are harmonic

* A function has Mean Value Property (MVP) iff real and imaginary parts have MVP.

Claim: A real, harmonic $g(x,y)$ is locally the real part of a holomorphic function $f(z)$, which is uniquely determined up to a constant.

Pf: $\dfrac{\partial^2 g}{\partial z \partial \bar z} = 0 \Rightarrow g$ holomorphic, thus $\dfrac{\partial g}{\partial z} dz$ locally has a holomorphic primitive $f(z)$ (i.e., $df = \dfrac{\partial g}{\partial z} dz$

$$df = \frac{\partial f}{\partial z} dz + \frac{\partial f}{\partial \bar z} d\bar z = \frac{\partial g}{\partial z} dz$$

$\iff \dfrac{\partial f}{\partial \bar z} = 0 \Rightarrow f$ holomorphic).

$$d\bar f = \underbrace{\frac{\partial g}{\partial \bar z} d\bar z}_{\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) g \, (dx - i dy)}$$

g real valued $\Rightarrow \bar g = g$.

Upshot: $d(f + \bar f) = dg$

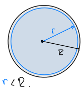
$$\Rightarrow \boxed{g = 2\,\mathrm{Re}(f) + \text{constant.}}$$

$\square$

Therefore, harmonic functions satisfy MVP, and, thus also satisfy the maximum module principle.

Now, a real-valued harmonic function $g(x,y)$ = real part of holomorphic $f(z)$.

! But $f(z)$ holom. $\Rightarrow$ $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in



some disk $|z| < R$. Assume $a_0 \in \mathbb{R}$. So,

$$g(r\cos\theta, r\sin\theta) = a_0 + \frac{1}{2}\sum_{n=1}^{\infty} r^n \left(a_n e^{in\theta} + \overline{a_n}\, e^{-in\theta}\right)$$

where

Integrate both sides

← Abs. and unif. convergent in θ.

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, d\theta$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\,\frac{1}{r^n e^{in\theta}}\, d\theta.$$

Thus,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\cdot \underbrace{\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{z}{re^{i\theta}}\right)^n\right]}_{\text{Rational function of the form } \frac{re^{i\theta}+z}{re^{i\theta}-z}} d\theta$$

for $|z| < r$.

Thus,

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, \frac{re^{i\theta}+z}{re^{i\theta}-z}\, d\theta$$

expresses a holomorphic function $f(z)$ in terms of the real part on the boundary.

* Equate real parts to get $g(x,y)$:

!

$$g(x,y) = \frac{1}{2\pi}\int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, \underbrace{\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}}_{\text{POISSON KERNEL}}\, d\theta$$

* DIRICHLET PROBLEM ON A DISK:

Assumed to be real

Given a continuous function $f(\theta)$ on the circle centred at 0 and radius r (i.e., given continuous 2π-periodic function $f(\theta)$). Then, we can find a function $F(z)$, continuous in $|z| \le r$ and harmonic $|z|<r$ such that $F(re^{i\theta}) = f(\theta)$.

and the solution is unique

Pf: (Uniqueness) By Max. Mod. Principle

(Existence) Define

$f(\theta) \rightsquigarrow g(r\cos\theta, r\sin\theta)$ from before.

$$F(z) := \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}\, d\theta.$$

$|z|<r$

NTS: harmonic on the disk $|z|<r$ and has the right boundary values.

$F(z)$ harmonic on $|z|<r$ b/c it's the real part of a holomorphic function:

$$F(z) = \mathrm{Re}\left[\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, \underbrace{\frac{re^{i\theta}+z}{re^{i\theta}-z}}_{\text{Rational function} \Rightarrow \text{holomorphic}}\, d\theta\right]$$

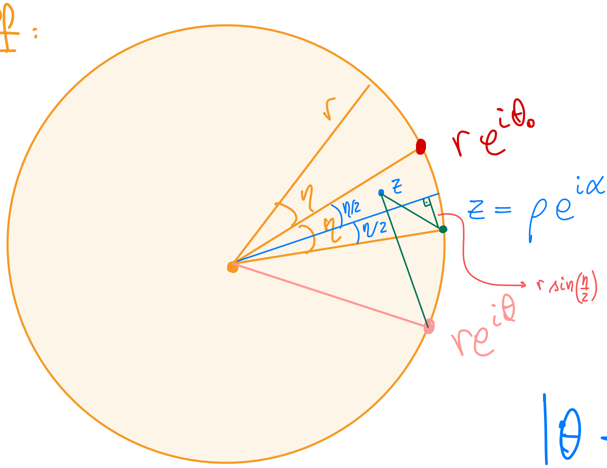
Note: $\frac{1}{2\pi}\int_0^{2\pi} \underbrace{\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}} d\theta = 1$ (Take $g=1$)

"Density" of mass distribution on circle w/ total mass $= 1$.

WTS: $\lim_{z\to re^{i\theta_0}} F(z) = f(\theta_0)$.

Lemma: $\forall \eta>0$, $\frac{1}{2\pi}\int_{|\theta-\theta_0|>\eta} \frac{r^2-|z|^2}{|re^{i\theta}-z|^2} d\theta \xrightarrow{z\nearrow re^{i\theta_0}} 0$

Pf:



$|\alpha-\theta_0|<\frac{\eta}{2}$ Then

$|\alpha-\theta_0| \geq \frac{\eta}{2}$ $\forall\theta$ s.t. $|\theta-\theta_0|>\eta$.

$\Rightarrow |re^{i\theta}-z| > r\sin\frac{\eta}{2}$. So, integral is bdd by

$\frac{1}{r^2\sin^2\frac{\eta}{2}}(r^2-\rho^2) \longrightarrow 0$ as $z\longrightarrow re^{i\theta_0}$.

i.e., $\rho\longrightarrow r$.

So

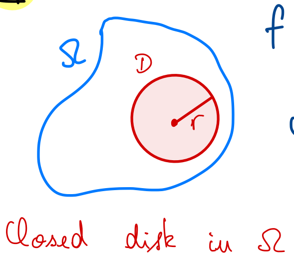
$$\left(F(z) - f(\theta_0)\right) = \frac{1}{2\pi}\int_0^{2\pi}\left(f(\theta) - f(\theta_0)\right)\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

$$= \frac{1}{2\pi}\int_{|\theta-\theta_0|\le\eta} + \frac{1}{2\pi}\int_{|\theta-\theta_0|>\eta}$$

given $\varepsilon > 0$, TO BE CONTINUED...

* **CoROLLARY**: Continuous function in open $\Omega \subset \mathbb{R}^2$ with MVP is harmonic.

Pf:



$f|_{\partial D}$ continuous so there exists F continuous on D (by Thm) which is harmonic on Int D and s.t. $F|_{\partial D} = f|_{\partial D}$.

But, in D, $F-f$ satisfies the Max. Mod. Principle on ∂D, but $F-f = 0$ on ∂D thus 0 on Int D.

$\square$

LECTURE 23:

* **DIRICHLET PROBLEM ON THE DISK:** Continue the proof ...

Given a continuous function $f(\theta)$, 2π-periodic, find a function $F(z)$ continuous on the closed disk $\{|z| \leq r\}$ and harmonic on the open disk $\{|z| < r\}$ s.t. $F(re^{i\theta}) = f(\theta)$.

Solution: For $|z| < r$, define

$$F(z) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) \frac{r^2 - |z|^2}{|re^{i\theta} - z|^2} \, d\theta .$$

Note that $F(z)$ is harmonic on $\{|z| < r\}$ because it's the real part of a holomorphic function.

Just left to show that $\lim_{z \to re^{i\theta}} F(z) = f(\theta)$,

$\varepsilon < r$.

Lemma: If $\eta > 0$, then

$$\frac{1}{2\pi}\int_{|\theta-\theta_0|>\eta} \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta \longrightarrow 0 \quad \text{as } z \nearrow re^{i\theta_0}$$

Pf: From where we left...

$\int$ Poisson kernel $= 1$.

$$F(z) - f(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi} [f(\theta) - f(\theta_0)]\,\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

$$= \frac{1}{2\pi}\int_{|\theta-\theta_0|\le\eta} (1) \;+\; \frac{1}{2\pi}\int_{|\theta-\theta_0|\geq\eta} (2)$$

Let $\varepsilon > 0$

$$(1) \le \sup_{|\theta-\theta_0|\le\eta} |f(\theta)-f(\theta_0)| < \frac{\varepsilon}{2}$$

since f is continuous, we can choose η s.t. this is $< \varepsilon/2$.

With this choice of η,

$$(2) \le \sup |f(\theta)-f(\theta_0)| \cdot \frac{1}{2\pi}\int_{|\theta-\theta_0|\ge\eta} \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

$$< \frac{\varepsilon}{2} \quad \text{for } z \text{ suff. near } re^{i\theta_0}.$$

Lemma

$\square$

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* **RUNGE'S APPROXIMATION THEOREM**

Can a holomorphic function be approximated uniformly by polynomial on a given compact set?

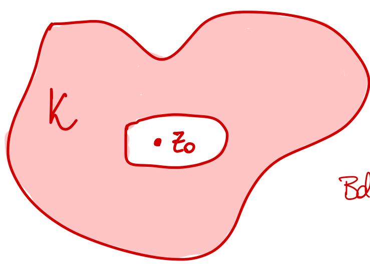
Ex 1: Holomorphic function in open disks has a convergent power series expansion so it is equal to the uniform limit of partial sums on any compact subset.

Ex 2: $f(z) = 1/z$ cannot be uniformly approximated by polynomial on S^1:

$$\int_{S^1} \frac{dz}{z} = 2\pi i$$ but by Cauchy's thm

$$\int_{S^1} (\text{any poly.})\, dz = 0$$ (closed curve and holomorphic function)

$\Rightarrow$ It's necessary for $\mathbb{C} \setminus K$ be connected. Otherwise, $\mathbb{C} \setminus K$ has a bounded component:



Bdd component of $\mathbb{C} \setminus K$.

Suppose $\dfrac{1}{z - z_0}$ can be approximated by polynomials on K. Then, choose a polynomial $P(z)$ s.t., on K,

$$\left| \frac{1}{z-z_0} - P(z) \right| < \frac{1}{C}$$

Bound for $|z-z_0|$ on K:
$|z-z_0| < C$ (reasonable b/c K is compact).

This means

$$|(z-z_0)P(z) - 1| < 1 \quad \text{on } K.$$

Then

$$|(z-z_0)P(z) - 1| < 1$$

Rouché ???

in the component of $\mathbb{C} \setminus K$ that contains z_0 by the Max. Modulus Principle.

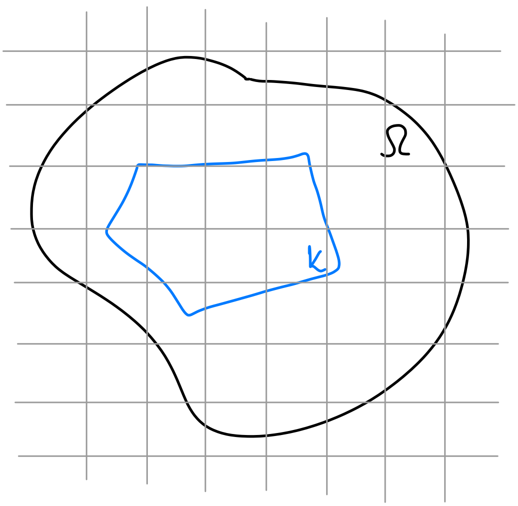
$\longleftrightarrow$ // $\longleftrightarrow$
Not true when $z = z_0$.

Thm: (RUNGE'S THEOREM) Let $\underset{\text{compact}}{K} \subset \underset{\text{open}}{\Omega} \subset \mathbb{C}$ and $f(z)$ be holomorphic on Ω. Then

1) f can be approximated on K uniformly by rational functions with poles in $\mathbb{C} \setminus K$.

2) if $\mathbb{C} \setminus K$ is connected, then f can be approximated uniformly on K by polynomials

Pf:



grid of squares of side length

$$d < \frac{1}{\sqrt{2}} d(K, \mathbb{C} \setminus \Omega)$$

so that any square that intersect K will lie inside Ω

Let $Q := \{Q_1, \_\_, Q_m\}$ squares that intersect K and let $\gamma_1, \_\_, \gamma_n$ be the boundary segments of Q_j which don't belong to 2 adjacent squares in Q, each $\gamma_i \subset \Omega$ doesn't intersect K.

w/ positively oriented bds.

If $z \in K$, then

$$f(z) = \sum_{l=1} \frac{1}{2\pi i} \int_{\gamma_l} \frac{f(\xi)}{\xi - z} \, d\xi$$

Cauchy's Integral Formula

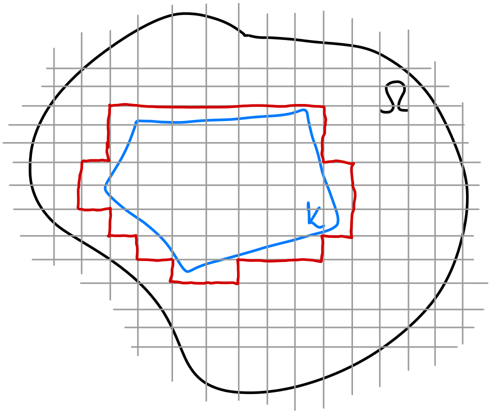
LECTURE 24: RUNGE'S APPROX. THM.

Recall: If $K \subset \Omega \subset \mathbb{C}$, then
compact open

(1) A holomorphic function f on Ω can be approximated uniformly on K by rational functions with poles in $\Omega \setminus K$

(2) If $\mathbb{C} \setminus K$ is connected, then f can be approximated by polynomials.

Pf: Grid by squares of side length



$$d < \frac{1}{\sqrt{2}} d(K, \mathbb{C} \setminus \Omega).$$

so any squares that intersect K lie completely inside Ω. Define

$$Q := \{Q_1, \ldots, Q_M\}$$

to be the squares that intersect K w/ positively oriented boundaries $\underbrace{\{\gamma_1, \ldots, \gamma_N\}}$.

sides of the Q_j which don't lie in 2 adjacent squares.

Then, for each $\gamma_n \subset \Omega$, $\gamma_n \cap K = \emptyset$.

Then,

$$f(z) = \frac{1}{2\pi i} \sum_{n=1}^{N} \int_{\gamma_n} \frac{f(\xi)}{\xi - z} \, d\xi$$

for $z \in K$. By Cauchy's Thm:

Consider $z \in Q = Q_1 \cup \cdots \cup Q_M$ not on the boundary of any Q_j.

If $z \in Q_j$,

$$\frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\xi)}{\xi - z} \, d\xi = \begin{cases} f(z), & \text{if } m = j \\ 0, & \text{else} \end{cases}.$$

So,

$$f(z) = \frac{1}{2\pi i} \sum_{m=1}^{M} \int_{\partial Q_m} \frac{f(\xi)}{\xi - z} \, d\xi$$

$$= \frac{1}{2\pi i} \sum_{n=1}^{N} \int_{\gamma_n} \frac{f(\xi)}{\xi - z} \, d\xi$$

Because $\int$'s of sides shared by 2 adjacent squares will cancel. Then also true for $z \in Q$ by continuity.

Approximate each one of these individually and then add everything up.

(1) Enough to prove:

Lemma: take γ to be a line segment in $\Omega \setminus K$. Then $\int_\gamma \frac{f(\xi)}{\xi - z} \, d\xi$ can be approximated uniformly on K by rational functions with poles on γ.

Pf: For $\gamma : [0,1]_t \to \Omega$ (parametrize γ). Then

$$\int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi = \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z}\, \gamma'(t)\, dt$$

Integrand $F(z,t)$ is continuous on $K \times [0,1]$ so uniformly continuous.

continuous on a compact set

So, $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$\sup_{z \in K} |F(z,t_1) - F(z,t_2)| < \varepsilon$$

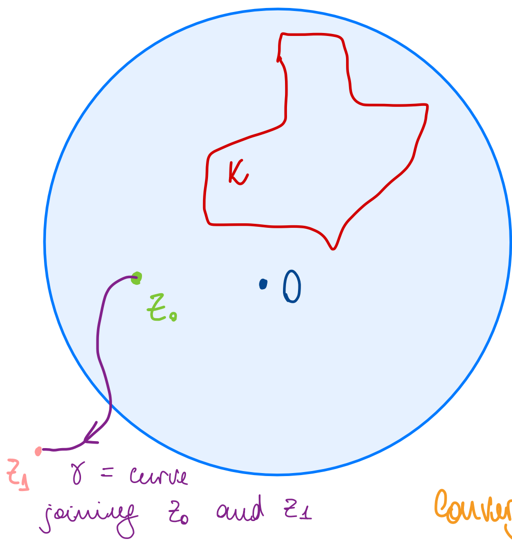
whenever $|t_1 - t_2| < \delta$. It follows that the Riemann sums of $\int_0^1 F(z,t)\,dt$ approximate $\int$ uniformly on K as mesh of parameters $\to 0$.

But the Riemann sums are rational functions w/ poles on γ.

(2) Enough to prove:

Lemma: If $\mathbb{C} \setminus K$ is connected and $z_0 \notin K$ then $\frac{1}{z-z_0}$ can be approximated uniformly on K by polynomials.

Pf: $D =$ "big disk containing K centred at 0".



• Case 1: z_0 outside D:

$$\frac{1}{z-z_0} = \frac{-1}{z_0} \frac{1}{1 - z/z_0}$$

$$= - \sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}}$$

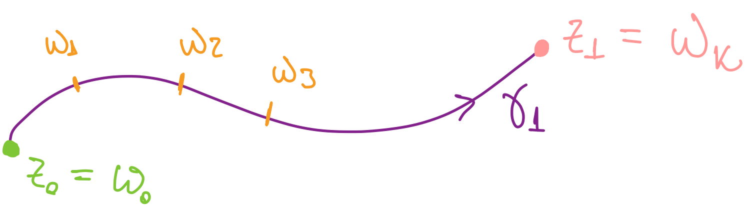
Power series expansion at the origin.

converges uniformly on K

So, the partial sums give a sequence of polynomials converging to $\frac{1}{z-z_0}$ uniformly on K.

. Case 2: $z_0 \in D$. Take z_1 outside of D. We will show that $\frac{1}{z-z_0}$ can be approximated uniformly in K by polynomials in $\frac{1}{z-z_1}$.

Take a partition of γ:



s.t. $|w_{i+1} - w_i| < \frac{1}{2} d(\gamma, K) =: \delta$, say.

Claim: If $w \in \gamma$ and $|w - w'| < \delta$, then $\frac{1}{z-w}$ can be approximated uniformly on K by polys in $\frac{1}{z-w'}$.

Pf:
$$\frac{1}{z-w} = \frac{1}{z-w'-(w-w')} = \frac{1}{z-w'} \cdot \frac{1}{1 - \frac{w-w'}{z-w'}}$$
$$= \sum_{n=0}^{\infty} \frac{(w-w')^n}{(z-w')^{n+1}}$$

sum converges in K

Approximation by partial sums gives the result. Apply this step by step from W_1 to W_2, etc. ...

$\square$