Lecture 1: Introduction

* Dictionary of Topology:

• Topological Space: set X together w/ a topology on X: collection S of subsets of X which are
open sets
closed under finite union, arbitrary intersections, and includes ϕ and X.

→ A set $Z \subset X$ is closed if $X \setminus Z$ is open

* If $f: X \to Y$ is a function between topological spaces, then f is continuous if $f^{-1}(V)$ is open for every $V \subset Y$.

(Continuous: preimage of open is open)

* A continuous mapping $f: X \to Y$ is called a homeomorphism if it has an inverse which is continuous.

* **Subspace of X**: subset Z of X with the induced (or inherited) topology; i.e., open set $Z \cap U$, $U \subset X$ open.

   $\Rightarrow$ Inclusion map: $Z \hookrightarrow X$ (continuous)

* **Quotient space**: $Y = X/\sim$ is the set of equivalence classes given by an equivalence relation $\sim$ on X.

   This induces a mapping $p: X \to Y$ that gives Y the topology that for every open $V \subset Y$, $p^{-1}(V)$ open in X. This way, p is continuous (take smallest topology s.t. p is continuous).

   Quotient mapping: surjective mapping s.t. $V \subset Y$ is open if and only if $p^{-1}(V)$ is open in X.

   $\Rightarrow$ Every quotient map is continuous.

Q: Is every continuous surjection a quotient map?

No! Take $X = (-\infty, 0) \sqcup \underbrace{[0, \infty)}$ disjoint union $\Longrightarrow$ open in X

$p \downarrow$ but not p^{-1}(open)

$Y = \mathbb{R}$

* **Basis for the topology of X:** collection $\mathcal{B}$ of open subsets of X s.t. every open set is a union of elements of $\mathcal{B}$.

Ex: Open balls or open rectangles are bases for the usual Euclidean space.

* **Neighborhood of a point $x \in X$:** is a subset of X which includes an open set containing x.

* **Neighborhood basis of $x \in X$:** is a collection of neighborhoods of x s.t. every neighborhood of x includes an element of the collection.

Ex: open balls centred at x for Euclidean space.

* Property of X holds locally on X if there is a neighborhood basis of x on which the property holds; i.e., every neighborhood of x contains a neighborhood on which the property holds.

* X is compact if every open covering of X has a finite subcovering.

* Subset of X is compact if it is compact on the subspace topology.

* The continuous image of compact sets is compact.

Therefore, a continuous function $f: X \to \mathbb{R}$ on a compact set X takes on a maximum and a minimum value.

Remark: (Heine - Borel) In $\mathbb{R}^n$, compact is equivalent to closed and bounded.

! Careful that boundedness might not make sense.

Make sense here

Metric Spaces: Topological space X with metric (notion of distance) d: (X, d), $d: X \times X \to \mathbb{R}$ s.t.

1) $d(x,y) \geq 0$, zero iff $x = y$;

2) $d(x,y) = d(y,x)$ $\forall x, y \in X$;

3) $d(x,y) \leq d(x,z) + d(z,y)$.

* Metric spaces have topologies w/ basis given by open balls $B(x,r) := \{ y \in X : d(x,y) < r \}$.

* Topological spaces are metrizable if its underlying set has a metric which defines the topology.

* X is disconnected if it is a disjoint union of 2 nonempty closed sets. Otherwise, it is connected.

A subset of X is connected if it is connected in the subspace topology.

* Connected components of X: maximal connected subsets of X.

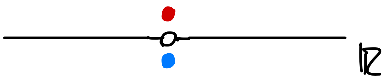
↳ They form a partition of X

↳ Every connected component is closed (since closure of connected set is connected)

Note: connected component need not be open; e.g., $\mathbb{Q} \subset \mathbb{R}$ with subspace topology. The connected components are single points (irrational number between two rational numbers)

* A topological space is a MANIFOLD if it locally looks like $\mathbb{R}^n$. More specifically, every point has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

EXAMPLE: $\mathbb{R}$ with a double origin



The open neighborhoods of • or • consist of $\bigcup \mathcal{U}\setminus\{0\}$ where $\mathcal{U}$ consist of the neighborhoods of $\mathbb{R}$.

NOT HAUSDORFF !!

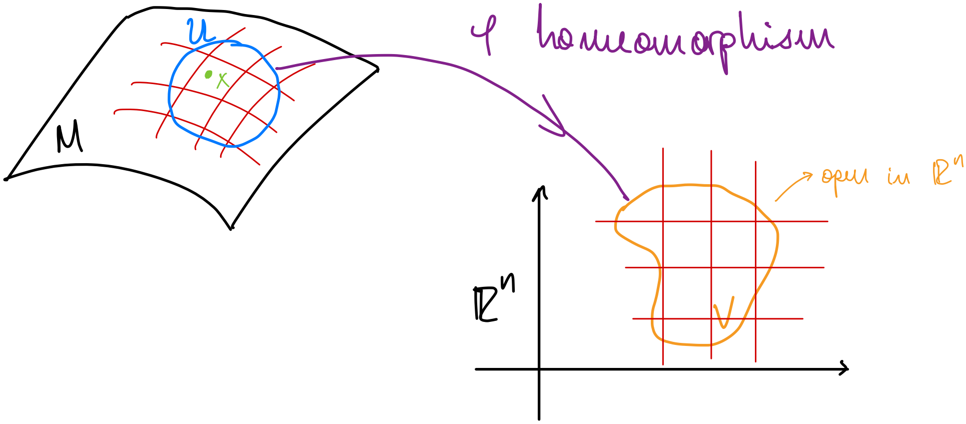
$\Rightarrow$ The double origin line does not look like $\mathbb{R}$ around the origin.

$\Rightarrow$ Homeomorphic but does not look like $\mathbb{R}$

Another way of visualizing this is taking $\mathbb{R} \sqcup \mathbb{R}/\sim$ where $x \sim y$ if $x = y$ as points of $\mathbb{R}$, $x \neq 0$.

Def: (HAUSDORFF SPACE) A topological space is Hausdorff if any 2 distinct points lie in disjoint open sets.

Def: (n-Manifold) A n-dimensional topological manifold M is a Hausdorff topological space s.t. every point $x \in M$ has an open neighborhood that is homeomorphic to an open subset of $\mathbb{R}^n$.



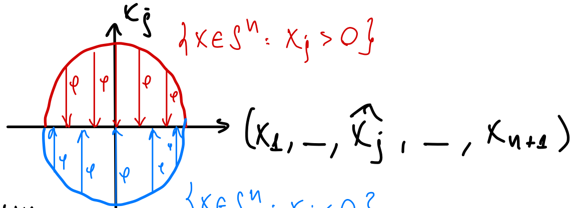
* Every manifold is locally compact and locally connected (locally looks like open balls of $\mathbb{R}^n$); i.e., in particular, every connected component is open.

EXAMPLES OF MANIFOLDS:

1) $S^n \subset \mathbb{R}^{n+1}$ unit sphere: $x_1^2 + \cdots + x_{n+1}^2 = 1$

* Stereographic projection (one for the northern hem. and one for the southern hem.) defines an atlas for S^n.

* Another way:



"φ, φ" are the homeomorphism

Upper hemisphere is the graph of

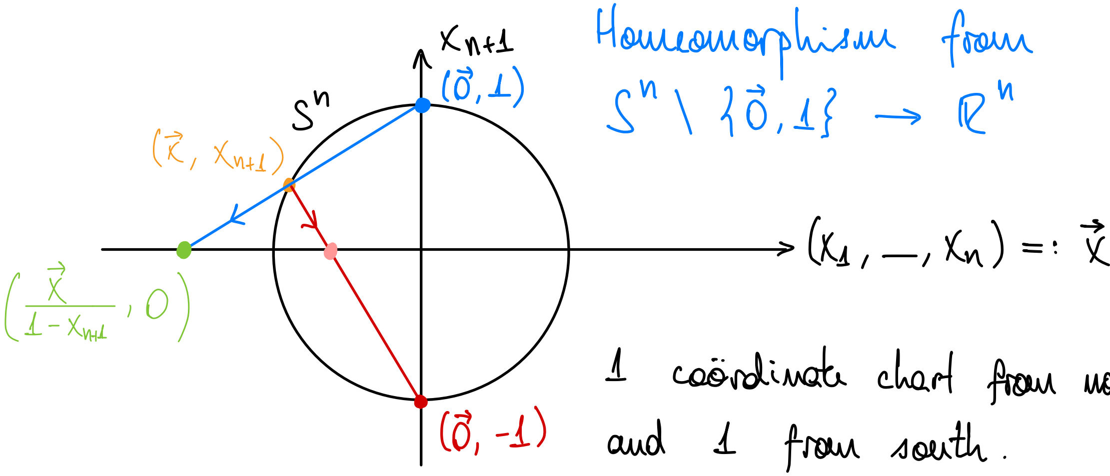
$$x_j = \sqrt{1 - \sum_{i \neq j} x_i^2}$$

Lower hemisphere is the graph of

$$x_j = -\sqrt{1 - \sum_{i \neq j} x_i^2}$$

Proceeding in this way, we cover S^n by $2(n+1)$ coördinate charts.

* Stereographic projection: cover S^n by only 2 coördinate charts.



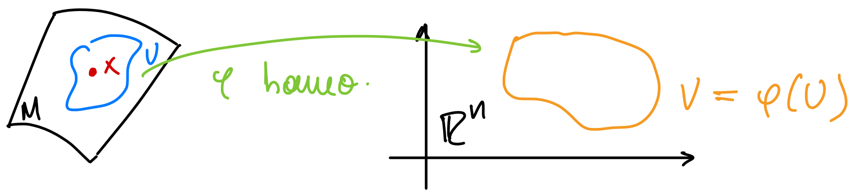
Homeomorphism from $S^n \setminus \{\vec{0}, 1\} \to \mathbb{R}^n$

1 coördinate chart from north and 1 from south.

Lecture 2: n-Manifolds

n-dimensional topological manifold M: second countable Hausdorff space such that every point $x \in M$ has an open neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.

Def: (Second Countable) Topological space is second countable if it has a countable basis.



* If M satisfies "Hausdorff s.t. every point has an open neigh. homeo to subset of $\mathbb{R}^n$" then the following are equivalent:

1) Every component of M is σ-compact (countable union of compact sets);

2) Every component is 2nd countable;

3) M is metrizable;

4) M is paracompact (every open cover has locally finite open refinement)

→ $p \in M$ lie in the intersection of finitely many sets of the refinement

5) There is a C^0 Partition of Unity subordinate to any open cover.

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EXAMPLES OF MANIFOLDS:

1) An open subset of a manifold is also a manifold.

2) If M_1 and M_2 are topological manifolds, then so is $M_1 \times M_2$.

2nd Countable & Hausdorff: immediate

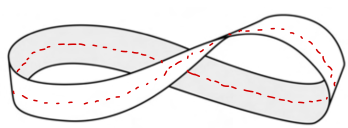
Homeo.: for $(x_1, x_2) \in M_1 \times M_2$, if U_1 and U_2 are coordinate charts at x_1 and x_2, respectively, then

$U_1 \times U_2$ is a coordinate chart for $M_1 \times M_2$ at (x_1, x_2)

Note: $\dim(M_1 \times M_2) = \dim M_1 + \dim M_2$.

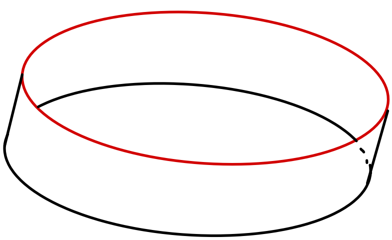
E.g.: Torus $S^1 \times S^1$ is a manifold of dimension 2 $\Rightarrow$ submanifold of $\mathbb{R}^4$. In fact, the torus is a submanifold of $\mathbb{R}^3$.

3) MÖBIUS STRIP



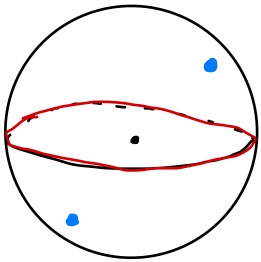
Cutting in the middle, we get a band with 2 sides (a "genuine" band)

get $S^1 \times (-1, 1)$ (with 2 twists)

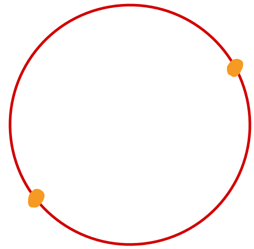


get back the Möbius strip by identifying antipodal points on one of the edges.

4) REAL PROJECTIVE PLANE: $S^2/\sim$ where $\sim$ is the equivalence relation: identify antipodal points.



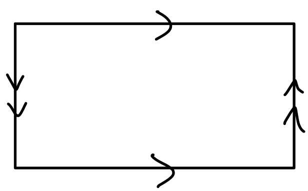
OR



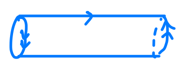
Identify antipodal points on the boundary of a closed disk.

$\Rightarrow$ Obtain $\mathbb{RP}^2$ by gluing the boundary of a closed disk to the boundary of a Möbius strip.

5) **KLEIN BOTTLE**: gluing together 2 Möbius strips along their boundary.

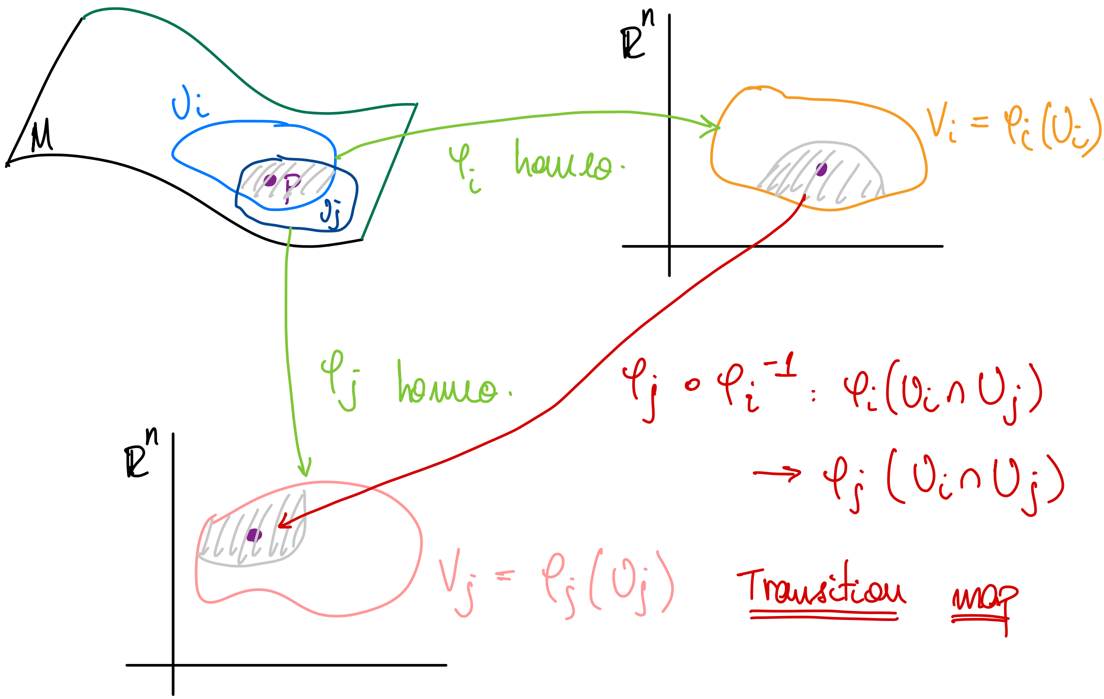


$\leftarrow$ In $\mathbb{R}^3$, cannot do this without making it self intersect:



LECTURE 3: DIFFERENTIABLE MANIFOLDS

Recall: A TOPOLOGICAL manifold M is a 2^{nd}-countable Hausdorff space such that $\forall p \in M$, there is an open neighborhood U of p that is homeomorphic to $\mathbb{R}^n$.



$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

Transition map

(φ_i, U_i), (φ_j, U_j) are called coördinate charts.

Def: (n-MANIFOLD w/ SMOOTH ATLAS) 2^{nd} countable Hausdorff space M together with a covering of M by n-dimensional coördinate charts which are C^∞-related.

SMOOTH ATLAS

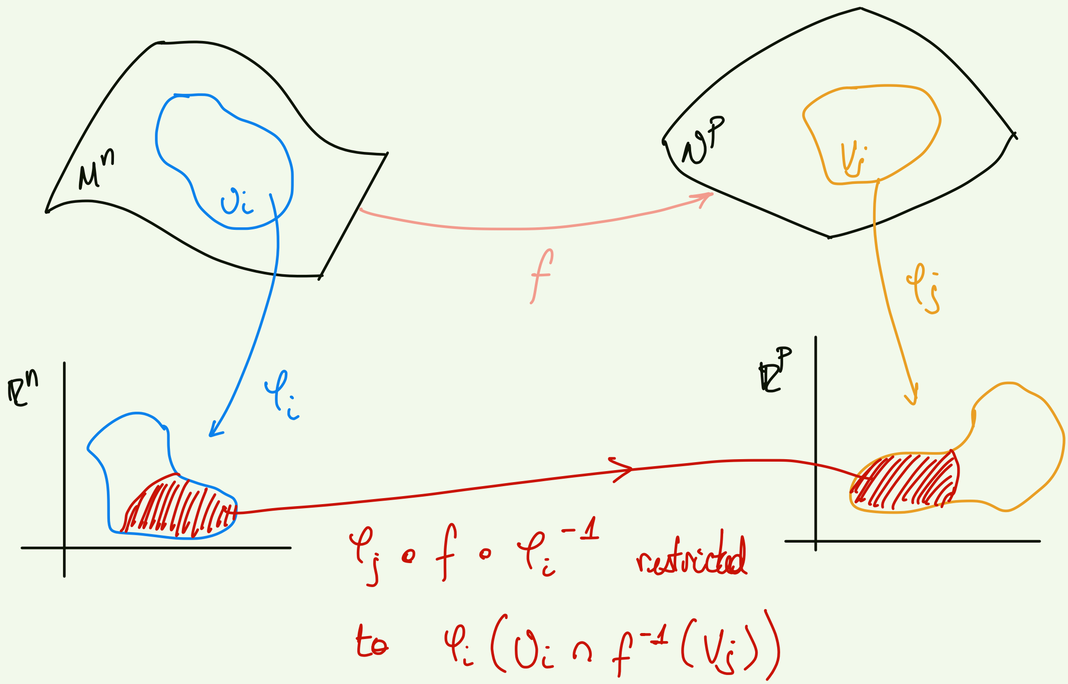
$\bigcup_i U_i = M$

n-dim manifold M w/ smooth atlas

Def: (SMOOTH FUNCTIONS ON M) A function $f: M \to \mathbb{R}$ is smooth if $f \circ \varphi^{-1}$ is C^∞ for every coördinate chart $(\varphi: U \to \mathbb{R}^n, U)$.

Def: (SMOOTH MAPS BETWEEN MANIFOLDS) A continuous function $f: M^n \to N^p$ between manifolds with smooth atlases is smooth if $\varphi_j \circ f \circ \varphi_i^{-1}$ is smooth for every i and j.

As seen below



Def: (DIFFEOMORPHISM) Smooth mapping with smooth inverse.

Lemma: Consider n-dimensional smooth manifolds w/ maximal atlases: $(M, \mathcal{A})$ and $(N, \mathcal{B})$. Suppose $f: M \to N$ is a continuous and bijective mapping. Then, the following are equivalent:

(1) f is a diffeomorphism;

(2) $\psi \circ f \in \mathcal{A}$ iff $\psi \in \mathcal{B}$;

(3) A function g on $(N, \mathcal{B})$ is smooth iff $g \circ f$ is smooth on $(M, \mathcal{A})$.

EXAMPLE: $M = \mathbb{R}$

$\mathcal{A}$: all open subsets $U \subset \mathbb{R}$ with $\varphi : U \to \mathbb{R}$ given by the identity / inclusion.

$\mathcal{B}$: $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$, $x \longmapsto x^3$ together w/ charts that are C^∞ related to φ.

Are $(\mathbb{R}, \mathcal{A})$ and $(\mathbb{R}, \mathcal{B})$ diffeomorphic?

Yes! $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$.

Are there differentiable manifolds which are homeo. but <u>not</u> diffeom.?

No, up to dim. 3! (Milnor, 60s) Exotic diff'ble structures on S^7 (28 diffeo. classes).

In dim 4, uncountably many open subsets of $\mathbb{R}^4$ pairwise <u>not</u> diffeo., but all homeo. to $\mathbb{R}^4$. (Donaldson, Freedman).

Now, say that 2 smooth atlases on a topological manifold M are equivalent if the identity map of M is a diffeomorphism.

Lemma: An n-dimensional smooth manifold is a manifold with an equivalence class of maximal atlases.

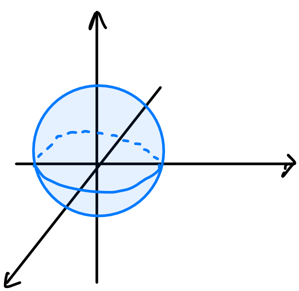
Point of this: 2 smooth atlases are equivalent iff they lie in some max. atlas.

Can a topological space have atlases of different dimensions? No! Open sets in $\mathbb{R}^n$, $\mathbb{R}^p$ cannot be diffeo. if $n \neq p$ (by Inverse Function Theorem).

C^0 case: No! By the <u>Invariance of domain</u>; i.e., $f : \underset{\mathbb{R}^n}{\underset{\cap}{U}} \to \mathbb{R}^n$ cont. injec. has open image.

EXAMPLES:

1) $S^n \subset \mathbb{R}^{n+1}$: $x_1^2 + \dots + x_{n+1}^2 = 1$



$$x_i = \pm \sqrt{1 - \sum_{j \neq i} x_j^2}$$

<u>C^∞ ATLAS</u>:

$$U_i^{\pm} := \{ x \in S^n : x_i \gtrless 0 \}$$

The corresponding coördinate charts are

$$\varphi_i^{\pm} : U_i^{\pm} \longrightarrow \mathbb{R}^n$$

$$x = (x_1, -, x_{n+1}) \longmapsto (x_1, \_\_, \hat{x}_i, \_\_, x_{n+1}) .$$

Note that $\varphi_i^{\pm} : U_i^{\pm} \xrightarrow{\text{homeo.}} \{\text{open unit disk}\}$. Define

$V_i^{\pm} = \varphi_i^{\pm}(U_i^{\pm})$. The transition map from

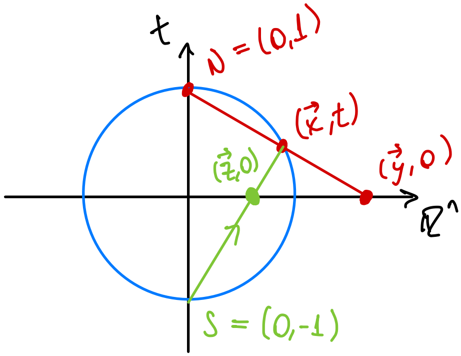
$\varphi_i^{\pm}$ to $\varphi_j^{\pm}$, $j \neq i$ looks like

$$(x_1, -, \hat{x}_i, -, x_n) \longmapsto (y_1, -, \hat{y}_j, -, y_n)$$

where $y_k = x_k$ for all but one k, namely, $k = i$, which is given by $\pm\sqrt{1 - \sum_{l \neq i} x_l^2}$.

C^∞ ATLAS GIVEN BY THE STEREOGRAPHIC PROJ.

$$S^n = \{ x_1^2 + \cdots + x_n^2 + t^2 = 1 \} .$$



$$y = \frac{\vec{x}}{1-t}$$

$$z = \frac{\vec{x}}{1+t}$$

Transition maps: $z = \dfrac{1-t}{1+t}\, y$ and they are C^∞ on the overlap $\mathbb{R}^n \setminus \{0\}$. Show the two atlases are equivalent (i.e., the identity is a diffeo.).

Lecture 4: More Examples of Manifolds

(1) An open subset of a smooth manifold M has the manifold structure of a manifold.

If $\bigcup_i (\varphi_i, U_i)$ is an atlas for M ($\varphi_i : U_i \to \mathbb{R}^n$)

then $\bigcup_i (\varphi_i|_{U_i \cap \mathcal{U}}, U_i \cap \mathcal{U})$ is an atlas for $\mathcal{U}$.

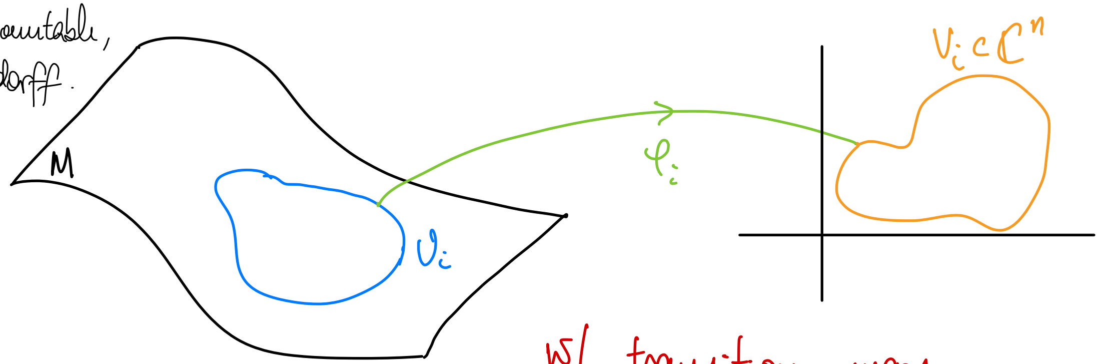
(2) If M_1, M_2 are manifolds with atlases $\mathcal{A}_1$ and $\mathcal{A}_2$, then $M_1 \sqcup M_2$ is a manifold with atlas $\mathcal{A}_1 \cup \mathcal{A}_2$.

(3) The product $M^n \times N^p$ of manifolds of dimensions n, p is a manifold of dim. $n+p$.

If $\varphi : \mathcal{U} \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^p$ are coördinate charts for M, N

then $\varphi \times \psi : \mathcal{U} \times V \longrightarrow \varphi(\mathcal{U}) \times \psi(V) \subset \mathbb{R}^n \times \mathbb{R}^p$ is a coördinate chart for $M \times N$.

(4) Can define COMPLEX MANIFOLDS (modeled in $\mathbb{C}^n$); i.e., manifolds w/ a complex atlas.



w/ transition maps that are complex analytic.

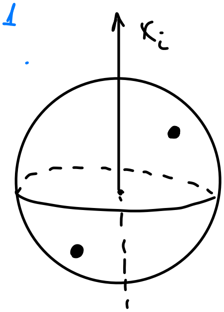
(5) REAL PROJECTIVE SPACE $\mathbb{RP}^n$

dim = n

geometrically, the space of lines through the origin in $\mathbb{R}^{n+1}$.

1st DEF.: $S^n/\sim$, where

$S^n \subset \mathbb{R}^{n+1}$



$$(x_0, x_1, \ldots, x_n) \sim (-x_0, -x_1, \ldots, -x_n)$$

Identify antipodal points of S^n

Denote this equiv. class by $((x_0, x_1, \ldots, x_n))$.

• Take the "standard" atlas for S^n. This induces an atlas for $\mathbb{RP}^n$:

$$\begin{array}{ccc} U_i = \{((x_0, x_1, \ldots, x_n)) : x_i \neq 0\} & & ((x_0, x_1, \ldots, x_n)) \\ \downarrow & & \downarrow \\ \mathbb{R}^n & \ni & \operatorname{sign}(x_i)\,(x_0, x_1, \ldots, \hat{x}_i, \ldots, x_n) \end{array}$$

<u>2nd DEF.</u>:

$\mathbb{RP}^n$ = set of lines through 0 in $\mathbb{R}^{n+1}$

$\Rightarrow$ $\mathbb{RP}^n = \left(\mathbb{R}^{n+1} \setminus \{0\}\right) / \sim$, where $(x_0, x_1, \ldots, x_n) \sim (x_0', x_1', \ldots, x_n')$ if there is nonzero $\lambda \in \mathbb{R}$ s.t.

$$(x_0, x_1, \ldots, x_n) = \lambda (x_0', x_1', \ldots, x_n')$$

Denote this equivalence class by $[x_0, x_1, \dots, x_n]$ in homogeneous coördinates

Atlas:

$$
\begin{tikzcd}
U_i = \{ [x_0, x_1, \dots, x_n] \in \mathbb{RP}^n : x_i \neq 0 \} \arrow[d] & \arrow[d, mapsto, "\varphi_i"] \\
\mathbb{R}^n \ni \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \dots, \widehat{\frac{x_i}{x_i}}, \dots, \frac{x_n}{x_i} \right) &
\end{tikzcd}
$$

Note: $\operatorname{Im} \varphi_i = \mathbb{R}^n$

Inverse:

$$\varphi_i^{-1} : (y_1, \dots, y_n) \longmapsto [y_1, \dots, y_i, 1, y_{i+1}, \dots, y_n]$$

Verify that $\mathbb{RP}^n$ is a smooth manifold need to check that all transition functions are C^∞.

Transition maps are of the form

$$y_i \to y_i \; ; \quad y_i \to 1/y_i \; ; \quad y_i \to y_i/y_j \, .$$

Lecture 5:

* More Examples of Manifold Structures

From last time:

!

1) $\mathbb{RP}^n$:

(1) $S^n/\sim$ → Identify antipodal points

(2) $(\mathbb{R}^{n+1}\setminus\{0\})/\sim$ → Identify points on the same line through the origin.

Equivalent when both atlases lie in the same maximal atlas or when identity is a diffeo.

2) $\mathbb{CP}^n$ (Complex Projective Space) !

↳ Space of complex lines through the origin in $\mathbb{C}^{n+1}$

(1) $(\mathbb{C}^{n+1}\setminus\{0\})/\sim$ → Identify points on the same line through 0

$[x_0, x_1, \dots, x_n]$ equiv. class of $(x_0, \dots, x_n) \in \mathbb{C}^{n+1}\setminus\{0\}$ in homogeneous coördinates.

Cover by coördinate charts:

$$\begin{array}{ccc} \mathcal{U}_i & = & \{[x_0, \ldots, x_n] : x_i \neq 0\} \\ \text{Homeo. onto} \Big\downarrow \varphi_i & & \Big\downarrow \\ \mathbb{C}^n & \ni & \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_n}{x_i}\right) \end{array}$$

Upshot: This is a complex manifold structure and transition maps are $\mathbb{C}$-analytic.

Analytic in each coördinate

In fact, for this $\mathbb{C}P^n$, the transition mappings are rational!

→ Even better!
⇒ Complex algebraic structure.

Obs:

$$\dim_{\mathbb{C}} \mathbb{C}P^n = n$$

$$\dim_{\mathbb{R}} \mathbb{C}P^n = 2n$$

Remark: Every complex manifold is orientable but this is not true for real manifolds

$$\Rightarrow \mathbb{C}P^n \not\simeq \mathbb{R}P^{2n}.$$

- **SPECIAL CASE**: $\mathbb{C}P^1 = \mathbb{C}^2 \setminus \{0\} / \sim$

Coördinate charts:

$$\begin{array}{ccccccc}
U_0 = \{ [x_0, x_1] : x_0 \neq 0 \} & & ; & U_1 = \{ [x_0, x_1] : x_1 \neq 0 \} & \\
\varphi_0 \downarrow \quad\quad \downarrow & & & \varphi_1 \downarrow \quad\quad \downarrow & \\
\mathbb{C} \quad\quad x_1/x_0 =: z & & & \mathbb{C} \quad\quad x_0/x_1 =: w &
\end{array}$$

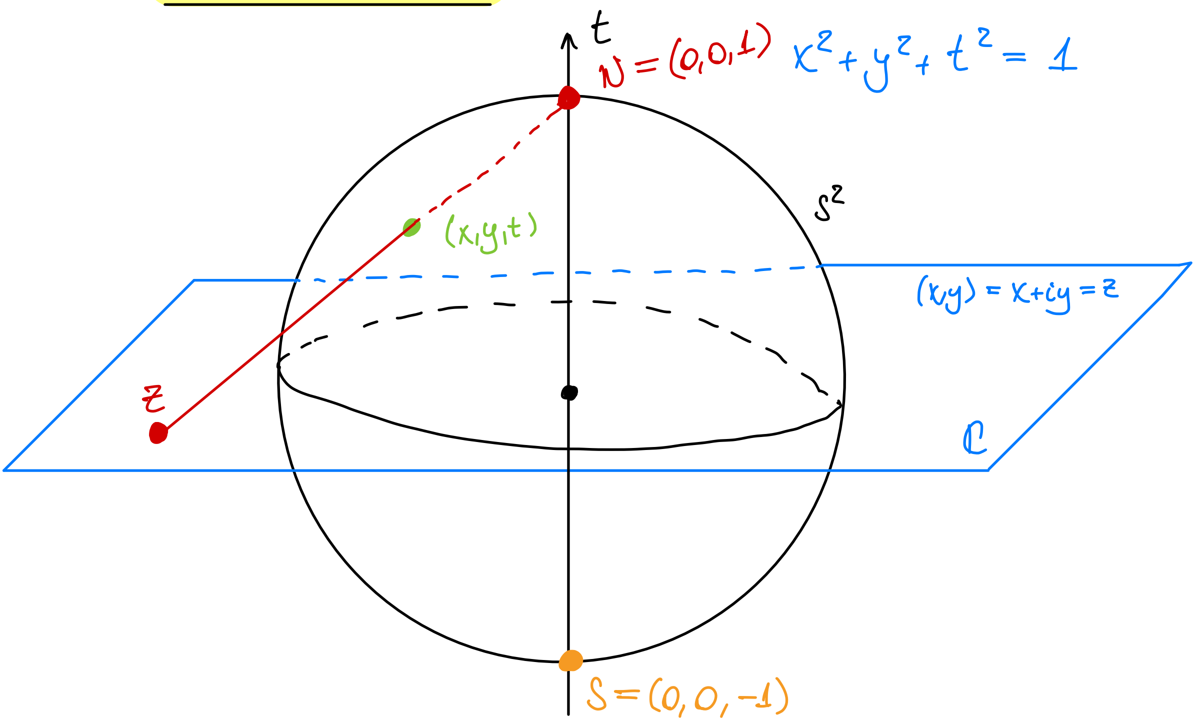
$$w = \frac{1}{z} \quad \Rightarrow \quad \mathbb{C}P^1 = \mathbb{C} \cup \{[0,1]\}$$

point in homog. coördinates

So, U_1 only misses the point $[0,1]$ from U_0.

This is the same as it was for the sphere!

⇒ **RIEMANN SPHERE**



- Stereographic proj. from N:

$$z = \frac{x+iy}{1-t}$$

- Stereographic proj. from S:

$$w = \frac{x-iy}{1-t}$$

$$\Rightarrow \quad zw = \frac{x^2+y^2}{1-t^2} = 1 \quad \Rightarrow \quad \boxed{w = 1/z}$$

Therefore, $\mathbb{C}P^1 \simeq S^2$.

Aside: $\mathbb{R}P^1 \simeq S^1$

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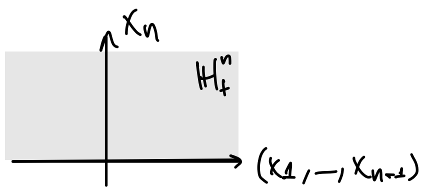
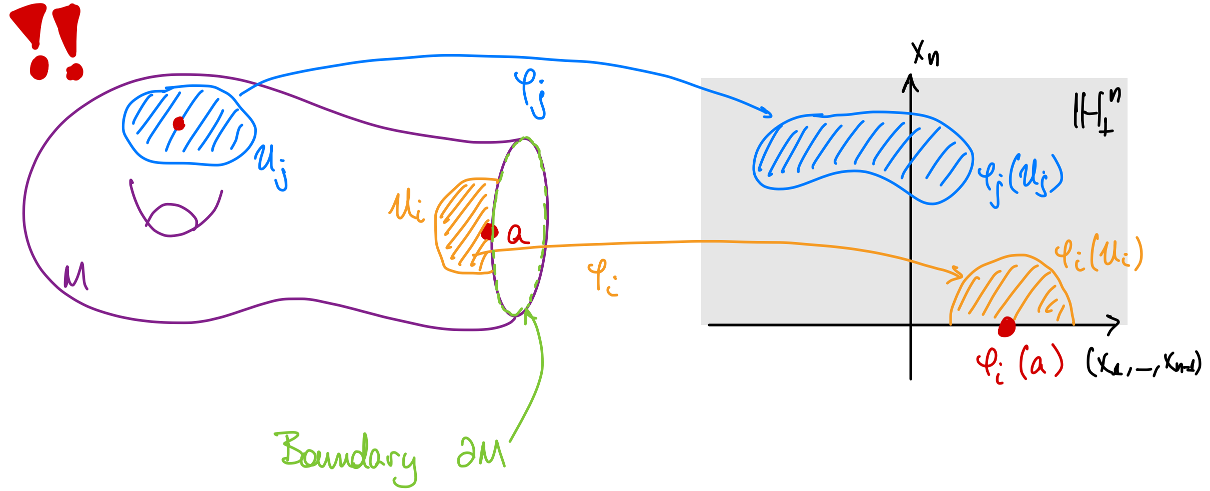
* **SMOOTH MANIFOLDS**: locally modelled on $\mathbb{R}^n$.

* **SMOOTH MANIFOLDS w/ BOUNDARY**: locally modelled by $\mathbb{R}^n$ **or** $\mathbb{H}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$



Def: (SMOOTH MANIFOLDS w/ BOUNDARY) 2^{nd} countable Hausdorff space covered by coördinate charts $U_i \xrightarrow[\text{homeo.}]{\varphi_i} \mathbb{H}^n_+$ with C^∞-related transition maps.

C^∞ on the intersections between charts.



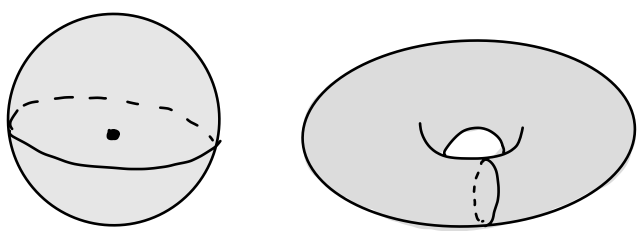
Remark: In order for a function to be C^∞ in $\mathbb{H}^n_+$:

(1) Restriction of C^∞ function on $\mathbb{R}^n$

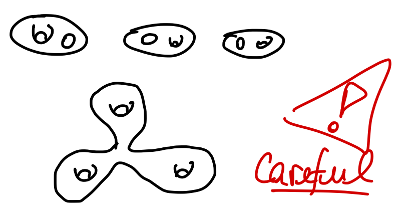
or, equivalently,

(2) C^∞ function on $\{x_n > 0\}$ such that all partial derivatives extend continuously to the boundary.

* Classification of Compact Orientable 2-Manifolds



n-Torus:



In general, the boundary ∂M of a smooth manifold M w/ boundary has an open neighborhood U diffeo. to $\partial M \times [0,1)$.

---

* **ORIENTATION** !

Def: A smooth manifold M is orientable if it has a smooth atlas such that all transition mappings are orientation preserving i.e., Jacobian det > 0.

---

* **SMOOTH FUNCTIONS & SMOOTH MAPPINGS**

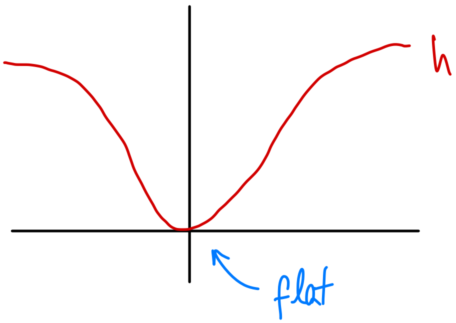
BUMP FUNCTION LEMMA: If M is a smooth-manifold and $C \subset U \subset M$, then there exists

compact open

C^∞ function $f : M \to [0,1]$ such that $f \equiv 1$ on C and 0 on $M \setminus U$.

Pf: Construct some smooth functions: !!

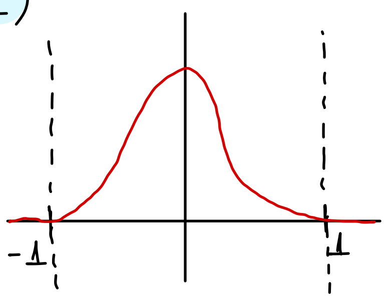
1)



$h : \mathbb{R} \to \mathbb{R}$

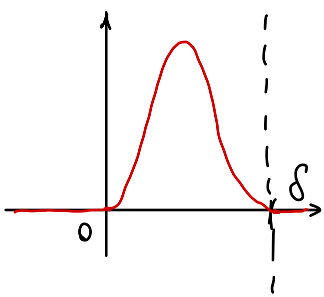
$$h(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

2)



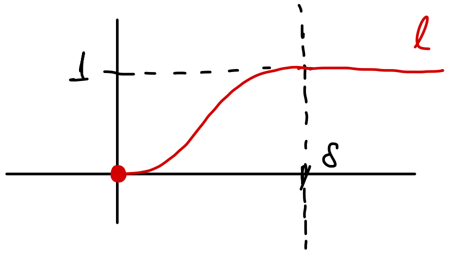
$$j(x) = \begin{cases} e^{-1/(x-1)^2} e^{-1/(x+1)^2}, & x \in (-1,1) \\ 0, & x \notin (-1,1) \end{cases}$$

3)



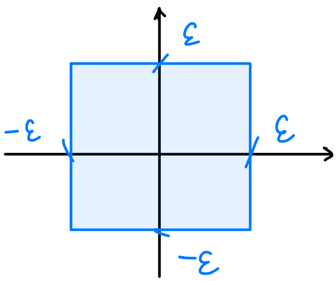
$$b(x) \begin{cases} > 0 & \text{on } (0,\delta) \\ = 0 & \text{outside} \end{cases}$$

4)



$$\ell(x) = \int_0^x b \Big/ \int_0^\delta b$$

5)



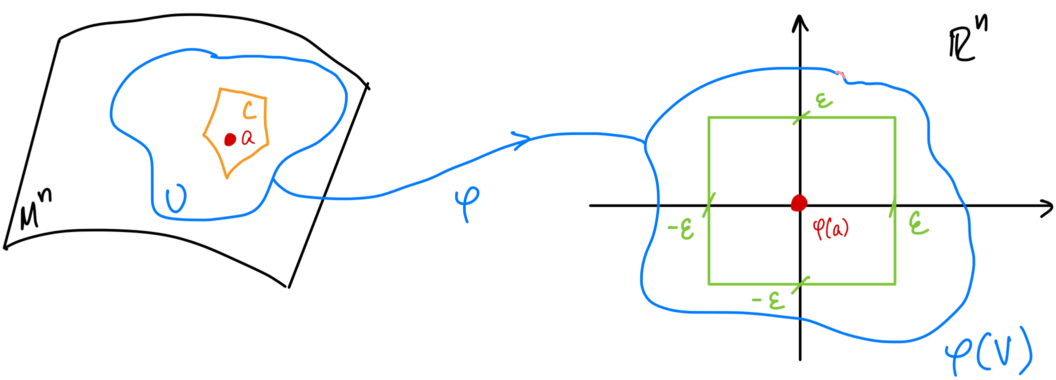
$$g : \mathbb{R}^n \to \mathbb{R}$$

$$g(x_1, \ldots, x_n) := j\left(\frac{x_1}{\varepsilon}\right) \cdots j\left(\frac{x_n}{\varepsilon}\right)$$

is C^∞ and

$$g \begin{cases} > 0 & \text{on } (-\varepsilon, \varepsilon)^n \\ = 0 & \text{outside} \end{cases}$$

Now, prove the Lemma:



Given $a \in C$, choose a coördinate chart $\varphi : V \to \mathbb{R}^n$ s.t. $a \in V$, $\bar{V} \subset U$ and $\varphi(a) = 0$.

Choose $\varepsilon > 0$ s.t. $(-\varepsilon, \varepsilon)^n \subset \varphi(V)$.

as above in pg. 35

Now, $g \circ \varphi$ extends to C^∞ function f_a on M that is > 0 on an open nbhd W_a of a where the closure lies in U.

There are finitely many points $a_1, \dots, a_k$ s.t. W_{a_k} cover C.

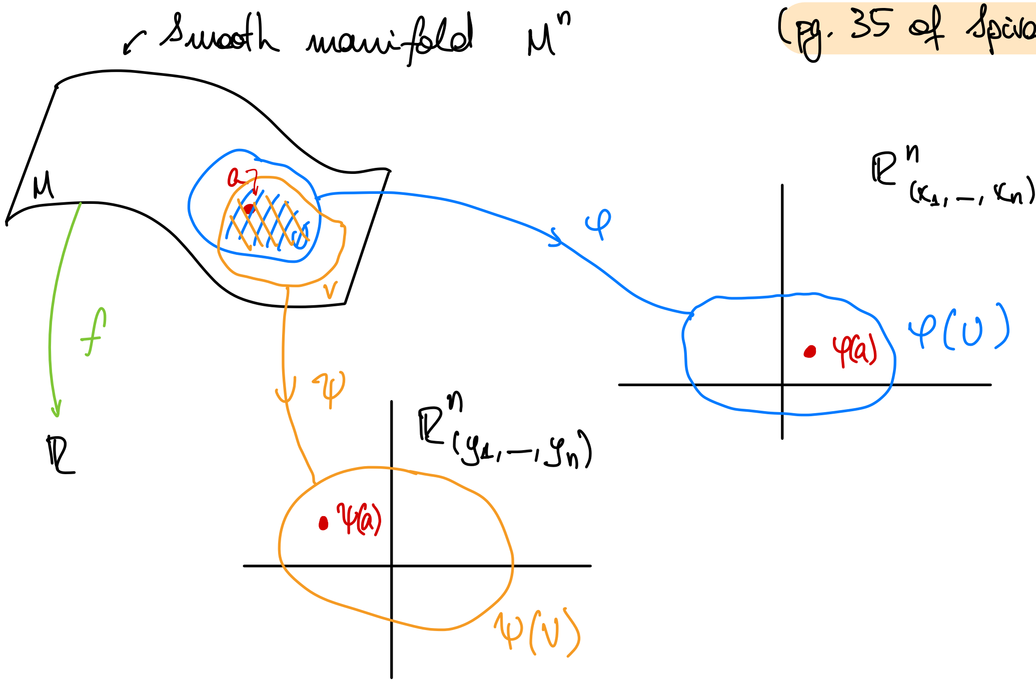
Obs: $f_{a_1} + \dots + f_{a_k}$ lies in U; i.e., $\mathrm{supp}(f_{a_1} + \dots + f_{a_k}) \subset U$ and $f_{a_1} + \dots + f_{a_k} > 0$ on C. So, $f_{a_1} + \dots + f_{a_k} \geq \delta$ on C for some $\delta > 0$.

previous page

Take $f := l \circ (f_{a_1} + \dots + f_{a_k})$.

Lecture 6: Differentiation in Local Coördinates

(pg. 35 of Spivak)



Def: $f: M \to \mathbb{R}$ is <u>differentiable at a</u> if $f \circ \varphi^{-1}$ is differentiable at $\varphi(a)$ for one (or every) coördinate chart $\varphi: U \to \mathbb{R}^n$ at a.

Define

$$\frac{\partial f}{\partial x_i}(a) := \frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(a)).$$

So,

$$\frac{\partial f}{\partial y_i}(a) = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) \frac{\partial x_j}{\partial y_i}(a).$$

Pf: $\dfrac{\partial f}{\partial y_i}(a) = \dfrac{\partial(f\circ\psi^{-1})}{\partial y_i}(a) = \dfrac{\partial(f\circ\varphi^{-1}\circ\varphi\circ\psi^{-1})}{\partial y_i}(\psi(a))$

$$= \sum_{j=1}^{n}\left[\frac{\partial(f\circ\varphi^{-1})}{\partial x_j}(\varphi(a))\cdot\underbrace{\frac{\partial(\varphi\circ\psi^{-1})}{\partial y_i}(\psi(a))}_{\frac{\partial x_j}{\partial y_i}(a)}\right]$$

$\square$

Upshot: Sponsored by the Chain Rule

$$\frac{\partial}{\partial y_i} = \sum_{j=1}^{n}\frac{\partial x_j}{\partial y_i}\frac{\partial}{\partial x_j}$$

$$\left.\frac{\partial}{\partial y_i}\right|_a = \sum_{j=1}^{n}\frac{\partial x_j}{\partial y_i}(a)\left.\frac{\partial}{\partial x_j}\right|_a$$

Note: $\left.\dfrac{\partial}{\partial y_i}\right|_a$ is a DERIVATION ∂ at a; i.e., — from the ring of C^∞-func. on M.

$\partial : C^\infty(M) \longrightarrow \mathbb{R}$ linear such that

$$\partial(fg) = f(a)\,\partial g + g(a)\,\partial f$$

Moreover, the matrix

$$\left(\frac{\partial x_j}{\partial y_i}(a) \right)_{ij}$$

is the Jacobian matrix of $\varphi \circ \psi^{-1}$ at $\psi(a)$ with inverse

$$\left(\frac{\partial y_i}{\partial x_j}(a) \right)_{ij} .$$

FUNCTION BETWEEN MANIFOLDS:

given a smooth function $f: M^n \to N^p$, $a \in M$, $\varphi: U \longrightarrow \mathbb{R}^n_{(x_1, \ldots, x_n)}$ coördinate system for M at a; and $\psi: V \longrightarrow \mathbb{R}^p_{(y_1, \ldots, y_p)}$ coördinate system for N at $f(a)$, then the $p \times n$ matrix

$$\left(\frac{\partial (y_i \circ f)}{\partial x_j}(a) \right)_{ij}$$

is the Jacobian matrix of $\psi \circ f \circ \varphi^{-1}$ at $\varphi(a)$.

The rank of this matrix doesn't depend on the choice of coördinate charts. We call this the rank of f at a.

Def: We say that $a \in M$ is a
- CRITICAL POINT of f: if rank of f at a is $< p$;
- REGULAR POINT: otherwise.

If $b \in N$ is a CRITICAL VALUE of f if $b = f(a)$, where a is a critical point. It is called a REGULAR VALUE otherwise.

↪ In particular $b \notin \varphi(M)$ is a regular value.

EXAMPLE:

$f: \mathbb{R}^n \to \mathbb{R}$ is C^1 and all points are critical $\Longrightarrow$ f is constant.

Def: (MEASURE 0) $A \subset \mathbb{R}^n$ is called MEASURE 0 if for every $\varepsilon > 0$, A can be covered by countably many rectangles $\{R_i\}$ such that

$$\sum_i \mathrm{vol}(R_i) < \varepsilon.$$

LEMMA 1: If $A \subset \mathbb{R}^n$ is a rectangle, $f: A \to \mathbb{R}^n$ is C^1, $f = (f_1, \ldots, f_n)$, assume

$$\left| \frac{\partial f_i}{\partial y_j} \right| \leq M \text{ on } A.$$

Then, $\forall x, y \in A$, $|f(x) - f(y)| \leq n^2 M |x - y|$.

This says that a rectangle of diameter at most ε gets mapped to a set of diameter at most $n^2 M \varepsilon$.

Lemma 2: If $f: \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and $A \subset \mathbb{R}^n$ has measure 0, then $f(A)$ has measure zero.

Lecture 7: Sard & Rank Theorem

Lemma: If $f: \mathbb{R}^n \to \mathbb{R}^n$ is C^1 and A has measure 0, then $f(A)$ has measure 0.

Implies Baby Sard.

Def: (Measure 0 on manifolds) If M has measure 0, then $A \subset M$ has measure 0 if M can be covered by countably many charts $\varphi_i : U_i \to \mathbb{R}^n$ such that $\varphi_i(A \cap U_i)$ has measure 0 for all i.

since M is 2^{nd} countable

$\varphi(A \cap U) = \bigcup_i (\varphi \circ \varphi_i^{-1})(\varphi_i(A \cap U_i))$

Equivalently, if for every coördinate chart $\varphi : U \to \mathbb{R}^n$, $\varphi(A \cap U)$ has measure 0.

COROLLARY: If $f: M \to N$ is a C^1 function between manifolds of same dimension and $A \subset M$ has measure 0, then $f(A) \subset N$ has measure 0 as well.

Pf: WTS: Atlas of $M \to \{(\varphi_i, U_i)\}$

Atlas of $N \to \{(\psi_i, V_i)\}$

$\psi_i(f(A) \cap V_i)$ has measure 0 for all i.

* TRANSLATE THIS TO EUCLIDEAN SPACE THROUGH TRANSITION MAPS:

IMPORTANT TECHNIQUE

$$f(A) \cap V = \bigcup_i f(A \cap U_i) \cap V$$

$$\psi\left(f(A \cap U_i) \cap V\right) = \left(\psi \circ f \circ \varphi_i^{-1}\right)\Big|_{(f \circ \varphi_i^{-1})^{-1}(V)} \left(\varphi_i(A \cap U_i)\right).$$

this is where the map is def.

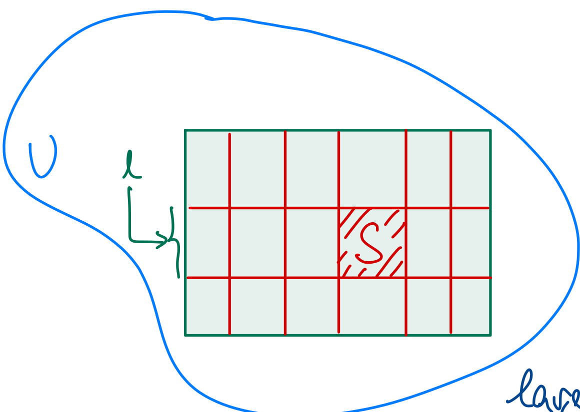
$\square$

THM: (EQUIDIMENTIONAL SARD) If $f: M \to N$ is a C^1 mapping between manifolds of same dimension n, then the set of critical values of f has measure 0.

Pf: WLOG, we can restrict $f: \underset{\text{open}}{U} \subset \mathbb{R}^n \to \mathbb{R}^n$ because we can restrict to coördinates charts.

Let $A = \{x \in U : \det f'(x) = 0\}$ be the set of critical pts. of f on U.

<u>Enough to show</u>: $f(A \cap R)$ has measure 0, where R is a closed rectangle in U.



Subdivide R into N^n subcubes S of side length l/N.

Given $\varepsilon > 0$, choose N large enough s.t. if $x \in S$, then $\forall y \in S$,

$$\| f(y) - f(x) - Df(x)(x-y) \| < \varepsilon \| x - y \|$$
$$\leq \varepsilon \sqrt{n} \cdot \frac{l}{N}.$$

If $S \cap A \neq \emptyset$, take $x \in S \cap A$. So,

$$\{ Df(x)(y-x) : y \in S \} \subset \underline{V}$$

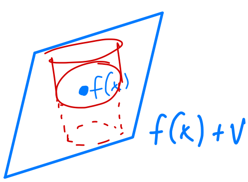
(n-1)-dimensional lin. subspace of $\mathbb{R}^n$

This means that $\{ f(y) : y \in S \}$ lies within

$\varepsilon \sqrt{n} \frac{l}{N}$ of the hyperplane $f(x) + V$.

Now, "$C^1 \Rightarrow$ Lipschitz" $\exists M$ s.t.

$$\| f(x) - f(y) \| \leq M \| x - y \| \quad \forall y \in S.$$



So, if $x \in S \cap A$, then

$\{ f(y) : y \in S \}$ lies in the cylinder of height $\varepsilon\sqrt{n}\,\frac{l}{N}$ whose base is a ball in $f(x)+V$ of radius $M\sqrt{n}\,\frac{l}{N}$.

$\Rightarrow$ Volume of the cylinder is

$$(\text{constants})\cdot\varepsilon\left(\frac{l}{N}\right)^n$$

But R is covered by N^n cubes S. So, $f(R\cap A)$ lies in a set of volume

$$(\text{constants})\cdot N^n\varepsilon\left(\frac{l}{N}\right)^n = (\text{constants})\,l^n\varepsilon$$

$\forall\,\varepsilon>0$. So, $f(R\cap A)$ has measure 0.

Just refine the subdivision of R

$\square$

THM (RANK THEOREM) Suppose $f: M^n \to N^p$ is C^∞.

(1) If f has rank k at $a\in M$, then there are coördinate systems $(x_1,\ldots,x_n)$ at a and $(y_1,\ldots,y_p)$ at $f(a)$ in which f has the form

$$(y_1, \dots, y_p) = (x_1, \dots, x_k, f_{k+1}(x), \dots, f_p(x)).$$

(2) If f has rank k on a neighborhood of a, then we can choose coördinate systems in which $(y_1, \dots, y_n) = (x_1, \dots, x_n, \underbrace{0, \dots, 0}_{n-p \text{ zeros}})$.

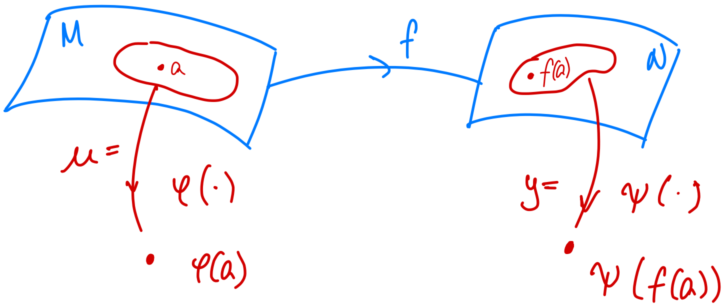
Pf:

(1) Consider coördinate systems $\begin{cases} (u_1, \dots, u_n) \text{ at } a \\ (y_1, \dots, y_p) \text{ at } f(a) \end{cases}$

After permuting coördinates (i.e., columns of f), we can assume that the k-lin. indep columns are the first k ones; i.e.,

$$\det \left(\frac{\partial y_i}{\partial u_j}(a) \right)_{\substack{i=1,\dots,k \\ j=1,\dots,k}} \neq 0 .$$

Define a new coördinate system



Define

$$x_i := \begin{cases} y_i \circ f, & i = 1, \ldots, k \\ u_i, & i = k+1, \ldots, p \end{cases}$$

Check that this is a coordinate change; i.e., check it's Jacobian is invertible.

$$\left(\frac{\partial x_i}{\partial u_j}(a)\right) = \underbrace{\left(\begin{array}{c|c} \left(\frac{\partial y_i}{\partial u_j}(a)\right)_{k\times k} & * \\ \hline 0_{(p-k)\times(n-k)} & Id_{(p-k)\times(p-k)} \end{array}\right)_{p\times n}}_{\text{Invertible}}$$

Invertible.

(b/c $\frac{\partial y_i}{\partial u_j}(a)$ is a change of variables)

Thus, by the Inverse Function theorem, $x = x(u)$ is a coördinate change at a and $(y_1, \dots, y_n) = (x_1, \dots, x_k, f_{k+1}(x), \dots, f_p(x))$.

(2) By (1), we can choose coördinate systems x at a and v at $f(a)$ in which f has the form

$$(v_1, \dots, v_p) = (x_1, \dots, x_k, f_{k+1}(x), \dots, f_p(x)).$$

Thus, the Jacobian is:

$$\left(\frac{\partial v_i}{\partial x_j}\right) = \left(\begin{array}{c|c} Id_{k\times k} & 0 \\ \hline * & \frac{\partial f}{\partial x} \end{array}\right)$$

$$\begin{pmatrix} \frac{\partial f_{k+1}}{\partial x_{k+1}} & \cdots & \frac{\partial f_{k+1}}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial x_{k+1}} & \cdots & \frac{\partial f_p}{\partial x_n} \end{pmatrix}$$

$\leftarrow$ But this has to be zero (else, rank would be greater than k).

$\Rightarrow \quad f_i(x) = f_i(x_1, \ldots, x_k)$, for every $i = k+1, \ldots, p$.

Depend only on the first k variables.

Define the change of coördinates

$$y_i := \begin{cases} v_i, & i = 1, \ldots, k \\ v_i - f_i(v_1, \ldots, v_k), & i = k+1, \ldots, p \end{cases}.$$

Check that this is a coördinate change:

$$\left(\frac{\partial y_i}{\partial v_j} \right) = \left(\begin{array}{c|c} Id_{k \times k} & 0 \\ \hline * & Id \end{array} \right).$$

Therefore, in these coördinates, f has the form

$$y_i = v_i = x_i, \qquad i = 1, \ldots, k$$

$$\begin{aligned} y_i &= v_i - f(v_1, \ldots, v_k), \qquad i = k+1, \ldots, p \\ &= f_i(x_1, \ldots, x_k) - f_i(x_1, \ldots, x_k) \\ &= 0 \end{aligned}$$

$\square$

Lecture 8: Submersion & Immersions

Recall: (Rank Theorem) If $f: M^n \to N^p$ is C^∞ and $a \in M$, then

(1) If f has rank k at the point a, then there are coördinate systems

$$(x_1, \ldots, x_n) \text{ at } a, \text{ and}$$
$$(y_1, \ldots, y_p) \text{ at } f(a)$$

in which f has the form

$$(y_1, \ldots, y_p) = (x_1, \ldots, x_k, f_{k+1}(x), \ldots, f_p(x))$$

↪ Permute the entries if needed.

(2) If f has rank k at a neighborhood of a, then we can choose coördinate systems in which f is of the form

$$(y_1, \ldots, y_p) = (x_1, \ldots, x_k, \underbrace{0, \ldots, 0}_{n-p})$$

COROLLARIES:

THM: (SUBMERSION THEOREM) If $p \leq n$ and $f: M^n \to N^p$ has rank p at a, then for any coördinate system $(y_1, \ldots, y_p)$ at $f(a)$ we can find a coördinate system $(x_1, \ldots, x_n)$ at a in which f has the form

$$(y_1, \ldots, y_p) = (x_1, \ldots, x_p).$$

By Rank Thm because no perm of columns is needed to pick out first the k lin. indep. columns.

THM: (IMMERSION THEOREM) If $n \leq p$ and f has rank n at a, then for any coördinate system $(x_1, \ldots, x_n)$ at a, we can choose a coördinate system $(y_1, \ldots, y_p)$, in which f is of the form

$$(y_1, \ldots, y_p) = (x_1, \ldots, x_n, \underbrace{0, \ldots, 0}_{n-p})$$

Claim: We can write

$$(z_1, \dots, z_n) = (u_1, \dots, u_n, 0, \dots, 0)$$

after coördinate change $z = z(y)$.

Pf: We have

$$(y_1, \dots, y_n) = (x_1(u), \dots, x_n(u), 0, \dots, 0)$$

where $x = x(u)$ is invertible.

Define $(z_1(y), \dots, z_p(y))$ by

$$(z_1, \dots, z_n) = x^{-1}(y_1, \dots, y_n)$$

$$(z_{n+1}, \dots, z_p) = (y_{n+1}, \dots, y_p)$$

This is a coördinate change b/c the Jacobian is invertible:

$$\left(\begin{array}{c|c} \text{Invertible b/c it is a coörd. change} & 0 \\ \hline * & Id \end{array}\right).$$

In these coördinates, f is given by $(u, 0)$. $\square$

DEF: (**IMMERSION**) A smooth mapping $f: M^n \to N^p$ is an <u>immersion</u> if $\operatorname{rank} f = n$ at every point of M. $\longrightarrow$ Only possible if $n \leq p$.

Immersions are <u>LOCALLY</u> 1-1 (by Rank Thm). <u>NOT</u> necessarily 1-1 globally.

Moreover, 1-1 smooth map <u>not</u> necessarily an immersion.

EXAMPLES:

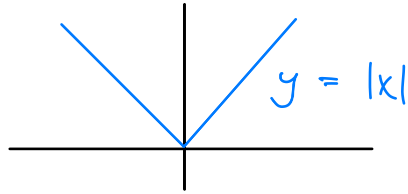
NOT immersion

1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$, is 1-1 rank is $\neq 1$ at 0 (i.e., smooth 1-1 but <u>NOT</u> immersion)

NOT IMMERSION

2) Now, consider $h: \mathbb{R} \to \mathbb{R}^2$ smooth and

1-1 with image:



Take

$$h(t) := (f(t), g(t)),$$

where

$$f(t) = \begin{cases} e^{-1/t^2}, & t > 0 \\ 0, & t = 0 \\ -e^{1/t^2}, & t < 0 \end{cases}; \qquad g(t) = \begin{cases} e^{-1/t^2}, & t \neq 0 \\ 0, & t = 0 \end{cases}$$

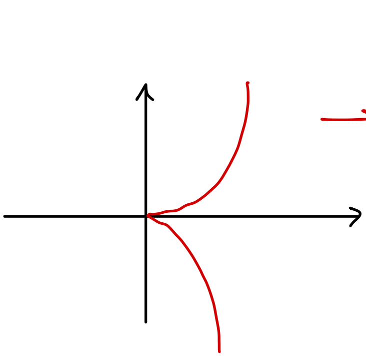
$\Rightarrow$ BUT **NOT** IMMERSION B/C rank $h \neq 1$ at 0.

NOT IMMERSION

3) $(x, y) = (t^2, t^3)$

$Df(t) = \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix}$

rank $Df(t)(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = 0 \neq \dim \mathbb{R}$



$\longrightarrow$ 1-1, smooth but <u>not</u> immersion b/c rank $\neq 1$ at 0.

4) The curve $y^2 = x^2(x+1)$

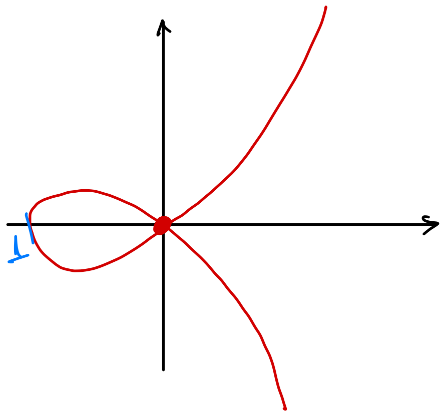


Image of $f: \mathbb{R}_t \to \mathbb{R}^2$ given by:

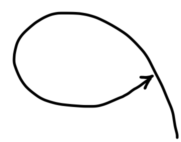
$$x = t^2 - 1$$
$$y = t(t^2 - 1)$$

THIS is an IMMERSION: just compute the rank.

$\operatorname{rank} Df(t)(x) = \operatorname{rank} \begin{pmatrix} 2t \\ 3t^2 - 1 \end{pmatrix} = 1 = \dim \mathbb{R}$ everywhere

Now, consider

$$f\big|_{(-\infty, 1)} : \mathbb{R} \to$$



This is an immersion b/c has rank 1 at every point.

5)

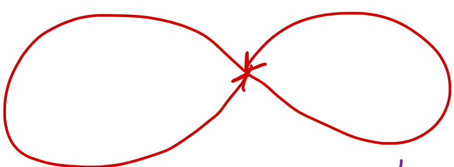


Image of

$$\mathbb{R}_\theta \longrightarrow \mathbb{R}^2$$
$$\theta \overset{f}{\longmapsto} (\sin\theta, \sin(2\theta))$$

IMMERSION

$$\operatorname{rank} Df(\theta) = \operatorname{rank} \begin{pmatrix} \cos\theta \\ 2\cos 2\theta \end{pmatrix}$$
$$= \operatorname{rank} \begin{pmatrix} \cos\theta \\ 0 \end{pmatrix} = 1 = \dim \mathbb{R} \ \checkmark$$

Also the image of the non-injective immersion $S^1 \to \mathbb{R}^2$
$(x,y) \mapsto (y, 2xy)$

Also image of $f\big|_{S^1 \setminus \{(1,0)\}}$

6) Consider a subset of the torus:

$$P \subset S^1 \times S^1$$

image of $\theta \xrightarrow{f} (e^{i\theta}, e^{i\alpha\theta})$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$

The map f is an immersion but P is dense in the torus $S^1 \times S^1$.

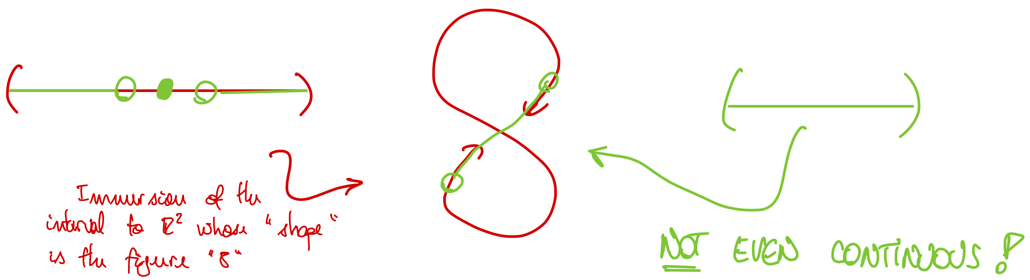
Lecture 9: Immersed & Embedded Manifolds and Partitions of Unity

Def (Immersed Manifold) If M is a smooth manifold. Then $P \subset M$ is an immersed manifold if P is the image of an injective immersion; i.e., P can be given the differential structure of a smooth manifold such that the inclusion $P \hookrightarrow M$ is an immersion.

REMARK: Suppose $P \subset M$ is an immersed submanifold and $f: N \to M$ is a smooth map such that $f(N) \subset P$.

Q: Is f smooth as a map $N \to P$?

A: No !! Take the figure "8" on the plane:



DEF: (EMBEDDING) Immersion which is a homeomorphism onto its image (with the subspace topology).

DEF: (SMOOTH SUBMANIFOLD) $P \subset M$ being a (smooth) submanifold of M means that it is the image of an embedding.

EQUIVALENTLY, an immersed submanifold $P \subset M$ is a submanifold when the inclusion map $P \hookrightarrow M$ is an embedding.

NOTE: charts for the submanifold obtained by "intersecting" atlases.

PROP: Suppose $f : M^n \to N^p$ has constant rank k in some neighborhood of a fibre $f^{-1}(b)$, $b \in N$. Then,

(1) $f^{-1}(b)$ is a closed submanifold of M of dimension $n-k$. (Rank-Thm)

(2) In particular, if f is a submersion, then $f^{-1}(b)$ is a closed submanifold of dimension $n-p$. (Submersion Thm)

PARTITIONS OF UNITY

LEMMA 1: If M is a smooth manifold (or topological manifold) and $\mathcal{O}$ is an open cover of M. Then there is an open cover $\mathcal{O}'$ which is locally finite and refines $\mathcal{O}$ (i.e., M is PARACOMPACT)

Each pt. has a neighborhood that intersects finitely many elements of the cover.

Every open set in $\mathcal{O}'$ is contained in some open set in $\mathcal{O}$

can make every $U \in \mathcal{O}'$ diffeo. (or homeo.) to $\mathbb{R}^n$.

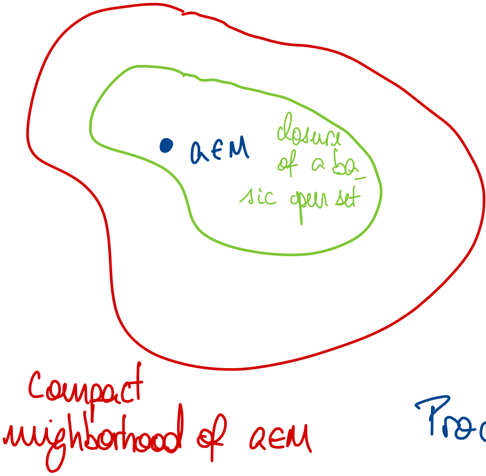
Pf: WLOG, assume M is connected (else do the following proof for each component).

We claim that M is a countable union of compact sets. i.e., "M is σ-COMPACT"

B/c each point has a compact neighborhood

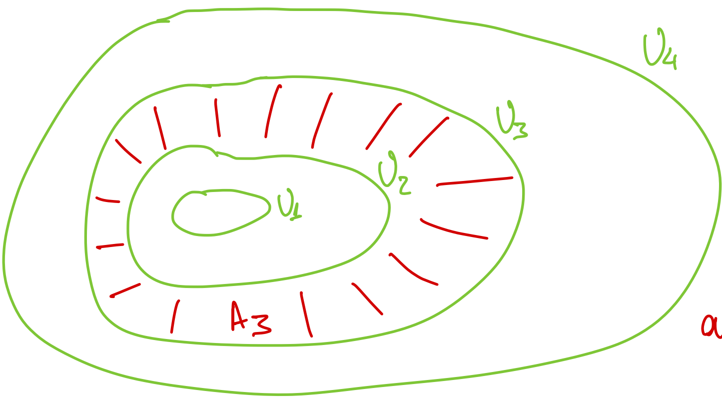
$C_1, C_2, \ldots$

C_1 has an open neighborhood U_1 with compact closure in M. Moreover, $\overline{U_1} \cup C_2$



has also an open neigh. U_2 with compact closure. Thus, $\overline{U}_2 \cup C_3$ has open neigh. U_3 w/ compact closure. Etc...

Proceed this way until M is covered by open U_i such that $\overline{U}_i$ compact and $\overline{U}_i \subset U_{i+1}$. So, we get that M is the union of compact annuli $A_i := \overline{U}_i \setminus U_{i-1}$



We can cover A_i by finitely many open sets; each in some element of $\mathcal{O}$ and each in $U_{i+1} \setminus \overline{U}_{i-2}$.

This gives $\mathcal{O}'$ (locally finite since U_i intersects only finitely many of the $U_{j+1} \setminus \overline{U}_{j-2}$).

at most 3...

□

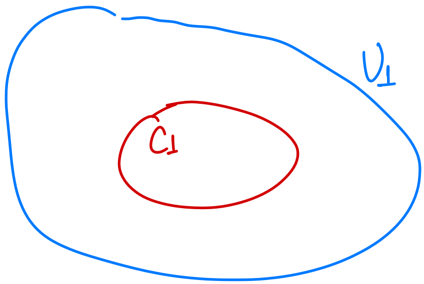
Note: A locally finite open cover is countable. This is b/c there is a finite covering by compact sets but these only need finitely many sets to cover.

Lemma 2: Given a locally finite open cover $\mathcal{O}$ of M, then, for every $U \in \mathcal{O}$, there is an open $U' \subset \overline{U'} \subset U$ such that the U' still form an open cover.

Pf: Again, WLOG, assume M is connected and $\mathcal{O} = \{U_1, U_2, \ldots\}$ covers M.

Define
$$C_1 := U_1 \setminus (U_2 \cup U_3 \cup \ldots).$$
This is closed and contained in U_1.

So, there exists U_1' such that



$C_1 \subset U_1' \subset \overline{U_1'} \subset U_1$.

↳ Easy topological argument

So, do the same:

$C_2 := U_2 \setminus (U_1' \cup U_3 \cup \dots)$ and note that

$C_2 \subset U_2' \subset \overline{U_2'} \subset U_2$. Again,

$C_3 := U_3 \setminus (U_1' \cup U_2' \cup U_4 \cup U_5 \cup \dots)$, etc.

and proceed like this.

Clearly $\{U_i'\}$ is still locally finite. Now, $\{U_i'\}$ covers M because: Let $a \in M$, then there is k such $a \in U_k$ (since locally finite). By construction, $\bigcup_i U_i' = M$, hence $a \in \bigcup_i U_i'$ $\forall i$.

$\square$

Preliminary of the "big" PoI theorem

THM: Given a locally finite open cover $\mathcal{O}$ of a smooth manifold M, then, for all $U \subset \mathcal{O}$, there is a collection of C^∞ functions $\varphi_U : M \longrightarrow [0,1]$ such that

(1) $\operatorname{supp} \varphi_U \subset U$

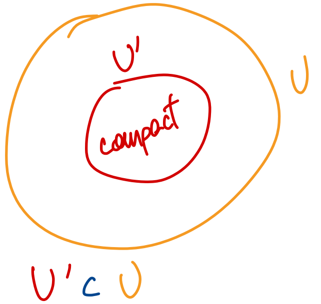
(2) $\displaystyle\sum_{U \in \mathcal{O}} \varphi_U(a) = 1$ for all $a \in M$.

Pf: Recall the Lemma from before:

Lemma (pg. 33) There exists C^∞ function $\varphi : M \to [0,1]$ s.t. $\varphi = 1$ on C & $\operatorname{supp} \varphi \subset U$.

Case 1: First, assume every $U \in \mathcal{O}$ has compact closure. Choose $U' \subset U$ as in Lemma 2. By the Bump Function Lemma (pg. 33), there is a C^∞ function $\psi_U : M \to [0,1]$ such that

$\psi_U = 1$ on $\bar{U}'$ and $\operatorname{supp} \psi_U \subset U$.



Note that

$$\sum_{V \in \mathcal{O}} \psi_V > 0$$

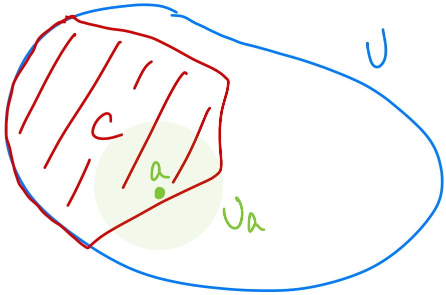
∞-sum makes sense b/c, by local finiteness, only finitely many terms of the sum are non zero.

everywhere b/c $\{U_i'\}$ covers M.

Take

$$\varphi_U := \frac{\psi_U}{\sum_{V \in \mathcal{O}} \psi_V}.$$

<u>Case 2</u>: <u>General case</u>: we can use the same argument if we have the Bump Function Lemma for <u>closed</u>, instead of compact, sets.



For every $a \in C$, take an open neighborhood U_a w/ compact closure in U. Also, cover $M \setminus C$ by open sets $\{V_\beta\}$ with compact closure. Then, the collection $\{U_a, V_\beta\}$ is an open cover of M. So, it has a locally finite refinement $\mathcal{O}$.

Apply Case 1 to $\mathcal{O}$ to get φ_U, $U \in \mathcal{O}$ from Case 1.

Let

$$f := \sum_{W \in \mathcal{O}'} \varphi_W, \text{ where } \mathcal{O}' = \{U_a\}.$$

Then, f is

- C^∞ b/c $\sum$ is locally finite.
- 1 on C because $\sum \varphi_U = 1$ everywhere but φ_U vanishes on C whenever $U \subset V_\beta$.
- $\operatorname{supp} f \subset U$ by construction taking the refinements.

$\square$

COROLLARY: (EXISTENCE OF PARTITIONS OF UNITY) Let $\mathcal{O}$ be an open cover of M. Then, there is a countable collection of C^∞ functions

$$\{\varphi_i : M \to [0,1]\}$$

such that

These 2 conditions define a Partition of Unity

(1) each point $a \in M$ has an open neighborhood on which only <u>finitely</u> many φ_i are nonzero;

(2) $$\sum_i \varphi_i(a) = 1 \quad \forall a \in M.$$

(3) For all i, there exists $U \in \mathcal{O}$ such that $\operatorname{supp} \varphi_i \subset U$.

With this, we have a Partition of Unity SUBORDINATE TO $\mathcal{O}$.

Pf: Immediate from Lemma 1 and the Theorem. $\square$

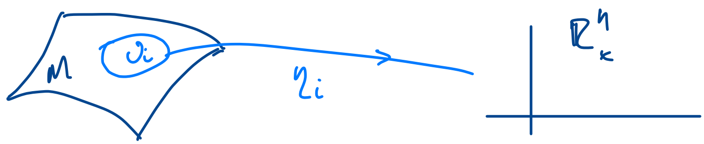
Lecture 10: Whitney Embedding Theorem

can easily generalize to non-compact (WEAK)

THM: (Weak Whitney Embedding Theorem) Every compact smooth n-manifold has an embedding in $\mathbb{R}^N$, for some N.

Best estimate is $N = 2n$ (strong Whitney)

Pf: Cover M^n by finitely many (since compact) coördinate charts $\eta_i : U_i \to \mathbb{R}^n$ with coördinate $(x_{i_1}, \ldots, x_{i_n})$, $i = 1, \ldots, k$.



We can shrink U_i to $U_i' \subset \overline{U_i'} \subset U_i$ so that $\{U_i'\}$ still covers M (Lemma from last time).

Choose the PO1 subordinate to $\{U_i'\}$:

$\varphi_i : M \to [0,1]$ C^∞, $\varphi_i = 1$ on $\overline{U_i'}$, $\operatorname{supp} \varphi_i \subset U_i$

Define $f : M \to \mathbb{R}^{kn+k}$

$$f = (\varphi_1 \eta_1, \_\_\_, \varphi_k \eta_k, \varphi_1, \_\_\_, \varphi_k)$$

Need to check: f immersion and homeom.

• Immersion: (i.e., rank $f = n$ at all $a \in M$)

take $a \in U_i'$ for some i. On U_i', $\varphi_i = 1$, so $f = (\_\_, \eta_i, \_\_)$. On U_i', the Jacobian of f includes the Jacobian matrix of η_i as submatrix: $\left(\frac{\partial x_{ij}}{\partial x_{ik}}\right) = Id \Rightarrow$ rank $= n$.

• 1-1: suppose $f(a) = f(b)$, $a \in U_i'$ for some $\varphi_i(a) = 1$; therefore $\varphi_i(b) = 1$. So, $b \in U_i$.

But then

$$\eta_i(a) = \varphi_i(a)\eta_i(a) = \varphi_i(b)\eta_i(b)$$
$$= \eta_i(b)$$

But η_i is 1-1 (b/c coord. chart), thus $a = b$.

- Homeomorphism: because INJECTIVE map $g: X \to Y$ (X, Y Hausdorff spaces) where X is COMPACT is a homeomorphism.

$\square$

TANGENT BUNDLE !!

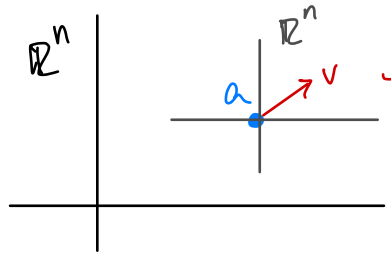
What is the TANGENT SPACE TM_a at a point $a \in M$ of a n-manifold?

Given some smooth $f: M^n \to N^p$, we want an induced map $TM_a \to TN_{f(a)}$ $\forall a \in M$. This induced map is the DERIVATIVE at a $f_{*a}: TM_a \to TN_{f(a)}$.

RECALL: If $a \in U \subset \mathbb{R}^n$ (open), then

(TANGENT SPACE @ a) TU_a or $T\mathbb{R}^n_a$ or $\mathbb{R}^n_a$

means just a copy of $\mathbb{R}^n$ "centred" at a.



$\rightsquigarrow (a, v) =$ "vector v beginning at a" $= v_a$

If we have $f: U \longrightarrow \mathbb{R}^p$, we get a map

$$f_{*a}: T\mathbb{R}^n_a \longrightarrow T\mathbb{R}^p_{f(a)}$$
$$v_a \longmapsto (Df(a)v)_{f(a)}$$

!

TANGENT BUNDLE OF U: $TU = U \times \mathbb{R}^n$

The points here are PAIRS
a vector x of U and a "tangent vector" v from

the copy of $\mathbb{R}^n$ at x : (x, v)

$$\begin{array}{cc} \cap & \cap \\ U & \mathbb{R}^n \end{array}$$

Now, natural map

$$\begin{array}{ccc} TU = U \times \mathbb{R}^n & & (x, v) \\ \Big\downarrow \pi & & \Big\downarrow \\ U & \ni & x \end{array}$$

PROJECTION ONTO THE ELEMENTS OF U

TU_a means the **FIBRE** $\pi^{-1}(a)$!

Now, for a smooth $f: U \to \mathbb{R}^p$, we get the map

$$f_*: TU \longrightarrow T\mathbb{R}^p \quad !!$$

$$f(x, v) = (f(x), Df(x)v)$$

Now, if we have

$$U \xrightarrow{f} \mathbb{R}^p \xrightarrow{g} \mathbb{R}^q$$

$$(g \circ f)_* = g_* \circ f_*,$$

<u>Proof:</u>

$$(g \circ f)_* (x, v) = \big((g \circ f)(x),\ D(g \circ f)(x) v \big)$$

$$= \big(g(f(x)),\ Dg(f(x))\, Df(x) v \big)$$

!

$$= g_* \big(f(x),\ Df(x) v \big)$$

$$= (g_* \circ f_*)(x, v)$$

<u>NOTE</u>: Tangent vectors v_a act on differentiable functions: $v_a(f) = D_v f(a) = Df(a) v$ DIRECTIONAL DERIVATIVE OF f AT a IN THE DIRECTION OF v

$v_a(f) = e_{i,a}$? Then $D_{e_i} f(a) = \frac{\partial f}{\partial x_i}(a)$

$$= \frac{\partial}{\partial x_i}\Big|_a (f).$$

Lecture 11: Tangent Bundle

This lecture is in Chapter 3 of Intro. to Smooth Man. by J. Lee.

$T\mathbb{R}^n_a$ = Tangent vectors $v_a \in T\mathbb{R}^n_a$ operate on differentiable functions:

$$v_a(f) := D_v f(a) = Df(a)\cdot v$$

Directional derivative of f at a in the direction of v.

E.g.: $e_{i,a}(f) = D_{e_i} f(a) = Df(a)\, e_i$

e_i selects the i-th column of $Df(a)$

$$= \frac{\partial f}{\partial x_i}(a)$$

$$= \left(\frac{\partial}{\partial x_i}\Big|_a\right)(f).$$

THUS, IDENTIFY $\frac{\partial}{\partial x_i}\Big|_a$ w/ TANGENT VECTORS $e_{i,a}$.

THEN, IF $f: \mathbb{R}^n \to \mathbb{R}$,

$$f_{*a}\left(\frac{\partial}{\partial x_i}\Big|_a\right) = f_{*a}(e_{i,a}) = \left(Df(a)\, e_i\right)_{f(a)} = \frac{\partial f}{\partial x_i}(a)$$

so, if $f : \mathbb{R}^n \to \mathbb{R}^p$, then we get

$$f_{*a} : T\mathbb{R}^n_a \to T\mathbb{R}^p_{f(a)}$$

such that

$$f_{*a}\left(\frac{\partial}{\partial x_i}\bigg|_a \right) = \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i}(a) \frac{\partial}{\partial y_j}\bigg|_{f(a)}$$

compute

$$\text{LHS} = f_{*a}\left(\frac{\partial}{\partial x_i}\bigg|_a \right)$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(a) & \cdots & \frac{\partial f_n}{\partial x_n}(a) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow i\text{th} \\ \\ \\ \end{array}$$

$$= \begin{pmatrix} \frac{\partial f_1}{\partial x_i}(a) \\ \vdots \\ \frac{\partial f_n}{\partial x_i}(a) \end{pmatrix}_{f(a)} = \text{RHS}.$$

DEF: (DERIVATION) $\left.\dfrac{\partial}{\partial x_i}\right|_a$ is called a DERIVATION

at a; i.e., it is a LINEAR MAPPING

$$\ell : \underbrace{\boxed{\begin{matrix}C^\infty \text{ functions on} \\ \text{a neighborhood of } a\end{matrix}}}_{C^\infty_{\mathbb{R}^n,a}} \longrightarrow \mathbb{R}$$

such that

$$\ell(f \cdot g) = \ell(f)\,g(a) + f(a)\,\ell(g).$$

!!! We use Leibniz Rule as a DEFINITION here!.

DEF: Denote by C^∞_a or $C^\infty_{\mathbb{R}^n,a}$ the RING OF GERMS OF C^∞ FUNCTIONS AT a.

A GERM OF A C^∞ FUNCTION AT a is defined to be the equivalence class of pairs (U, f) where U is an open neighborhood of a and

f is C^∞ on U. The equivalence relation is: $(U, f) \sim (V, g)$ if $f = g$ on a neighborhood W of a; i.e., $W \subset U \cap V$.

!

PROP: The SPACE $\mathrm{Der}(C^\infty_{\mathbb{R}^n, a})$ of linear derivations at a is an n-dimensional vector space spanned by $\left.\frac{\partial}{\partial x_i}\right|_a$.

LEMMA: (HADAMARD'S LEMMA) If $f(x_1, \ldots, x_n)$ is C^r, then

$$f(x) - f(0) = \sum_i x_i \, g_i(x),$$

where $g_i \in C^{r-1}$ $\forall i$. $\rightsquigarrow g_i(0) = \frac{\partial f}{\partial x_i}(0)$

Pf: (Hadamard's Lemma) Use Fundamental thm of Calculus:

$$f(x) - f(0) \overset{FTC}{=} \int_0^1 \underbrace{\frac{\partial f(tx)}{\partial t}}_{\sum \frac{\partial f}{\partial x_i}(tx)\, x_i} dt$$

$$= \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx)\, dt}_{=: g_i(x)} \qquad \square$$

"Loses" our diff. order

Remark: can also write Hadamard w/

$$f(x) - f(a) = \sum_i (x_i - a_i)\, g_i(x)$$

$$\Rightarrow g_i(a) = \frac{\partial f}{\partial x_i}(a).$$

Pf: (Proposition about $Der(C^\infty_{\mathbb{R}^n,a})$)

Need to show $\left(\frac{\partial}{\partial x_j}\Big|_a\right)$ are linearly independent and span $Der(C^\infty_{\mathbb{R}^n,a})$.

SPAN: Let $\ell \in Der(C^\infty_{\mathbb{R}^n,a})$.

<u>Aside:</u> $\ell(1) = \ell(1\cdot 1) = \ell(1)\cdot 1 + 1\cdot \ell(1)$
$$= 2\ell(1) \Rightarrow \ell(1) = 0$$

$\Rightarrow \ell(c) = 0$ for any constant by linearity.

Now, given $f \in C^\infty_{\mathbb{R}^n,a}$, we have

$$\ell(f) \overset{\ell(f(a)) \text{ constant by linearity}}{=} \ell(f - f(a))$$

$$\overset{\text{Hadamard's lemma}}{=} \ell\left(\sum_{i=1}^n (x_i - a_i) g_i(x)\right)$$

$$\overset{\text{linearity}}{=} \sum_{i=1}^n \left(\underbrace{(a_i - a_i)\,\ell(g_i(x))}_{=0} + g_i(a)\,\ell(x_i - a_i)\right)$$

$$= \sum_{i=1}^n g_i(a)\,\ell(x_i - a_i)$$

$$\overset{\text{Conclusion of Hadamard}}{=} \sum_{i=1}^n \ell(x_i - a_i) \left.\frac{\partial}{\partial x_i}\right|_a (f).$$

This means that $\left.\frac{\partial}{\partial x_i}\right|_a$ span $\operatorname{Der}(C^\infty_{\mathbb{R}^n,a})$.

LINEARLY INDEPENDENT: Need to show that

$$\sum_i c_i \left(\frac{\partial}{\partial x_i}\Big|_a \right) = 0 \Rightarrow c_i = 0$$

Just apply this to x_j.

$\square$

CONCLUSION:

$$\mathrm{Der}\left(C^\infty_{\mathbb{R}^n, a}\right) \simeq T\mathbb{R}^n_a$$

Therefore.. define:

DEF: (TANGENT SPACE) The TANGENT SPACE OF M AT $a \in M$ is the LINEAR SPACE $\mathrm{Der}(C^\infty_{M,a})$ of derivations of the ring $C^\infty_{M,a}$ of germs of C^∞ functions on M at a.

* INDUCED MAPS

!!!

Suppose $f: M \to N$ is smooth and $a \in M$.

Then, this induces a RING HOMOMORPHISM

$$f_a^*: C^\infty_{N,f(a)} \longrightarrow C^\infty_{M,a}$$
$$h \longmapsto h \circ f$$

In turn, this induces a map !

$$f_{*a}: \underset{\displaystyle \mathrm{Der}(C^\infty_{M,a})}{\underset{\|}{TM_a}} \longrightarrow \underset{\displaystyle \mathrm{Der}(C^\infty_{N,f(a)})}{\underset{\|}{TN_{f(a)}}}$$

! !

given by !

$$f_{*a}(\lambda)(h) := \lambda(f_a^*(h)) = \lambda(h \circ f)$$

$\in \mathrm{Der}(C^\infty_{M,a})$

$\in C^\infty_{N,f(a)}$

DIFFERENTIAL OF f AT a

Claim: $f_{*a}(\lambda)$ is a derivation at $f(a)$.

Pf: Clearly linear. So, only need to check it satisfies the Leibniz Rule:

Take $g, h \in C^\infty_{N, f(a)}$. Then

$(g \circ f) \cdot (h \circ f)$

$$f_{*a}(\lambda)(g \cdot h) \overset{\text{def}}{=} \lambda\big((g \cdot h) \circ f\big)$$

$$= g(f(a))\, \lambda(h \circ f) + h(f(a))\, \lambda(g \circ f)$$

$$= g(f(a))\, f_{*a}(\lambda)(h) + h(f(a))\, f_{*a}(\lambda)(g)$$

Def. of Leibniz Rule for derivation.

!!

$\square$

LEMMA:

(1) f_{*a} is a linear mapping;

(2) If $M \xrightarrow{f} N \xrightarrow{g} P$, then

$$(g\circ f)_{*a} = g_{*f(a)} \circ f_{*a}$$

with

$$TM_a \xrightarrow{f_{*a}} TN_{f(a)} \xrightarrow{g_{*f(a)}} TP_{g(f(a))}$$

Pf: (2) Compute

$$(g\circ f)_{*a}(\lambda)(h) \overset{\text{def}}{=} \lambda(h\circ g\circ f)$$

$\lambda \in TM_a$

$h \in C^\infty_{P,\,g(f(a))}$

$$= \underbrace{f_{*a}(\lambda)}_{\text{derivation at } f(a)}(h\circ g)$$

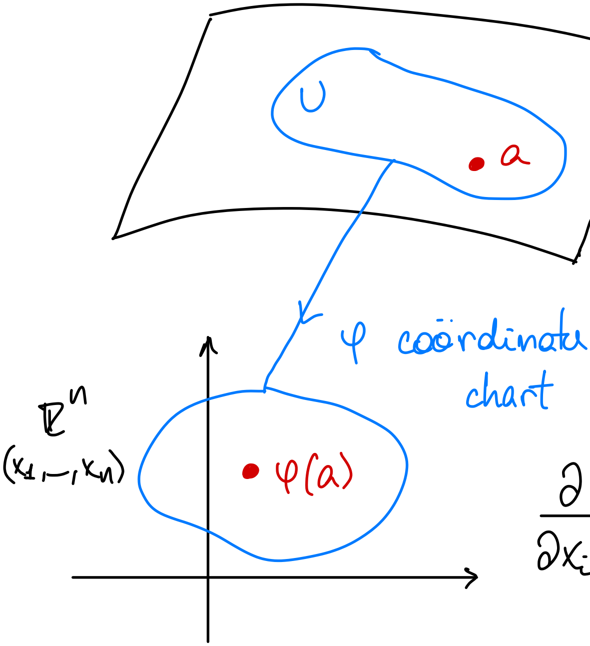
$$= (g_{*f(a)} \circ f_{*a})(\lambda)(h),$$

as desired. $\square$

In particular, if f is a diffeomorphism then f_{*a} is an isomorphism between tangent spaces.

— // —

* COMPUTATION IN LOCAL COÖRDINATES



$\left.\frac{\partial}{\partial x_i}\right|_a$ (on M)

function on M

$\frac{\partial f}{\partial x_i}(a)$ means

$$\frac{\partial (f \circ \varphi^{-1})}{\partial x_i}(\varphi(a))$$

$$\left.\frac{\partial}{\partial x_i}\right|_a (f) \longleftrightarrow \left.\frac{\partial}{\partial x_i}\right|_{\varphi(a)} (f \circ \varphi^{-1})$$

So,

$$\left.\frac{\partial}{\partial x_i}\right|_a = (\varphi^{-1})_{*\varphi(a)} \left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(a)} \right)$$

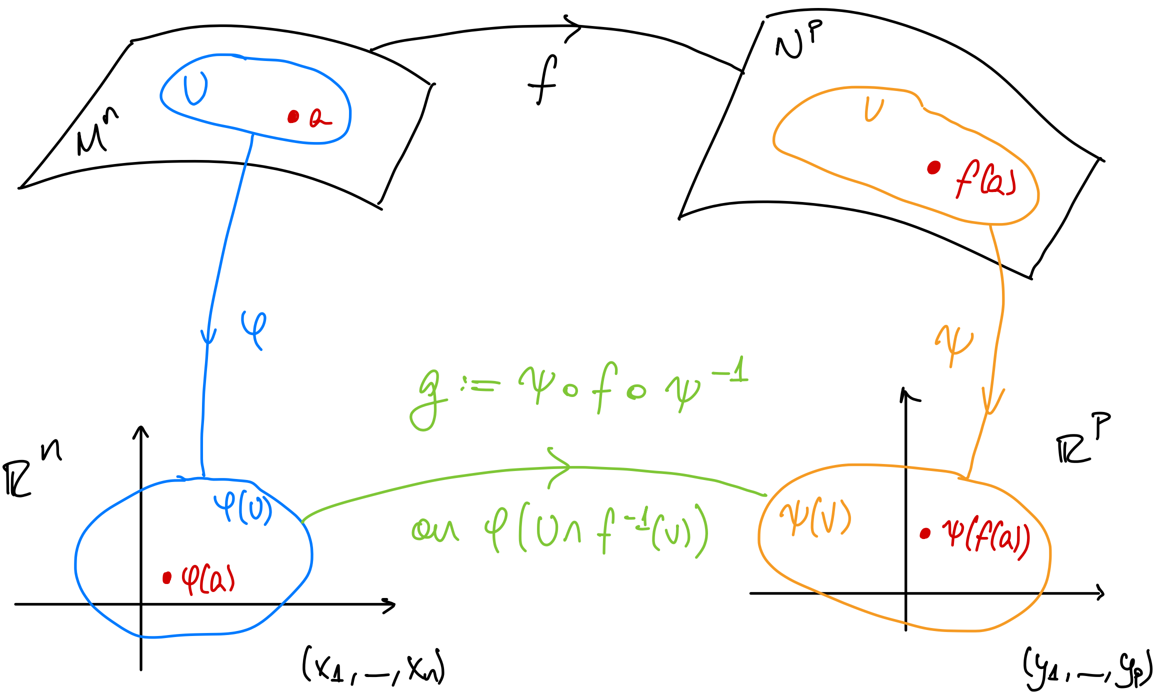
$$\varphi_{*a}\left(\left.\frac{\partial}{\partial x_i}\right|_a \right) = \left(\left.\frac{\partial}{\partial x_i}\right|_{\varphi(a)} \right)$$

$$\Rightarrow \quad (\varphi^{-1})_{*\varphi(a)} = (\varphi_{*a})^{-1}.$$

This means that ... !!!

LEMMA: TM_a is an n-dimensional vector space with basis given by the $\left.\frac{\partial}{\partial x_i}\right|_a$ (from a given coördinate chart).

* MAKE SENSE OF THE CHAIN RULE



! **Upshot**: We have the following commutative diagram: ↱ by defs.

$$\begin{tikzcd} TM_a \arrow[r, "f_{*a}"] \arrow[d] & TN_{f(a)} \arrow[d] \\ T\mathbb{R}^n_{\varphi(a)} \arrow[r, "g_{*\varphi(a)}"'] & T\mathbb{R}^p_{\psi(f(a))} \end{tikzcd}$$

We claim that $\boxed{g_{*\varphi(a)} = Dg(\varphi(a))}$.

Indeed, the derivative at $g(\varphi(a)) = \psi(f(a))$

$$g_{*\varphi(a)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(a)}\right)(h) \overset{\text{def}}{=} \frac{\partial}{\partial x_i}\Big|_{\varphi(a)}(h \circ g)$$

$$= \frac{\partial (h\circ g)}{\partial x_i}(\varphi(a)) = \sum_{j=1}^{n} \frac{\partial h}{\partial y_j}(g(\varphi(a)))\frac{\partial g_j}{\partial x_i}(\varphi(a))$$

$$= \frac{\partial}{\partial y_j}\Big|_{\psi(f(a))}(h) \Rightarrow g_{*\psi(a)}\left(\frac{\partial}{\partial x_i}\Big|_{\psi(a)}\right) = \sum_{j=1}^{n} \frac{\partial g_j}{\partial x_i}(\varphi(a))\frac{\partial}{\partial y_j}\Big|_{\psi(f(a))}$$

$$= Dg(\varphi(a))\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(a)}\right). \square$$

Lecture 12: Tangent Bundle (Finally)!

Recall: For $U \underset{open}{\subset} \mathbb{R}^m$, we defined the tangent bundle $TU := U \times \mathbb{R}^m$, where the elements
TRIVIAL BUNDLE
are (a, v), $a \in U$, $v \in \mathbb{R}^m$

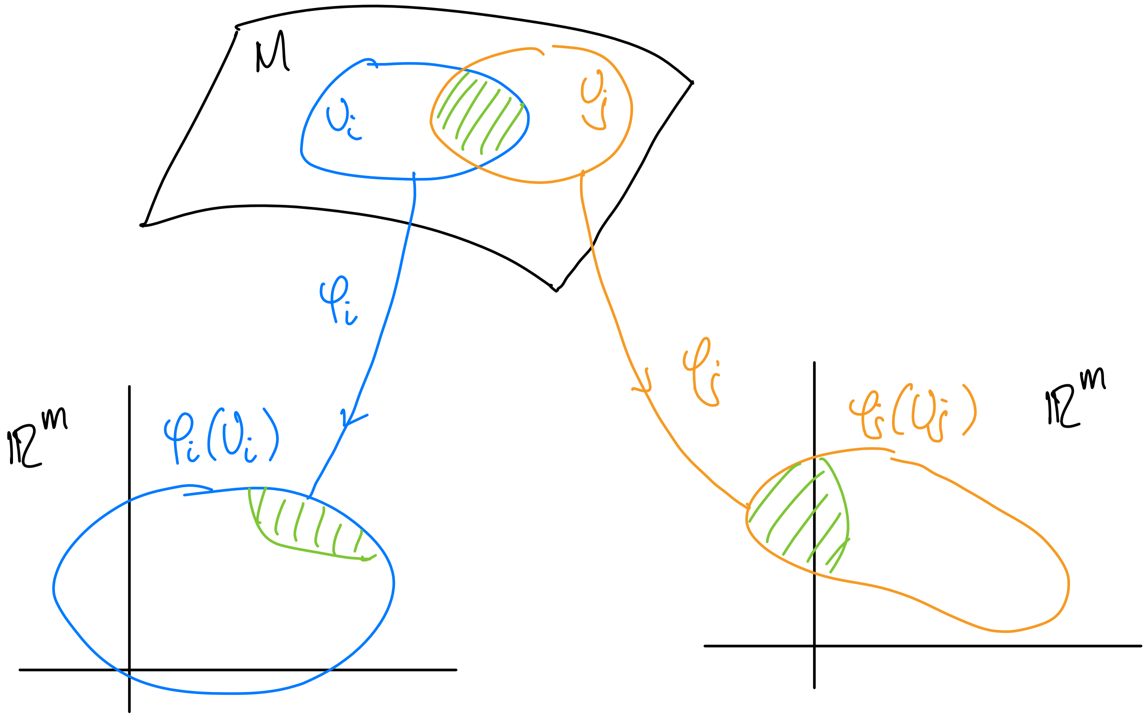
Could do this b/c for stuff inside $\mathbb{R}^m$, the tangent spaces are just a $\mathbb{R}^m$ "glued" at the vector $a \in U$.

Then, if we have a map $f : U \to \mathbb{R}^n$, $f \in C^\infty$, then we get an induced map

$$f_* : TU \longrightarrow T\mathbb{R}^n$$
$$(x, v) \mapsto (f(x), \underbrace{Df(x)(v)}_{f_{*x}(v)})$$

!

As a set, TM will be the disjoint union of all TM_x , $x \in M$



Now,

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \longrightarrow \varphi_j(U_i \cap U_j)$$

Induces

$$T\varphi_i(U_i \cap U_j) \xrightarrow{(\varphi_j \circ \varphi_i^{-1})_*} T\varphi_j(U_i \cap U_j)$$

|| Trivial bundle in $\mathbb{R}^m$ Trivial bundle in $\mathbb{R}^m$ ||

$\varphi_i(U_i \cap U_j) \times \mathbb{R}^m$ $\varphi_j(U_i \cap U_j) \times \mathbb{R}^m$

where, as before:

TRANSITION MAPS FOR THE BUNDLE ATLAS !

$$(\varphi_j \circ \varphi_i^{-1})_*(x, v) = \left((\varphi_j \circ \varphi_i^{-1})(x),\ D(\varphi_j \circ \varphi_i^{-1})(x)(v)\right)$$

As shown in HW0207, this "transition map structure" defines a manifold whose charts are these.

This structure defines a "bundle atlas":
the transition maps define an equivalence relation on the disjoint union of all

$$\varphi_i(U_i) \times \mathbb{R}^m = T\varphi_i(U_i)$$

where $\begin{cases} (x, v) \in \varphi_i(U_i) \times \mathbb{R}^m \\ (y, w) \in \varphi_j(U_j) \times \mathbb{R}^m \end{cases}$ are <u>EQUIVALENT</u>

if $x \in \varphi_i(U_i \cap U_j)$ and

$$(y, w) = \left((\varphi_j \circ \varphi_i^{-1})(x),\ D(\varphi_j \circ \varphi_i^{-1})(x)(v)\right).$$

So, the quotient of this disjoint union by the equivalence relation defined above with the quotient topology defines the manifold TM, provided that the quotient space is 2^{nd} countable and Hausdorff.

Countable union of 2^{nd} countable spaces.

You can always seperate pts b/c:

- if $x \neq y$, then use that M is Hausdorff
- if $x = y$, then v and w are just two vector in Euclidean pace $\Rightarrow$ can separate

Remark: this atlas is more than special: it involves the map of special form

$$\begin{aligned} & x \longmapsto D(\varphi_j \circ \varphi_i^{-1})(x) \\ C^\infty \ni \ & \varphi_i(U_i \cap U_j) \longrightarrow \mathrm{Hom}(\mathbb{R}^m, \mathbb{R}^m) \end{aligned}$$

This defines a "BUNDLE ATLAS" (atlas for the tangent bundle whose transition maps have the special form above)... for now

Upshot: TM is a manifold with

$$\dim TM = 2 \dim M.$$

STRUCTURE OF TM: $TM \rightsquigarrow$ manifold of dimension $2 \cdot \dim M$ with a **PROJECTION OPERATOR** that is smooth and given by

$$TM \xrightarrow{\pi_M} M$$

$$\pi_M^{-1}(a) = TM_a$$

There is also C^∞ sections $s : M \longrightarrow TM$ which are C^∞ maps such that

$$\pi_M \circ s = id_M$$

"Fiber of TM at x is TM_x"

Now, if we have $f: M \to N$, we get induced maps

$$TM_x \xrightarrow{f_{*x}} TN_{f(x)}$$

$$TM \xrightarrow{f_*} TN$$

LECTURE 13: TANGENT BUNDLE (ctd.)

Recall: If we have $U \subset \mathbb{R}^m$ open, then the tangent bundle TU to U is the trivial bundle $TU = U \times \mathbb{R}^m$ composed of pairs (x, v), where $x \in U$ and $v \in \mathbb{R}^m \Rightarrow (x, v) \in TU_x$.

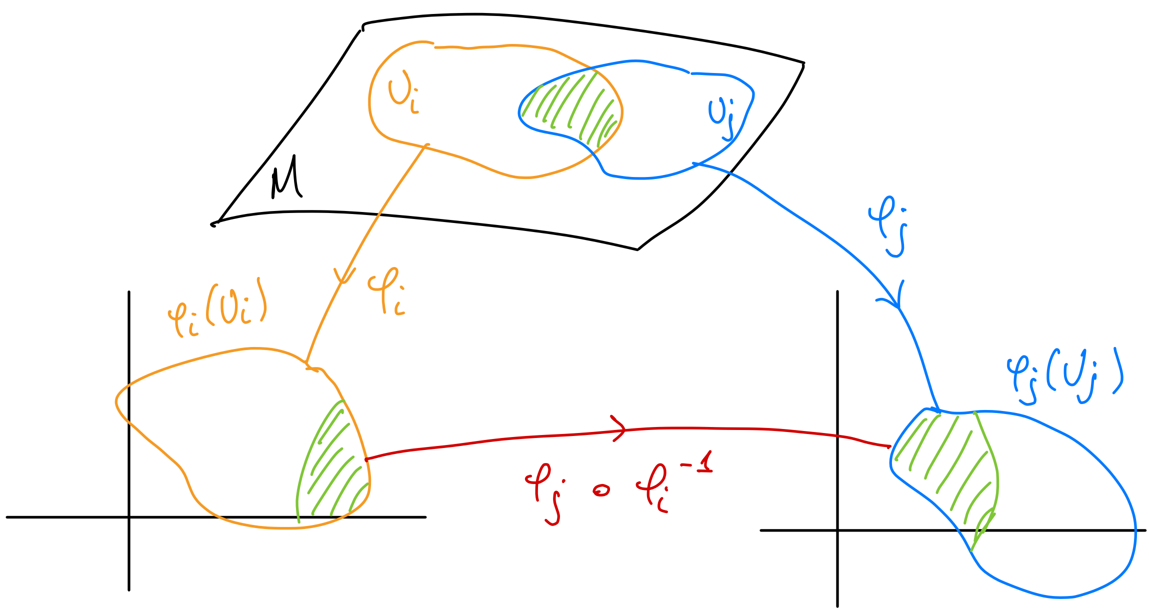
In addition, if we have a map $f: U \to \mathbb{R}^n$ that is smooth, we have an induced map

$$f_* : TU \longrightarrow T\mathbb{R}^n$$
$$(x, v) \longmapsto (f(x), \underbrace{Df(x)}_{f_{*x}} v)$$

Now, for a manifold M. The <u>tangent bundle</u> TM of M can be viewed as the disjoint union of all TM_x $\forall x \in M$.

<u>Claim</u>: Can write coördinate charts for this.

<u>Indeed</u>: CHARTS FOR TM ! We have the coördinate charts for M. They are all C^∞, so they



induce maps:

$$\varphi_i(U_i\cap U_j)\times\mathbb{R}^n \underset{\text{bundles}}{\overset{\text{Trivial}}{\longleftrightarrow}} \varphi_j(U_i\cap U_j)\times\mathbb{R}^n$$

$$\begin{aligned}(\varphi_j\circ\varphi_i^{-1})_*:\ T\varphi_i(U_i\cap U_j) &\longrightarrow T\varphi_j(U_i\cap U_j)\\ (x,v) &\longmapsto \Big((\varphi_j\circ\varphi_i^{-1})(x),\ \underbrace{(\varphi_j\circ\varphi_i^{-1})_{*x}v}_{D(\varphi_j\circ\varphi_i^{-1})(x)v}\Big)\end{aligned}$$

Moreover, the map $x\longmapsto(\varphi_j\circ\varphi_i^{-1})_{*x}$ is a C^∞-mapping from $\varphi_i(U_i\cap U_j)\longrightarrow \mathrm{Hom}(\mathbb{R}^n,\mathbb{R}^n)$.

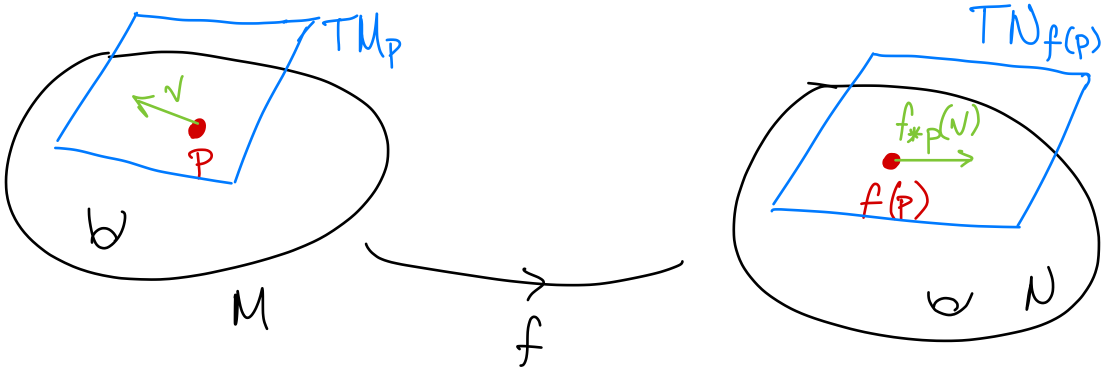
! This structure defines a "BUNDLE ATLAS". The transition mappings define an equivalence relation on the disjoint union of the spaces $T\varphi_i(U_i)$ given by: **!**

$$T\varphi_i(U_i)\ni(x,v)\sim(y,w)\in T\varphi_j(U_j)$$

if and only if

$$x\in\varphi_i(U_i\cap U_j) \quad\text{and}$$

$$(y,w)=\big((\varphi_j\circ\varphi_i^{-1})(x),\ D(\varphi_j\circ\varphi_i^{-1})(x)v\big)$$



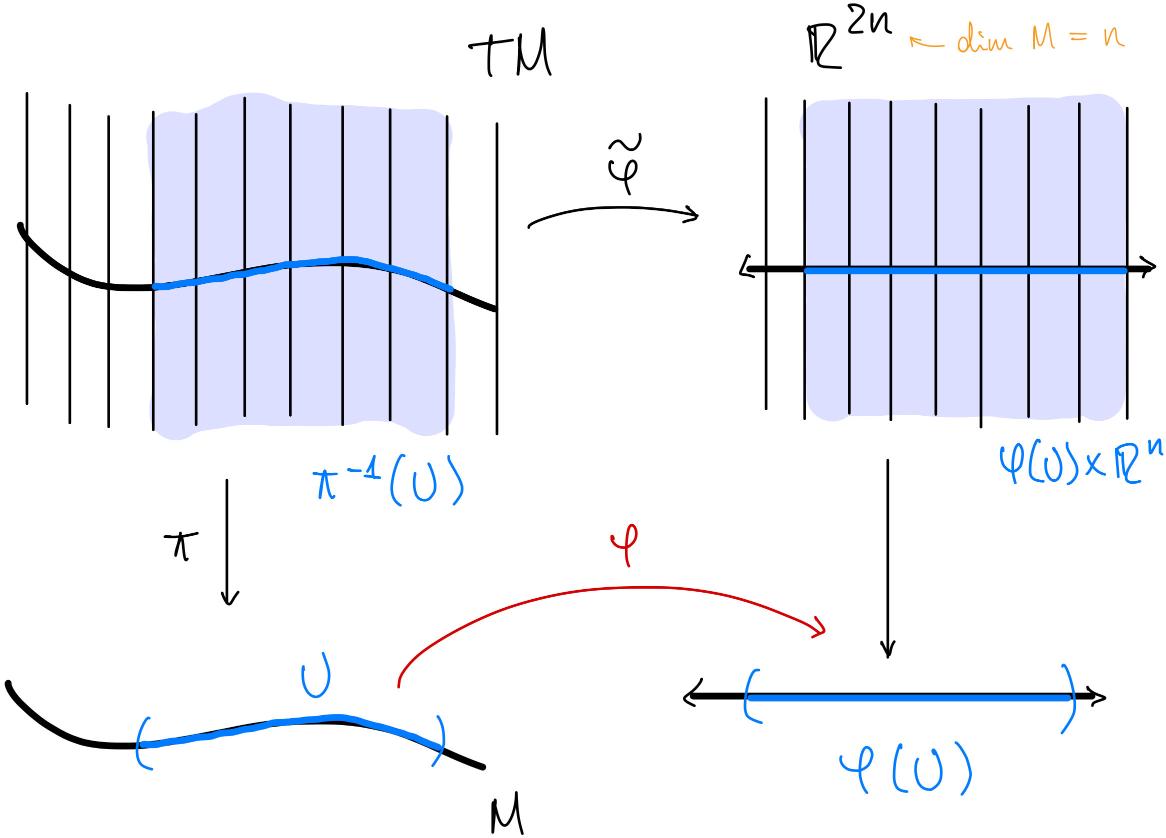
<u>Def</u>: The quotient of $\bigsqcup_i T\varphi_i(U_i)$ by the equivalence relation described above defines a C^∞ manifold denoted TM (since the quotient is Hausdorff and 2^{nd} countable):

$$\underset{\substack{\downarrow \\ p\in M}}{(p,}\ \underset{\substack{\downarrow \\ v\in TM_p}}{v)} \in \boxed{TM := \bigsqcup_{p\in M} TM_p}.$$

Note that $\boxed{\dim TM = 2\cdot \dim M}$. Moreover, TM comes equipped with a natural C^∞ <u>projection map</u> $\pi: TM \to M$, $\pi(p,v) = p$.

Note: π is a submersion and the fibre $\pi^{-1}(p) = TM_p$ is a vector space.

Coördinate Charts for the Tangent Bundle:



Def: (C^∞-Section of Tangent Bundle TM) A C^∞-section of the tangent bundle TM is a smooth map $X : M \longrightarrow TM$ such that

$$\pi_M \circ X = Id_M.$$

In other words, $X(a) \in TM_a \ \forall a \in M$.

If $a \in (\varphi, U)$, coördinate chart for M with local coördinates $(x_1, \_\_\_, x_m)$ then

$$X(a) := \sum_i c_i(a) \left.\frac{\partial}{\partial x_i}\right|_a .$$

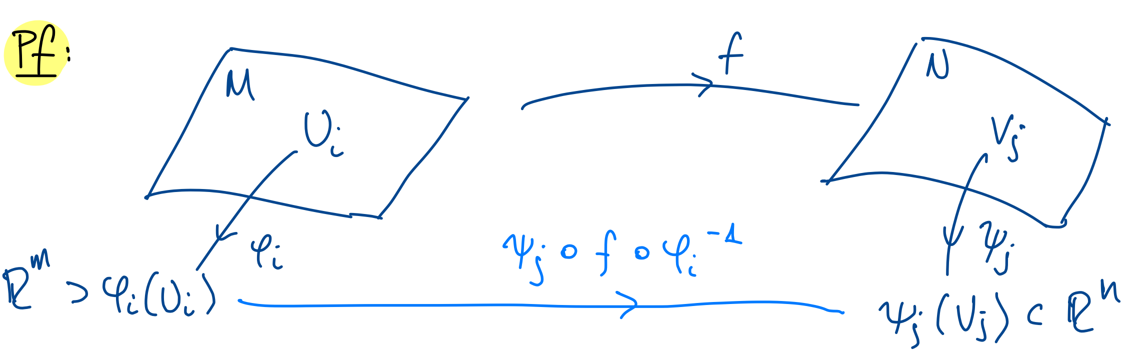
Then X is smooth if and only if all the c_i's are smooth. These mappings X are called "VECTOR FIELDS" on M.

Lemma: If $f : M^m \longrightarrow N^n$ is smooth, then there exists an INDUCED BUNDLE MAPPING

$$f_* : TM \longrightarrow TN$$

that restricts to f_{*x} on TM_x $\forall x \in M$.

Pf:



Then, we have the following:

$$\begin{tikzcd}
U_i \cap f^{-1}(V_j) \arrow[r, "f"] \arrow[d, "\varphi_i"'] & V_j \arrow[d, "\psi_j"] \\
\varphi_i(U_i \cap f^{-1}(V_j)) \arrow[r, "\psi_j \circ f \circ \varphi_i^{-1}"'] & \psi_j(V_j)
\end{tikzcd}$$

The above (commutative) diagram induces the following:

$$\begin{tikzcd}
TM \supset \pi_M^{-1}(U_i \cap f^{-1}(V_j)) \arrow[r, "f_*"] \arrow[d] & \pi_N^{-1}(V_j) \subset TN \arrow[d] \\
\varphi_i(U_i \cap f^{-1}(V_j)) \times \mathbb{R}^m & \psi_j(V_j) \times \mathbb{R}^n
\end{tikzcd}$$

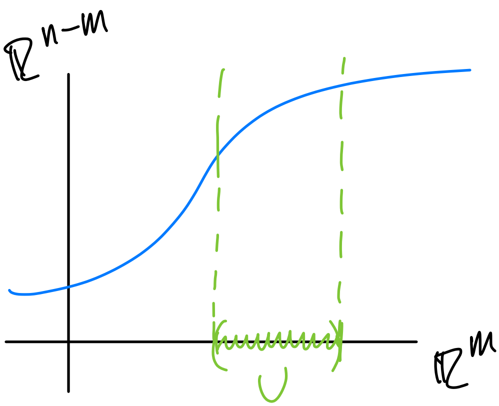
$$(x, v) \longmapsto \Big((\psi_j \circ f \circ \varphi_i^{-1})(x), \underbrace{(\psi_j \circ f \circ \varphi_i^{-1})_{*x} v}_{\text{LINEAR MAP}} \Big)$$

as desired. $\square$

EXAMPLE: Consider a smooth manifold $M^n \subset \mathbb{R}^n$. As we saw before, this means that the inclusion map $\iota : M \hookrightarrow \mathbb{R}^n$ is an embedding and induces the map $\iota_* : TM \to T\mathbb{R}^n$.

This means that TM is a SUBBUNDLE of $T\mathbb{R}^n$. So, in particular, TM_a is a linear subspace of $T\mathbb{R}^n_a$.

Now, suppose M is locally the graph of a C^∞ function $g : U \to \mathbb{R}^{n-m}$



$$\Gamma(g) : U \longrightarrow \overbrace{\mathbb{R}^m \oplus \mathbb{R}^{n-m}}^{\mathbb{R}^n \,=}$$

$$x \longmapsto (x, g(x))$$

Can take the "derivative" of $\Gamma(g)$ as follows:

$$\Gamma(g)_{*x} : T\mathbb{R}^m_x \longrightarrow T\mathbb{R}^n_{(x,g(x))}$$

$$v \longmapsto \begin{pmatrix} I \\ Dg(x) \end{pmatrix} v$$

So that

$$\Gamma(g)_* : \underbrace{TU}_{= U \times \mathbb{R}^{n-m}} \longrightarrow T\mathbb{R}^n$$

$$(x, v) \longmapsto \big((x, g(x)), (v, Dg(x)v) \big)$$

Finally, we have that

$$TM \cap (U \times \mathbb{R}^{n-m}) \times \mathbb{R}^m$$

is the graph of

$$(*) \quad \begin{aligned} U \times \mathbb{R}^{n-m} &\longrightarrow \mathbb{R}^{n-m} \times \mathbb{R}^{n-m} \\ (x, v) &\longmapsto (g(x), Dg(x)v) \end{aligned}$$

Note that the graph of $(*)$ is

$$(x, v) \longmapsto (x, v, g(x), Dg(x)v).$$

!!! **Def**: (VECTOR BUNDLE) A C^∞ k-dimensional VECTOR BUNDLE (or k-plane bundle) is a five-tuple

$$\xi := (E, \pi, B, \oplus, \odot),$$

where

(1) E ("fibre") is the "TOTAL SPACE" of ξ and B is the "BASE SPACE" of ξ;

(2) $\pi : E \longrightarrow B$ is a continuous surjection ("BUNDLE PROJECTION");

(3) $\oplus$ and $\odot$ are maps

$$\oplus : \bigcup_{p \in B} \pi^{-1}(p) \times \pi^{-1}(p) \longrightarrow E$$

$$\odot : \mathbb{R} \times E \longrightarrow E$$

with $\oplus(\pi^{-1}(p) \times \pi^{-1}(p)) \subset \pi^{-1}(p)$ and $\odot(\mathbb{R} \times \pi^{-1}(p)) \subset \pi^{-1}(p)$, which makes each fibre

$\pi^{-1}(p)$ into an k-dimensional real vector space

such that the following condition is satisfied:

(LOCAL TRIVIALITY) for each $p \in B$, there is a neighborhood U of p and a homeomorphism $t : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ which is a vector space isomorphism from each $\pi^{-1}(q)$ onto $\{q\} \times \mathbb{R}^k$ for all $q \in U$.

Simplify notation: denote vector bundles by $\pi : E \to B$ (or even just E) and $\oplus(v, w) = v + w$ and $\odot(\lambda, v) = \lambda v$.

! In order words, a VECTOR BUNDLE $\pi : E \to M$ of fibre-dimension k over a C^∞ manifold M is a submersion $\pi : E \to M$ of corank k and each fibre $E_a = \pi^{-1}(a)$ $(= TM_a)$ is a vector space of dimension k together with a bundle atlas.

BUNDLE ATLAS: Let $\{(\varphi_i, U_i)\}$ be an atlas of M. Then, we get an atlas for E by $\{(\underbrace{\Phi_i}_{\text{local trivializations}}, \pi^{-1}(U_i))\}$, where

$$\begin{tikzcd}
\pi^{-1}(U_i) \arrow[r, "\Phi_i", "\text{diffeo.}"'] \arrow[d, "\pi"'] & \varphi_i(U_i) \times \mathbb{R}^k \subset \mathbb{R}^m \times \mathbb{R}^k \arrow[d, "\text{projection}"] \\
M \supset U_i \arrow[r, "\varphi_i"'] & \varphi_i(U_i) \subset \mathbb{R}^m
\end{tikzcd}$$

such that, for all i, j,

$$\begin{tikzcd}
& U_i \cap U_j \arrow[dl, "\varphi_i"'] \arrow[dr, "\varphi_j"] & \\
\varphi_i(U_i \cap U_j) \arrow[rr, "\varphi_j \circ \varphi_i^{-1}"'] & & \varphi_j(U_i \cap U_j)
\end{tikzcd}$$

and over this:

$$\begin{tikzcd}
 & \pi^{-1}(U_i \cap U_j) \arrow[dl, "\Phi_i"'] \arrow[dr, "\Phi_j"] & \\
\varphi_i(U_i \cap U_j) \times \mathbb{R}^k \arrow[rr] & & \varphi_j(U_i \cap U_j) \times \mathbb{R}^k
\end{tikzcd}$$

$$(x, v) \longmapsto \left((\varphi_j \circ \varphi_i^{-1})(x), \Lambda_{ij}(x) v \right)$$

where $\Lambda_{ij} : \varphi_i(U_i \cap U_j) \longrightarrow \underbrace{GL(k, \mathbb{R})}$ is a C^∞-mapping.

Invertible linear transformations $\mathbb{R}^k \to \mathbb{R}^k$.

!!

Def: (BUNDLE MAPPING) A C^∞ <u>bundle mapping</u> from $\xi_1 = (E_1, \pi_1, M_1)$ (fibre dimension $= k$) to $\xi_2 = (E_2, \pi_2, M_2)$ (fibre dimension $= \ell$) is a <u>pair of continuous maps</u> (F, f) with $F: E_1 \longrightarrow E_2$ and $f : M_1 \longrightarrow M_2$ such that

(1) the following diagram commutes:

$$\begin{tikzcd} E_1 \arrow[r, "F"] \arrow[d, "\pi_1"'] & E_2 \arrow[d, "\pi_2"] \\ M_1 \arrow[r, "f"'] & M_2 \end{tikzcd}$$

(2) $F : \pi_1^{-1}(p) \longrightarrow \pi_2^{-1}(f(p))$ is a linear map.

In other words, (F, f) such that there are bundle atlases as above over $\{(\varphi_i, U_i)\}$ for M_1 and over $\{(\psi_j, V_j)\}$ for M_2 so that, for all i, j,

$$\begin{tikzcd} E_1\big|_{U_i \cap f^{-1}(V_j)} \arrow[r, "F"] \arrow[d, "\Phi_i"', "\text{diffeo}"] & E_2\big|_{V_j} \arrow[d, "\Psi_j", "\text{diffeo}"'] \\ \varphi_i(U_i \cap f^{-1}(V_j)) \times \mathbb{R}^k \arrow[r] & \psi_j(V_j) \times \mathbb{R}^l \end{tikzcd}$$

$$(x, v) \longmapsto \left((\psi_j \circ f \circ \varphi_i^{-1})(x), \underbrace{\Lambda_{ij}(x)}\, v \right)$$

$\Lambda_{ij} : \varphi_i(U_i \cap f^{-1}(V_j)) \longrightarrow \mathrm{Hom}(\mathbb{R}^k, \mathbb{R}^l)$ is C^∞

Lecture 14: Tangent Bundle (some examples)

Note: A C^∞ vector bundle isomorphism is, as the name suggests, a C^∞ vector bundle mapping with an inverse which is itself a C^∞ vector bundle mapping; i.e., it is an isomorphism of C^∞ vector bundles.

Example 1: Say M^m is a smooth manifold and $A : M \longrightarrow \underbrace{\mathcal{M}(p,m)}$. ↪ space of $p \times m$ matrices

Then

$$\begin{array}{ccc} M \times \mathbb{R}^m & \longrightarrow & M \times \mathbb{R}^p \\ (x,v) & \longmapsto & (x, A(x)v) \\ \downarrow & & \downarrow \\ M & = & M \end{array}$$

This defines a C^∞ vector bundle mapping if and only if A is C^∞.

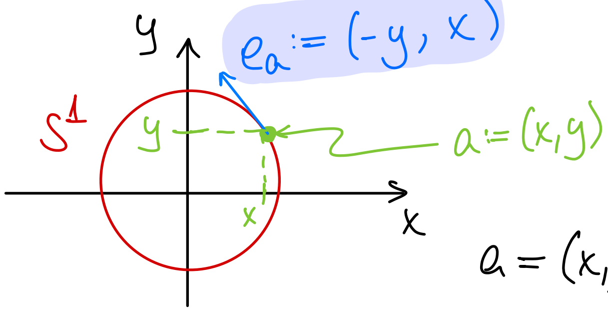
Similarly, this defines a C^∞ vector bundle isomorphism if and only if $p = m$ and A has image in $GL(n, \mathbb{R})$.

$\square$

<u>EXAMPLE 2</u>: TS^1 is trivial; i.e., $TS^1 \simeq S^1 \times \mathbb{R}$:

$$\begin{tikzcd} TS^1 \arrow[rr, "\simeq"] \arrow[dr] & & S^1 \times \mathbb{R} \arrow[dl] \\ & S^1 & \end{tikzcd}$$

In order to show that $TS^1 \simeq S^1 \times \mathbb{R}$, we must define an isomorphism between the two spaces.



Note that for $a := (x,y) \in S^1$ we get a basis for the tangent space at $a = (x,y)$ given by $e_a := (-y, x)$

This gives a nowhere-vanishing vector field on S^1 given by: $a \mapsto e_a$.

Define $f : TS^1 \longrightarrow S^1 \times \mathbb{R}$ by

$$f(\lambda e_a) := (a, \lambda).$$

This f is clearly invertible, hence $TS^1 \simeq S^1 \times \mathbb{R}$; i.e., TS^1 is trivial. $\square$

Q: Is TS^2 trivial?

A: No! Hairy-ball theorem

Only TS^1, TS^3, and TS

Sponsored by a procrastinating Marcello

ASIDE: (Hairy Ball Theorem) There are no non-vanishing continuous vector fields on <u>even</u> dimensional spheres.

(First proof for S^2 due to Poincaré in 1885 and generalization to S^n, n even, by Brouwer in 1912)

For any topological space, the Euler characteristic can be written as

$$\chi = b_0 - b_1 + b_2 - b_3 + \cdots,$$

where the b_n's are the Betti numbers. The n-th Betti number b_n is the rank (# of linearly independent generators) of the n-th singular homology group $H_n = \ker \delta_k / \operatorname{im} \delta_{k+1}$ (where the δ_k's are the boundary maps of the simplicial complex).

Suppose you could comb a n-sphere, n even (i.e., that S^n, n even, had a nonvanishing vector field V). Then $\mathrm{Fix}(V) = \emptyset$ so $\sum_{p \in \mathrm{Fix}(V)} \mathrm{Ind}(V,p) = 0$. But, by Poincaré-Hopf,

$$\sum_{p \in \mathrm{Fix}(V)} \mathrm{Ind}(V,p) = \chi(S^n)$$

Betti numbers

$$= \underbrace{1}_{b_0} - 0 + \cdots - 0 + \underbrace{1}_{b_n} - 0 \cdots$$

n even $\rightarrow$ $= 2$ $\hookrightarrow \not\hookleftarrow$

True for any compact manifold M with $\chi(M) \neq 0$. $\rightarrow$ $\square$

EXAMPLE 3: **TAUTOLOGICAL LINE BUNDLE** over $\mathbb{RP}^n$ (or $\mathbb{CP}^n$), when we interpret $\mathbb{RP}^n$ as the space of lines through $0 \in \mathbb{R}^{n+1}$. Clearly, we want

$$E \longrightarrow \mathbb{RP}^n$$

$$E_\lambda \longmapsto \lambda = [\lambda_0, \lambda_1, \ldots, \lambda_n]$$

line in $\mathbb{R}^{n+1}$ corresponding to λ $\rightarrow$ E_λ

Naturally, we can choose

$$E \longrightarrow \mathbb{RP}^n$$

$$\{(\lambda, x) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : \underbrace{x \in \lambda}_{x_i \lambda_j = x_j \lambda_i \ \forall i,j}\} \longmapsto \lambda = [\lambda_0, \ldots, \lambda_n]$$

Now, $\mathbb{RP}^n$ is covered by coördinate charts U_i,

$i = 0, 1, \dots, n$, given by

$$\varphi_i : U_i = \{\lambda \in \mathbb{RP}^n : \lambda_i \neq 0\} \longrightarrow \mathbb{R}^n$$

$$[\lambda_0, \dots, \lambda_n] = \lambda \longmapsto \frac{1}{\lambda_i}(\lambda_0, \dots, \widehat{\lambda_i}, \dots, \lambda_n)$$

w/ inverse φ_i^{-1} given by

$$(y_0, \dots, \widehat{y_i}, \dots, y_n) \longmapsto [y_0, \dots, 1, \dots, y_n].$$

Now, consider what E looks like over U_i:

$$E\big|_{U_i} : \qquad x_j = x_i \frac{\lambda_i}{\underbrace{\lambda_j}_{\text{coördinates on } U_i}} \rightarrow \text{Graph of a function}$$

$$(\lambda, x) \longmapsto \left(\left(\frac{\lambda_0}{\lambda_i}, \dots, \widehat{\frac{\lambda_i}{\lambda_i}}, \dots, \frac{\lambda_n}{\lambda_i}\right), x_i\right)$$

with inverse given by

$$((y_0, \dots, \widehat{y_i}, \dots, y_n), t) \longmapsto ([y_0, \dots, 1, \dots, y_n], x)$$

where $x = (x_0, \dots, x_n)$ and $x_i = t$, $x_j = t y_j \ \forall j \neq i$.

Now, $E \xrightarrow{p} \mathbb{RP}^{n+1}$, $(\lambda, x) \longmapsto x$ which gives

$$p^{-1}(x) = \begin{cases} \text{singleton}, & x \neq 0 \\ \mathbb{RP}^n, & x = 0 \end{cases} \rightsquigarrow \text{Clearly, \underline{not} a vector bundle!}$$

LECTURE 15: VECTOR BUNDLES

* THE GRASSMANIAN !!

DEF: (GRASSMANIAN)

$Gr(k,n)$ = space of k dimensional linear subspaces of $\mathbb{R}^n$ of dimension $k(n-k)$

E.g.: $Gr(1,n) = \mathbb{R}P^{n-1}$

TAUTOLOGICAL BUNDLE: fibre dimension (or corank) k.

TAUTOLOGICAL BUNDLE

$$\begin{array}{ccc} E & = & \{(\lambda, x) \in Gr(k,n) \times \mathbb{R}^n : x \in \lambda\} \\ {\scriptstyle \pi}\big\downarrow & & \big\uparrow \\ Gr(k,n) & & \lambda \end{array}$$

$\rightarrow$ k-dim subspace of $\mathbb{R}^n$

So, given $\lambda \in Gr(k,n)$, choose basis $w_1, \ldots, w_k$ for λ. Define the matrix:

$$W := \begin{pmatrix} | & | & & | \\ w_1 & w_2 & \cdots & w_k \\ | & | & & | \end{pmatrix}$$

Homogeneous coördinates

Note: W and W' give the same element λ of $Gr(k,n)$ if $W' = WA$, for some $A \in GL(k, \mathbb{R})$.

In general, if the rows $i_1, \ldots, i_k$ are linearly independent, let $W_{i_1,\ldots,i_k}$ be the corresponding submatrix. Thus, by right-multiplying $W_{i_1,\ldots,i_k}^{-1}$ to W, we obtain that $W W_{i_1,\ldots,i_k}^{-1}$ reduces to a $k(n-k) \times$

$k(n-k)$ submatrix of the identity.

- Let $U_{i_1,\dots,i_k}$ denote the equivalence class of matrices W such that $W_{i_1,\dots,i_k}$ is invertible.

Corresponds to the k-dimensional linear subspace λ of $\mathbb{R}^n$ whose orthogonal projection to the linear subspace $\mathbb{R}^k_{i_1,\dots,i_k}$ of $\mathbb{R}^n$ given by $x_i = 0$, $i \neq i_1,\dots,i_k$ is <u>invertible</u>

k nonzero coordinate vectors

If $W \in U_{i_1,\dots,i_k}$, let

$$A_{i_1,\dots,i_k} = W \cdot W_{i_1,\dots,i_k}^{-1}.$$

then, the sumatrix $B_{i_1,\dots,i_k}$ given by all the rows of $A_{i_1,\dots,i_k}$ <u>except</u> the rows

$i_1, \ldots, i_k$ defines a homeomorphism

$$U_{i_1,\ldots,i_k} \longrightarrow \mathbb{R}^{k(n-k)} = \mathcal{M}(n-k, k)$$

$(n-k) \times k$ matrices

Transition maps from coördinate charts

$U_{i_1,\ldots,i_k}$, $U_{j_1,\ldots,j_k}$: If $W \in U_{i_1,\ldots,i_k} \cap U_{j_1,\ldots,j_k}$, then

$$A_{i_1,\ldots,i_k} W_{i_1,\ldots,i_k} = A_{j_1,\ldots,j_k} W_{j_1,\ldots,j_k}.$$

$\Rightarrow$ Transition maps are rational functions.

Vector Bundle charts (i.e., trivialization of $E|_{U_{i_1},\ldots,U_{i_k}}$) :

$$E|_{U_{i_1},\ldots,U_{i_k}} \longrightarrow \mathcal{M}(n-k,k) \times \mathbb{R}^k_{i_1,\ldots,i_k}$$

$$(\lambda, x) \longmapsto \left(B_{i_1,\ldots,i_k}, P_{i_1,\ldots,i_k}(x)\right)$$

P = orthogonal proj. onto $\mathbb{R}^k_{i_1,\dots,i_k}$.

Now, to show this is indeed a trivialization, we need to write an inverse:

$t = (x_{i_1}, \dots, x_{i_k})$

$$\left(A_{i_1,\dots,i_k}, \boxed{\begin{array}{l}\text{pt. of a that}\\ \text{projects to } t\end{array}}\right) \longleftarrow\!\!| \; (B_{i_1,\dots,i_k}, t)$$

Point $(x_1, \dots, x_n)$ such that $x_{i_j} = t_{i_j}$, $j = 1, \dots, k$.

Then

$$(x_1, \dots, \widehat{x_{i_1}}, \dots, \widehat{x_{i_k}}, \dots, x_n) = B_{i_1,\dots,i_k} \cdot t$$

Thus, this is a vector space isomorphism hence $\boxed{T\,Gr(k,n) \simeq \mathcal{M}(n-k,k) \times \mathbb{R}^k_{i_1,\dots,i_k}}$. !

! * ORIENTATION OF MANIFOLDS / VECTOR BUNDLES

Def: (ORIENTATION PRESERVING) $T: V \to V$ is orientation preserving if $\det T > 0$.

"reversing" if $\det T < 0$.

w.r.t. given basis

Def: Two ordered bases $(v_1, \ldots, v_n)$ of V and $(w_1, \ldots, w_n)$ of V are equivalent if

$$\det \begin{pmatrix} \text{change of} \\ \text{basis matrix} \end{pmatrix} > 0.$$

Def: (ORIENTATION PRESERVING) Given oriented vector spaces (V, μ) and (W, ν), we say that a linear isomorphism $T: V \to W$ is orientation preserving if

$$[Tv_1, Tv_2, \ldots, Tv_n] = \nu$$

whenever $[v_1, \ldots, v_n] = \mu$.

↳ If true for a particular ordered basis, then it is true any ordered basis.

ORIENTATION OF TRIVIAL BUNDLE: given a trivial bundle of M:

$$TM = M \times \mathbb{R}^k \longrightarrow M,$$

we can "put" the standard orientation of $\mathbb{R}^n$ on every fibre.

If $f: M \times \mathbb{R}^k \longrightarrow M \times \mathbb{R}^k$ is an isomorphism from TM to itself, then f is orientation-preserving or orientation-reversing on all fibres (provided that M is connected since $\det A(x)$ can not change sign).

↳ More generally

ORIENTATION OF A VECTOR BUNDLE: If we have a smooth vector bundle **!**

$$\pi : E \longrightarrow M,$$

an orientation μ of (E, π, M) is the collection of orientations μ_x of the fibres E_x which are compatible.

Being compatible means that if

$E|_U \xrightarrow{\Phi} U \times \mathbb{R}^k$, U connected, is a local trivialization and $\mathbb{R}^k$ has the standard orientation, then Φ is either orientation-preserving or orientation-reversing on all fibres.

Remark: If E has orientation μ, then it has another: $-\mu = \{-\mu_x\}$.

But not every vector bundle is orientable...

Def: A bundle is <u>ORIENTABLE</u> if it has an orientation (otherwise non-orientable).

Def: "Oriented bundle" means a vector bundle with an orientation.

Def: The orientation of TM is called the orientation of M. !

Def: An oriented manifold is a pair (M, μ) of a manifold M and orientation μ of M.

Prop: A manifold M is orientable if and only if there is an atlas $\{(\varphi_i, U_i)\}$ such that, for all i, j,

$$\det D(\varphi_j \circ \varphi_i^{-1})(x) > 0, \quad x \in \varphi_i(U_i \cap U_j).$$

!!

Pf: ($\Rightarrow$) Given an orientation μ of M, we can take all charts (φ, U) such that

$$TM|_U \longrightarrow \varphi(U) \times \mathbb{R}^n \leftarrow \text{standard orientation}$$

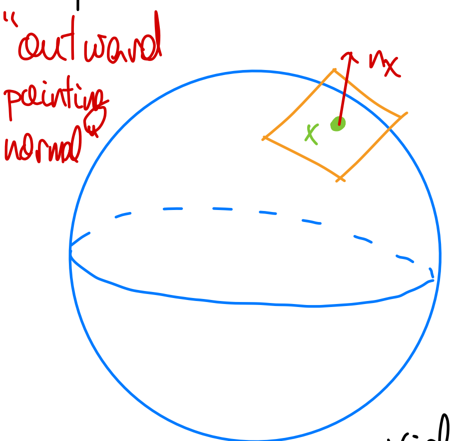
is orientation preserving.

($\Leftarrow$) Conversely, given such an atlas we can orient fibres of TM such that each φ_* is orientation preserving.

$\square$

EXAMPLE 1: $S^{n-1} \subset \mathbb{R}^n$ is ORIENTABLE !

This is true because S^{n-1} is the boundary of the unit ball in $\mathbb{R}^n$, and you can always orient the boundary of a manifold in $\mathbb{R}^n$.



Given $(v_1, \dots, v_{n-1}) \in TM_x$, linearly independent, we can define an orientation μ_x as $[v_1, \dots, v_{n-1}]$ provided that $[n_x, v_1, \dots, v_{n-1}]$ gives the standard orientation of $\mathbb{R}^n$.

$\square$

EXAMPLE 2: (OPEN) MÖBIUS STRIP !

The (open) Möbius strip is a vector bundle E of fibre dimension 1 over

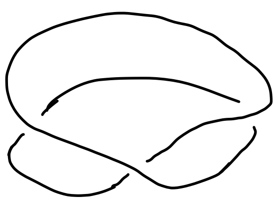
S^1: E is the quotient $\mathbb{R} \times \mathbb{R}$ by the equivalence relation:

$$(x,t) \sim (x+1, -t)$$

$$\begin{array}{ccc} E & & (x,t) \\ \pi \downarrow & & \downarrow \\ S^1 & & (\cos 2\pi x, \sin 2\pi x) \end{array}$$

E is non-orientable (either as a vector bundle or as a manifold)

Now, E has no nonvanishing section but we can pick 2 points in each fibre to get



$$A = \mathbb{R} \times \{-1, 1\} / \sim$$

<u>Claim</u>: If E had an orientation $\{\mu_p\}$, $p \in S^1$, then we could define a non-

vanishing section $s(p)$ by taking $s(p)$ as the unique point $s(p) \in A \cap E_p$ such that $[s(p)] = \mu_p$.

$\square$

Example 3: **ORIENTATION OF $\mathbb{RP}^n$** !!

In general,

$$\mathbb{RP}^n \begin{cases} \text{is orientable if } n \text{ odd} \\ \text{is nonorientable if } n \text{ even} \end{cases}$$

$\hookrightarrow$ $\mathbb{RP}^n$ is orientable if and only if the antipodal map of S^n is orientation preserving. $\hookrightarrow$ Only true for n odd.

Lecture 16: Orientations & Whitney

- $\mathbb{RP}^2$ is NOT ORIENTABLE:

Consider the antipodal map

$$A : S^2 \longrightarrow S^2$$
$$x \longmapsto -x$$

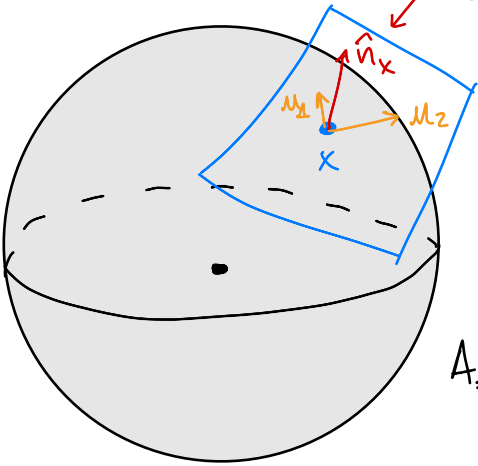
Induced by antipodal map of $\mathbb{R}^3$: orientation reversing.

Now, recall $\mathbb{RP}^2 = S^2/\sim$, where $x \sim A(x)$.

Now, this induces the tangent map:

$$A_{*x} : TS^2_x \longrightarrow TS^2_{A(x)=-x}$$
$$(x, v) \longmapsto (A(x), A(v))$$

A is linear.



outward pointing unit normal at x.

$(u_1, u_2) =$ basis for TS^2_x.

Note that

$$A_{*x}(n_x, u_1, u_2) = (\underbrace{-n_x}_{= n_{-x}}, \underbrace{-u_1, -u_2}_{})$$

$\Leftarrow$ gives opposite orientation.

A is orientation reversing!

Now, consider

$$g : S^2 \longrightarrow \mathbb{R}P^2$$
$$x \longmapsto [x]$$

Suppose $\mathbb{R}P^2$ has orientation $\{\mu_y\}_{y \in \mathbb{R}P^2}$. Then we can define an orientation $\{\nu_x\}_{x \in S^2}$ by requiring that g is orientation preserving. Thus, since we are doing "things in a smooth way," this would mean that the antipodal map is orientation preserving. ~~$\Rightarrow$~~

$\Rightarrow$ $\mathbb{R}P^2$ is <u>NOT</u> orientable.

* $\mathbb{RP}^3$ <u>is</u> ORIENTABLE

Now, by a similar argument as before, the antipodal map $A: S^3 \to S^3$, $x \mapsto -x$, <u>is</u> orientation preserving.

Thus, defining
$$g: S^3 \longrightarrow \mathbb{RP}^3$$
$$x \longmapsto [x]$$
we can define an orientation for $\mathbb{RP}^3$ by requiring that g is orientation preserving.

$\Rightarrow$ $\mathbb{RP}^3$ <u>is</u> orientable !

$\square$

In general: !!!

$$\mathbb{RP}^n \text{ is } \begin{cases} \underline{\text{ORIENTABLE}} & \text{if } n \text{ ODD} \\ \underline{\text{NONORIENTABLE}} & \text{if } n \text{ EVEN} \end{cases}$$

$\square$

THM: (STRONG-ISH WHITNEY EMBEDDING) Every compact smooth manifold M of dimension n has an embedding on $\mathbb{R}^{2n+1}$.

Remark: • Easy to generalize this for non-compact manifolds as well.

• Since we are dealing with M compact, it suffices to show that there is an injective immersion (injective immersion of compact manifolds is an embedding)

Lemma: If M has an injective immersion onto $\mathbb{R}^N$, $N \geq 2n+2$, then it has an injective immersion into $\mathbb{R}^{N-1}$

Pf: Consider $g : M \times M \setminus \Delta \longrightarrow S^{N-1}$

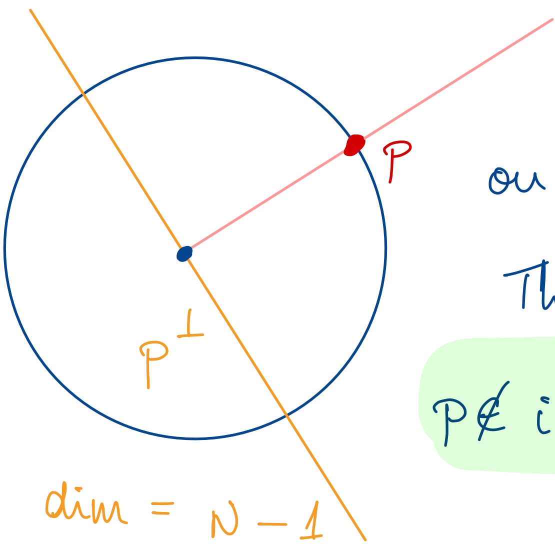
$$(x,y) \longmapsto \frac{i(x)-i(y)}{|i(x)-i(y)|},$$

where $i: M \to \mathbb{R}^N$ is the given injective immersion.

This induces $h: TM \setminus \{0\text{-section}\} \to S^{N-1}$

$$v \longmapsto \frac{i_*(v)}{|i_*(v)|}$$

Then, we have:



Consider the projection $\pi_p: \mathbb{R}^N \to p^\perp$.

Then

$$\boxed{p \notin \operatorname{im} g \iff \pi_p \circ i \text{ injective}}$$

If $p \in \operatorname{im} g$, then $p = \dfrac{i(x)-i(y)}{|i(x)-i(y)|}$

$\Leftrightarrow \; i(x) - i(y) = |i(x) - i(y)|\,p$

$\Leftrightarrow \; i(x) - i(y)$ is a multiple of p

$\Leftrightarrow \; \pi_p \circ i(x) - \pi_p \circ i(y) = 0.$

$\Leftrightarrow \; \pi_p \circ i$ <u>not</u> injective.

Now,

$$p \notin \operatorname{im} h \Leftrightarrow (\pi_p \circ i)_* \text{ injective},$$

since

$$p \in \operatorname{im} h \Leftrightarrow i_*(v) = |i_*(v)|\,p$$

$$\Leftrightarrow i_*(v) \text{ is a multiple of } p$$

$$\Leftrightarrow (\pi_p)_* \circ i_*(v) = 0, \text{ some } v$$

$$\Leftrightarrow (\pi_p)_* \circ i_* = (\pi_p \circ i)_* \text{ \underline{not} injective.}$$

Therefore,

$p \notin \operatorname{im} g \cup \operatorname{im} h$

$$\iff \pi_p \circ i : M \to p^{\perp} \simeq \mathbb{R}^{N-1}$$

is an injective immersion!

Finally, $M \times M$ and TM are manifolds of dimension $2n$, $N-1 > 2n$. So, $p \notin \operatorname{im} g \cup \operatorname{im} h$

iff p is a regular values for g and h.

By Sard's theorem, set of regular values is dense (i.e., set of critical values is a zero set).

$\square$

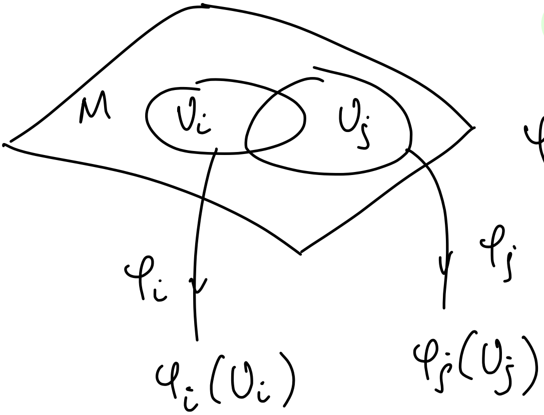
LECTURE 17: VECTOR FIELDS & DIFFERENTIAL FORMS

First, need to define some duals...

Def: (COTANGENT BUNDLE) The cotangent bundle T^*M is the bundle dual to TM. The fibres of T^*M are the dual vector spaces of the fibres of TM; i.e.,

$$T^*M_a \overset{\text{def.}}{=} (TM_a)^* .$$

CHARTS FOR T^*M: Suppose $\{(U_i, \varphi_i)\}$ is a smooth atlas for M. Then, the <u>bundle charts of TM are</u>:



Transition maps on M:

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

Induces the tangent mapping

$$(\varphi_j \circ \varphi_i)_* : \underbrace{T\varphi_i(U_i \cap U_j)}_{\text{open sets in } \mathbb{R}^n} \longrightarrow \underbrace{T\varphi_j(U_i \cap U_j)}$$

$$\parallel$$

$$\varphi_i(U_i \cap U_j) \times \mathbb{R}^n \longrightarrow \varphi_j(U_i \cap U_j) \times \mathbb{R}^n$$

$$(x, v) \mapsto \Big((\varphi_j \circ \varphi_i^{-1})(x), \underbrace{(\varphi_j \circ \varphi_i^{-1})_*(x)}_{D(\varphi_j \circ \varphi_i^{-1})(x)} v \Big)$$

* **Note**: the map $x \longmapsto (\varphi_j \circ \varphi_i^{-1})_*$ is a map $\varphi_i(U_i \cap U_j) \longrightarrow \mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$, which

is C^∞.

Now, consider the **DUALIZATION**: if we have a linear map between vector spaces $\lambda : V \to W$; i.e., $\lambda \in \mathrm{Hom}(V, W)$.

Then, this induces the dual map

$$\lambda^* : W^* \longrightarrow V^*$$

$$\lambda^*(\eta)(v) = \eta(\lambda(v))$$

In particular, for the charts, we have that

$$(\varphi_j \circ \varphi_i^{-1})_{*x} : \overset{\simeq V}{\mathbb{R}^n} \longrightarrow \overset{\simeq W}{\mathbb{R}^n}$$

induces

$$(\varphi_j \circ \varphi_i^{-1})^*_x : \overset{\simeq W^*}{(\mathbb{R}^n)^*} \longrightarrow \overset{\simeq V^*}{(\mathbb{R}^n)^*}.$$

Thus, the transition mappings between bundle charts of T^*M are

$$(\varphi_j \circ \varphi_i^{-1})^* : T^*\varphi_j(U_i \cap U_j) \longrightarrow T^*\varphi_i(U_i \cap U_j)$$

$$\begin{array}{ccc} \| & & \| \\ \varphi_j(U_i \cap U_j) \times (\mathbb{R}^n)^* & \longrightarrow & \varphi_i(U_i \cap U_j) \times (\mathbb{R}^n)^* \end{array}$$

$$(y, \eta) \longmapsto \Big(\underbrace{(\varphi_i \circ \varphi_j^{-1})(y)}_{=: x},\ (\varphi_j \circ \varphi_i^{-1})^*_x(\eta) \Big)$$

Note: the function $y \mapsto (\varphi_j \circ \varphi_i^{-1})^*$

$x = (\varphi_i \circ \varphi_j)^{-1}(y)$

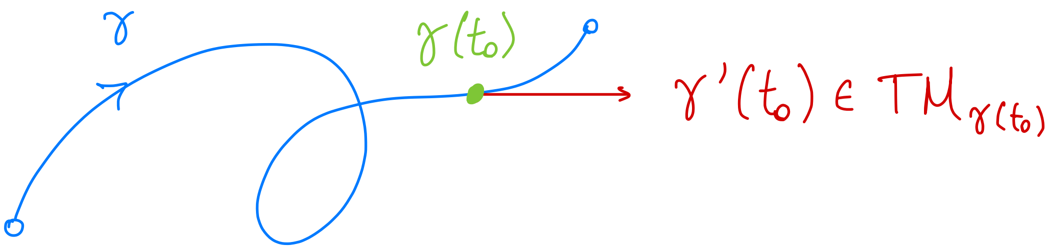
is a C^∞ function that maps

$$\varphi_j(U_i \cap U_j) \longrightarrow \mathrm{Hom}\big(\underset{W^*}{\underset{\|}{(\mathbb{R}^n)^*}}, \underset{V^*}{\underset{\|}{(\mathbb{R}^n)^*}}\big)$$

* VECTOR FIELDS

First, consider the example:

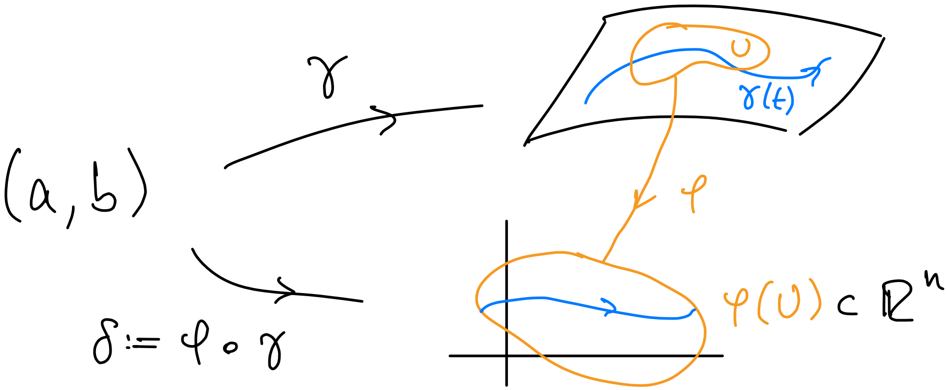
• **EXAMPLE**: Tangent vector to a smooth curve $\gamma : (a,b) \longrightarrow M$ at $t_0 \in (a,b)$



is defined as $\gamma_{*t_0}(e_{1,t_0}) = \dfrac{d}{dt}\Big|_{t_0}$.

We denote this by $\gamma'(t_0)$

In local coördinates



Set the coordinates in $\varphi(U) \subset \mathbb{R}^n$ to be $x_i = \varphi_i$. Then, $\delta = (\delta_1, \_, \delta_n)$. So, very <u>informally</u>

$$\delta_i = \varphi_i \circ \gamma = x_i \circ \gamma \text{ ``}= \gamma_i\text{''}.$$

$\square$

!!

Tangent vectors act on smooth functions:

for $X \in TM_a$ and $f \in C^\infty(M)$, we have

$\frac{\partial}{\partial x_i}(\varphi_j \circ f)(a)$

$$X(f) = f_{*a}(X) = \sum_i \xi_i \frac{\partial f}{\partial x_i}(a).$$

In coördinates,

$$X_a = \sum_i \xi_i \left.\frac{\partial}{\partial x_i}\right|_a .$$

In particular, from the example:

Chain Rule

$$\underbrace{\gamma'(t_0)}_{\in TM_{\gamma(t_0)}}(f) = f_{*\,\gamma(t_0)}\,\gamma_{*t_0}(e_{1,t_0})$$

$$= (f\circ\gamma)_{*t_0}(e_{1,t_0})$$

$$= \frac{d(f\circ\gamma)}{dt}(t_0)$$

In coordinates

$$= \sum_i \frac{\partial f}{\partial x_i}(\gamma(t_0))\,\frac{d\gamma_i}{dt}(t_0)$$

$$= \sum_i \left(\frac{\partial \gamma_i}{\partial t}(t_0)\,\frac{\partial}{\partial x_i}\bigg|_{\gamma(t_0)} \right)(f)$$

Def: (C^∞ 1-forms) A C^∞ 1-form ω on M is defined to be a C^∞ section of T^*M. So, if we have a C^∞ 1-form on M and a C^∞ vector field X on, then we get a C^∞ function on M given by

$$\omega(X)(x) = \omega(x)\, X(x).$$

$\omega(X)$ takes a point on M and sets it to a real number.

special example

Def: If $f \in C^\infty(M)$, we define a C^∞ 1-form denote df (differential of f)

$$df(x)(X) = f_{*x}(X) = X(f).$$

$TM_x \ni X$

E.g., if $\varphi : U \to \mathbb{R}^n$ is a coordinate chart for M, then we get a section of $T^*M|_U$ given by $dx_i = d\varphi_i$. In this case,

$$dx_i(x)\left(\frac{\partial}{\partial x_j}\bigg|_x\right) = \frac{\partial x_i}{\partial x_j}\bigg|_x$$

$$= \delta_{ij}.$$

Thus, $dx_i(x)$ form a basis for the fibre $T^*M_x = (TM_x)^*$ of the co-tangent bundle. It is the dual basis to $\frac{\partial}{\partial x_i}\Big|_x$ (basis for TM_x).

Therefore, a C^∞ 1-form ω can be expressed as

$$\omega = \sum_i \omega_i \, dx_i .$$

In local coordinate chart U,

$$\omega(x) = \sum_i \omega_i(x) \, dx_i(x)$$

ω C^∞ 1-form $\Longleftrightarrow$ Coefficient functions ω_i are C^∞

Thus:

Prop: (Differential df in local coördinates) If $f \in C^\infty(M)$, then in a local coördinate chart $\varphi_i : U_i \to \mathbb{R}^n$, the differential of f is

$$df = \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i} dx_i$$

!!

Pf: Let $X \in TM_x$; i.e.,

$$X = \sum_i \xi_i \frac{\partial}{\partial x_i}\Big|_x$$

so, $\underbrace{\xi_i = X(x_i) = dx_i(x)(X)}_{*}$; hence

$$\boxed{df(x)}(X) = X(f)$$

$$= \sum_i \xi_i \frac{\partial f}{\partial x_i}(x)$$

$$= \left(\boxed{\sum_i \frac{\partial f}{\partial x_i}(x)\, dx_i(x)} \right)(X)$$

$\forall x \in M$. $\square$

!!

* IMPORTANT REMARK: A C^∞ function $f: M \to N$ between manifolds induces a bundle map $f_*: TM \to TN$. In particular, for every $x \in M$, we have $f_{*x}: TM_x \longrightarrow TN_{f(x)}$, which is a linear map. Since linear, it induces the dual

$$\underbrace{(f_{*x})^*}_{=:\, f_x^*} : T^*N_{f(x)} \longrightarrow T^*M_x$$

Q: Do we have a bundle map f^*: $T^*N \to T^*M$? **No** (not necessarily injective...)

But! given a C^∞ section ω of T^*N (i.e., a C^∞ 1-form on N), we get a C^∞ 1-form $f^*\omega$ on M by pullback.

!!

Def: (PULLBACK)

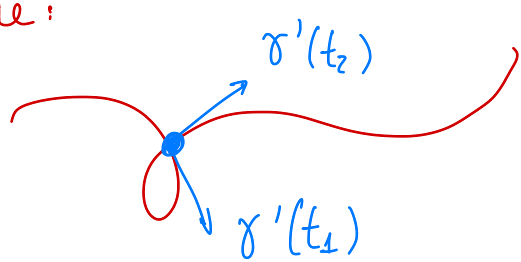
$$(f^*\omega)(x)(X) = \omega(f(x))(f_{*x}(X))$$

$M \ni$ $\in TM_x$

Now,

Q: Does f_* induce a map from vector fields on M to vector fields on N?

No! Could have:



$\rightsquigarrow$ greater mathematical phenomenon that, in order to study a geometrical object, we can "limit" ourselves to studying all the functions on this object.

Leave differential forms aside for a while

* INTEGRATION

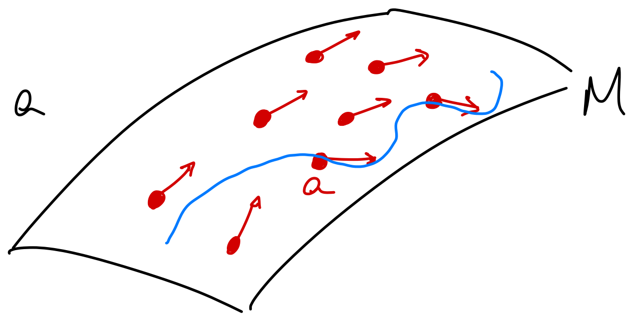
Given a C^∞ vector field X on M and a point $a \in M$.

QUESTION: Is there a smooth curve

$\gamma : (-\varepsilon, \varepsilon) \to M$

such that $\gamma(0) = a$ and such that

$$\gamma'(t) = X(\gamma(t)) \quad ?$$



In the case of YES: call γ an integral curve of X with initial condition $\gamma(0) = a$.

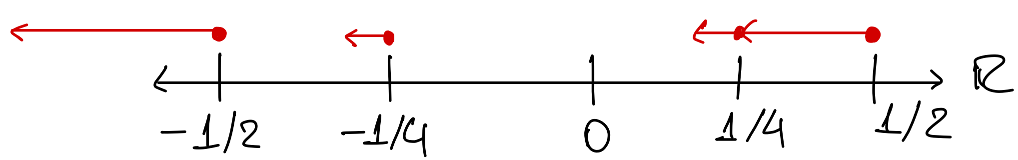
E.g.: In $\mathbb{R}^n$ (or equivalently in local coördinate charts $\varphi: U \to \mathbb{R}^n$), it reduces to solving

$$\gamma'(t) = f_i(\gamma_i(t), \_\_, \gamma_n(t)),$$

$i = 1, \_, n$, a system of ODEs w/ initial condition $\gamma(0) = a$.

EXAMPLE 1: Find $\gamma: \mathbb{R} \to \mathbb{R}$ such that $\gamma'(t) = -\gamma(t)^2$.

We want to integrate the vector field in $\mathbb{R}$ given by $X(a) = -a^2$; i.e.,



Solve it the "practical" way: separation of variables

$$\frac{dy}{dx} = -y^2 \rightsquigarrow -\frac{dy}{y^2} = dx$$

$$\rightsquigarrow \frac{1}{y} = x + C$$

So, $y = \dfrac{1}{x+C}$. If initial condition is $y(0) = a$, then $C = 1/a$. Thus

$$y(x) = \frac{1}{x + 1/a}.$$ Not defined for $x = -\frac{1}{a}$.

Also, $y \equiv 0$ is solution ....

curve escapes to ∞ as $x \to -1/a$

Thus, there is no integral curve that can be defined for all t, even though X is defined everywhere.

LECTURE 18: FLOWS ON MANIFOLDS

* FUNDAMENTAL EXISTENCE & UNIQUENESS THEOREM

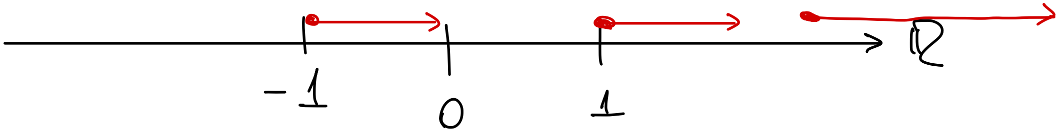
or C^r, $r \geq 1$

Thm: Let X be a C^∞ vector field on an open $U \subset \mathbb{R}^n$, and let $K \subset U$ be compact. Then, for all $\varepsilon > 0$, there is a neighborhood U' of K in U and a unique C^∞ mapping $g : U' \times (-\varepsilon, \varepsilon) \longrightarrow U$ s.t.

$$\frac{\partial g(x,t)}{\partial t} = X(g(x,t)), \quad g(x,0) = x.$$

EXAMPLE: On $\mathbb{R}$, consider the following vector field

$$X(a) = a^{2/3} \left.\frac{d}{dt}\right|_{t=a}.$$



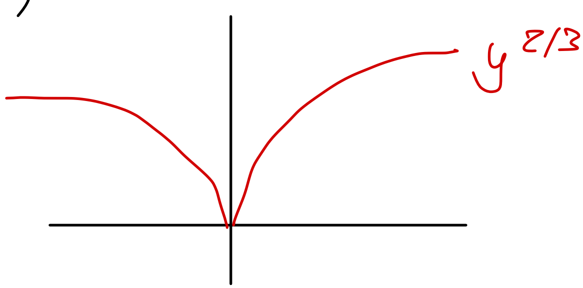
Then, we are solving $\frac{dy}{dt} = y^{2/3}$

From MAT267, there are two solutions w/ $y(0) = 0$

$$y \equiv 0 \quad \text{and} \quad y(t) = \frac{1}{27} t^3$$

$\Rightarrow$ Solution not unique

Indeed, $y^{2/3}$ is not C^1 (not even Lipschitz...)



Def: (Flow) $\Phi_t(x) := g(x,t)$

From the thm in pg. 149

Corollary: Conditions to be considered a flow

(i) $\Phi_0 = id$

(ii) $\Phi_{t+s}(x) = (\Phi_t \circ \Phi_s)(x)$ whenever both sides are defined; i.e., $|s|, |t|$ s.t. $|s+t| < \varepsilon$ and $\forall x$, $\Phi_t(x) \in U'$.

Pf: (ii) Fix t.

$$\frac{d}{ds} g(x, s+t) \overset{\text{ODE}}{=} X(g(x, s+t)).$$

Now,

$$\Phi_{s+t}(x) = g(x, s+t)$$

Uniqueness of the solution of

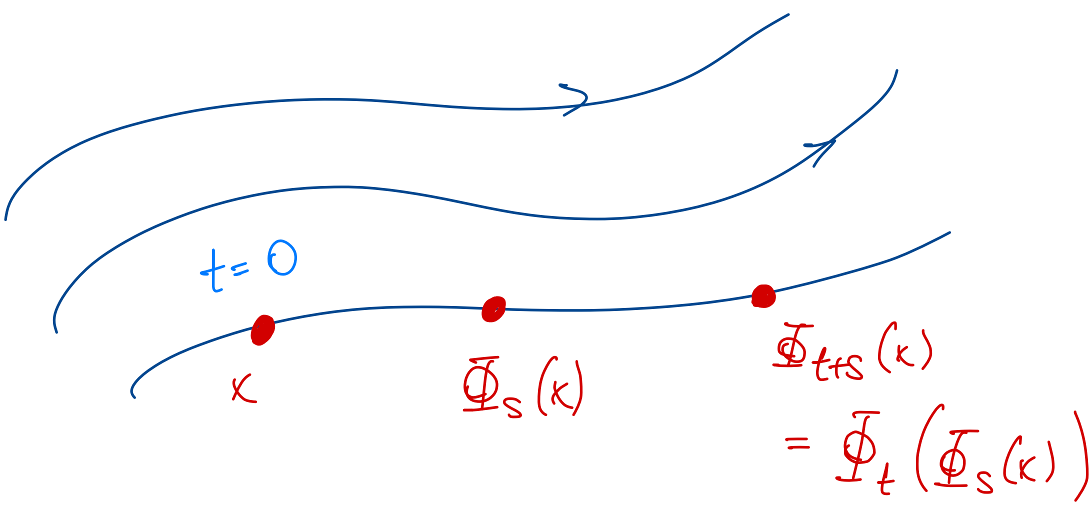
$$= g(g(x,t), s)$$

$$\frac{\partial}{\partial t} g(x,t) = X(g(x,t)), \quad g(x,0) = x.$$

At $s = 0$, we need $g(x,t)$. Here, when $s=0$, we need such that

$$g(g(x,t), s=0) = g(g(x,t), 0)$$

$$= g(x,t)$$



$\square$

Thm: (Existence & Uniqueness on Manifolds)

Let X be a C^∞ vector field on a manifold M. Given $a \in M$, there exists an open neighborhood V of a, an $\varepsilon > 0$, and a unique collection of diffeomorphisms

$$\Phi_t : V \longrightarrow \Phi_t(V) \subset M, \quad |t| < \varepsilon.$$

such that

(1) the function

$$\Phi : V \times (-\varepsilon, \varepsilon) \longrightarrow M$$
$$(x, t) \longmapsto \Phi_t(x)$$

Analogous to $g(x,t)$

is C^∞;

(2) $\Phi_{t+s}(x) = (\Phi_t \circ \Phi_s)(x)$ if $|s|, |t|,$

then $|t+s|<\varepsilon$ and, $\forall x$, $\Phi_t(x) \in V$.

"ODE"

(3) For all $x \in V$, $X(x)$ is a tangent vector at $t=0$ of the curve

$$t \mapsto \Phi_t(x).$$

<u>Obs:</u> Since the Φ_t are diffeomorphisms, from (2) w/ $s=0$, the initial condition $\Phi_0 = id$ follows: $\Phi_t = \Phi_0(\Phi_t(x))$

✳ IMPORTANT ! The support of X needs to be the closure of $\{x \in M : X(x) \neq 0\}$

<u>Thm:</u> If X has compact support (e.g., if M is compact), then there are diffeos. $\Phi_t(x)$, $t \in \mathbb{R}$, satisfying (1)–(3).

1-parameter group of diffeos of M.

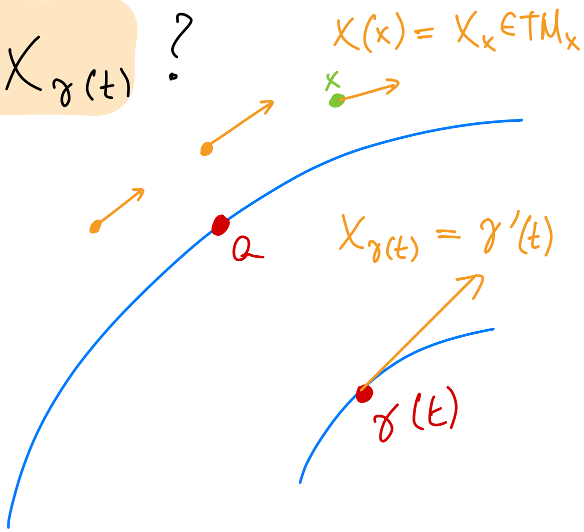
Lecture 19: More on Flows & Lie Stuff

Recall: Let X be a C^∞ vector field on an n-manifold M.

Notation: $X(x) = X_x \in TM_x$ and the space of all C^∞ vector fields on M is denoted $\mathfrak{X}(M)$.

Given $a \in M$, can we find an integral curve γ of X with initial condition $\gamma(0) = a$ and

$$\gamma'(t) = X(\gamma(t)) = X_{\gamma(t)} \,?$$



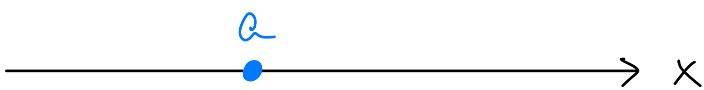
E.g.: on $\mathbb{R}$, consider the following (C^∞-) vector field

$$X(x) = -x^2 \frac{\partial}{\partial x}.$$

(local) basis for $T\mathbb{R}_x$.

Now, we look for a curve $\gamma(t)$ such that

$$\gamma'(t) = X(\gamma(t)), \quad \gamma(0) = a.$$



$$x = \gamma(t)$$

So, we need to solve

$$\gamma'(t) = \frac{dx}{dt} = -x^2.$$

Obs: $\gamma'(t) \in \underbrace{TM_{\gamma(t)}}$

!

spanned by $\left.\frac{d}{dx}\right|_{\gamma(t)}$,

so we can write this "in full" w.r.t. the bases:

This equation is the same as $\gamma'(t) = X(\gamma(t))$

$$\frac{dx}{dt}\left.\frac{d}{dx}\right|_{\gamma(t)} \overset{!}{=} -x^2\left.\frac{d}{dx}\right|_{\gamma(t)}$$

Solving the ODE: we find that

$$-\frac{dx}{x^2} = dt \rightsquigarrow x = \frac{1}{t+C}.$$

$$\gamma(0) = a \implies C = 1/a.$$

Thus,

$$\boxed{x = \frac{1}{t + 1/a} = \gamma(t).}$$

$\square$

Thm: (INFORMAL) given a C^∞ vector field X on an open $U \subset \mathbb{R}^n$, then, locally there exists $g(x,t)$ (for $|t| < \varepsilon$ and $x \in U$) with values in $\mathbb{R}^n$ such that

$$\frac{\partial g(x,t)}{\partial t} = X(g(x,t)), \quad g(x,0) = x$$

↳ From the previous example,

$$g(x,t) = \frac{1}{t + 1/x}$$

Not the "same x" as in the solution of the example.

This g is C^∞ w.r.t. both t and x (locally). The g needs to contain information about the initial parameter.

$g(x,t) = \phi_t(x)$

Now, Fix t and consider $g(x, s+t)$.

Then

$$\frac{\partial}{\partial s} g(x, s+t) = X\big(g(x, s+t)\big).$$

Thus, $g(x, s+t)$ is the unique solution w/ initial condition

$$g(x, s+t)\Big|_{s=0} = g(x, t)$$

Now, consider $g(g(x,t), s)$. Then,

$$\frac{\partial}{\partial s} g(g(x,t), s) = X\big(g(g(x,t), s)\big).$$

So, $g(g(x,t), s)$ is the unique solution w/ initial condition

$$g(g(x,t),0) = g(x,t).$$

Remark: $g(y,s)$ is the unique solution w/ initial condition

$$g(y,0) = y.$$

In flow notation,

$$g(x,s+t) = g(g(x,t),s)$$

$$\phi_{t+s}(x) = \phi_s(\phi_t(x)).$$

Upshot: Always use the equation, no matter the variables...

Thm: (FUNDAMENTAL THEOREM OF EXISTENCE OF INTEGRAL CURVES) given $X \in \mathfrak{X}(M)$, for all $a \in M$, $\exists$ open neighborhood V of a and, for all $\varepsilon > 0$, a unique collection of diffeomorphisms

$$\phi_t : V \longrightarrow \phi_t(V) \subset M, \quad -\varepsilon < t < \varepsilon,$$

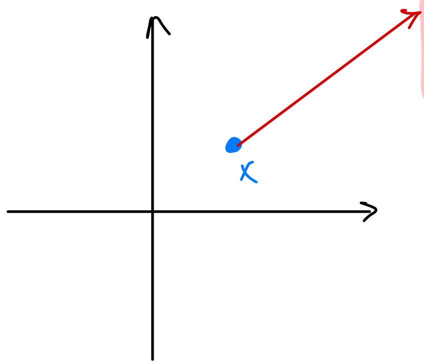
such that

(1) The map
$$\begin{aligned} (-\varepsilon, \varepsilon) \times V &\longrightarrow M \\ (t, x) &\longmapsto \phi_t(x) \end{aligned}$$
is C^∞;

(2) $\phi_{s+t}(x) = \phi_s(\phi_t(x))$ when $|s|$, $|t|$, $|s+t| < \varepsilon$, $x \in V$, $\phi_t(x) \in V$.

(3) If $x \in V$, then $X_x = X(x)$ is the tangent vector at $t = 0$ of the curve $t \longmapsto \phi_t(x)$.

! EXAMPLE IN $\mathbb{R}^n$: Consider the RADIAL VECTOR FIELD on $\mathbb{R}^n$:



$$X_x = X(x) = \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i}$$

* Find $\gamma(t)$ such that

$$\gamma'(t) = \gamma(t),$$

where $\gamma : x = \gamma(t)$, $\dfrac{dx_i}{dt} = x_i$

with initial condition $\gamma(0) = a$.

$X_i = a_i e^t \rightsquigarrow \phi_t(x) = e^t x$

components of the vectors

supp X = closure$(\{x : X_x \neq 0\})$

Thm: If $X \in \mathfrak{X}(M)$ has compact support (e.g., if M itself is compact), then there exist finitely many diffeomorphisms $\phi_t(x) : M \to M$ satisfying, for all $t \in \mathbb{R}$

(1) The map
$$\begin{aligned}(-\varepsilon, \varepsilon) \times V &\longrightarrow M \\ (t, x) &\longmapsto \phi_t(x)\end{aligned}$$
is C^∞;

(2) $\phi_{s+t}(x) = \phi_s(\phi_t(x))$ when $|s|, |t|, |s+t| < \varepsilon$, $x \in V$, $\phi_t(x) \in V$.

(3) If $x \in V$, then $X_x = X(x)$ is the tangent vector at $t = 0$ of the curve $t \longmapsto \phi_t(x)$.

Pf: Cover $\operatorname{supp} X$ by finitely many $V_1, \ldots, V_p$; each with corresponding $\varepsilon_1, \ldots, \varepsilon_p$ and corresponding $\phi_t^1, \ldots, \phi_t^p$.

compactness

Let $\varepsilon := \min\limits_i \varepsilon_i$. If $x \in V_i$ and V_j (i.e., $x \in V_i \cap V_j$), then, still for $|t| < \varepsilon$,

$$\phi_t^i(x) = \phi_t^j(x)$$

by uniqueness!

So, define

$$\phi_t(x) = \begin{cases} \phi_t^i(x), & x \in V_i \subset \operatorname{supp} X \\ x, & x \notin \operatorname{supp} X \end{cases}$$

since $X(x) \equiv 0$ outside of $\operatorname{supp} X$, it is the constant vector field. So, $\phi_t(x) = x$ is chosen to satisfy the initial condition.

Then,

$$\begin{aligned} \phi : (-\varepsilon, \varepsilon) \times M &\longrightarrow M \\ (t, x) &\longmapsto \phi_t(x) \end{aligned}$$

is C^∞. Moreover,

$$\phi_{s+t} = \phi_s \circ \phi_t, \quad |s|, |t|, |s+t| < \varepsilon.$$

Lastly, clearly each of the ϕ_t's is a diffeo..

To define ϕ_t when $|t| \geq \varepsilon$:

write $t = k \cdot \frac{\varepsilon}{2} + r$, $k \in \mathbb{Z}$, $|r| < \frac{\varepsilon}{2}$.

Define

$$\phi_t := \begin{cases} \overbrace{\phi_{\varepsilon/2} \circ \phi_{\varepsilon/2} \circ \cdots \circ \phi_{\varepsilon/2}}^{k \text{ times}} \circ \phi_r, & k \geq 0 \\ \underbrace{\phi_{-\varepsilon/2} \circ \cdots \circ \phi_{-\varepsilon/2}}_{-k \text{ times}} \circ \phi_r, & k < 0 \end{cases}$$

$\square$

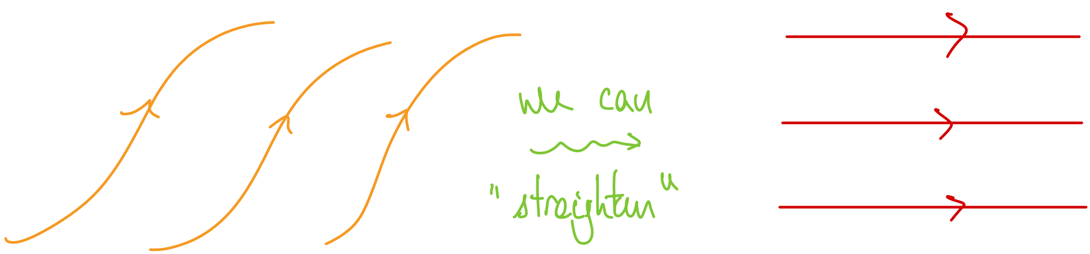
!!

Thm: (FLOW BOX THEOREM or CURVE STRAIGHTENING LEMMA) Let $X \in \mathfrak{X}(M)$ such that $X(a) \neq 0$. Then, there exists a coördinate system (U, φ), $\varphi : U \to \mathbb{R}^n_{(x_1, \dots, x_n)}$, such that

$$X = \frac{\partial}{\partial x_1} \quad \text{on } U;$$

i.e.,
$$\varphi_* X = \frac{\partial}{\partial x_1}.$$

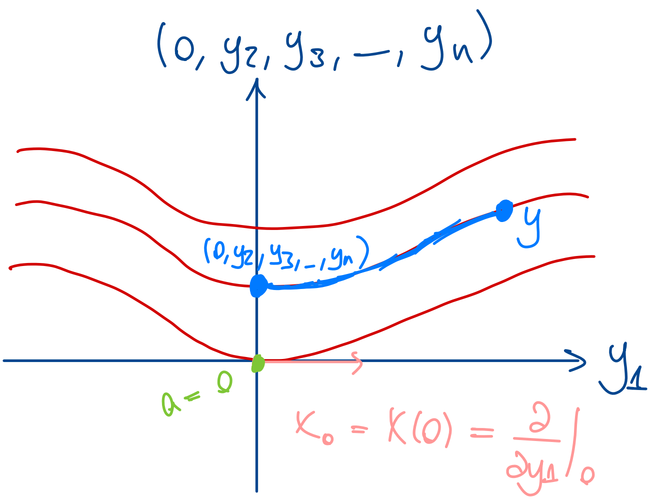
Remark 1: This means that, if we have a smooth flow:



Remark 2: Local result.

Pf: WLOG, assume $M = \mathbb{R}^n$ with coördinates $(y_1, \ldots, y_n)$ and suppose $a = 0$, so that we have

$$X_0 = \frac{\partial}{\partial y_1}\Big|_0$$



Let ϕ_t be the flow given by the vector field X.

Consider the map defined in a neighborhood of 0 in $\mathbb{R}^n$ by

$$\xi(x_1, \ldots, x_n) = \phi_{x_1}(0, x_2, \ldots, x_n)$$

w/ values in $M = \mathbb{R}^n$.

Then, check this works: !

Recall: THE ACTION OF A TANGENT VECTOR OF A CURVE γ AT t ON A FUNCTION f:

fixed

$$\frac{d\gamma}{dt}(f)(t_0) = \frac{d\, f(\gamma(t))}{dt}(t_0) = (f\circ\gamma)'(t_0).$$

Now, X_x is the tangent vector of the curve $t \mapsto \phi_t(x)$ at the point $t=0$.

So, !!

$$(Xf)(x) = X_x(f)$$

$$= \frac{d}{dt}\Big|_{t=0} (f\circ\phi_t)(x)$$

$\phi_0(x) = x$.

$$= \lim_{h\to 0} \frac{f(\phi_h(x)) - f(x)}{h}$$

Thus, compute **IMPORTANT !!**

$$\xi_*\left(\frac{\partial}{\partial x_1}\Big|_x\right)(f) \overset{\text{def}}{=} \frac{\partial}{\partial x_1}\Big|_x (f\circ\xi)$$

$$= \lim_{h\to 0} \frac{f(\xi(x_1+h, x_2, \dots, x_n)) - f(\xi(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(\phi_{x_1+h}(0, x_2, \dots, x_n)) - f(\xi(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(\phi_h(\underbrace{\phi_{x_1}(0, x_2, \dots, x_n)}_{\overset{\text{def}}{=}\xi(x)})) - f(\xi(x))}{h}$$

$$= \lim_{h\to 0} \frac{f(\phi_h(\xi(x))) - f(\xi(x))}{h}$$

$$= (Xf)(\xi(x)) = X_{\xi(x)} f.$$

Upshot:

$$\xi_{*x}\left(\frac{\partial}{\partial x_1}\Big|_x\right) = X_{\xi(x)} = X(\xi(x))$$

Thus, as a mapping between vector bundles,

$$\xi_*\left(\frac{\partial}{\partial x_1}\Big|_x\right) = X \circ \xi.$$

But this is precisely the statement of the theorem. We wanted a diffeo. that maps X to $\frac{\partial}{\partial x_1}\Big|_x$, or, in the other direction, that maps $\frac{\partial}{\partial x_1}\Big|_x$ to X. In other words, we want $\psi : U^{\subset M = \mathbb{R}^n} \longrightarrow \mathbb{R}^n$.

$\psi = \xi^{-1}$, $\quad \mathbb{R}^n \xrightarrow{\xi} U$

So, finally, we need to check that ξ^{-1} (or ξ) are indeed diffeos (i.e., local coördinates).

So, compute: for some $i > 1$, we have

$$\xi_{*0}\left(\frac{\partial}{\partial x_i}\Big|_0\right)(f) \overset{\text{def}}{=} \frac{\partial}{\partial x_i}\Big|_0 (f \circ \xi)$$

$$= \lim_{h \to 0} \frac{f(\xi(0, -, \overset{i\text{-th place}}{h}, -, 0)) - f(\xi(\underbrace{0}_{=0}))}{h}$$

$\phi_0(\bullet) = \bullet$! $\phi_0 = id$

$$= \lim_{h \to 0} \frac{f(0, -, h, -, 0) - f(0)}{h}$$

$$= \frac{\partial f}{\partial y_1}(0)$$

But, we know that

$$X_0 = \left.\frac{\partial}{\partial y_1}\right|_0$$

$\Longrightarrow$ $\xi_{*0} = Id$, hence local diffeo. by Inverse Function theorem.

Therefore, $\psi = \xi^{-1}$ is indeed a local coördinate system.

$\square$

More calculus w/ vector fields

Lecture 20: Lie Derivative

n-manifold

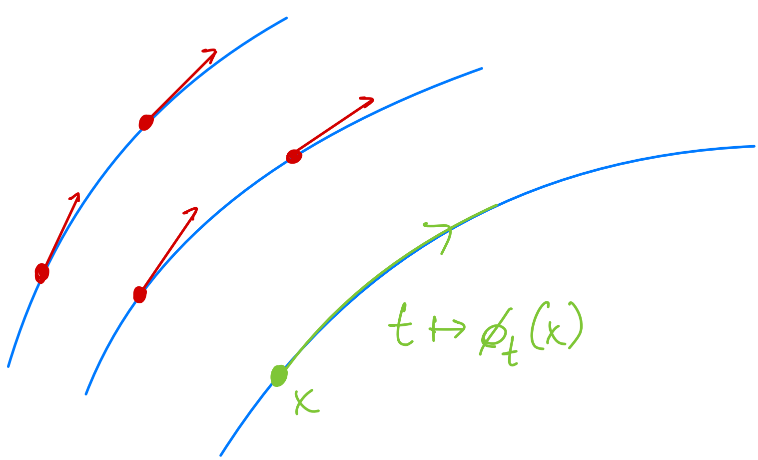
Recall that a smooth vector field $X \in \mathfrak{X}(M)$ generates the flow $\phi_t(x)$ for each "initial condition at $x = a$". Then, for $f \in C^\infty(M)$

IMPORTANT !

$$(Xf)(a) = \left.\frac{d}{dt}\right|_{t=0} (\phi_t^* f)(a)$$

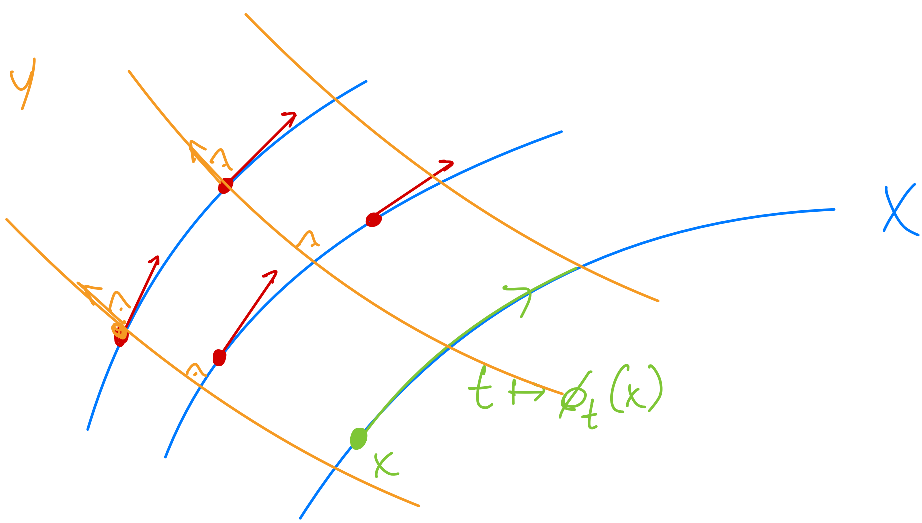
$$= \lim_{h \to 0} \frac{f(\phi_h(a)) - f(a)}{h}.$$

"FOLIATION"



Recall the **FLOW BOX THEOREM**: We can choose coördinates $(x_1, \_\_, x_n)$ in which $X = \dfrac{\partial}{\partial x_1}$.

(1) Suppose we have a 2^{nd} vector field $Y \in \mathfrak{X}(M)$, linearly independent to X. Can we find a coördinate system in which $X = \dfrac{\partial}{\partial x_1}$ and $Y = \dfrac{\partial}{\partial x_2}$?



To be continued later

IMPORTANT !!

Def: (LIE DERIVATIVE) Define the Lie derivative of $f \in C^\infty(M)$ with respect to $X \in \mathfrak{X}(M)$ as

$$(L_X f)(a) := (Xf)(a) = \frac{d}{dt}\bigg|_{t=0} (\phi_t^* f)(a)$$

$$= \lim_{h \to 0} \frac{f(\phi_h(a)) - f(a)}{h}$$

Analogous for differential 1-forms ω:

$$(L_X \omega)(a) = \frac{d}{dt}\bigg|_{t=0} (\phi_t^* \omega)(a)$$

This limit is in T^*M_a →

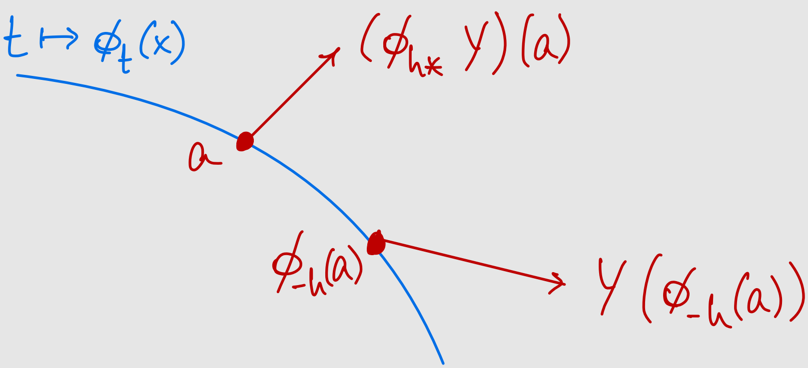
$$= \lim_{h \to 0} \frac{(\phi_h^* \omega)(a) - \omega(a)}{h}.$$

Lie derivative of a vector field $Y \in \mathfrak{X}(M)$ with respect to $X \in \mathfrak{X}(M)$:

$$(L_X Y)(a) := \lim_{h \to 0} \frac{Y(a) - (\phi_{h*} Y)(a)}{h}$$

This limit is taken in TM_a

$$= \lim_{h \to 0} \frac{Y(a) - \phi_{h*}\left(Y(\phi_{-h}(a))\right)}{h}$$



This only makes sense because $\phi_t(x)$ is a diffeo.

Loosely speaking $L_X Y$ measures the amount by which Y changes along the integral curves of X.

REMARK: If $g: M \to N$ is a diffeomorphism $Y \in \mathfrak{X}(M)$, then we can define $g^* Y \in \mathfrak{X}(M)$ as

$$\boxed{(g^* Y)(a) = (g^{-1})_* Y(g(a))}$$

$$\iff \quad g^* Y = (g^{-1})_* Y.$$

LEMMA:

$$L_X Y = \lim_{h \to 0} \frac{(\phi_h^* Y)(a) - Y(a)}{h}$$

Pf:

$$RHS = \lim_{h \to 0} \frac{Y(a) - (\phi_h^* Y)(a)}{-h}$$

$-h =: k$

$$= \lim_{t \to 0} \frac{Y(a) - (\phi_{-t}^{*} Y)(a)}{t}$$

$$= \lim_{t \to 0} \frac{Y(a) - (\phi_{t*} Y)(a)}{t}$$

$$= L_X Y .$$

$\square$

* **COMPOSING VECTOR FIELDS**: Suppose

$$XY : f \longmapsto X(Yf)$$

Not a vector field b/c not a derivation b/c of 2nd partial derivatives (e.g., $X = \frac{d}{dt}$, then $X^2 = \frac{d^2}{dt^2}$ is not a derivation)

LEMMA: The commutator

LIE BRACKET $[X, Y] = XY - YX$

is a derivation.

Pf: Compute $[X,Y](fg)$:

$$(XY)(fg) = X(Y(fg))$$

$$= X\left(f\,Y(g) + g\,Y(f)\right)$$

$$= f(XY)(g) + \underline{X(f)Y(g)}$$

$$+ g(XY)(f) + \underline{X(g)Y(f)}$$

Compute

$$(YX)(fg) = \cdots \quad \underline{\quad} \quad \underline{\quad}$$

Subtract the two and the terms in red cancel. $\square$

❗ * **EXAMPLE**:

$$X = \frac{\partial}{\partial x_1} \quad \text{and} \quad Y = (1 + x_1^2)\frac{\partial}{\partial x_2}$$

Then Product rule when apply X to Y

$$[X, Y] = \left(\frac{\partial}{\partial x_1}(1 + x_1^2)\right)\frac{\partial}{\partial x_2} + \cancel{(1+x_1^2)\frac{\partial}{\partial x_1}\frac{\partial}{\partial x_2}}$$

$$- \cancel{(1+x_1^2)\frac{\partial}{\partial x_2}\frac{\partial}{\partial x_1}}$$

$$= 2x_1 \frac{\partial}{\partial x_2}.$$

$\square$

More generally: in local coördinates, let

$$X = \sum_i \xi_i \frac{\partial}{\partial x_i} \quad \text{and} \quad Y = \sum_i \eta_i \frac{\partial}{\partial x_i}$$

Then,

$$(XY)(f) = \sum_i \xi_i \frac{\partial}{\partial x_i}\left(\sum_j \eta_j \frac{\partial f}{\partial x_j}\right)$$

$$= \sum_{i,j} \xi_i \left(\frac{\partial \eta_j}{\partial x_i} \frac{\partial f}{\partial x_j} + \eta_j \frac{\partial^2 f}{\partial x_i \partial x_j} \right)$$

$$= \sum_{i,j} \xi_i \left(\frac{\partial \eta_j}{\partial x_i} \frac{\partial f}{\partial x_j} \right)$$

$$+ \sum_{i,j} \xi_i \eta_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

So, the commutator is (without the f)

$$[X, Y] = XY - YX \quad !!$$

$$= \sum_{i,j} \left(\xi_i \frac{\partial \eta_j}{\partial x_i} - \eta_i \frac{\partial \xi_j}{\partial x_i} \right) \frac{\partial}{\partial x_j}$$

Scenes of the next chapter: $L_X Y = [X, Y]$.

Lecture 21: Lie Derivative & Lie Bracket

Recall that the Lie derivative of a vector field $Y \in \mathfrak{X}(M)$ with respect to another vector field $X \in \mathfrak{X}(M)$ is defined as: at $a \in M$

$\phi_t(x)$ flow of X

$$(L_X Y)(a) := \lim_{h \to 0} \frac{Y(a) - (\phi_{h*} Y)(a)}{h}$$

as is $\phi_t(x)$

If g is a diffeo, we can define $g^* Y = (g^{-1})_* Y$

$$= \lim_{h \to 0} \frac{(\phi_h^* Y)(a) - Y(a)}{h}$$

Now, we prove the following result using Hadamard's Lemma

If $f : (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ is C^∞, $f(0, x) = 0$

for all $x \in M$, then $\exists$ C^∞ $g: (-\varepsilon, \varepsilon) \times M \to \mathbb{R}$ s.t. $f(t,x) = t\, g(t,x)$ and

$$\frac{\partial f}{\partial t}(0,x) = g(0,x).$$

Thm: $\boxed{L_X Y = [X,Y]}$. !!

! **IMPORTANT CALCULATION** ?

Pf: Let $\phi_t(x)$ be the flow generated by X for $|t| < \varepsilon$. Then, for $f \in C^\infty(M)$, by Hadamard's Lemma $\exists$ family of C^∞ functions $g_t(x)$ s.t.

$$f \circ \phi_t - f = t\, g_t,$$

where $g_t(x)$ is C^∞ and

$$g_0(x) = \frac{\partial}{\partial t}\Big|_{t=0} \phi_t^* f = X(f)$$

So, compute

$$(\phi_{h*} Y)(f) = \phi_{h*}\Big(Y(\phi_{-h}(a))\Big)(f)$$

dif. of lower star acting on tangent vectors

$$= Y_{\phi_{-h}(a)}(f \circ \phi_h)$$

from Hadamard.

$$= Y_{\phi_{-h}(a)}(f + h g_h).$$

Now, plug the above into the limit of the definition of $L_X Y$:

at $a \in M$

$$(L_X Y)_a(f) = \lim_{h \to 0} \frac{1}{h}\Big(Y(f)(a) - \big[Y(f)(\phi_{-h}(a)) + Y(h g_h)(\phi_{-h}(a))\big]\Big)$$

applied to C^∞ function f

Y is linear

$$= \underbrace{\lim_{h \to 0} \frac{Y(f)(a) - Y(f)(\phi_{-h}(a))}{h}}_{L_X Y(f)(a)}$$

Y is linear

$$- \lim_{h\to 0} \frac{\cancel{h}\, Y(g_h)(\phi_{-h}(a))}{\cancel{h}}$$

$\phi_0(a) = a$

$$= \underbrace{L_X \overbrace{Y(f)(a)}^{\text{function!}}}_{\text{Lie derivative of a function}} - Y(g_0)(a)$$

$$= X_a(Y(f)) - Y_a(X(f))$$

$$= [X, Y](f)(a).$$

$\square$

Using the above & properties of the bracket,

- $L_X Y = - L_Y X$,
- $L_X X = 0$, etc...

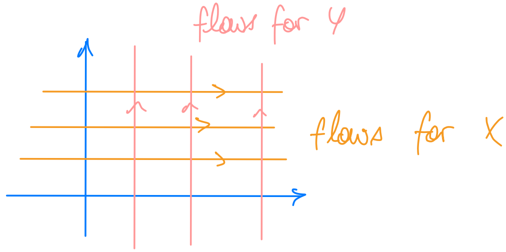
Recall: (FLOW BOX THEOREM) If $X(a) \neq 0$, $\in \mathfrak{X}(M^n)$
then there is a coördinate system $x = (x_1, \dots, x_n)$ in a neighborhood of a in which

$$X = \frac{\partial}{\partial x_1}.$$

If $Y \in \mathfrak{X}(M)$ linearly independent of X, is there a coördinate system s.t

$$X = \frac{\partial}{\partial x_1} \quad \text{and} \quad Y = \frac{\partial}{\partial x_2} \, ?$$

No! Not true unless $[X, Y] = 0$.



Remark: If $g: M \to N$ is a diffeo, then $g_*[X,Y] = [g_* X, g_* Y]$.

E.g.: $X = \frac{\partial}{\partial x_1}$, $Y = (1+x_1^2)\frac{\partial}{\partial x_2}$, then $[X,Y] = 2x_1 \frac{\partial}{\partial x_2} \neq 0$ but X lin indep of Y.

Thm: Given $X_1, \dots, X_k \in \mathfrak{X}(M^n)$ linearly independent C^∞ vector fields in a neighborhood of $a \in M$ such that $[X_i, X_j] = 0$ for all i, j, then there is a coördinate chart $(U, \varphi =: x)$ around a in which

$$X_i = \frac{\partial}{\partial x_i} \text{ on } U, \quad i = 1, \dots, k.$$

Pf: As in the Flow Box Thm, WLOG, suppose $M = \mathbb{R}^n_{(y_1, \ldots, y_n)}$ and $a = 0$,

and

$$X_i(0) = \frac{\partial}{\partial y_i}\Big|_0 , \quad i = 1, \ldots, k.$$

Let X_i generate flow ϕ^i_t. Now,

define

$$\xi(x_1, \ldots, x_n) \overset{\varphi^{-1}}{:=} \phi^1_{x_1}(\phi^2_{x_2}(\cdots(\phi^k_{x_k}(0, \ldots, 0, x_{k+1}, \ldots, x_n)\cdots))$$

As before, compute

$$\xi_*\left(\frac{\partial}{\partial x_i}\Big|_0\right) = \begin{cases} X_i(0) = \frac{\partial}{\partial y_i}\Big|_0 , & i = 1, \ldots, k \\ \frac{\partial}{\partial y_i}\Big|_0 , & i = k+1, \ldots, n \end{cases}$$

So, as before, we can also see that

$$X_1 = \frac{\partial}{\partial x_1}.$$

To show the rest of the vec. fields $X_i = \frac{\partial}{\partial x_i}$, we need the following:

LEMMA 1: If $g: M \to N$ is a diffeo. and $X \in \mathfrak{X}(M)$ w/ $\phi_t(x)$ as its flow, then $g_* X$ generates the flow $g \circ \phi_t \circ g^{-1}$.

Corollary: If $g: M \to M$ is a diffeo. then $g_* X = X$ if and only if

$$g \circ \phi_t = \phi_t \circ g.$$

$\to g \circ \phi_t \circ g^{-1} = \phi_t \circ g \circ g^{-1}$
$= \phi_t \rightsquigarrow$ flow of X.

LEMMA 2: If $X, Y \in \mathfrak{X}(M)$ respectively generate the flows ϕ_t and ψ_t, then

$$[X, Y] = 0 \iff \phi_t \circ \psi_s = \psi_s \circ \phi_t \quad \forall s, t.$$

By Lemma 2, we can just commute the flows since the vec. fields commute w/ each other ($[X_i, X_j] = 0 \ \forall i,j$). Thus, by the last formula, we have that, commuting the flows:

$$\xi(x_1, \_, x_n) = \phi^i_{x_i}\left(\phi^1_{x_1}\left(\ldots(0, \_, 0, x_{k+1}, \_, x_n)\ldots\right)\right)$$

$$\Rightarrow \quad X_i = \frac{\partial}{\partial x_i} \quad \forall i = 1, \_, k, \text{ as desired.}$$

Pf: (Lemma 1) Compute:

$$(g_* X)_b(f) \overset{\text{def}}{=} g_*\left(X_{g^{-1}(b)}\right)(f)$$

$$\overset{\text{def}}{=} X_{g^{-1}(b)}(f \circ g)$$

If $X \in \mathfrak{X}(M)$ generates flow ϕ_t,

$(Xf)(a) = \frac{d}{dt}\Big|_{t=0} (\varphi_t^* f)(a)$

$= \lim_{h\to 0} \frac{f(\phi_h(a)) - f(a)}{h}$

$$= \lim_{h \to 0} \frac{1}{h}\left[(f\circ g)(\phi_h(g^{-1}(b)))\right.$$

$$- (f\circ g)(g^{-1}(b))\Big]$$

$$= \lim_{h\to 0} \frac{1}{h}\Big[f(g\circ\phi_h\circ g^{-1})(b) - f(g\circ g^{-1})(b)\Big]$$

$$= \lim_{h\to 0} \frac{1}{h}\Big[f(g\circ\phi_h\circ g^{-1})(b) - f(b)\Big]$$

$$= \frac{d}{dt}\Big|_{t=0} \big((g\circ\phi_t\circ g^{-1})^* f\big)(b).$$

$\square$

LEMMA 2: $(\Leftarrow)$ If, for fixed t

$$\phi_t\circ\psi_s = \psi_s\circ\phi_t \quad \forall s,$$

then $\phi_{t*} Y = Y$ by the Corollary of Lemma 1. But, since this is true for all t, $L_X Y = 0 = [X, Y]$.

limit definition of Lie derivative

($\Rightarrow$) Conversely, suppose $[X, Y] = 0$, i.e.,

$$L_X Y = \lim_{h \to 0} \frac{Y_b - (\phi_{h*} Y)_b}{h} = 0$$

Given $a \in M$, consider the curve

$$\gamma : (-\varepsilon, \varepsilon) \to TM_a$$

$$\gamma(t) = (\phi_{t*} Y)_a .$$

Compute $\gamma'(t)$:

$$\gamma'(t) = \lim_{h \to 0} \frac{\gamma(t+h) - \gamma(t)}{h}$$

$$= \lim_{h \to 0} \frac{(\phi_{(t+h),*} Y)_a - (\phi_{t*} Y)_a}{h}$$

Since for diffeo. g:

$(g_* Y)_a = g_{*\varphi^{-1}(a)} Y_{\varphi^{-1}(a)}$

$$= \lim_{h \to 0} \frac{\overbrace{[\phi_{t*}(\phi_{h*} Y)]_a}^{= \phi_{t*}(\phi_{h*} Y)_{\phi_t^{-1}(a)}} - \overbrace{(\phi_{t*} Y)_a}^{\phi_{t*}(Y_{\phi_t^{-1}(a)})}}{h}$$

ϕ_{t*} is linear

$$= \phi_{t*} \left[\underbrace{\lim_{h \to 0} \frac{(\phi_{h*} Y)_{\phi_h^{-1}(a)} - Y_{\phi_t^{-1}(a)}}{h}}_{-L_X Y = 0} \right]$$

$$= 0.$$

Thus, $\gamma'(t) = 0$, so for all t

$\gamma(t) = \gamma(0)$; i.e., $\phi_{t*} Y = Y$; thus, By the Corollary to Lemma 2, we find that $\phi_t \circ \psi_s = \psi_s \circ \phi_t$, as desired. $\square$

SCENES OF THE NEXT CHAPTER...

Define a k-dimensional DISTRIBUTION Δ on M as $\Delta : a \mapsto \Delta_a$ ← k-dimensional subspace of TM_a

Δ is C^∞ if every point of M has a neighborhood U on which there are C^∞ vector fields $X_1, \ldots, X_k$ s.t. $X_i(a)$ form a basis for Δ_a, for all $a \in U$.

Then, a k-dimensional submanifold N of M is an INTEGRAL SUBMANIFOLD of Δ if for all $a \in N$, $i_{*a} TN_a = \Delta_a$, where $i : N \hookrightarrow M$. ← N needn't exist even locally; for example:

$X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}$ in $\mathbb{R}^3$ $\Rightarrow [X, Y] = \frac{\partial}{\partial z}$ → linearly independent from X and Y

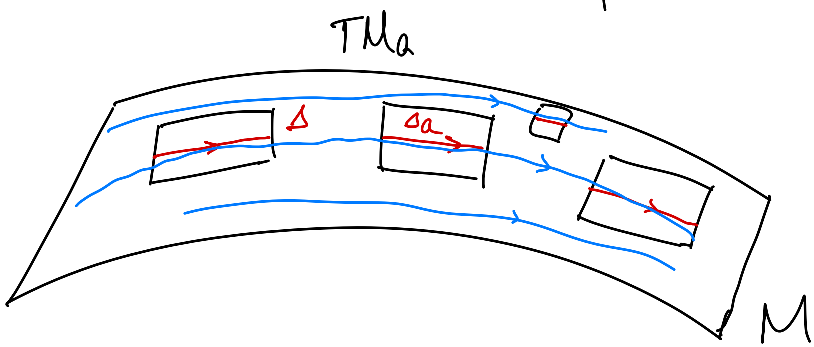
LECTURE 22: FINAL SHENANIGANS

Final: Definitely Frobenius integrability thm.

LECTURE 23: FROBENIOUS' INTEGRABILITY THEOREM

k-DIMENSIONAL DISTRIBUTION: A map that assigns a k-dimensional vector space to each pt. of a manifold M; i.e.,

$$\Delta : a \mapsto \Delta_a , \quad k\text{-dimensional subspace of } TM_a .$$



We say a distribution Δ is C^∞ if every pt. $a \in M$ has a neighborhood U on which there are C^∞ vector fields $X_1, \dots, X_k \in \mathcal{X}(M)$ s.t.

$$\Delta_a = \operatorname{span}\{X_1, \dots, X_k\}$$

for every $a \in U$.

INTEGRAL SUBMANIFOLD: A k-dimensional submanifold N of M is an integral submanifold of Δ if

$$\imath_{*a} T N_a = \Delta_{\imath(a)}$$

for every $a \in N$, where $\imath : N \hookrightarrow M$ is the inclusion map.

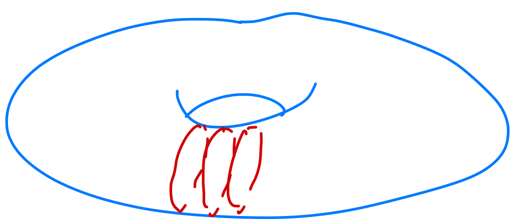
Remark: Integral submanifolds needn't exist even locally!

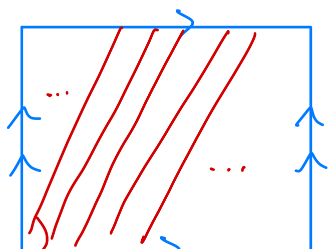
e.g., in $\mathbb{R}^3$: $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$

$\Rightarrow$ Δ spaned by X and Y is

$$[X, Y] = \frac{\partial}{\partial z}$$ (not lin. indep.)

REMARK. We should allow immersed submanifolds N as integral manifolds of Δ:





irrational angle

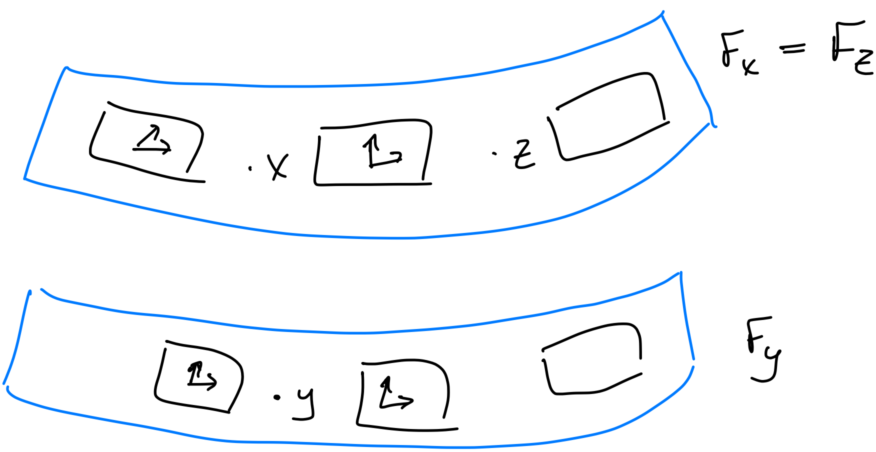
If N is an integral submanifold of a distribution on M, then

$$[X_i, X_j]_a \in TN_a$$

whenever the $X_\bullet$'s are as above (i.e., locally span Δ locally).

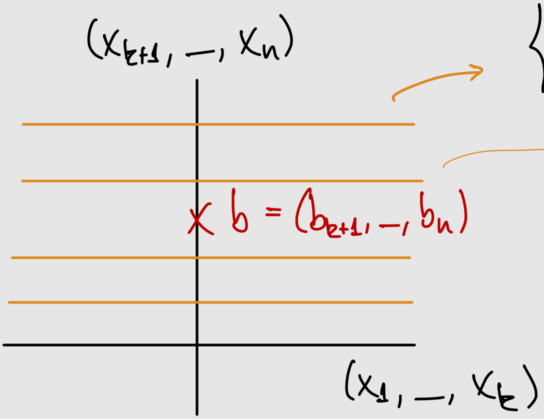
* **Δ INVOLUTIVE**: if $\forall X, Y \in \Delta$, we have $[X, Y] \in \Delta$.

* **Δ INTEGRABLE**: if $\forall a \in M$, there exists a k-dimensional submanifold $F_a \subset M$ such that $T(F_a)_a = \Delta_a$



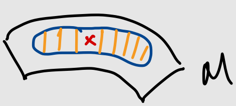
Thm: (LOCAL INTEGRABILITY THEOREM) If Δ is a rank k integrable distribution on M, then

(1) For all $a \in M$ there is a coordinate chart $\varphi : U \longrightarrow (-\varepsilon, \varepsilon)^n \subset \mathbb{R}^n$, $\varepsilon > 0$, s.t. $\varphi(a) = 0$ in which



$\{x_i = b_i : i = k+1, \ldots, n\}$ ← called "LEAVES"

are integral manifolds of Δ in U for all $(b_{k+1}, \ldots, b_n) \in (-\varepsilon, \varepsilon)^{n-k}$.



(2) Any connected integral manifold of $\Delta|_U$ is contained in one of the leaves.

DEF: (f-RELATED VECTOR FIELDS) Vector fields are **related** by a C^∞ function $f: M \to N$ if $X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$ s.t.

$$f_{*a} X_a = Y_{f(a)} \quad \forall a \in M.$$

Means that: $\forall g \in C^\infty(N)$,

$$\underbrace{(f_{*a} X_a)(g)}_{\text{Equivalent to } X_a(g \circ f) \text{ or } X(g \circ f)_a} = \underbrace{Y_{f(a)}(g)}_{\text{Equivalent to } Y(g)(f(a))}$$

i.e.,
$$\boxed{X(g \circ f) = Y(g) \circ f \quad \forall g \in C^\infty(N).}$$

EXAMPLES:

(1) If f is a diffeo., then $X \in \mathfrak{X}(M)$ is f-related to $f_* X$.

(2) If f is an immersion, $Y \in \mathfrak{X}(N)$ and $Y_{f(a)} \in f_{*a} TM_a$ $\forall a \in M$, then there exists a <u>unique</u> $X \in \mathfrak{X}(M)$ s.t. X and Y are f-related (by the Immersion Thm)

Lemma: If X_i, Y_i are f-related, $i = 1, 2$, then $[X_1, X_2]$ is f-related to $[Y_1, Y_2]$.

Pf: Going to use

$$X(g \circ f) \overset{(*)}{=} Y(g) \circ f \quad \forall g \in C^\infty(N).$$

So, for all $g \in C^\infty(N)$,

$$Y_i(g) \circ f = X_i(g \circ f).$$

Now,

$$[Y_1, Y_2](g)\circ f$$

these are functions!

$$= Y_1(\underbrace{Y_2(g)})\circ f - Y_2(\underbrace{Y_1(g)})\circ f$$

identity (*)

$$= X_1(Y_2(g)\circ f) - X_2(Y_1(g)\circ f)$$

$$= X_1(X_2(g\circ f)) - X_2(X_1(g\circ f))$$

$$= [X_1, X_2](g\circ f),$$

as desired. □

LEMMA: Suppose $X_1, \_\_, X_k$ span a rank k distribution Δ in some neighborhood U of $a\in M$. Then

$c_{ij}^\ell \in C^\infty(U)$

$$\Delta \text{ involutive on } U \iff [X_i, X_j] = \sum_{\ell=1}^{n} c_{ij}^\ell X_\ell$$

Pf: ($\Rightarrow$) <u>WTS</u>: if $X \in \Delta$ on U then

$$X = \sum_{i=1}^{\ell} c_i X_i , \quad c_i \in C^\infty(U) .$$

We can complete $X_1, \ldots, X_k$ to a "full" list $X_1, \ldots, X_k, X_{k+1}, \ldots, X_n$ of n linearly independent sections of $TM|_U$. In local coördinates of U,

C^∞ from the local coördinate maps

Cramer's Rule

$$X_i = \sum_j a_{ij} \frac{\partial}{\partial x_j} .$$

($\Leftarrow$) <u>WTS</u>: $X, Y \in \Delta \Rightarrow [X, Y] \in \Delta$.

But we know that

$$X = \sum f_i X_i , \quad Y = \sum g_i X_i ,$$

$f_i, g_i \in C^\infty$. So, it is enough to show that $[f_i X_i, g_j X_j] \in \Delta \quad \forall i, j$.

Recall: From Leibniz Rule

$$[fX, gY] = fg[X,Y] + fX(g)Y - gY(f)X$$

So, LHS $\in \Delta$ if $X, Y, [X,Y] \in \Delta$. $\square$

Pf: (Local Integrability Theorem)

IDEA: Reduce to the Flow Box Thm.

Assume $M = \mathbb{R}^n_{(y_1, \dots, y_n)}$ → by means of local charts and suppose that $a = 0$, and $\Delta_0 \subset T\mathbb{R}^n_a$ spanned by $\frac{\partial}{\partial y_i}\Big|_0$, $i = 1, \dots, n$.

Consider the projection

$$\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^k$$
$$y \longmapsto (y_1, \dots, y_k)$$

onto 1st k entries.

Then π_* is an injection on Δ_a (clearly true at $a = 0$ and true by continuity for pts. near a).

We can choose $X_1, \ldots, X_k$ belonging to Δ near O s.t.

$$\pi_*(X_i(a)) = \left.\frac{\partial}{\partial y_i}\right|_{\pi(a)}$$

near O.

[Why? B/c we can choose $X_1, \ldots, X_k$ s.t.

$$X_i \overset{(*)}{=} \sum_{j=1}^{k} a_{ij} \frac{\partial}{\partial y_j} + \sum_{j=k+1}^{n} b_{ij} \frac{\partial}{\partial y_j}$$

where $(a_{ij}(0)) = I$, $b_{ij}(0) = 0$. Just define X_j', $j = 1, \ldots, k$, by the formula

Apply inverse matrix of (a_{ij}) to the X_i's. $\rightarrow$ $X_i = \sum a_{ij} X_j'$, $i = 1, \ldots, k$.

$$\Longrightarrow \quad X_i' = \sum_{j=1}^{k} \delta_{ij} \frac{\partial}{\partial y_j} + \sum_{j=k+1}^{n} b_{ij}' \frac{\partial}{\partial y_j}.$$

Apply $(a_{ij})^{-1}$ to X_i on $(*)$

Finally, when we project onto the 1^{st} k coördinates, we find that

$$\pi_*\left(X_i'(a)\right) = \frac{\partial}{\partial y_i}\bigg|_{\pi(a)} \Big]$$

This means that X_i and $\frac{\partial}{\partial y_i}$ are π-related for $i = 1, \ldots, k$. By a Lemma from before, the brackets are π-related:

linearly indep.

$$\pi_*[X_i, X_j]_a = \left[\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}\right]_{\pi(a)} = 0$$

$\Rightarrow [X_i, X_j]_a = 0$ b/c π is injective.

By the souped up Flow Blox Theorem, we can choose NEW different local coördinates $(x_1, \dots, x_n)$ s.t. $X_i = \frac{\partial}{\partial x_i}$, $i = 1, \dots, k$.

But these clearly span the "horizontal lines". Thus

Δ is integrable b/c, at every pt a, Δ_a is the tangent spaces are spanned by $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_k}$.

(Left to show (2): next time)

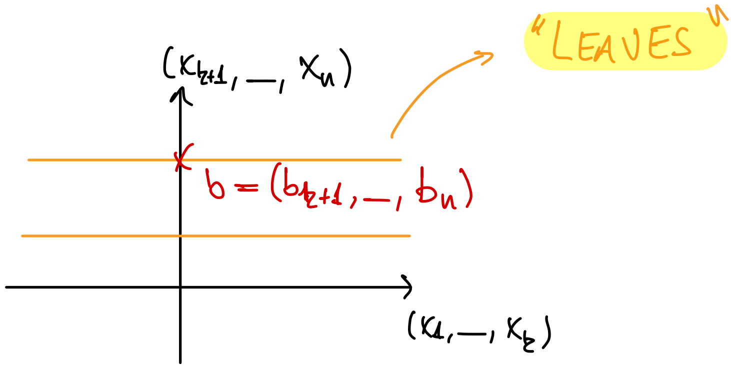
$\square$

Lecture 24: Frobenius' Integrability Theorem

Recall the local integrability theorem: if Δ is a rank k distribution on M^n, then

(1) For all $a \in M$, there is a coördinate

chart $\varphi : U \to (-\varepsilon, \varepsilon)^n \subset \mathbb{R}^n$ such that $\varphi(a) = 0$ and $\{x_i = b_i : i = k+1, \dots, n\}$ are integral manifolds of $\Delta|_U$.



(2) Any connected integral manifold N of $\Delta|_U$ lies in one of the leaves.

Pf of (2): $\iota : N \hookrightarrow U$, need to show $x_j \circ \iota = \text{constant} =: b_j$, $j = k+1, \dots, n$.

Compute its differential $d(x_j \circ \iota)$ at some X_b (element of TN_b):

$$d(x_j \circ z)(X_b) = X_b(x_j \circ z)$$

$$= \underbrace{z_{*b}(X_b)}_{\in \Delta_b \,:\, \text{spanned by } \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_k}.}(x_j)$$

$\square$

Def: (k-dimensional Foliation) A k-dim. foliation of M is

$$\mathcal{F} := \{F_a : a \in M\}.$$

from def. of integrable distribution in pg. 199.

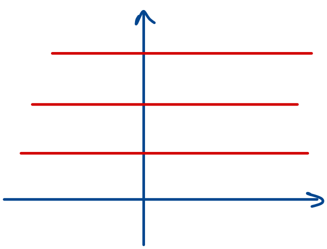
!!

We have that

$$\begin{tikzcd}
x \arrow[d, mapsto] & \in & M \arrow[d, "\pi"] & = & \bigcup_{x \in M} F_x \\
F_x & \in & M/\mathcal{F} & = & \text{leaf space}
\end{tikzcd}$$

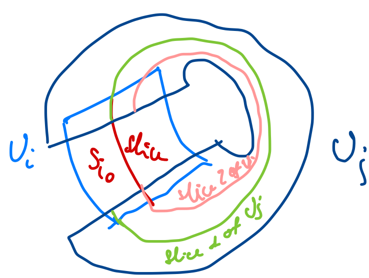
THM: (GLOBAL INTEGRABILITY THEOREM) If Δ is a C^∞ integrable k-dimensional distribution on M, then M is foliated by integral submanifolds of Δ.

Pf: M is covered by countably many coördinate charts (U_i, φ_i) as in (1) of the local thm. Denote by $x^i = (x^i_1, \dots, x^i_n)$ the local coördinates in (1).



Slices

Obs: Slices S of U_i may intersect U_j in more than 1 slice of U_j: e.g.



But: $S \cap U_j$ has only countably many components each in a single slice of U_j, by (2).

So, $S \cap U_j$ contained in at most countably many slices of U_j.

Fix $a \in M$. Choose i_0 s.t. $a \in U_{i_0}$ and let S_{i_0} be the slice of U_{i_0} containing a

Say slice S of U_i, for some i, is joined (or "joined to a") to S_{i_0} if $\exists$ $i_0, i_1, \ldots, i_q =: i$ and slices $S_{i_0}, S_{i_1}, \ldots, S_{i_q} =: S$ s.t.

$$S_{i_j} \cap S_{i_{j+1}} \neq \emptyset, \quad j = 0, \ldots, q-1.$$

Note that for each sequence $i_0, i_1, \ldots, i_q = i$, there are only countably many chains $S_{i_0}, S_{i_1}, \ldots, S_{i_f} = S$. But there are, again, only countably many $i_0, \ldots, i_f = i$. So, there is only countably many chains $S_{i_0}, \ldots, S_{i_f} = S$.

So, there are only countably many slices joined to a. The union of these slices is an <u>immersed</u> submanifold of M.

Given a different pt. $b \neq a$, correspon-ding union of slices is either equal or disjoint.

Thus, M is foliated by the disjoint union of all those submanifolds.

$\square$