LECTURE 1

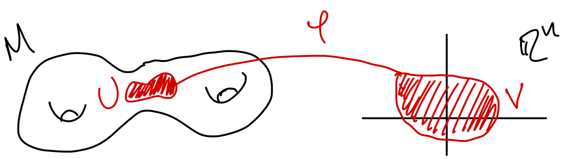
Sep 5th, 2024

INTRODUCTION

TOPOLOGICAL MANIFOLDS

Def: (Topological Mnfd) A real n-dimensional topological mnfd is a Hausdorff, 2nd countable topological space that is locally homeomorphic to $\mathbb{R}^n$.

i.e., $\forall p \in M \; \exists (U, \varphi)$ → $U \subset M$ open, $U \ni p$; → $\varphi : U \to V \subset \mathbb{R}^n$ homeomorphism (i.e., φ is continuous w/ continuous inverse)



Such (U, φ) is called a COORDINATE CHART around p b/c the "standard fcts" $x^1, \dots, x^n : \mathbb{R}^n \to \mathbb{R}$ define "coordinates".

Ex: $\emptyset$, $\mathbb{R}^n$

Ex: Any open subset of $\mathbb{R}^n$ or any another top. mnfd.

$\mathrm{Mat}(n, \mathbb{R}) := \{ n \times n \text{ matrices w/ real entries} \} \simeq \mathbb{R}^{n^2}$.

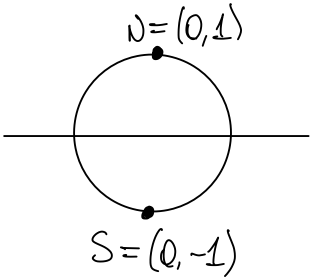
$\cup$

$GL(n, \mathbb{R}) = \{ X \in \mathrm{Mat}(n, \mathbb{R}) : \underbrace{\det X}_{\text{polynomial in the entries of } X} \neq 0 \}$

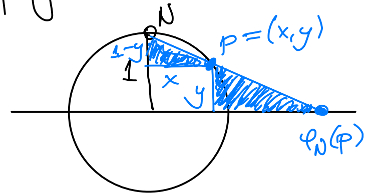
i.e., $\det^{-1}(\underbrace{\mathbb{R}^n \setminus \{0\}}_{\text{open}})$ is open

Ex: Circle $S^1 = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 = 1\} \subset \mathbb{R}^2$.

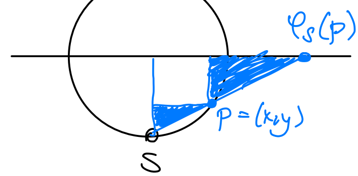
Define an atlas for S^1 via stereographic projection



$S^1 \supset \underset{\text{open}}{U_N} := S^1 \setminus \{N\}$



$S^1 \supset \underset{\text{open}}{U_S} := S^1 \setminus \{S\}$



Homeomorphisms:
(Stereographic projection)

$$\begin{cases} \varphi_N : U_N \to \mathbb{R}, & \varphi_N(x,y) = \dfrac{x}{1-y} \\[2ex] \varphi_S : U_S \to \mathbb{R}, & \varphi_S(x,y) = \dfrac{x}{1+y} \end{cases}$$

Ex: Cartesian products: If M^n, N^m are mnfds, then $M \times N$ (in the product top.) is a mnfd as well.

$\{(U_\alpha, \varphi_\alpha)\} \to$ Atlas for M

$\{(V_\beta, \psi_\beta)\} \to$ Atlas for N

$\Longrightarrow \{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}$

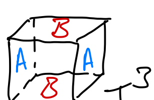
$(\varphi_\alpha \times \psi_\beta)(p,q) = (\varphi_\alpha(p), \psi_\beta(q))$

$\cap$

$\mathbb{R}^n \times \mathbb{R}^m$

Result: mnfd of dim $n+m$

Ex: $T^n := \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}$



Ex: $S^n = \{x \in \mathbb{R}^{n+1} : \|x\|^2 = 1\}$. $\sum_{i=0}^{n} x_i^2 = \|x\|^2$

Atlas: $\{(U_N, \varphi_N), (U_S, \varphi_S)\}$ $N := (1, 0, \ldots, 0)$, $S := (-1, 0, \ldots, 0)$

$$U_N := S^n \setminus \{N\} \qquad U_S := S^n \setminus \{S\}$$

$$\varphi_N : U_N \to \mathbb{R}^n \qquad \varphi_S : U_S \to \mathbb{R}^n$$

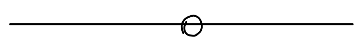
$$\varphi_N(x) = \frac{(x^1, \ldots, x^n)}{1 - x^0} \qquad \varphi_S(x) = \frac{(x^1, \ldots, x^n)}{1 + x^0}$$

stereographic proj$^{\wedge}$.

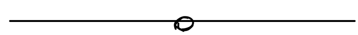
Obs: If M^n, N^n are n-mnfds then $M \sqcup N$ is an n-mnfd.

Ex: (Why Hausdorff?) Consider

$$\mathbb{R} =: \mathbb{R}_1 \supset \mathbb{R}_1^* := \mathbb{R}_1 \setminus \{0\}$$

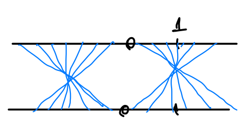


$$\mathbb{R} =: \mathbb{R}_2 \supset \mathbb{R}_2^* := \mathbb{R}_2 \setminus \{0\}$$



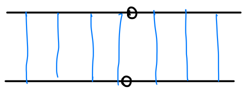
Glue $\mathbb{R}_1$ to $\mathbb{R}_2$ along gluing region $\mathbb{R}_1^*, \mathbb{R}_2^*$. To do this, we need a homeomorphism $\mathbb{R}_1^* \to \mathbb{R}_2^*$.

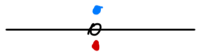
Option 1: use $\mathbb{R}_1^* \xrightarrow{\varphi} \mathbb{R}_2^*$
$t \mapsto t^{-1}$



Then $\mathbb{R}_1 \sqcup \mathbb{R}_2 \big/ \mathbb{R}^* \ni x \sim \varphi(x) \in \mathbb{R}_2^* = S^1$.

Option 2: Use $\mathbb{R}_1^* \xrightarrow{\varphi} \mathbb{R}_2^*$, $t \mapsto t$. Then $\mathbb{R}_1 \sqcup \mathbb{R}_2 \big/ \underset{\mathbb{R}_1^* \quad \mathbb{R}_2^*}{x \sim \varphi(x)} = \mathcal{B}$



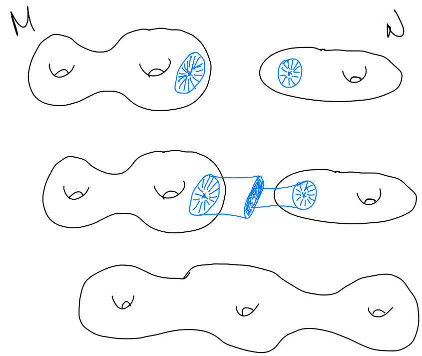


Line with 2 origins

PROBLEM: Is loc. homeom. to Euclidean space but has this exotic behavior that can be problematic.

WARNING: When building mnfds via gluing, really need to pay attention to whether the Hausdorff condition is satisfied (b/c it is not inherited by quotient topology)!

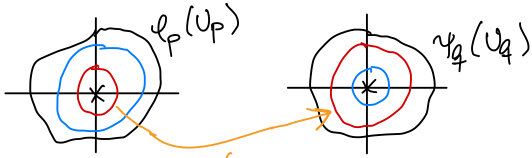
CONNECTED SUM: M^n, N^n mnfds, $p \in M$, $q \in N$



$M \# N$

Charts: (U_p, φ_p) and (V_q, ψ_q) s.t. $\varphi_p(p) = 0$ and $\psi_q(q) = 0$.

$$U_p \xrightarrow{\varphi_p} \qquad V_q \xrightarrow{\psi_q}$$



$$\phi(x) := \frac{2\varepsilon^2}{\|x\|^2} x$$

Choose balls of radius 2ε and annulus of radius ε.

Define

$$M \# N := \frac{\left(M \setminus \overline{\varphi^{-1}(B_\varepsilon)}\right) \sqcup \left(N \setminus \overline{\psi^{-1}(B_\varepsilon)}\right)}{x \sim \psi^{-1}(\phi(\varphi(x))) \ \forall x \in \varphi^{-1}(B_{2\varepsilon})}$$

LECTURE 2

PROJECTIVE SPACES, ETC.

Sep 6th, 2024

REMARK: Topological mnflds are objects in the category **Top**. The morphisms $X \to Y$ in **Top** are continuous maps.

Isomorphisms in **Top**:

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y$$

f, g are morphisms s.t. $fg = \mathbb{1}_Y$ and $gf = \mathbb{1}_X$.

f, g are called <u>homeomorphisms</u>.

(actually, **Top** is a subcategory of topological spaces)

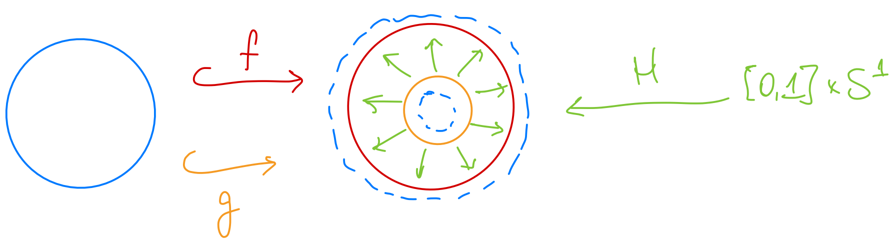
Feature of top. spaces: In top. spaces, two continuous maps may be related (equiv. relation) by a <u>homotopy</u>; i.e.:

$$X \overset{f}{\rightrightarrows} Y$$

f, g are <u>homotopic</u> if we can continuously interpolate them; i.e.:

$$\exists\, H : [0,1] \times X \longrightarrow Y \quad \text{continuous s.t.} \quad H_0(x) = f(x)$$
$$(t, x) \longmapsto H_t(x) \qquad\qquad H_1(x) = g(x)$$

e.g.: circle embedded in an annulus



Def: Top. spaces X, Y are **HOMOTOPY EQUIVALENT** when:

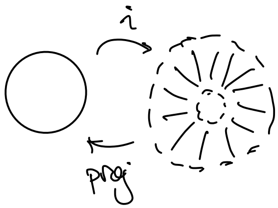
$$\begin{tikzcd} X \arrow[r, bend left, "f"] & Y \arrow[l, bend left, "g"] \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, bend left, "fg"{name=A}] \arrow[r, bend right, "\mathbb{1}_X"'{name=B}] \arrow[from=A, to=B, Rightarrow, "H_X"] & X \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[r, bend left, "gf"{name=C}] \arrow[r, bend right, "\mathbb{1}_Y"'{name=D}] \arrow[from=C, to=D, Rightarrow, "H_Y"] & Y \end{tikzcd}$$

$H_X : [0,1] \times X \to X$

$H_Y : [0,1] \times Y \to Y$

i.e., fg is homotopic to $\mathbb{1}_X$ and gf is homotopic to $\mathbb{1}_Y$ (not equal to identity but very close to it)

i.e., f and g are inverses to each other up to homotopy.

Ex:  are homotopy equivalent but <u>not</u> homeomorphic.

(Top. Poincaré)

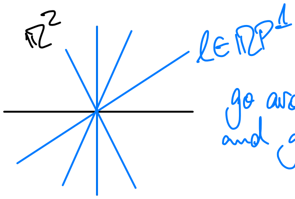
CONJECTURE: Is it possible for an n-dim top. mnfd X to be

- Homotopy equiv. to S^n
- not homeo. to it ?

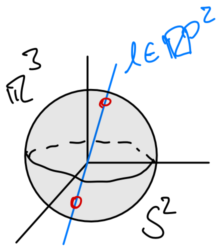
A: No.
- $n \geq 5$ Smale 60-70
- $n = 4$ Freedman 80-90
- $n = 3$ Hamilton-Perelman 00s
- $n = 1, 2$ conseq. of classification thm

* PROJECTIVE SPACES :

Over $\mathbb{R}$: $\mathbb{RP}^n$ = space of lines through 0 in $\mathbb{R}^{n+1}$



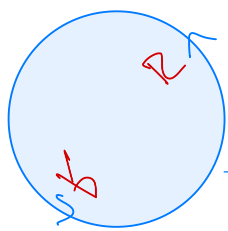
go around $180°$ and go back to l $\rightsquigarrow$ $\mathbb{RP}^1 = [0,\pi]/_{0\sim\pi} \simeq S^1$



$\mathbb{RP}^2 \not\simeq S^2$ (the antipodal pt. in the sphere produces the same line through 0)

$\mathbb{RP}^2 \simeq S^2/_{x\sim -x}$ $\rightsquigarrow$ Another way of doing this is just considering the upper hemisphere of S^2. Then the problematic pts. are on the bdry of the hemisphere

$\mathbb{RP}^2 \simeq$



Identify the antipodal pts on the bdry of this disk (which represents the upper hemisphere)

Obs: because we never get the "same $\mathbb{R}$" going around, we say $\mathbb{RP}^2$ is non-orientable.

Def: ($\mathbb{RP}^n$) Start with $X = \mathbb{R}^{n+1} \setminus \{0\}$. Define an equivalence relation

$$X \ni x \sim y \in X \iff \exists \lambda \in \mathbb{R} \text{ s.t. } \lambda x = y$$

(i.e., pts. are equiv iff they are colinear)

Then, define

$$\mathbb{RP}^n := X/_\sim \text{ with the quotient topology.}$$

Claim: $\mathbb{RP}^n$ is an n-dim. mnfd.

Pf: Let $\pi : X = \mathbb{R}^{n+1} \setminus \{0\} \longrightarrow \mathbb{RP}^n$ be the projection map.
Want to show that π is OPEN (i.e., $U \underset{\text{open}}{\subset} X \Rightarrow \pi(U) \underset{\text{open}}{\subset} \mathbb{RP}^n$),
This is true b/c

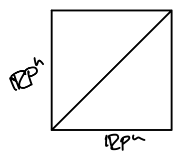
$U \underset{\text{open}}{\subset} X \Longrightarrow \pi^{-1}(\pi(U))$ is open. Why? B/c

$$\pi^{-1}(\pi(U)) = \overbrace{\bigcup_{\lambda \in \mathbb{R}} \lambda U}^{\text{open}}$$

Thus: $\mathbb{RP}^n$ is 2^{nd} countable.

NTS: $\mathbb{RP}^n$ is Hausdorff. $\Longleftrightarrow$ Diagonal in $\mathbb{RP}^n \times \mathbb{RP}^n$ is closed

$\Delta_{\mathbb{RP}^n} := \{(x,x) : x \in \mathbb{RP}^n\}$



But since we have $\pi : X \times X \longrightarrow \mathbb{RP}^n \times \mathbb{RP}^n$

$$\{(x,y) \in X \times X : x \sim y\} =: \Gamma_\sim = \text{preimage of } \Delta_{\mathbb{RP}^n}$$

to show $\mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta_{\mathbb{RP}^n}$ is open, we can just use that π is open and that

$$\mathbb{RP}^n \times \mathbb{RP}^n \setminus \Delta_{\mathbb{RP}^n} = (\pi \times \pi)\left(X \times X \setminus \Gamma_\sim\right).$$

So,

$$\mathbb{RP}^n \text{ is Hausdorff} \Longleftrightarrow \Gamma_\sim \subset X \times X \text{ is closed.}$$

How to show $\Gamma_\sim$ is closed? Express $\Gamma_\sim$ as the zero set/preimage of a pt. of a continuous map.

Have $\mathbb{R}^{n+1} \setminus \{0\} \ni x \sim y \in \mathbb{R}^{n+1} \setminus \{0\}$. Need to define a map that detects whether x and y

$$\begin{array}{cc} \| & \| \\ (x^i)_{i=0}^n & (y^i)_{i=0}^n \end{array}$$

are lin. dependent or not.

$\Gamma_\sim = M^{-1}(0)$
where $M(x,y) = x \wedge y$.

Atlas for $\mathbb{RP}^n$: $U_i := \pi(\tilde{U}_i)$, where

$\tilde{U}_i := \{(x^0, \dots, x^n) \in \mathbb{R}^{n+1} \setminus \{0\} : \underbrace{x_i \neq 0}_{\text{deleting the } i\text{-th hyperplane.}}\}$ ← $n+1$ hyperplane compliments

$\varphi_i : U_i \to \mathbb{R}^n$

$$[\underbrace{(x^0, \dots, x^n)}_{\in \tilde{U}_i}] \longmapsto \left(\frac{x^0}{x^i}, \frac{x^1}{x^i}, \dots, \widehat{\frac{x^i}{x^i}}, \dots, \frac{x^n}{x^i}\right)$$

- φ_i well-defined ✓
- φ_i continuous ✓
- φ_i invertible ✓ w/ inverse $(y^1, \dots, y^n) \xmapsto[\text{of } \varphi_i \text{ and clearly continuous}]{\text{is the inverse}} [(y^1, \dots, \underset{i\text{th}}{1}, \dots, y^n)]$

Upshot: $\{(U_0, \varphi_0), \dots, (U_n, \varphi_n)\}$ is an <u>atlas</u> for $\mathbb{RP}^n$ since

$$\mathbb{RP}^n = \bigcup_{i=0}^n U_i$$

Same for $\mathbb{CP}^n$... but complex

Lecture 3

Flags & Gluings

Sep 12th, 2024

S^n, $\mathbb{RP}^n$, $\mathbb{CP}^n$, $Gr_k(V)$ = space of k-dim subspaces of n-dim vec. space

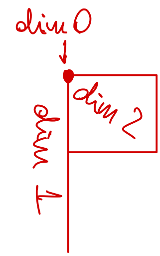
$\mathbb{RP}^n = Gr_1(\mathbb{R}^{n+1})$, $\mathbb{CP}^n = Gr_1(\mathbb{C}^{n+1})$

$\cap$

"Flag manifolds" $\rightsquigarrow$ $Fl_{\mathbb{R}}(n)$ = space of "full" flags in $\mathbb{R}^n$

$\cup$

$$\{\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^n\}$$
$$\dim V_i = i$$



<u>Ex</u>: $Fl(3) = \{\{0\} \subset V_1 \subset V_2 \subset \mathbb{R}^3\}$.

$$\begin{tikzcd}
 & Fl(3) \arrow[dl, "\text{forget } V_1"', "\pi_1"] \arrow[dr, "\text{forget } V_2"] & \\
Gr_2(\mathbb{R}^3) \simeq \mathbb{RP}^2 & & \mathbb{RP}^2
\end{tikzcd}$$

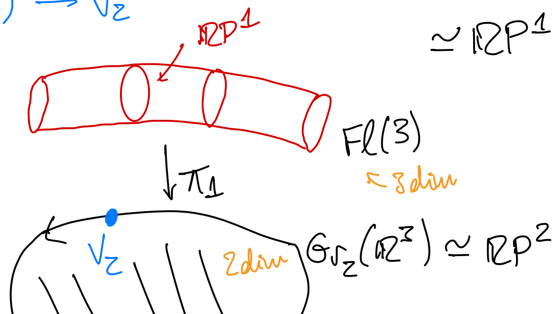
inclusion $V_2 \xrightarrow{\iota} \mathbb{R}^3$

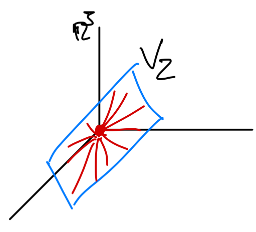
~~Annihilator~~ Annihilator(V_2) "=" $\underbrace{K_1}_{\text{kernel of } \iota^*} \longrightarrow (\mathbb{R}^3)^* \xrightarrow{\iota^*} V_2^*$

Fix $V_2 \subset \mathbb{R}^3$ and consider $\pi_1^{-1}(V_2)$

Have: $\pi_1^{-1}(V_2) = \{V_1 \text{ 1-dim subspace of } V_2\}$

$\simeq \mathbb{RP}^1$





$\Longrightarrow$ $Fl(3)$ is a <u>bundle</u> of $\mathbb{R}P^1$ over <u>base</u> $\mathbb{R}P^2$ with π_1 as the bundle projection.

fibers

base

Ex: (FIBER BUNDLES) Let F be a top. space (the <u>FIBER</u>). A fiber bundle with fiber F is (E, π, B) where

- E (<u>TOTAL SPACE</u>) and B (<u>BASE SPACE</u>) are top. spaces
- $\pi : E \to B$ continuous surjective (<u>BUNDLE PROJECTION MAP</u>)

<u>SUCH THAT</u>: $\forall p \in B$ $\exists$ a neighborhood $U \ni p$ and a homeo.

$$\begin{tikzcd} \Phi : \pi^{-1}(U) \arrow[r] \arrow[dr, "\pi"'] & U \times F \arrow[d, "\pi_1"] \\ & U \end{tikzcd}$$

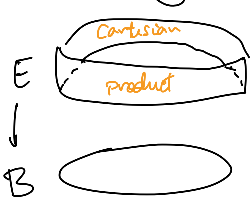
(LOCAL TRIVIALITY)

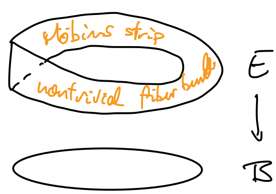
that commutes with the projections.

i.e., locally, the bundle is a Cartesian product. We just don't want it to be globally trivial...

<u>Claim</u>: If F, B are top. spaces then so is E.

<u>Rmk</u>: There are many fiber bundles with the same base & fiber





$F = (0, 1)$

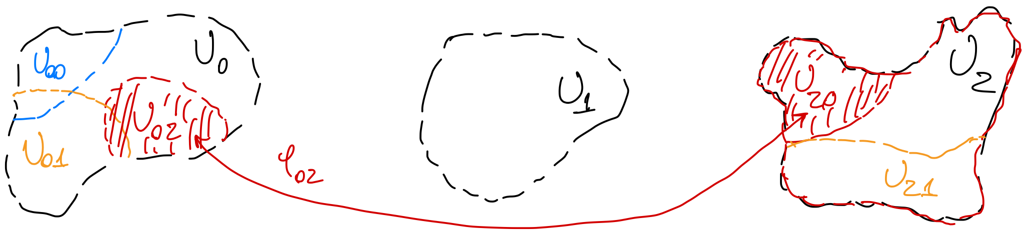
GLUING CONSTRUCTION OF MANIFOLDS

Begin with: a countable collection of open sets $\mathcal{A} = \{U_i\}_{i\in I}$

Idea: Quotient $\coprod_{i\in I} U_i$ by an equiv. relation to get

$$M = \coprod_{i\in I} U_i \Big/ \sim$$

Gluing: $\forall i$ choose finitely many opens $U_{ij} \subset U_i$



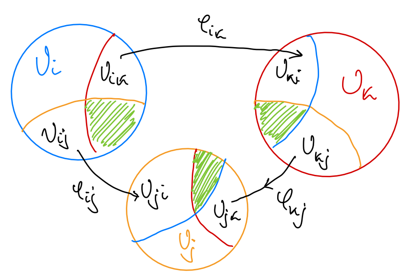
and gluing maps $\varphi_{ij} : U_{ij} \xrightarrow{\simeq} U_{ji}$ (homeos.).

Gluing maps define an equivalence relation iff:

(i) $\varphi_{ij}\varphi_{ji} = \mathbb{1}_{U_{ji}}$ (reflexivity)

$$\begin{tikzcd} U_{ji} & U_{ij} \arrow[l, "\varphi_{ij}"'] & U_{ji} \arrow[l, "\varphi_{ji}"'] \arrow[ll, bend left, "\mathbb{1}_{U_{ji}}"] \end{tikzcd}$$

(ii) $\varphi_{ij}(U_{ij} \cap U_{ik}) = U_{ji} \cap U_{jk} \quad \forall k$ (transitivity)



$$\varphi_{ki}\Big|_{U_{ki}\cap U_{kj}} \circ \varphi_{jk}\Big|_{U_{ji}\cap U_{jk}} \circ \varphi_{ij}\Big|_{U_{ik}\cap U_{ij}} = \mathbb{1}_{U_{ij}\cap U_{ik}}$$

This defines an equiv. relation on $\coprod_{i\in I} U_i$

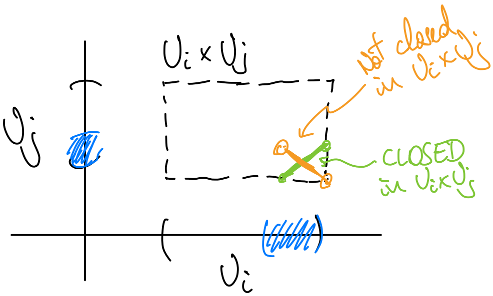
$$\Longrightarrow M = \bigsqcup_{i\in I} U_i \Big/ U_{ij} \ni x \sim \varphi_{ij}(x) \in U_{ji}$$

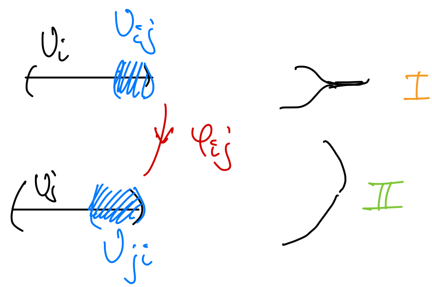
- top space
- 2nd countable
- locally homeo to $\mathbb{R}^n$

& Hausdorff under an additional assumption:

(iii) Graph of $\varphi_{ij} = \{(x, \varphi_{ij}(x)) : x \in U_{ij}\} \subset U_{ij} \times U_{ji} \subset U_i \times U_j$

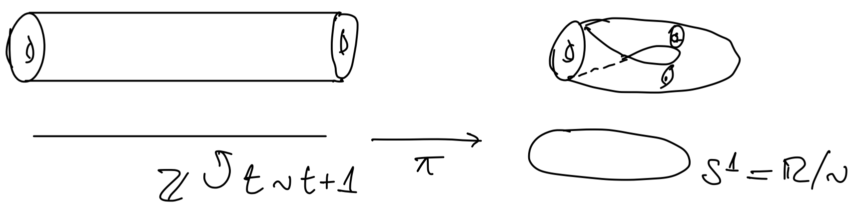
needs to be CLOSED in $U_i \times U_j$.





Ex: (MAPPING TORUS) M top. mnfd and $\phi: M \circlearrowleft$ Homeo

First: $M \times \mathbb{R}$



Quotient: $M_\phi := M \times \mathbb{R}/\sim$, where $\boxed{(x,t) \sim (\phi(x), t+1)}$

Upshot: The resulting mapping torus M_ϕ is a fiber bundle over S^1 with fiber M.

Lecture 4

Differential Structures

Sep 13th, 2024

We say $f: U \subset V \longrightarrow W$, V, W finite dim vector spaces, is differentiable at $p \in U$ if there exists a linear map $Df(p): V \longrightarrow W$ that approximates f at p:

$$\lim_{\substack{x \to 0 \\ x \neq 0}} \frac{|f(p+x) - f(p) - Df(p)(x)|}{|x|} = 0.$$

$Df(p)$ uniquely characterized by this limit.

$$Df(p) = \begin{pmatrix} \left.\frac{\partial f_1}{\partial x_1}\right|_p & \cdots & \left.\frac{\partial f_1}{\partial x_n}\right|_p \\ \vdots & & \\ \left.\frac{\partial f_m}{\partial x_1}\right|_p & \cdots & \left.\frac{\partial f_m}{\partial x_n}\right|_p \end{pmatrix};$$

Def: f is continuously differentiable if

$$Df: U \to \mathrm{Hom}(V, W)$$

is continuous. In this case, $f \in C^1(V, W)$.

Note: $C^\infty(U, W) := \bigcap_n C^n(U, W)$

Def: A smooth manifold is a topological manifold equipped with an equivalence class of smooth atlases.

Def: An atlas $\mathcal{A} = \{(U, \varphi)\}$ for a topological manifold is smooth when all transition functions

$$\varphi_j \circ \varphi_i^{-1}\big|_{\varphi_i(U_{ij})} : \varphi_i(U_{ij}) \longrightarrow \varphi_j(U_{ij})$$

are smooth maps; i.e., lie in $C^\infty(\varphi_i(U_{ij}), \mathbb{R}^n)$.
Two atlases $\mathcal{A}, \mathcal{A}'$ are equivalent if $\mathcal{A} \cup \mathcal{A}'$ is itself a smooth atlas.

Rmk: these transition maps only need to be smooth on an open subset $\varphi_i(U_i \cap U_j) \subset \mathbb{R}^n$ (not necessarily the whole $\mathbb{R}^n$).

LECTURE 5

Sep 19th, 2024

Category: collection of objects $\mathcal{C}$ and arrows $\mathcal{A}$. There are two natural maps source and target telling us what is the beginning and end of each arrow:

$$\begin{tikzcd} \mathcal{A} \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & \mathcal{C} \arrow[l, "id"'] \end{tikzcd}.$$

Also, there must be an identity $\mathbb{1}_X$:

$$X \circlearrowright \mathbb{1}_X.$$

Plus, an associative composition of arrows.

If X, Y are objects, define

$$\mathcal{A} \supset \mathrm{Hom}_{\mathcal{C}}(X, Y) = \text{morphisms (arrows) from } X \text{ to } Y.$$

Associative composition: $\mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \longrightarrow \mathrm{Hom}(X,Z)$.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend left, "g\circ f"] \arrow[rrr, bend left=50] & Y \arrow[r, "g"] \arrow[rr, bend left, "h\circ g"] & Z \arrow[r, "h"] & W \end{tikzcd}$$

$$(h \circ g) \circ f = h \circ (g \circ f)$$

Ex: (i) $\mathcal{C}$ = sets, $\mathcal{A}$ = maps of sets

(ii) $\mathcal{C}$ = groups, $\mathcal{A}$ = group homomorphisms

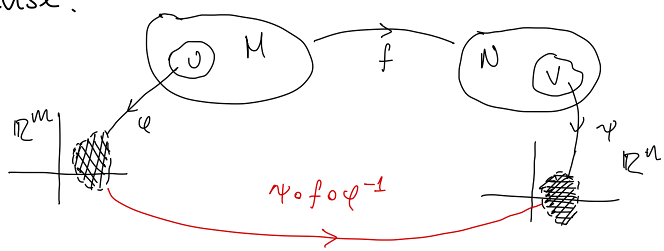
(iii) $\mathcal{C}$ = Vec. space over F, $\mathcal{A}$ = linear maps

(iv) $\mathcal{C}$ = top. spaces, $\mathcal{A}$ = continuous maps

* **MORPHISMS BETWEEN C^∞ MANIFOLDS**: If M, N are C^∞ mnfds, a map $f: M \to N$ is **SMOOTH** when it is continuous and C^∞ in charts; i.e., for any charts (U,φ) and (V,ψ) on M, N in the smooth atlas, we require

$$f_U^V := \psi \circ f \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \longrightarrow \psi(V) \subset \mathbb{R}^n$$

to be C^∞ in the usual sense.



e.g.: for 1^{st} derivative,
$f_U^V = ((f_U^V)^1, \ldots, (f_U^V)^n)$

$\left[\frac{\partial}{\partial x^i}(f_U^V)^j\right]_{n\times m}$ matrix.

- The set of C^∞ maps $M \to N$ is denoted

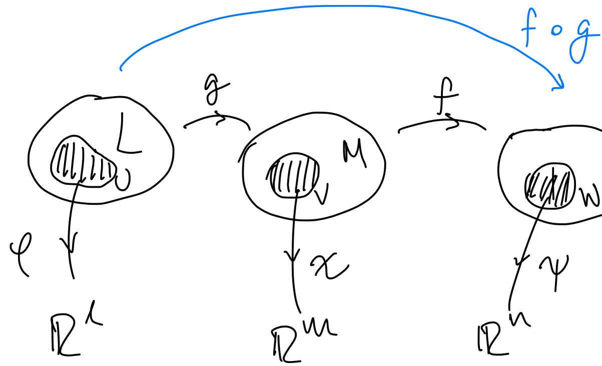
$$\mathrm{Hom}_{C^\infty}(M,N) = C^\infty(M,N).$$ ⇒ Identify arrows clearly

WARNING: $(\mathrm{Id}_M)_U^V = \psi \circ \mathrm{Id}_M \circ \varphi^{-1}$ need not be $\mathrm{Id}_{\mathbb{R}^n}$. It's only going to look like $\mathrm{Id}_{\mathbb{R}^n}$ if we use $(\mathrm{Id}_M)_U^U$.

- **Associative composition**: we already have a continuous associative composition inherited from the topological aspect of C^∞-mnfds. We only need to check that this composition is smooth:

Prop: If $L \xrightarrow{g} M$ and $M \xrightarrow{f} N$ are C^∞ maps, then $f \circ g$ is also C^∞.

Pf: Check on the charts: $(f\circ g)_U^W = \psi\circ(f\circ g)\circ\varphi^{-1}$



$$= \underbrace{(\psi\circ f\circ x^{-1})}_{C^\infty}\circ\underbrace{(x\circ g\circ \varphi^{-1})}_{C^\infty}$$

by assumption

$\mathbb{R}^n \longleftarrow \mathbb{R}^m \qquad \mathbb{R}^m \longleftarrow \mathbb{R}^l$

Composition of C^∞ is C^∞.

KEY: Chain rule for differentiation of maps $\mathbb{R}^l \to \mathbb{R}^m$ i.e.,

$$\underset{n\times l \text{ matrix}}{D_p(\psi f g \varphi^{-1})} = \underbrace{D_{x(g(\varphi^{-1}(p)))}(\psi f x^{-1})}_{n\times m \text{ matrix}}\ \underbrace{D_p(x g \varphi^{-1})}_{m\times l \text{ matrix}}$$

$\square$

ISOMORPHISMS IN C^∞ MANIFOLDS:

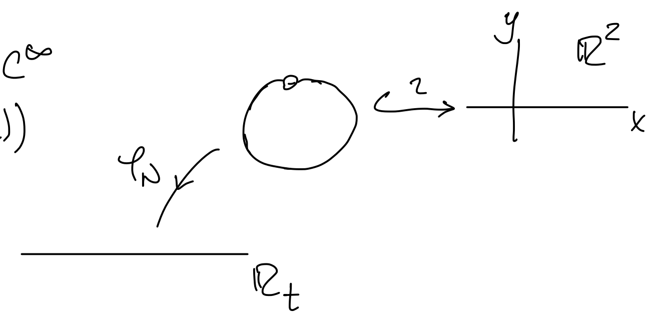
i.e., arrows $\underset{\text{Hom}(X,Y)}{f \in}$ with inverses $\underset{\text{Hom}(Y,X)}{g \in}$ $\qquad f\circ g = Id_Y$, $\quad g\circ f = Id_X$

Such smooth maps f w/ smooth inverse g are **DIFFEOMORPHISMS**

Ex: $S^1 \overset{\iota}{\hookrightarrow} \mathbb{R}^2$ inclusion is smooth

i.e., $\iota\circ\varphi_N^{-1}: \mathbb{R} \longrightarrow \mathbb{R}^2$ is C^∞

$t \longmapsto (x(t), y(t))$



Ex: $S^1 \times S^1 \xrightarrow{\iota \times \iota} \mathbb{R}^2 \times \mathbb{R}^2$ inclusion of T^2 into $\mathbb{R}^4$.

$(z_1, z_2) \longmapsto \left(\frac{\iota(z_1)}{\sqrt{2}}, \frac{\iota(z_2)}{\sqrt{2}}\right)$ so that it has length 1

$\Rightarrow S^1 \times S^1$ includes first through S^3 and then it includes into $\mathbb{R}^4$ via $S^3 \hookrightarrow \mathbb{R}^4$.

$$\begin{tikzcd} S^1 \times S^1 \arrow[rr, hook] \arrow[dr, "\tau"'] & & \mathbb{R}^4 \\ & S^3 \arrow[ur, hook] & \end{tikzcd}$$

Cl: We defined a $\tau \in C^\infty(T^2, S^3)$

Pf: This involves 4 charts in domain T^2 (products of charts of S^1)
2 charts in codomain S^3

$\Rightarrow$ Check 8 components. ◢

Ex: (LIE GROUP) Group G is a set w/ associative multiplication $m: G \times G \to G$ and identity $e \in G$, and inversion $i: G \to G$, $g \mapsto g^{-1}$

If we endow G with a C^∞ structure (i.e., an equivalence class of smooth atlases) and we require $m \in C^\infty(G \times G, G)$ and $i \in C^\infty(G, G) \Longrightarrow G$ is a Lie Group

e.g.: • $(\mathbb{R}, +, 0, i(t) = -t)$ is a Lie group $(x,y) \mapsto x+y$, $\mathbb{R}^2 \xrightarrow[C^\infty]{} \mathbb{R}$; $t \overset{i}{\underset{C^\infty}{\longmapsto}} -t$

• $(\mathbb{R}^k, +, 0, -1)$ is a Lie group

• $\mathbb{R}/\mathbb{Z} = S^1$ is a Lie group $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2} \rightsquigarrow z_1 \cdot z_2 = e^{i(\theta_1 + \theta_2)}$

- $GL(n,\mathbb{R}) \underset{\text{open}}{\subset} \mathbb{R}^{n^2}$ is a Lie group. Matrix multiplication is smooth b/c it is just a polynomial in the entries of the matrices.

- If G is a Lie group and $g \in G$ is fixed, we can define natural C^∞ maps $R_g : G \to G$, $x \mapsto xg$ and $L_g : G \to G$, $x \mapsto gx$

$\uparrow$ since multiplication is C^∞

Inverses: $R_{g^{-1}}$ and $L_{g^{-1}}$ (both C^∞)

$\Rightarrow R_g, L_g$ are DIFFEOMORPHISMS. ◢

⚠ **WARNING**: Do <u>not</u> confuse DIFFEO with EQUIV. OF ATLASES.

E.g.:

$$\begin{tikzcd} & (\mathbb{R}, x) \arrow[dl, "\alpha(x)=x"'] \arrow[dr, "\beta(x)=x^3"] & \\ \mathbb{R} & & \mathbb{R} \end{tikzcd}$$

α is a valid C^∞ atlas on $\mathbb{R}$
β is also a valid C^∞ atlas on $\mathbb{R}$ $\Big\}$ α, β represent two C^∞ structures on $\mathbb{R}$.

Is $[\alpha] = [\beta]$? <u>No</u>

Check: $\beta\alpha^{-1} : x \longmapsto x^3$ C^∞ ✓

$\alpha\beta^{-1} : t \longmapsto t^{1/3}$ is not C^∞

$\Longrightarrow$ $(\mathbb{R},\alpha)$ and $(\mathbb{R},\beta)$ are two different C^∞ manifolds.

But this is still acceptable b/c they are ISOMORPHIC:

$$\begin{tikzcd}
(\mathbb{R},\alpha) \arrow[r, bend left, "\lambda"] & (\mathbb{R},\beta) \arrow[l, bend left, "\lambda^{-1}"]
\end{tikzcd}$$

Define λ as: $\lambda(x) = x^{1/3}$ $\longleftarrow$ doesn't look smooth but it is smooth with this structure and is invertible.

$$\begin{tikzcd}
 & (\mathbb{R}, x) \arrow[rr, "\lambda"] \arrow[dl, "\alpha"'] \arrow[d] & & (\mathbb{R}, y) \arrow[d, "\beta"'] \arrow[dr] & \\
\mathbb{R} & x & & \mathbb{R} & y^3
\end{tikzcd}$$

Obs: S^7 has 28 non-isomorphic C^∞ structures lol (Kervaire-Milnor)

$\mathbb{R}^4$ has uncountably many non-iso. C^∞

$\mathbb{R}^n$ $n \neq 4$ has only 1

LECTURE 6

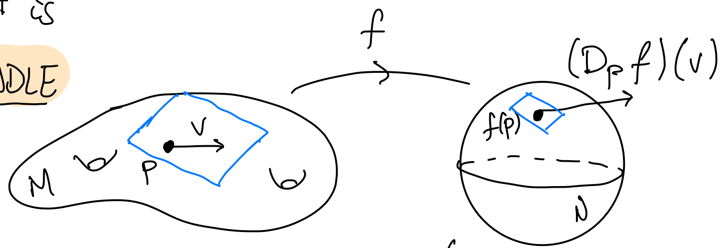
DERIVATIVES

Sep 20th, 2024

To define Df, the derivative of a C^∞ map f, we need to build the space where it is defined: the **TANGENT BUNDLE** TM

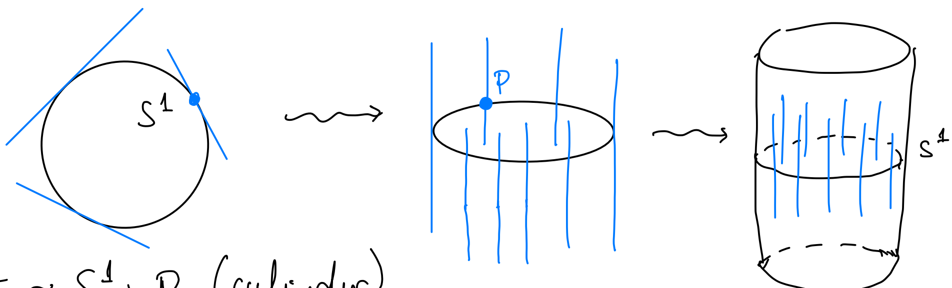
$$TM = \bigsqcup_{p \in M} T_pM$$

(union of all tangent spaces)



$v \in T_pM$ $\qquad$ $(D_pf)(v) \in T_{f(p)}N$

e.g.:



$TS^1 \simeq S^1 \times \mathbb{R}$ (cylinder)

2d mnfd

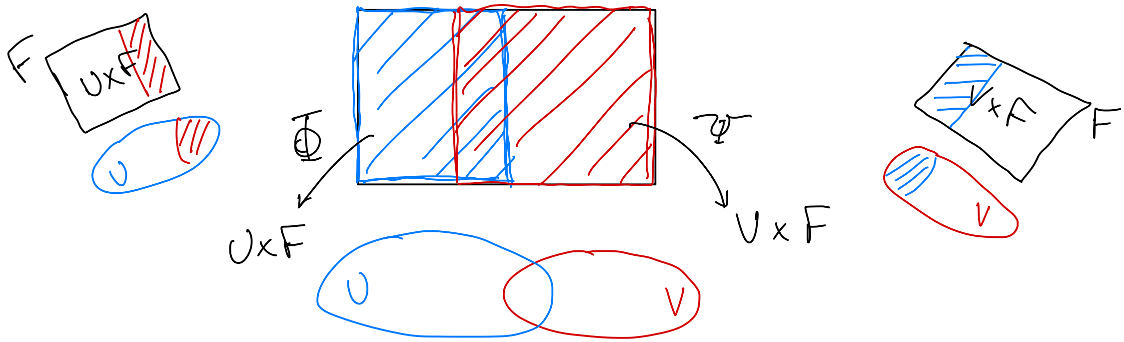
REMARK: In general $TM \not\simeq M \times \mathbb{R}^n$ but it is a $2n$-dim. mnfd. Moreover, TM has the following structure

(Bundle Projection) $\pi : TM \longrightarrow M$
$\quad (p,v) \longmapsto p$

$TM \xrightarrow{\pi} M$ is a **FIBER BUNDLE** with fiber $\mathbb{R}^n$ (often nontrivial)

REMARK: Fiber bundles have charts

$U, V \subset B \qquad \pi^{-1}(U) \xrightarrow[\simeq]{\Phi} U \times F, \qquad \pi^{-1}(V) \xrightarrow[\simeq]{\Psi} V \times F$



<u>Transition map</u>: $\Psi \Phi^{-1} : (U \cap V) \times F \longrightarrow (U \cap V) \times F$

is a family of homeomorphisms $F \to F$ parametrized by $U \times V$.

- If $F = \mathbb{R}^n$ we can ask that $\Psi \Phi^{-1}$ respect the vec. space structure of the fiber, i.e., <u>require</u>

$$\Psi \Phi^{-1} : U \cap V \longrightarrow GL(n, \mathbb{R}).$$

With this additional constraint, the fiber bundle is called a **<u>VECTOR BUNDLE</u>**.

<u>GOAL</u>: 1) Any C^∞ mnfd M has a natural vector bundle

$$TM \xrightarrow{\pi} M.$$

2) Any smooth map $f : M \to N$ has natural **<u>DERIVATIVE</u>** Df

$$\begin{tikzcd} (TM)^{2m} \arrow[r, "Df"] \arrow[d, "\pi_M"'] & (TN)^{2n} \arrow[d, "\pi_N"] \\ M^m \arrow[r, "f"'] & N^n \end{tikzcd}$$

which is a linear map of vector bundles.

Obs: This association

$$\begin{tikzcd} M \arrow[r, mapsto] \arrow[d, "f"'] & TM \arrow[d, "Df = \mathbf{T}f"] \\ {} \arrow[r, mapsto] & {} \\ N \arrow[r, mapsto] & TN \end{tikzcd}$$

Tangent functor

$$\text{Smooth Manifolds} \xrightarrow[\mathbf{T}]{} \text{Smooth Vector Bundles}$$

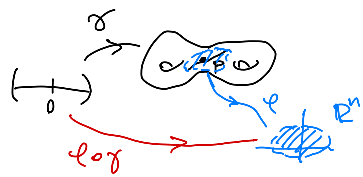
Tangent Functor

* **CONSTRUCT THE TANGENT BUNDLE**:

Def: T_pM = equivalence classes of paths through p.

Def: Let $I \subset \mathbb{R}$ open interval containing 0. A PATH in M through p is

$$\gamma : I \to M \text{ smooth}$$
$$\gamma(0) = p$$



Def: Two paths are equivalent $\gamma_1 \sim \gamma_2$ when they have the same velocity at zero. (in any fixed chart)

i.e., in any chart (U, φ) about p

$$\frac{d}{dt}\Big|_{t=0} (\varphi\circ\gamma : I \longrightarrow \mathbb{R}) \in \mathbb{R}^n$$

Velocity of γ in chart (U,φ)

Note: This definition is independent of chart.

$\Longrightarrow$ T_pM is a vector space identified with $\mathbb{R}^n$ by a chart

Changing coordinates $\psi\circ\varphi^{-1} : \varphi(U\cap V) \longrightarrow \psi(U\cap V)$ implies

$$\underbrace{D_{\varphi(p)}(\psi\circ\varphi^{-1})}_{\text{Jacobian linear map}} \overbrace{\frac{d}{dt}\Big|_{t=0}(\varphi\circ\gamma)}^{\text{velocity relative to } \varphi} = D_0(\psi\circ\cancel{\varphi^{-1}}\circ\cancel{\varphi}\circ\gamma)$$

$$= \underbrace{\frac{d}{dt}\Big|_{t=0}(\psi^{-1}\gamma)}_{\text{velocity relative to } \psi}$$

$\mathbb{R}^n \longleftarrow \mathbb{R}^n \longleftarrow \mathbb{R}^n$

Chain rule for vector valued fcts.

SUMMARY: Def: Let $(U,\varphi), (V,\psi)$ smooth charts on M containing $p\in M$

$$u \in T_{\varphi(p)}\varphi(U) = \varphi(U)\times\mathbb{R}^n$$
$$v \in T_{\psi(p)}\psi(V) = \psi(V)\times\mathbb{R}^n$$

$$\boxed{((U,\varphi),u) \sim ((V,\psi),v) \quad \text{when } D_{\varphi(p)}(\psi\circ\varphi^{-1})(u) = v}$$

The space of equivalence classes is T_pM

Def: $TM \overset{\text{set}}{=} \bigsqcup_{p\in M} T_pM$ equipped with

(TM as a set)

$$\pi : TM \longrightarrow M$$
$$(p, [((U,\varphi), u)]) \longmapsto p$$

Prop: TM is equipped with a smooth (mnfd) structure. Vector Bundle

Pf: Any chart (U,φ) for M defines a bijection

(TM as a C^∞ mnfd)

$$T\varphi(U) := \varphi(U) \times \mathbb{R}^n \longrightarrow \pi^{-1}(U).$$

In this way, each chart (U,φ) endows $\pi^{-1}(U)$ with topology and chart into $\mathbb{R}^n \times \mathbb{R}^n$

$$\pi^{-1}(U) \xrightarrow{\Phi} \mathbb{R}^{2n}.$$

Given another chart (V,ψ) on M, it lifts to

$$\pi^{-1}(V) \xrightarrow{\Psi} \mathbb{R}^{2n}.$$

Transition:

$$\Psi \circ \Phi^{-1} : \varphi(U\cap V) \times \mathbb{R}^n \longrightarrow \psi(U\cap V) \times \mathbb{R}^n$$
$$(p, u) \longmapsto \Big(\underbrace{\psi\varphi^{-1}(p)}_{C^\infty \text{ by assumption}}, \underbrace{D_p(\psi\varphi^{-1})(u)}_{\text{smooth}}\Big)$$

TM inherits:

- Topology: $W \subset TM$ open $\Leftrightarrow$ $W \cap \pi^{-1}(U)$ open $\forall U$ chart on M

- Hausdorff: separate $p, q \in TM$
  - if p,q lie in same chart ✓
  - if p,q don't lie in same chart, separate by charts ✓

$\blacksquare$

* **MORE USEFUL CONSTRUCTION OF TM ALONG W/ M**:

Choose a countable atlas $\underbrace{\{(U_i, \varphi_i)\}_{i\in I}}_{=: \mathcal{A}}$ for M^n. Then

$$TM = \bigsqcup_{i\in I} \left(\varphi_i(U_i) \times \mathbb{R}^n\right) \Big/ \begin{array}{l}(x,u) \sim (y,v) \\ \Leftrightarrow y = \varphi_j \varphi_i^{-1}(x) \ \& \\ \quad v = D_x(\varphi_j \varphi_i^{-1})(u)\end{array}$$

Can verify that the general gluing construction holds.

- While this depends on the atlas, if another atlas (smoothly equiv-to $\mathcal{A}$) is used, then there is a canonical diffeomorphism between the tang. bundles coming from the different atlases:

$$\left(\varphi_j^{\mathcal{A}'} \circ (\varphi_i^{\mathcal{A}})^{-1},\ D\left(\varphi_j^{\mathcal{A}'} \circ (\varphi_i^{\mathcal{A}})^{-1}\right)\right).$$

LECTURE 7

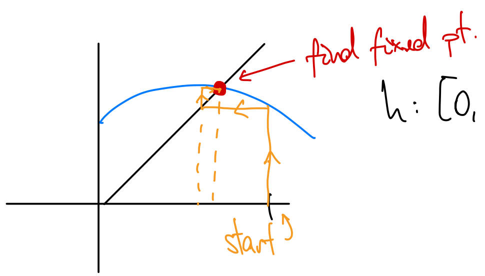
INVERSE FUNCTION & CONSTANT RANK THEOREMS

* INVERSE FUNCTION THEOREM:

Thm: (IFT) If $f:(M,p)\to(N,q)$ is a smooth map of n-manifolds such that $Df(p): T_pM \to T_qN$ is invertible, then f has a local smooth inverse.

i.e., $\exists$ open neighborhoods $U \ni p$ and $V \ni q$ and a smooth $g: U \to V$ s.t. $fg = Id_V$ and $gf = Id_U$

Pf: **Step 0** (BANACH FIXED PT THM) If $h: X \to X$ is s.t. $d(h(x),h(y)) \le \frac{1}{2} d(x,y)$ and X is complete, then $\exists!$ fixed pt.



$h:[0,1]\to[0,1]$ contraction map.

Setup: Reduce to case M = open in $\mathbb{R}^n$, $N = \mathbb{R}^n$, $p = q = 0$

also WLOG $Df(0) = Id: \mathbb{R}^n \to \mathbb{R}^n$ (can replace f by $Df(0)^{-1} \circ f$).

Step 1: (DEFINE INVERSE MAP) For each y $\overbrace{\text{suff. small}}^{\text{close to zero as above.}}$ we want x s.t. $f(x)=y$ to be the fixed pt of a contraction map.

$$f(x) = \underset{\text{linear}}{x} + \underset{\text{nonlinear part of } f}{\kappa(x)} \implies x+\kappa(x)=y \text{ as a fixed pt.}$$

$$y-\kappa(x) = x$$

for any y, defined map

$$h_y : x \longmapsto y-\kappa(x)$$

fixed pt of this map would be an inverse; i.e., x s.t. $f(x)=y$.

- Check h is a contraction map:

$Dh_y(0) = 0 \implies |Dh_y| \leq \frac{1}{2}$ in some ball $B_r(0)$.

MVT $\implies |h_y(x)-h_y(x')| \leq \frac{1}{2}|x-x'|$ for $x, x' \in B_r(0)$.

- h_y acts on a complete vector space:

$$|h_y(x)| = |h_y(x)-h_y(0)+h_y(0)| \leq |h_y(x)-h_y(0)| + |h_y(0)|$$
$$\leq \frac{1}{2}|x| + |y|.$$

So, as long as y is chosen in $B_{\frac{r}{2}}(0)$, $\overline{B_r(0)} \xrightarrow{h_y} \overline{B_r(0)}$.

BFPT $\implies \exists!$ fixed pt. of h_y in $\overline{B_r(0)}$ for each $y \in B_{\frac{r}{2}}(0)$

so we define:

$$\boxed{\begin{array}{rcl} g: B_{\frac{r}{2}}(0) & \longrightarrow & \overline{B_r(0)} \\ y & \longmapsto & \text{fixed pt. of } h_y \end{array}}$$ Inverse

Upshot: $f \circ g = \mathrm{Id}_{B_{\frac{r}{2}}(0)}$ $\longrightarrow$ since $h_y(g(y)) = g(y)$

$g \circ f = \mathrm{Id}_{f^{-1}(B_{\frac{r}{2}}(0)) \cap \overline{B_r(0)}}$ $\longrightarrow$ since fixed pt. in $\overline{B_r(0)}$ is unique

<u>but</u> $f^{-1}(B_{\frac{r}{2}}(0)) \cap \overline{B_r(0)}$ may not be open in M! So, we need to shrink $B_{\frac{r}{2}}(0)$.

Step 2: (CONTINUITY OF INVERSE)

$$\begin{aligned} |g(y) - g(y')| &= |h_y(g(y)) - h_{y'}(g(y'))| \\ &\leq |y - y'| + |\kappa(g(y)) - \kappa(g(y'))| \\ &\leq |y - y'| + \frac{1}{2}|g(y) - g(y')| \end{aligned}$$

$\Longrightarrow |g(y) - g(y')| \leq 2|y - y'| \Longrightarrow g$ is continuous.

Step 3: (f IS LOCAL HOMEO.)

$g(0) = 0$ and continuous $\Longrightarrow$ let $U \subset B_r(0)$ nhbd of zero and let $V = g^{-1}(0)$.

then $\begin{cases} f\circ g = Id_V & \text{as before} \\ g\circ f = Id_U & \text{by uniqueness of fixed pt.} \end{cases}$

Step 4: (g IS DIFFERENTIABLE AT y) If g is smooth, $Dg(y)$ must be $Df(g(y))^{-1}$ by Chain Rule. Now Df will be invertible on some nbhd of 0 so for this to make sense we should have chosen r small enough s.t. $Df(x)$ invertible for $x\in B_r(0)$.

Step 5: (g IS C^∞) $Dg(y) = Df(g(y))^{-1}$ since inversion is C^∞ g has as many derivatives as f does. ∎

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* CONSTANT RANK THEOREM

Thm: (CRT) If $f: M^m \to N^n$ is smooth and Df has constant rank k in a neighborhood of $p\in M$, then $\exists$ charts $(U,\varphi)\ni p$ and $(V,\psi)\ni f(p)$ s.t.

$$\psi\circ f\circ\varphi^{-1} : (x_1, \ldots, x_m) \longmapsto (x_1,\ldots,x_k,0,\ldots,0)$$

Lecture 8

Embeddings

Oct 2nd, 2024

* Regular (or Embedded) Submanifolds

Def: Let M^n be a mnfd. An embedded (or regular) submnfd of codimension k is a subspace $S \subset M$ such that every $p \in S$ is contained in a chart (U, φ) of the ambient mnfd M such that

$$\varphi(U \cap S) = \{x \in \varphi(U) : \underbrace{x_{n-k+1} = \cdots = x_n = 0}_{\text{last } k \text{ coordinates}}\}.$$

Rmk: (i) We call charts above "adapted" to S.

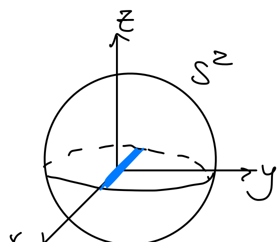
(ii) If the codimension $k = 1$, we say S is a hypersurface.

Ex: $M = S^2$, $S := \{(x,y,z) \in S^2 : y = 0\} \simeq S^1$

$U := \{z > 0\} \subset S^2$, $\varphi(x,y,z) = (x,y)$

$$\Rightarrow \varphi(U) = B_1(0) \quad \text{and} \quad \varphi(U \cap S) = \{(x,y) \in B_1(0) : y = 0\}$$

Using $\{z<0\}$ or x instead of z, we can cover S by adapted charts.



Alternative: stereographic projection.

Ex: $M = \mathbb{RP}^n$ and $S = \{[x_0, \dots, x_n] \in \mathbb{RP}^n : x_n = 0\} \simeq \mathbb{RP}^{n-1}$.

$U_i = \{x_i \neq 0\} \subset \mathbb{RP}^n$, $\varphi_i([x_0, \dots, x_n]) = \frac{1}{x_i}(x_1, \dots, \hat{x}_i, \dots, x_n)$

$\Rightarrow \varphi(U_i) = \mathbb{R}^n$ and $\varphi(U_i \cap S) = \{x \in \mathbb{R}^n : x_n = 0\}$,

Repeat for all $i = 0, \dots, n$ and thus get adapted charts covering S. ◢

Prop: Let $S^{n-k} \subset M^n$ be an embedded submnfd. The adapted charts (U, φ) induce charts for S given by

$$U \cap S \xrightarrow[\varphi]{\cong} \varphi(U) \cap (\mathbb{R}^{n-k} \times \{0\}) \xrightarrow{\cong} \text{open set in } \mathbb{R}^{n-k}$$

$$(x_1, \dots, x_n) \longmapsto (x_1, \dots, x_{n-k}).$$

These give S a mnfd structure.

---

* FIBERS & IMAGES OF MAPS

Prop: If $f: M \to N$ is a smooth map and $Df(x): T_xM \to T_{f(x)}N$ has constant rank k on M, then $f^{-1}(q) \subset M$ is an embedded submanifold of codim k $(\forall q \in f(M))$.

Pf: Given $x \in f^{-1}(q)$, by the Constant Rank Thm, there exists charts (U, φ) around x and (V, ψ) around q such that

(i) $f(U) \subset V$ and $\varphi(x) = 0$ and $\psi(q) = 0$

Can assume they are centered

(ii) $\psi \circ f \circ \varphi^{-1} : \varphi(U) \to \psi(V)$ is of the form

$$(x_1, \dots, x_m) \longmapsto (x_1, \dots, x_k, 0, \dots, 0)$$

Then $\varphi(U \cap f^{-1}(q)) = \{x \in \varphi(U) : x_1 = \dots = x_k = 0\}$

0 is the image of q under ψ

So, up to permutations of coordinates (which is a diffeo of $\mathbb{R}^m$...), (U, φ) is an adapted chart. ∎

Ex: $f: \mathbb{R}^n \to \mathbb{R}$, $(x_1, \dots, x_n) \longmapsto \sum_i x_i^2 \;\; = q = r^2$

Then $Df(x) = (2x_1, \dots, 2x_n)$ has rank 1 everywhere except at zero; i.e., $\forall x \in \mathbb{R}^n \setminus \{0\} = \mathbb{R}^n \setminus f^{-1}(0)$.

So, $f|_{\mathbb{R}^n \setminus f^{-1}(0)}$ has constant rank 1

$\Rightarrow f^{-1}(q) \simeq S^{n-1}(\sqrt{q})$ is an embedded submfd $\forall q > 0$. ◢

radius

⚠

Def: Let $f: M \to N$ be a smooth map. A point $x \in M$ is

- regular point (of f) if $Df(x)$ is surjective (i.e., if $\operatorname{rank} Df(x) = \dim N$)
- critical point (of f) otherwise

If all points in the fiber $f^{-1}(q)$ are regular, then we say q is a regular value. Otherwise, we say q is a critical value.

FACT: If $f: M \to N$ is a smooth map and $q \in f(M)$ is a regular value, then $f^{-1}(q) \subset M$ is an embedded submanifold of $\text{codim} = \dim N$.

Pf: B/c of the fact that

$$\text{rank } Df(x) = k \implies \text{rank } Df(y) \geq k \quad \forall y \text{ in an open nighb. of } x$$

So, $U := \{x \in M : Df(x) \text{ is surjective}\}$ is open in M around $f^{-1}(q)$. So we can apply the prop. from two pages ago. $\square$

Def: A smooth map $f: M \to N$ with constant rank is called

- a smooth **SUBMERSION** iff $Df(x)$ is surjective $\forall x \in M$ i.e., iff $\text{rank } Df(x) = \dim N$ at all points $x \in M$.
- a smooth **IMMERSION** iff $Df(x)$ is injective $\forall x \in M$ i.e., iff $\text{rank } Df(x) = \dim M$ at all points $x \in M$.
- a smooth **EMBEDDING** iff f is an injective immersion and it is a homeomorphism onto $f(M)$ (w.r.t. to the subspace topology of $f(M) \subset N$).

Prop: If $f: M \to N$ is a smooth embedding, then $f(M) \subset N$ is an embedded submanifold.

Pf: Let $x \in M$. By the Constant Rank Thm, there exists charts

$$x \in (U, \varphi) \text{ and } (V, \psi) \ni f(x)$$

s.t.

(i) $f(U) \subset V$ and $\psi \circ f \circ \varphi^{-1}(y_1, \dots, y_m) = (y_1, \dots, y_m, 0, \dots, 0)$

(ii) $\psi(f(U)) = \{ y \in \psi(V) : y_{m+1} = \cdots = y_n = 0 \}$

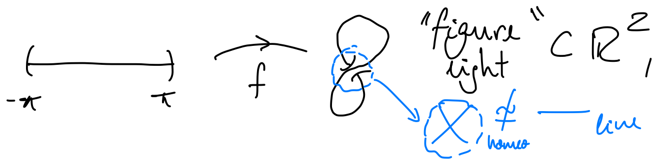
$\quad \overset{\shortparallel}{\psi(f(\varphi^{-1}(\varphi(U))))}$

Since $f: M \xrightarrow{\simeq} N$ is a homeo., $f(U) = f(M) \cap W$, where $W \underset{\text{open}}{\subset} N$. Set $V' := W \cap V$. Then $(V', \psi|_{V'})$ is a chart around $f(x)$ s.t.

$$\psi(V' \cap f(M)) = \psi(f(U))$$
$$= \{ y \in \psi(V') : y_{m+1} = \cdots = y_n = 0 \}$$

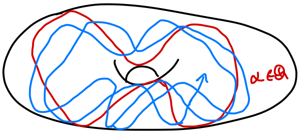
■

EXAMPLES: (for what can fail if $f: M \to f(M)$ is not homeo).



"figure eight" $\subset \mathbb{R}^2$, $\quad f(t) := (\sin 2t, t)$

is an injective immersion but not a homeo $M \to f(M)$



$f: \mathbb{R} \to S^1 \times S^1$, $f(t) = (e^{it}, e^{i\alpha t})$
with $\alpha \in \mathbb{R}$ fixed.

$\alpha \in \mathbb{Q} \Rightarrow f(\mathbb{R})$ is an embedded S^1

$\alpha \notin \mathbb{Q} \Rightarrow f(\mathbb{R})$ is dense and not open in $S^1 \times S^1$, so
it is not an embedded submnfd.

Lecture 9

Cobordisms

Oct 4^{th}, 2024

Facts: (1) If $S \subset N$ is an embedded submnfd, then $\exists!$ smooth structure on S making the inclusion map $S \hookrightarrow N$ an emb.

(2) If $f: M \to N$ is an embedding, then $f(M) =: S \subset N$ is an embedded submnfd and $f: M \to f(M) \subset N$ is a diffeo. w.r.t. the smooth structure on $f(M)$ as in (1).

—— // ——

* Manifolds w/ Boundary

Def: A mnfd w/ boundary is the same as a mnfd but that is locally modelled on $\mathbb{H}^m := \{(x_1, \dots, x_m) \in \mathbb{R}^m : x_m \geq 0\}$.

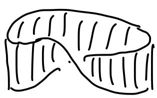
$\Rightarrow$ Can make sense of a smooth atlas here

Def: Let M be a smooth mnfd w/ boundary, then

- $x \in \operatorname{int} M$ if $\varphi(x)_m > 0$ for some (or equiv. any) chart (U, φ). ← m-th coordinate
- $x \in \partial M$ if $\varphi(x)_m = 0$ for some (or equiv. any) chart (U, φ).

Prop: Charts for M^m restrict to charts for ∂M making ∂M an $(m-1)$-dimensional mnfd w/out boundary. $\partial^2 = \emptyset$.

Ex: $M =$ Möbius band, $\partial M = S^1$.

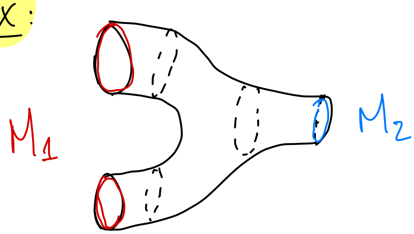
 $\longrightarrow S^1$ is a nontrivial fiber bundle

If it were, it would be $S^1 \times [0,1]$.
But $\partial(S^1 \times [0,1]) = S^1 \sqcup S^1 \neq \partial M = S^1$.

* **COBORDISMS:**

Def: Compact n-mnfds (without boundary) M_1 and M_2 are **cobordant** iff $\exists$ a compact (essential) $(n+1)$-mnfd N s.t. $\partial N \cong M_1 \sqcup M_2$

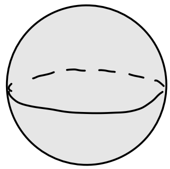
Ex:



$\Longrightarrow$ $\overset{M_1}{\overset{\shortparallel}{S^1 \sqcup S^1}}$ and $\overset{M_2}{\overset{\shortparallel}{S^1}}$ are cobordant.

Ex: $M_1 := \partial B$

$M_2 := \emptyset$

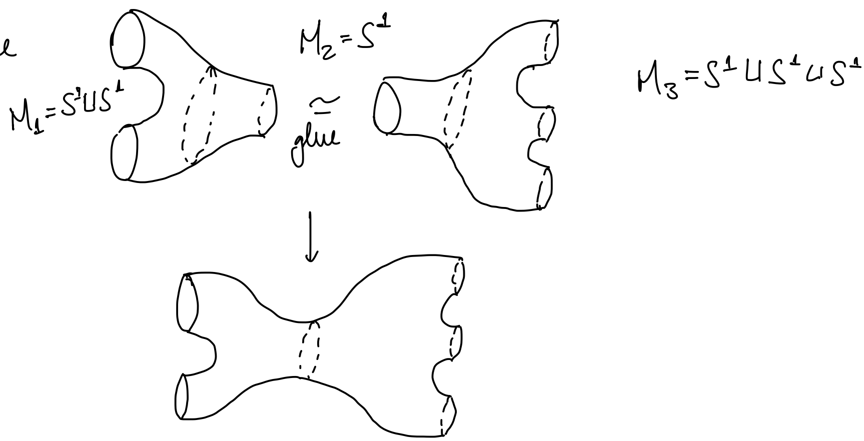


$\Rightarrow$ S^2 and $\emptyset$ are cobordant

We say S^2 is NULL-COBORDANT

Prop: Cobordance is an equivalence relation.

Pf:
- Reflexive ✓
- Transitive



∎

NOTATION: $\Omega^n :=$ cobordism classes of compact n-mnfds.

GROUP STRUCTURE: $[M_1] + [M_2] := [M_1 \sqcup M_2]$.

Well-defined: $\partial N_1 \simeq M_1 \sqcup M_1'$ and $\partial N_2 \simeq M_2 \sqcup M_2'$, then take

$$\partial(N_1 \sqcup N_2) = \partial N_1 \sqcup \partial N_2 \simeq (M_1 \sqcup M_1') \sqcup (M_2 \sqcup M_2')$$
$$\simeq (M_1 \sqcup M_2) \sqcup (M_1' \sqcup M_2').$$

Prop: $x + x = 0 \quad \forall x \in \Omega^n$.

Pf: $x = [M] \Rightarrow x + x = [\underbrace{M \sqcup M}_{\substack{\cong \\ \partial(M \times [0,1])}}] = [\emptyset] = 0$ ∎

NOTE: $\Omega^* := \bigoplus_{n \geq 0} \Omega^n$ becomes a graded ring with multiplication given by: $\underset{\substack{\cap \\ \Omega^{m_1}}}{[M_1]} \cdot \underset{\substack{\cap \\ \Omega^{m_2}}}{[M_2]} = \underset{\substack{\cap \\ \Omega^{m_1+m_2}}}{[M_1 \times M_2]}$

__Well-defined__: $\partial N_1 \simeq M_1 \sqcup M_1'$, $\partial N_2 \simeq M_2 \sqcup M_2'$

$[M_1 \times M_2] = [M_1' \times M_2]$ since $\partial(N_1 \times M_2) = (\partial N_1) \times M_2 \simeq M_1 \times M_2 \sqcup M_1' \times M_2$

Same for $[M_1' \times M_2] = [M_1' \times M_2']$

__Commutative ring__: b/c $M_1 \times M_2 \cong M_2 \times M_1$

__Unital__: unit $= [*]$ one-point space

Thm: (René Thom) The cobordism ring is a countably generated polynomial ring over $\mathbb{F}_2$ with generators in every degree $n \neq 2^k - 1$, $k \in \mathbb{N}$.

$\hookrightarrow \Omega^* = \mathbb{F}_2[x_2, x_4, x_5, x_6, x_8, \dots]$

deg of generator $\longleftrightarrow$ dimension

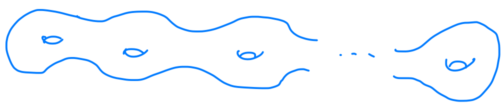
NOTE: (1) $\Omega^0 = \mathbb{Z}/2\mathbb{Z}$ consisting of $0 = [\emptyset]$ and $1 = [*]$.

(2) $\Omega^1 = 0$ $\leftarrow$ Classification of 1-dim compact mnflds w/out bdry $\Rightarrow$ only have S^1 which is the bdry of the disk $\Rightarrow$ null-cobordant.

(3) $\Omega^2 = \mathbb{Z}/2\mathbb{Z} = \langle x_2 \rangle$, $x_2 = [\mathbb{R}P^2]$.

Rmk: $[\Sigma] = 0$ $\forall \Sigma$ oriented surface.

Classification:



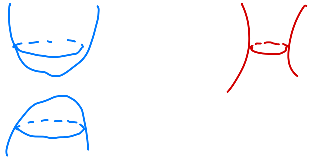
NOTE: $\Omega^4 = \{x_2^2, x_4, x_2^2 + x_4, 0\}$.

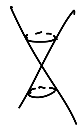
$$x_2^2 + x_4 = [\mathbb{R}P^2 \times \mathbb{R}P^2 \sqcup \mathbb{R}P^4] = [\mathbb{R}P^2 \times \mathbb{R}P^2 \# \mathbb{R}P^4].$$

* MAPS BETWEEN MANIFOLDS w/ BOUNDARY:

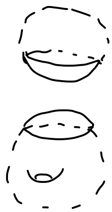
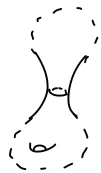
Prop: Let M be a mnfd $(\partial M = \emptyset)$ and $f: M \to \mathbb{R}$ smooth and $a \leq b$ both regular values of f. Then $f^{-1}([a,b])$ is a smooth n-mnfd embedded in M and <u>with</u> boundary $= f^{-1}(a) \sqcup f^{-1}(b)$.

Ex: $f: \mathbb{R}^3 \to \mathbb{R}$, $(x,y,z) \mapsto x^2 + y^2 - z^2$. Apply proposition to find that $f^{-1}([-1,1])$ is a mnfd with boundary given by $f^{-1}(-1) \sqcup f^{-1}(1) = \{x^2+y^2 = z^2 - 1\} \sqcup \{x^2 + y^2 = z^2 + 1\}$.



Note $f^{-1}(0) =$  singularity.

Cl: $[M_1 \sqcup M_2] = [M_1 \# M_2]$.

Pf: $M_1 \sqcup M_2 =$  $\overset{\text{prop.}}{=}$ 

LECTURE 10

Oct 10^{th}, 2024

Smooth maps $f: M^m \to N^n$ are modelled on $\mathbb{R}^m \xrightarrow[\text{linear}]{} \mathbb{R}^n$ if the rank Df is constant.

$\leftarrow$ If it's not constant near $p \in M$, then we cannot classify the form of the maps $\implies$ classification problem

Ex: (MORSE LEMMA) Let $f: M \to \mathbb{R}$ be C^∞ s.t. $\left\{ \begin{array}{l} \text{(i) } Df(x) = 0 \\ \text{(ii) } \left[\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right] \text{ is nondegenerate at } p \\ \text{of signature } (\underset{\uparrow \, \# \text{ of } +}{k}, \underset{\nwarrow \, \# \text{ of } - \text{ in the Hessian}}{n-k}) \end{array} \right.$

Then $\exists$ coords $(x^1, \ldots, x^n)$ about p s.t.

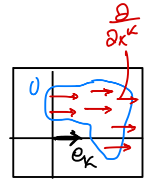
$$f(x^1, \ldots, x^n) = f(p) + \sum_{i=1}^{k} (x^i)^2 - \sum_{j=k+1}^{n} (x^j)^2.$$

$\hookrightarrow$ Allows us to infer the behavior of a fct. at the nbhd of a point p.

* **VECTOR FIELDS**: A vec. field on some $U \subset_{open} V$ (V is a $\mathbb{R}$-vec. space) is just $X : U \to V$ vector valued fct.

- If we choose a basis $(e_1, \ldots, e_n)$ for V, it induces a dual basis $\underbrace{(x^1, \ldots, x^n)}_{\text{coord. syst. on } U}$.

- Then we have constant vec. fields
$$\begin{aligned} U &\longrightarrow V \\ p &\longmapsto e_k \end{aligned}$$



These constant vec. fields are denoted $\dfrac{\partial}{\partial x^k}$

- **Reason for notation**: can identify $\frac{\partial}{\partial x^k}$ with the directional derivative of functions.

$$\left\{ \begin{matrix} \text{Vector fields} \\ \text{on } U \end{matrix} \right\} \overset{1-1}{\longleftrightarrow} \left\{ \begin{matrix} \text{Derivations} \\ \text{of } C^\infty(U, \mathbb{R}) \end{matrix} \right\}$$

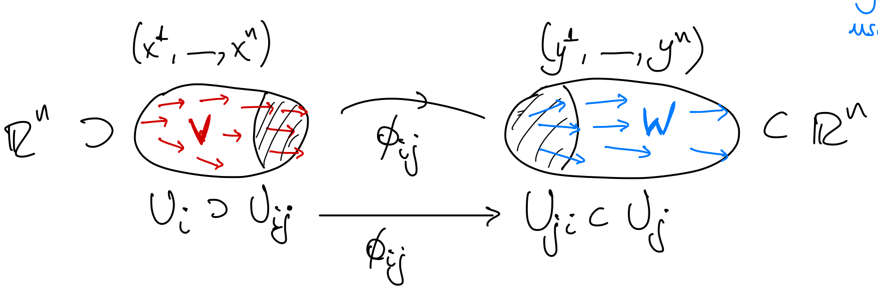
Def: A derivation of an algebra A is a linear map $D : A \to A$ s.t. $D(ab) = (Da)b + aDb$ (Leibniz Rule)

- **GLUING**: Under what conditions does
$$V = \sum_{i=1}^n V^i(x^1, \ldots, x^n) \frac{\partial}{\partial x^i} \text{ glue to } W = \sum_{i=1}^n W^i(y^1, \ldots, y^n) \frac{\partial}{\partial y^i}$$

$y^1 = y^1(x^1, \ldots, x^n)$ using ϕ_{ij}

Use $D\phi_{ij}$

Change basis: $\dfrac{\partial}{\partial y^i} = \displaystyle\sum_j \frac{\partial x^j}{\partial y^i} \frac{\partial}{\partial x^j}$



$\Longrightarrow$ Condition is that we can write:

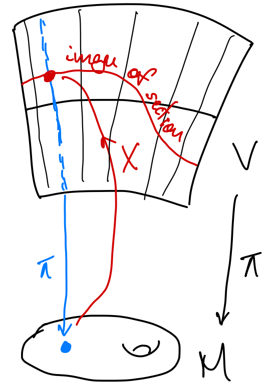
$$\boxed{V^i(x^1, \ldots, x^n) = \sum_{k=1}^{n} W^k\big(y^1(x^1,\ldots,x^n), \ldots, y^n(x^1,\ldots,x^n)\big) \frac{\partial x^i}{\partial y^k}(x^1,\ldots,x^n)}$$

Upshot: The vector field $X_i : U_i \subset \mathbb{R}^n \to \mathbb{R}^n$ is a "section" of $TU_i = U_i \times \mathbb{R}^n$

$$\begin{tikzcd} TU_i \arrow[d] \\ U_i \arrow[u, bend left, "X_i"] \end{tikzcd} \qquad U_i \ni p \overset{X_i}{\longmapsto} (p, X_i(p))$$

Def: A section of a tangent bundle $\pi : V \to M$ is a C^∞ map $X : M \to V$ s.t. $\pi \circ X = \mathrm{Id}_M$.

section hits each fiber precisely at 1 point



$$\boxed{\begin{array}{l}\text{Thus: given vector fields } X_i, X_j \text{ on } \\ U_i, U_j \text{, these glue to a global section} \\ \text{iff } D(\phi_{ij})_{p \in U_{ij}}(X_i(p)) = X_j(\phi_{ij}(p))\end{array}}$$

$$\begin{tikzcd} TU_i \supset TU_{ij} \arrow[r, "\simeq", "(\phi_{ij}, D\phi_{ij})"'] \arrow[d] & TU_{ji} \subset TU_j \arrow[d] \\ U_i \supset U_{ij} \arrow[r, "\simeq", "\phi_{ij}"'] \arrow[u, bend left, "X_i"] & U_{ji} \subset U_j \arrow[u, bend right, "X_j"'] \end{tikzcd}$$

Def: A **vector field** on a C^∞ mnfd M is a C^∞ section of TM.

$$\begin{tikzcd} TM \arrow[d, "\pi"'] \\ M \arrow[u, bend right, "X"'] \end{tikzcd} \qquad \pi \circ X = Id_M$$

The space of all vec. fields on M is denoted $\Gamma(M, TM) = \mathfrak{X}(M)$.

__Annoying Rmk__: Vector fields may not be pushed or pulled along C^∞ maps (unless f is invertible)

$$\begin{tikzcd} TM \arrow[r, "Df"] \arrow[d] & TN \arrow[d] \\ M \arrow[r, "f"'] \arrow[u, bend left, "X"] & N \arrow[u, bend right, "Y"'] \end{tikzcd}$$

$$f : M \xrightarrow{C^\infty} N$$

Def: $X \in \mathfrak{X}(M)$ is "f-related" to $Y \in \mathfrak{X}(N)$ when

$$Df \circ X = Y \circ f.$$

In the special case of f being a diffeomorphism, we can push and pull via

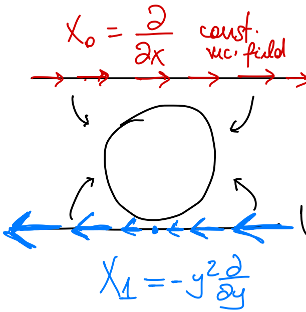
$$f_* X := Df \circ X \circ f^{-1}, \quad f^* Y := (Df)^{-1} \circ Y \circ f$$

Ex: (Vector field on S^1) Take $X_0 = \frac{\partial}{\partial x}$ const. vec field on U_0

$S^1 = U_0 \cup U_1$

$$\phi : x \longmapsto x^{-1}$$

$$D\phi : v \longmapsto \left[\frac{\partial \phi}{\partial x}\right] v = \left[-x^{-2}\right] v$$



$U_0 = \mathbb{R}_x \supset \mathbb{R}_{\neq 0}$

$\phi : x \mapsto x^{-1}$

$U_1 = \mathbb{R}_y \supset \mathbb{R}_{\neq 0}$

__Gluing__: $x \neq 0 \sim y \neq 0$ iff $y = x^{-1}$.

By the gluing construction for vec fields above,

$$(D\phi_{ij})_{p\in U_j}(X_i(p)) = X_j(\phi_{ij}(p)) \rightsquigarrow [-x^{-2}]\,1 = X_1$$

$$X_0 = 1\cdot\frac{\partial}{\partial x}$$

$$\Rightarrow X_1 = -x^{-2}\frac{\partial}{\partial y} = -y^2\frac{\partial}{\partial y}$$

Since smooth, we can extend it to all of U_1 (could be non-smooth and then we only have it on the gluing region and not on all of S^1)

LECTURE 11

Oct 11, 2024

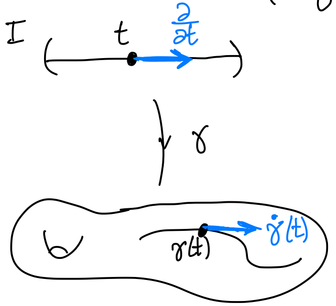
* FLOW OF A VECTOR FIELD

$I=(a,b)$ including (a,∞), $(-\infty,b)$, $(-\infty,\infty)$, ⋮

Def: A path or curve on M is a map $\gamma : I\subset\mathbb{R}\to M$.
The velocity of this path at time $t\in I$ is

$$(D\gamma)_t : T_tI \longrightarrow T_{\gamma(t)}M$$

$$T_tI = \mathbb{R} \ni 1 =: \frac{\partial}{\partial t} \longmapsto D\gamma_t\left(\frac{\partial}{\partial t}\right) = \dot\gamma(t)$$



NOTE: The velocity defines a lift of γ to a path in TM

$$
\begin{tikzcd}
 & TM \arrow[d, "\pi"] \\
I \arrow[ur, "\dot{\gamma}"] \arrow[r, "\gamma"'] & M
\end{tikzcd}
$$

i.e., the diagram commutes

Def. Let $X \in \mathfrak{X}(M)$. The path γ is called an **INTEGRAL CURVE** of X if its velocity coincides with X; i.e.,

$$\boxed{\dot{\gamma}(t) = X(\gamma(t))} \quad (*)$$

<u>In coordinates</u>: (U, φ) $(x^1, \ldots, x^n)$,

$$X = X^1 \frac{\partial}{\partial x^1} + \cdots + X^n \frac{\partial}{\partial x^n}, \quad X^i = X^i(x^1, \ldots, x^n) \text{ smooth}$$

$$\varphi \circ \gamma = (\gamma^1(t), \ldots, \gamma^n(t))$$

So, we can write $(*)$ as:

$$(*) \quad \boxed{\begin{aligned} \frac{d}{dt}\gamma^1(t) &= X^1(\gamma^1(t), \ldots, \gamma^n(t)) \\ &\vdots \\ \frac{d}{dt}\gamma^n(t) &= X^n(\gamma^1(t), \ldots, \gamma^n(t)) \end{aligned}}$$

System of n coupled 1^{st} order nonlinear ODEs

Thm: (Existence and uniqueness of solutions to ODEs)
Let $X \in \mathfrak{X}(V)$, $V \underset{\text{open}}{\subset} \mathbb{R}^n$. For each $x_0 \in V$, $\exists$ a nbhd U, $x_0 \in U \subset V$, and $\varepsilon > 0$ and a smooth map

← implies solution depends smoothly on initial conditions

$$\Phi : (-\varepsilon, \varepsilon) \times U \longrightarrow V$$
$$(t, x) \longmapsto \varphi_t(x)$$

such that $\forall x \in U$, the curve $t \mapsto \varphi_t(x)$ is an integral curve of X with initial condition x; i.e., $\varphi_0(x) = x$.

Uniqueness: if $(U', \varepsilon', \Phi')$ is another tuple satisfying the above, then $\Phi = \Phi'$ on a common domain.

Corollary: Let $X \in \mathfrak{X}(M)$. Then, there exists a nbhd U, $\{0\} \times M \subset U \subset \mathbb{R} \times M$, and a C^∞ map

$$\Phi : U \longrightarrow M$$

such that

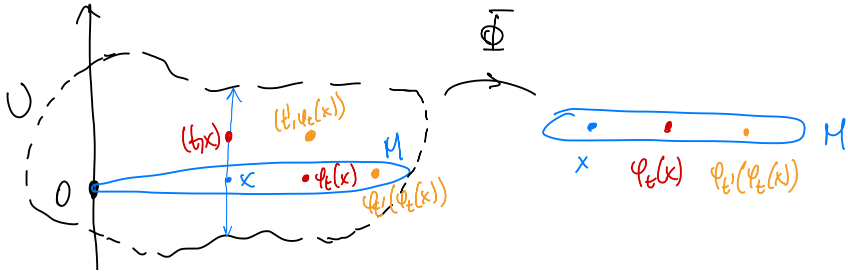
(i) $(\mathbb{R} \times \{x\}) \cap U$ is an interval about 0 in $\mathbb{R}$. (ensures U is "connected")

(ii) $t \longmapsto \varphi_t(y) = \Phi(t, y)$ is an integral curve of X

(iii) $\varphi_0(y) = y$ starting at y

(iv) if (t, x) and $(t', \varphi_t(x))$ and $(t + t', x)$ are in U, then $\varphi_{t+t'}(x) = \varphi_{t'}(\varphi_t(x))$.

Uniqueness: if (U', Φ') satisfies the above and satisfies (i), (ii), (iii), then it must satisfy (iv) and $\Phi = \Phi'$ on a common domain.



Pf: Using Exist. & Uniq. of ODEs, find an open cover $(U_i)_{i\in I}$ of M, $\varepsilon_i > 0$ $i \in I$, and maps $\Phi_i : (-\varepsilon_i, \varepsilon_i) \longrightarrow M$ as in thm.

By uniqueness Φ_i must agree with Φ_j on

$$\big((-\varepsilon_i, \varepsilon_i) \times U_i\big) \cap \big((-\varepsilon_j, \varepsilon_j) \times U_j\big).$$

Then, we get a well-def. map on the union:

$$\Phi : \underbrace{\bigcup_{i\in I} (-\varepsilon_i, \varepsilon_i) \times U_i}_{=: U} \longrightarrow M.$$

By construction, Φ satisfies (i), (ii), (iii).

Verify (iv), note that we can compare

$$\tau \longmapsto \varphi_\tau(\varphi_t(x))$$

$$\tau \longmapsto \varphi_{\tau+t}(x)$$

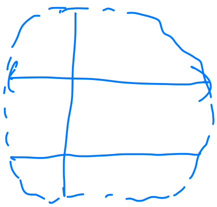
Both are integral curves w/ initial condition $\varphi_t(x)$. Thus, by uniqueness, they must coincide.

Similar analysis gives uniqueness of flows.

$\square$

Proposition: There exists a maximal flow (constructed by taking the union of all possible flows). This flow is called: maximal local 1-parameter group of diffeomorphisms
(bad terminology...)

Note: if $\{\pm\varepsilon\}\times M \subset U$ then $\varphi_\varepsilon : M \to M$ is defined and is a diffeomorphism (smooth inverse is $\varphi_{-\varepsilon} : M \to M$).



$\varepsilon \times M$

$$\mathfrak{X}(M) \longleftrightarrow \mathrm{Diff}(M)$$

Infinitesimal symmetries — Global symmetries

Def: A groupoid is a category in which every arrow has an inverse g^{-1}

$$\begin{tikzcd} y \arrow[r, bend right, "g^{-1}"'] & x \arrow[l, bend right, "g"'] \end{tikzcd}$$

For maximal flow: view elements of U as arrows, where

$U \ni (t,x)$

$\varphi_t(x) \longleftarrow x$

source $(t,x) = x$

target $(t,x) = \varphi_t(x)$

$Id_x = (0,x)$

inverse $(t,x) = (-t, \varphi_t(x))$

Composition:

$(t', \varphi_t(x))(t,x) = (t'+t, x)$

$$\boxed{\begin{tikzcd} U^X \arrow[r, shift left, "\text{target}"] \arrow[r, shift right, "\text{source}"'] & M \end{tikzcd} \quad \text{Unique flow groupoid of } X}$$

SPECIAL CASE:

Def: If $U^X = \mathbb{R} \times M$ then X is called a <u>COMPLETE</u> vector fields and we obtain a 1-parameter (sub-) group of diffeos.

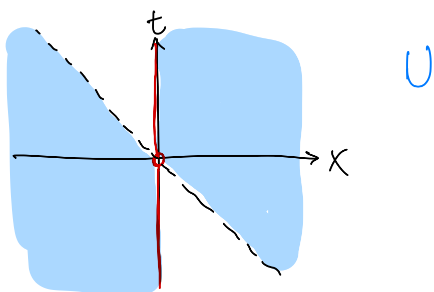
$$\mathbb{R} \ni t \longmapsto \varphi_t^X \in \mathrm{Diff}(M)$$

$$t + t' \longmapsto \varphi^X_{t+t'} = \varphi_t^X \circ \varphi_{t'}^X = \varphi_{t'}^X \circ \varphi_t^X$$

Sometimes written $\varphi_t^X = e^{tX}$ ← 1-param. group of diffeos. of complete vector fields

Ex: Constant vector field $X = \frac{\partial}{\partial x}$ is <u>complete in $\mathbb{R}$</u> with flow $\varphi_t^X(y) = t + y$.

<u>BUT</u> $X = \frac{\partial}{\partial x}$ is <u>incomplete in $\mathbb{R} \setminus \{0\}$</u>



Thm: If M is compact (w/out bdry), then all $X \in \mathfrak{X}(M)$ are complete.

Lecture 12

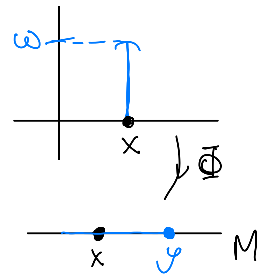
Oct 17^{th}, 2024

Thm: On a compact manifold, any $X \in \mathfrak{X}(M)$ is complete.

Pf: By contradiction, assume Φ is the maximal flow of X. If X is not complete, then

$\exists\, x \in M$ and WLOG $\omega > 0$ s.t.

$U \cap (\mathbb{R} \times \{x\})$ = open interval w/ upper boundary ω



WTS: this contradicts maximality.

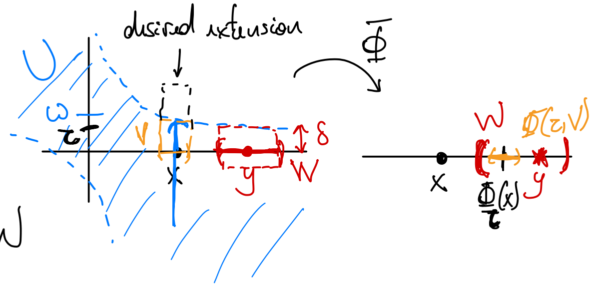
1. By compactness, as $t \longrightarrow \omega$, $\Phi(t,x) - y$ is an accumulation point.

(Idea: flow is defined at y! Extend flow at x by flowing almost to ω, continue using flow near y.)

2. $\exists\, \delta > 0$ and a nbhd of y s.t. $(-\delta, \delta) \times W \subset U$.

Since W is open, $\exists\, \tau \in (\omega - \delta, \omega)$ s.t. $\Phi(\tau, x) \in W$. Moreover, $\exists$ nbhd V of x s.t.

$\{\tau\} \times V \subset U$ and $\Phi(\tau, V) \subset W$



3. Given these choices, enlarge U by doing:

$$\tilde{U} = U \cup \big((\tau-\delta, \tau+\delta)\times V\big)$$

$$\tilde{\Phi}(t,z) = \begin{cases} \Phi(t,z), & \text{if } (t,z)\in U \\ \Phi(t-\tau, \Phi(\tau,z)) & \text{if } (t,z) \in \underbrace{(\tau-\delta, \tau+\delta)}_{\substack{\text{Extension of } U \text{ since} \\ \tau+\delta > \omega}} \times V \end{cases}$$

∎

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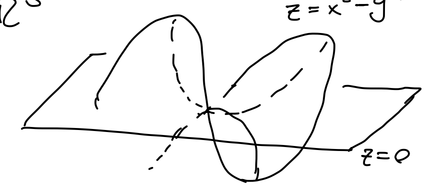
* **TRANSVERSALITY**: Unlike in the linear category, where $U_1, U_2 \subset V$ intersect in a linear subspace $U_1 \cap U_2$, for manifolds, <u>this</u> <u>fails</u>! ← Unless we require a special relationship between submnflds

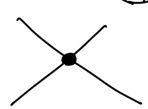
↳ Manifolds are not "closed under intersection".

<u>Ex</u>: $\{z = x^2 - y^2\} \cap \{z = 0\}$ in $\mathbb{R}^3$

surface codim 1   surface codim 1



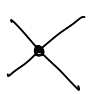
<u>Intersection</u>:



← <u>Not</u> a mnfd

<u>Note</u>: $\{z^2 = x^2 - y^2\} \cap \{z = t\}$

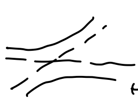
$t = 0$:



singular

$t = 1$:

$t = -1$:



} smooth 1-d submanifolds (codim 2) (Transverse intersections)

Thm: If K, L are smooth submanifolds of M^n of codim $k + l$ are TRANSVERSE, then $K \cap L$ is either empty or a submanifold of codim $k+l$.

Idea: submnfd of codim k = k indep constraints in M
" " " l = l " " " "

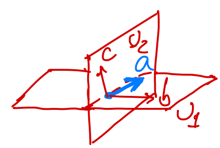
If k-constraints and l-constraints are independent (transversality), then $K \cap L$ imposes all $k+l$ constraints gives a submnfd of codim $k+l$.

Def: K, L submnfds of M are transverse if $\forall p \in K \cap L$, T_pK and T_pL are transverse in T_pM.

From linear algebra: two linear subspaces $U_1, U_2 \subset V$ are transverse if $U_1 + U_2 = V$.

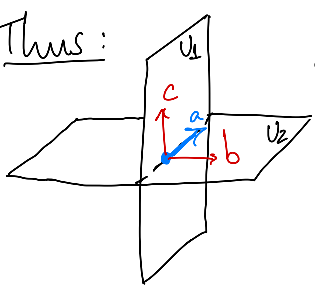
not direct sum (intersection need not be zero)

e.g.: $V + V = V$ transverse



$$\dim U_1 \cap U_2 = a = (a+b) + (a+c) - (a+b+c)$$
$$= \dim U_1 + \dim U_2 - \dim V$$

Thus:

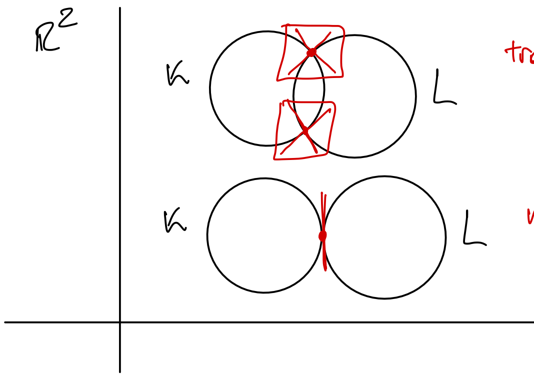


$$\dim U_1 \cap U_2 = \dim U_1 + \dim U_2 - \dim V$$

$$\underbrace{\operatorname{codim} U_1 \cap U_2}_{\#\text{ constraints def. } U_1 \cap U_2} = b + c = \underbrace{\operatorname{codim} U_1}_{\#\text{ constraints def. } U_1} + \underbrace{\operatorname{codim} U_2}_{\#\text{ constraints def. } U_2}$$

$\Rightarrow$ "dimension of transverse intersection is the expected dim."

⚠ **WARNING**: $K \cap L$ may be a smooth submnfd even if intersection is not transverse.



transverse $K \cap L =$: smooth submnfd ✓

not transverse $K \cap L = \cdot$ smooth submnfd ✓

$\Rightarrow$ "Transversality gives control over intersection # "
in the case $\dim K \cap L = 0$.

intersection mod 2

<u>Note</u>: in order for two closed loops in the plane to be transverse they must intersect in an even # of pts.

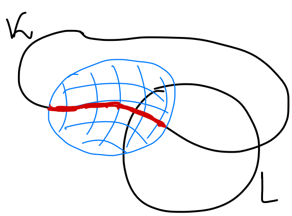
Pf: (of thm) K, L are submnfds of codim k, l. So, $\exists$ submnfd charts $(U, (x^1, \ldots, x^n))$ and $(V, (y^1, \ldots, y^n))$ in M s.t.

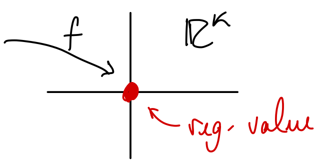
$K = \{x^1 = \cdots = x^k = 0\}$, $L = \{y^1 = \cdots = y^l = 0\}$.

In other words, $\exists$ $f: U \to \mathbb{R}^k$ s.t. 0 is a regular value
$g: V \to \mathbb{R}^l$

$K \cap U = f^{-1}(0)$

$L \cap V = g^{-1}(0)$





Now, on $U \cap V$, combine the constraint functions

$$(f,g) : U \cap V \longrightarrow \mathbb{R}^k \times \mathbb{R}^l .$$

Cl: $(K \cap L) \cap (U \cap V) = (f,g)^{-1}(0,0)$ and (f,g) is a submersion, so $K \cap L$ is a submnfd.

Only remains to show that $(0,0) \in \mathbb{R}^{k+l}$ is a reg. value of (f,g)

WTS: $D_x(f,g)$ is onto for all $x \in U \cap V \cap K \cap L$

$D_x(f,g) : T_x M \xrightarrow{\cong \mathbb{R}^n} \mathbb{R}^{k+l}$

$$\ker D_x(f,g) = (\underbrace{\ker D_x f}_{= T_x K}) \cap (\underbrace{\ker D_x g}_{= T_x L})$$

$$\begin{aligned} \text{Transversality} \Rightarrow \dim \operatorname{im}(D_x(f,g)) &= \operatorname{codim} \ker(D_x(f,g)) \\ \text{transversality} \rightarrow &= \operatorname{codim}(\ker D_x f) \\ &\quad + \operatorname{codim}(\ker D_x g) \\ &= k + l \end{aligned}$$

∎

LECTURE 13

Oct 18, 2024

Thm: $K, L \underset{\text{submnfd}}{\subset} M$ are transverse, then either $K \cap L$ is empty or a submnfd of $\operatorname{codim}(K \cap L) = \operatorname{codim} K + \operatorname{codim} L$.

Ex: In $\mathbb{R}^6$

K	L
0	0
1	1
2	2
3	3
4	4
5	5
6	6

etc

$\ast$ nonempty $\pitchfork$

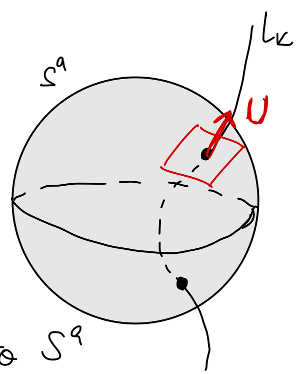
Ex: $\mathbb{R}^{10} \setminus \{0\} = \mathbb{C}^5 \setminus \{0\} =: M$

$K := S^9 = \{(z_1, \dots, z_5) : \sum_{i=1}^{5} |z_i|^2 = 1\}$ hypersurface (codim 1)

$L_\kappa := \{(z_1, \dots, z_5) : \underbrace{z_1^2 + z_2^2 + z_3^2 + z_4^3 + z_5^{6\kappa-1}}_{f} = 0\}, \ \kappa \in \mathbb{N}$

(think of $f : \mathbb{R}^{10} \to \mathbb{R}^2$, $f^{-1}(0)$ is a reg. value)

Cl: $K \pitchfork L$ (transverse intersection)



$$U = \frac{1}{2} z_1 \partial_{z_1} + \frac{1}{2} z_2 \partial_{z_2} + \frac{1}{2} z_3 \partial_{z_3} + \frac{1}{3} z_4 \partial_{z_4} + \frac{1}{6\kappa - 1} z_5 \partial_{z_5}$$

Now, $Re(U)$ is tangent to L_κ but not to S^9

Apply it to the fcts that define L_κ and S^9

Note: $z = x + iy \rightsquigarrow dz = dx + i\,dy \rightsquigarrow \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\right)$.

Brieskorn (60s): $K \cap L_k$ for $k = 1, \ldots, 28$ are homeomorphic to S^7.

Kervaire - Milnor: these are all the different smooth structures on S^7. The smooth structures on S^7 form a group by connected sums

$$S^7_{top} \# S^7_{top} = S^7_{top}.$$

In this case $C^\infty\text{-str}(S^7) = \mathbb{Z}/28\mathbb{Z}$

$$(K \cap L_1) \# (K \cap L_1) \underset{C^\infty}{\cong} K \cap L_2$$

We can rephrase transversality of $K, L \subset M$ in terms of the embeddings

$$\iota_K : K \underset{\boxed{\text{embedding}}}{\hookrightarrow} M \hookleftarrow L : \iota_L$$

(injective immersion homeo. onto image)

Need to map the tangent spaces to K and L to M. $\forall (k, \ell) \in K \times L$ s.t. $\iota_K(k) = \iota_L(\ell) =: p \in M$

((k, ℓ) is an "intersection point")

$$\begin{tikzcd} T_k K \arrow[dr, "D_k \iota_K"'] & & T_\ell L \arrow[dl, "D_\ell \iota_L"] \\ & T_p M & \end{tikzcd}$$

are transverse; i.e.,

$$\operatorname{im} D_k \iota_K + \operatorname{im} D_\ell \iota_L = T_p M$$

Transversality in terms of maps now (instead of submfds)

Def: Two **maps** $f: K \to M$ and $g: L \to M$ of manifolds are **TRANSVERSE** if $\forall$ intersection pts.

$$\text{i.e., } (a,b) \in K \times L \text{ s.t. } f(a) = g(b) =: p$$

we have $D_a f$, $D_b g$ are transverse

$$\text{i.e., } \operatorname{im} D_a f + \operatorname{im} D_b g = T_p M.$$

$\downarrow$ slightly improved thm

Thm: If $f \in C^\infty(K, M)$ and $g \in C^\infty(L, M)$ are transverse, then their intersection, called the fiber product,

$$K {}_f\times_g L = \{ (k,\ell) \in K \times L : f(k) = g(\ell) \}$$

is a smooth submanifold of $K \times L$ equipped with commuting maps

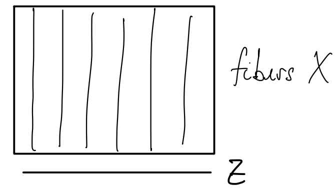
$$\begin{tikzcd}
& K \times L \arrow[ddl, bend right, "\pi_K"'] \arrow[ddr, bend left, "\pi_L"] & \\
& K {}_f\times_g L \arrow[u, hook, "\iota"'] \arrow[dl, "\pi_K \circ \iota"'] \arrow[dr, "\pi_L \circ \iota"] \arrow[dd, "f \cap g :="'] & \\
K \arrow[dr, "f"'] & & L \arrow[dl, "g"] \\
& M &
\end{tikzcd}$$

(When maps g, f are understood, we write $K {}_f\times_g L = K \underset{M}{\times} L$)

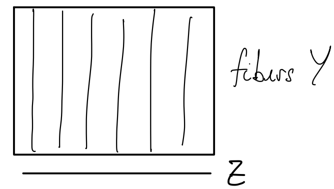
Ex: $K = Z \times X$ $\qquad$ $X, Y, Z \in C^\infty$-mnfd

$L = Z \times Y$

K:



L:

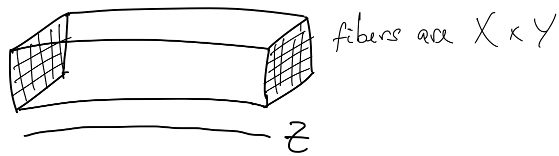


$$\begin{tikzcd}[column sep=small] & K \times_Z L \arrow[dl] \arrow[dr] & \\ K \arrow[dr, "\pi_Z"'] & & L \arrow[dl, "\pi_Z"] \\ & Z & \end{tikzcd}$$

π_Z is a submersion (i.e., $D\pi_Z$ is onto) so it is transverse to any map to Z

$$K \times_Z L = \{ (z,x), (\tilde{z}, y) : z = \tilde{z} \} = Z \times X \times Y.$$

$K \times_Z L$



Ex: $\pi : S^3 \to S^2$ Hopf fibration (S^1 fiber bundle)

$$\begin{tikzcd}[column sep=small] & S^3 \times_{S^2} S^3 \arrow[dl] \arrow[dr] & \\ S^3 \arrow[dr, "\pi"'] & & S^3 \arrow[dl, "\pi"] \\ & S^2 & \end{tikzcd}$$

fiber dim 1 $\qquad$ fiber dim 1

since π is continuous, we have that $\pi \pitchfork \pi$.

This gives a fiber bundle with fiber $T^2 = S^1 \times S^1$

$$S^3 \times_{S^2} S^3 \xrightarrow{\pi \cap \pi} S^2$$

fiber dim $1+1 = 2$

4d mnfd.

Ex: $L \overset{i_L}{\hookrightarrow} M$ submnfd

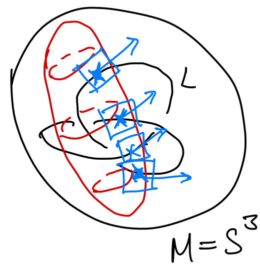
$K \xrightarrow{f} M$ smooth map <u>transverse to L</u>

e.g., if f is a submersion but this is not always the case

$$\begin{tikzcd} & K \,{}_f\!\times_{i_L} L \arrow[dl] \arrow[dr] & \\ K \arrow[dr] & & L \arrow[dl] \\ & M & \end{tikzcd}$$

$\leftarrow$ smooth fiber product

In order for f to be transverse, it needs to provide the tangent directions that are missing from L to complete into T_pM.



$$K \,{}_f\!\times_{i_L} L = \{(k,l) : K \times L \text{ s.t. } f(k) = i_L(l)\}$$
$$= f^{-1}(L)$$

$\leftarrow$ Generalizes the regular value theorem.

LECTURE 14

Oct 24th, 2024

<u>Transversality</u>

- <u>of manifolds</u>:
- <u>of C^∞ maps</u>:

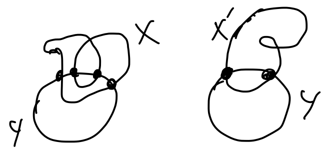
$X, Y \subset Z \qquad X \pitchfork Y$

$$\begin{tikzcd} & X \,{}_f\!\times_g Y \arrow[dl] \arrow[dr] & \\ X \arrow[dr, "f"] & & Y \arrow[dl, "g"'] \\ & Z & \end{tikzcd}$$

INTERSECTION NUMBER (mod 2): the intersection # of two submnfds X, Y in Z should be "= # of intersec. pts."

want to get rid of this

1. Want X, Y to have $\pitchfork$ intersec. in 0-dim submnfds
   i.e., want X, Y to have complimentary dim: $\dim X + \dim Y = \dim Z$
2. Want X, Y compact so the # is finite.

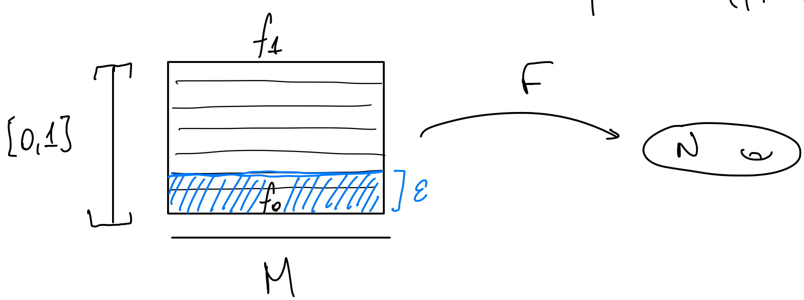


PROBLEM: # not well-def. but it is mod 2.

"Cl: # mod 2 is independent of perturbations of X, Y." (i.e., it's stable)

Def: (1) A smooth map $F: M \times [0,1] \to N$ is a smooth homotopy from f_0 to f_1, where

$$f_t = F \circ j_t \qquad j_t : M \longrightarrow M \times [0,1]$$
$$p \longmapsto (p, t)$$



(2) A property of smooth maps $f \in C^\infty(M, N)$ is STABLE under perturbations if $\forall$ smooth homotopies f_t, $t \in [0,1]$, we have

property holds for $f_0 \Longrightarrow$ property holds for f_t for some $\varepsilon > 0$, $0 \le t < \varepsilon$.

Def: (INTERSECTION #) Actual definition of $I_2(X,Y)$ removes the requirement of X, Y being transverse.

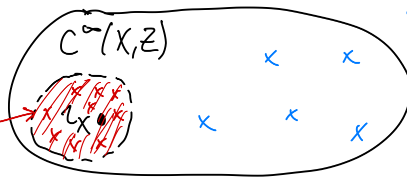
Strategy: **1.** If X, Y compact are given and are s.t. $\dim X + \dim Y = \dim Z$, deform X via a smooth homotopy until it becomes transverse to Y.

This depends on the fact that $\pitchfork$ is generic in the space of maps [SARD'S THM]

non-transverse maps are measure zero

$$\begin{tikzcd} & Y \arrow[d, "i_Y"] \\ X \arrow[r, hook, "i_X"'] & Z \end{tikzcd}$$

Given any map i_X we can find transverse maps in a nbhd of it

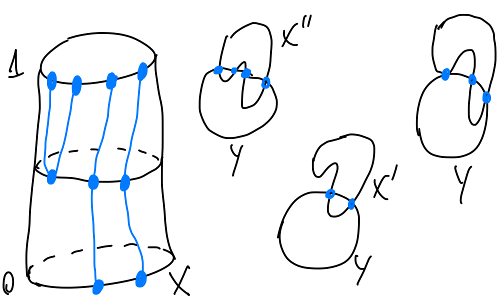


maps that fail to be $\pitchfork Y$ are measure zero

Rmk: the set of maps $\pitchfork Y$ are dense and have full measure.

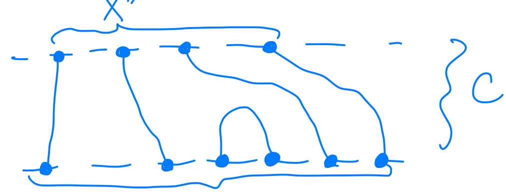
After deformation from X to X', $\# X' \cap Y \in \mathbb{Z}$ is well-defined. But this depends on the deformation.

2. Given two deformations X', X'' of X, each $\pitchfork Y$, what is the relation between $X' \cap Y$ and $X'' \cap Y$?



A: THEY ARE COBORDANT 0-DIM COMPACT MNFDS

cobordism of 0-dim mnfds



X'

C = compact 1-mnfd w/ boundary

$$\boxed{\partial C = X' \sqcup X''}$$

$$O \sqcup O \sqcup \cdots \sqcup O$$
$$\sqcup$$
$$[0,1] \sqcup [0,1] \sqcup \cdots \sqcup [0,1] \implies \#\text{ bdry pts. of } C \text{ must be even} \implies \#X' = \#X'' \text{ mod } 2$$

How to produce such cobordism? We need a C^∞ homotopy

$$J : X \times [0,1] \longrightarrow Z$$

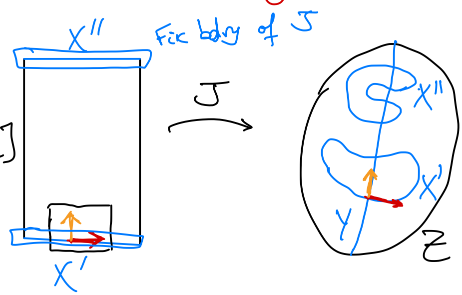
$$J_0 = \iota_{X'} \text{ and } J_1 = \iota_{X''}$$

$$\begin{tikzcd}
(X \times [0,1]) {}_J\!\times_{\iota_Y} Y \arrow[r] \arrow[d] & Y \arrow[d, "\iota_Y"] \\
X \times [0,1] \arrow[r, "J"'] & Z
\end{tikzcd}$$

Hope: $(X \times [0,1]) {}_J\!\times_{\iota_Y} Y$ is a cobordism.

Problem: We don't know that $(X \times [0,1]) {}_J\!\times_{\iota_Y} Y$ is smooth (e.g., it would be if $J \pitchfork Y$ but we have no guarantee of that). BUT: J_0, J_1 are $\pitchfork Y$.

$J \pitchfork Y$ along the boundary of the cylinder



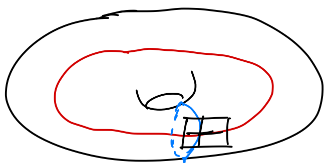
<u>Solution</u>: Same strategy using genericity of $\pitchfork$.

<u>Modify</u> J: find a homotopy from J to J' s.t. $J' \pitchfork \partial Y$.

<u>But</u>: domain of J is a mnfd w/ ∂ and so we want to fix $J|_{\partial(M\times[0,1])}$ (version of genericity allows us to do that)

[SARD II] Thm: Existence of $\pitchfork$ perturbation w/ bdry condition

Ex: T^2

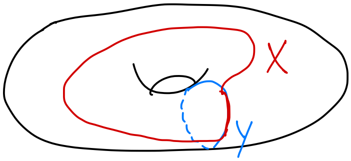


$X \cap Y = \{(1,1)\}$

But $X \pitchfork Y$ since $T_{(1,1)}X + T_{(1,1)}Y = T_{(1,1)}(T^2)$

$\Longrightarrow I_2(X,Y) = 1 \bmod 2$.

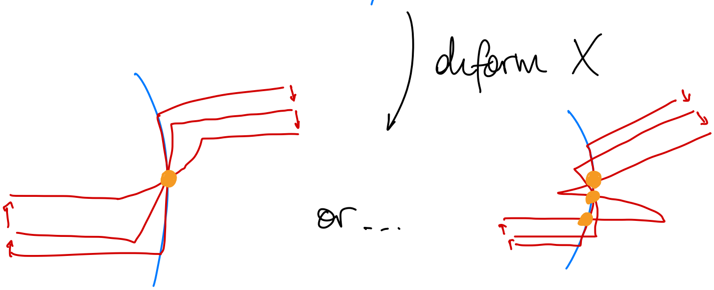
T^2



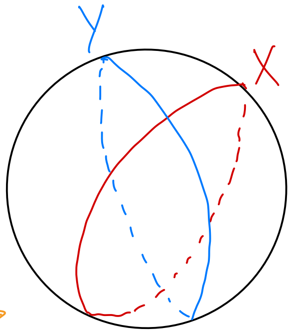
$X \cap Y$ not transverse, ∞-many intersection pts.

} Apply construction above

deform X



Ex: S^2



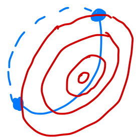
X, Y great circles

$$I_2(X,Y) = 2 = 0 \mod 2$$

or deform X to a point by smooth homotopy until

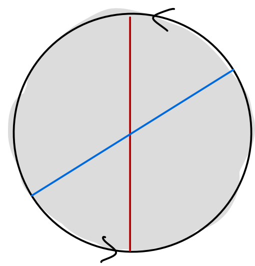
$$X' \cap Y = \phi.$$

Same works for any embedded circles



Ex: $\mathbb{R}P^2$ image of distinct great circles give paths

$$\# X \pitchfork Y = 1 \mod 2$$



LECTURE 15

Oct 25^{th}, 2024

Def: (INTERSECTION #) X, Y compact f, g smooth s.t. (not necessarily transverse)
$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$
$\dim X + \dim Y = \dim Z$. Then

$$I_2(f,g) := \#\left(X {}_{f}\!\times_{g} Y\right) \mod 2, \text{ where}$$

f' is any map smoothly homotopic to f and transverse to g.

$$\begin{tikzcd} X \times Y \underset{\substack{\text{smooth}\\ \text{submfd}}}{\supset} X {}_{f'}\!\times_g Y \arrow[r] \arrow[d] & Y \arrow[d, "g"] \\ X \arrow[r, "f'"'] & Z \end{tikzcd}$$

Rmk: $\dim X + \dim Y = \dim Z$

$$\Downarrow$$

$$\dim(X {}_{f'}\!\times_g Y) = 0$$

Need to prove:

Claim 1: It is always possible to find $f' \sim f$ (C^∞ homotopy)

Claim 2: For $f' \sim f$ and $f'' \sim f$, $X {}_{f'}\!\times_g Y$ is cobordant to $X {}_{f''}\!\times_g Y$

so that the definition is valid.

Obs: Both claims follow from Sard's theorem.

Ex: A pt $p \in S^1$ can be seen as a map

$$\begin{tikzcd} & \{p\} \arrow[dl] \arrow[dr] & \\ \{*\} \arrow[dr, "p"'] & & S^1 \arrow[dl, "Id"] \\ & S^1 & \end{tikzcd}$$

Note: Id is a submersion, hence $Id \pitchfork \left(\substack{\text{any map}\\ X \to S^1}\right)$.

So, we have a single point of intersection.

$\Rightarrow$ Id is not smoothly homotopic to any constant map.

If $\mathrm{Id} \sim$ const. map $g: x \mapsto q \in S^1$, $q \neq p$.

$$\Rightarrow \ast \, {}_p\!\times_{\mathrm{Id}} S^1 \text{ is cobordant to } \ast \, {}_p\!\times_g S^1 = \emptyset$$

$$\begin{array}{ccc} \ast \,{}_p\!\times_{\mathrm{Id}} S^1 & & \ast \,{}_p\!\times_g S^1 \\ \parallel & & \parallel \\ \ast & \longleftarrow \text{cobordant} \longrightarrow & \emptyset \\ & \hookrightarrow\!\!/\!\!\!\hookleftarrow & \end{array}$$

SAME HOLDS FOR ANY COMPACT MANIFOLD ! $\longrightarrow$

WEIERSTRASS APPROXIMATION: Any continuous function $f: [0,1] \to \mathbb{R}$ can be approximated by polynomials in the sup norm $\|f\|_\infty = \sup_{x \in [0,1]} |f(x)|$.

Generalize

STONE – WEIERSTRASS: for any compact Hausdorff top. space X (e.g., compact mnfds), if $A \subset C^0(X, \mathbb{R})$ is a subalgebra s.t.

1. A separates pts
2. A contains at least one nonzero continuous function

Then A is dense in the sup norm.

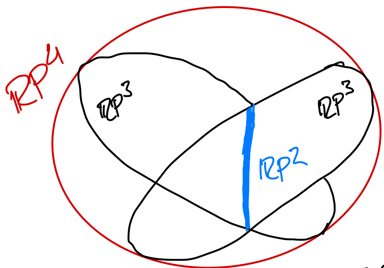
$\Rightarrow$ Since we can approximate all homotopies by smooth homotopies, the example above gives us that

COMPACT MANIFOLDS ARE NOT CONTRACTIBLE !

Ex: $\mathbb{RP}^4 = (\mathbb{R}^5 \setminus \{0\})/\sim$. In $\mathbb{R}^5$, choose two hyperplanes H_1 and H_2 s.t. $H_1 \pitchfork H_2$ (i.e., $H_1 + H_2 = \mathbb{R}^5$). Then

4-dim

$$\underbrace{\mathbb{P}(H_1)}_{\mathbb{RP}^3 \cong} \pitchfork \overset{\text{Projectivization}}{\underbrace{\mathbb{P}(H_2)}_{\cong\, \mathbb{RP}^3}} = \mathbb{P}(\overbrace{H_1 \cap H_2}^{3\text{-dim}}) \simeq \mathbb{RP}^2$$



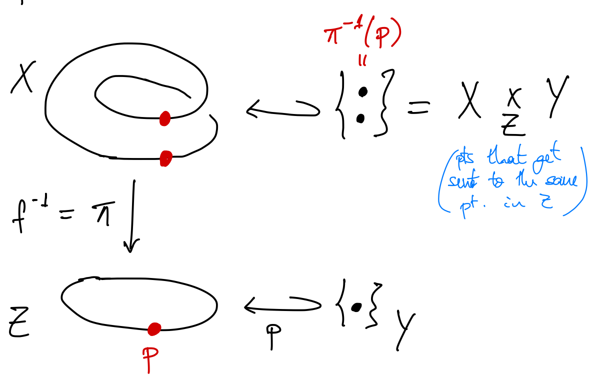
Q: Is it possible to remove intersection by deforming $2\ \mathbb{RP}^3$?

A: No! Since $\mathbb{RP}^2$ is not null-cobordant. If we remove this intersection, $\mathbb{RP}^2$ would be cobordant to $\emptyset$. Thom's classification of cobordism ring gives that $\mathbb{RP}^2$ is a generator in dim-2, hence cannot be null-cobordant. ◢

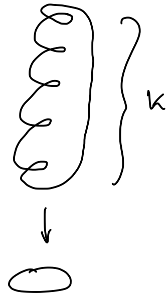
Def: (DEGREE OF MAP) Let $f: M^d \to N^d$ be smooth and $\dim M = \dim N$ with M compact and N connected, then the degree of f is

$$\deg_2(f) := I_2(\{p\}, f),$$

$p \in N$

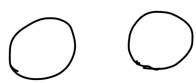


Ex: $f: S^1 \to S^1$, $z \mapsto z^k$ k-fold cover



$\deg_2 f = k \bmod 2$.

Ex: Use intersection # / degree to study mfds that <u>don't intersect</u>. Say M^m, N^n are mfds of dim m, n and they sit inside $\mathbb{R}^{m+n+1}$. Example: 2 circles in $\mathbb{R}^3$


$\neq$


Hopf Link ($m, n = 1$)   Unlink

Q: Can we detect if mfds are linked or not?

Strategy: Convert this into an intersection problem in an auxiliary space. Define

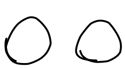
$$\lambda: \underset{\text{dim}=m+n}{M \times N} \longrightarrow \underset{\text{dim}=m+n}{S^{m+n}} \qquad \text{sphere at } \infty \text{ in } \mathbb{R}^{m+n}$$

$$(x, y) \longmapsto \frac{x-y}{\|x-y\|} \qquad \text{direction of } x \text{ from } y$$

well-def. b/c M, N don't intersect

$\Rightarrow$ Can compute the degree of λ

For Hopf Link, $\deg_2 \lambda = 1$, $\lambda: S^1 \times S^1 \to S^2$

$\Rightarrow$  $\underset{\text{not homotopic}}{\neq}$ 

LECTURE 16

Nov 14th, 2024

SARD'S THEOREM: Transversality is generic.

Almost all pairs $(f,g) \in \mathrm{Hom}(X,Z) \times \mathrm{Hom}(Y,Z)$

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Rmk: Transversality may be expressed as the existence of regular value of a map.

$X, Y \subset Z$ submnfds

$X \pitchfork Y \iff 0$ reg. value of $(f,g): Z \longrightarrow \mathbb{R}^{\mathrm{codim} X + \mathrm{codim} Y}$

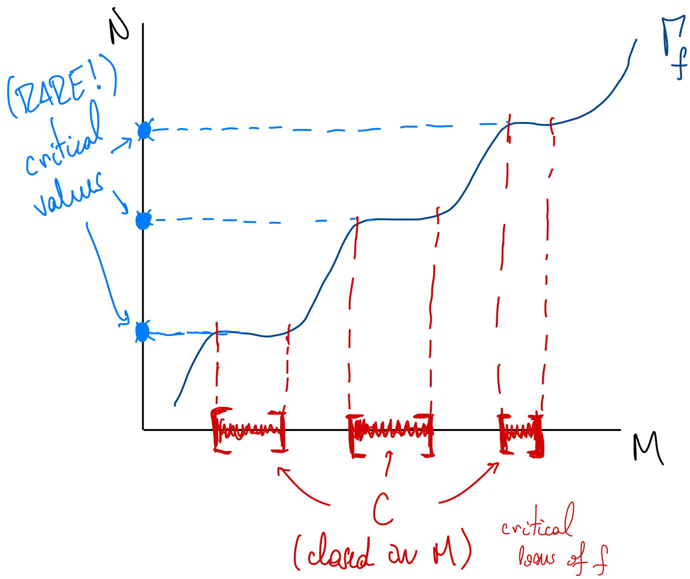
Ex:

$$\begin{tikzcd} M \arrow[d, "f"'] & \\ N & \{*\} \arrow[l, "p = i_p"] \end{tikzcd}$$

$f: M \to N$

$p \in N$ is reg. value $\iff f \pitchfork p$.

Sard's Theorem: Let $f \in C^\infty(M,N)$ and let $C \subset M$ be the set of critical points of f. Then $f(C) \subset N$ has measure 0.



Rmk: if f is not smooth, we need $f \in C^k(M,N)$ where

$k > \dim M - \dim N + 1$

Def: A subset $X \subset M$ of a mnfd M has measure 0 if, in each chart (U, φ), $\varphi(X \cap U)$ has Lebesgue measure 0 in $\mathbb{R}^n$.

← that's why we need mnfds to be 2nd countable...

i.e., $\forall \varepsilon > 0$, $\varphi(X \cap U)$ can be covered by a countable seq. of balls w/ total volume $< \varepsilon$.

Guarantees that the def. above is indep. of charts

Lemma: Let $I^m = [0,1]^m$ and $f : I^m \longrightarrow \mathbb{R}^n$ be C^1.

(a) If $m < n$, then $f(I^m)$ has measure 0

(b) If $m = n$ and $A \subset I^m$ has measure 0 then $f(A)$ has measure zero

$$\begin{tikzcd} I^m \times I^{n-m} \arrow[r, "\tilde{f}"] \arrow[dr, "\pi"'] & \mathbb{R}^n \\ & I^m \arrow[u, "f"'] \end{tikzcd}$$

$A = I^m \times \{0\} \subset_{\text{meas. } 0}$

$f(I^m) = \tilde{f}(A)$

Pf: (b) A meas. $0 \Rightarrow \forall \varepsilon > 0$ $\exists$ seq. of balls B_k w/ radius r_k covering A s.t.

$$C_m \sum_{k \in \mathbb{N}} (r_k)^m < \varepsilon$$

Idea: Cover $f(A)$ w/ $f(B_k)$. So, find balls $B_k' \subset \mathbb{R}^n$ s.t. $f(B_k) \subset B_k'$ w/ $\sum_{k \in \mathbb{N}} \mathrm{vol}(B_k') < \varepsilon$.

Need: Estimate for expansion factor of f

If $f \in C^1(I^m, \mathbb{R}^n)$, I^m compact $\Rightarrow$ derivative is bounded

LIPSCHITZ CONSTANT OF f ← $\exists K > 0$

$\forall x, y \in I^m$, $\|f(x) - f(y)\| \leq K \|x - y\|$.

$\Rightarrow$ $f(A)$ covered by balls of radius $K r_k$

$\Rightarrow$ volume $\leq C_m \sum_k (K r_k)^m < K^m \varepsilon$

$\Rightarrow$ $\varepsilon \to 0$, volume $\to 0$

$\Rightarrow$ $f(A)$ is measure zero. $\blacksquare$

Corollary: 1) $f: M \to N$ C^1 $\dim M = \dim N$ (uses 2nd count.)
then A meas. zero $\Rightarrow f(A)$ meas. zero

2) (Baby Sard) If $\dim M < \dim N$, $C =$ crit. pts. $= M$, $f \in C^1$
then $f(C) = f(M) \subset N$ has meas. zero (uses 2nd countable)

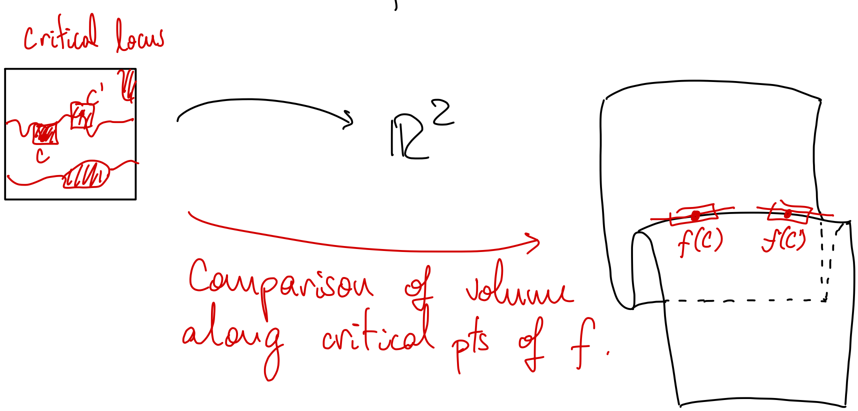
Thm: (Equidimentional Sard) $f: M \longrightarrow N$ C^1
$\dim M = \dim N$, $C =$ crit. pts. of $f \subset M$, $f(C) \subset N$
has meas. zero.

Pf: Suffices to prove for $I^n \to \mathbb{R}^n$.

(1) Let K be the Lipschitz const. of f

$$\| f(x) - f(y) \| \leq K \| x - y \|.$$

(2) Critical Vol. Comparison: $c \in I^n$ critical $\Rightarrow$ im Df_c is proper subspace of $\mathbb{R}^n$.



Choose hyperplane H_c centered at $f(C)$ containing image.

Then $d(f(x), H_c) \leq \| f(x) - f_c^{lin}(x) \|$,
where $f_c^{lin}(x) = f(c) + D_c f\,(x-c)$

<u>By def. of derivative</u>: $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$\| f(x) - f_c^{lin}(x) \| < \varepsilon \| x - c \| \quad \forall x \text{ s.t. } \| x - c \| < \delta$$

Use I^n compact $f \in C^1 \Rightarrow$ unif δ in c for any ε

δ indep. of c

Combining (1) + (2): $\|x - c\| < \delta \Rightarrow f(x)$ is within $\varepsilon\delta$ of H_c.

$d(f(x), H_c) \leq \| f(x) - f_c^{lin}(x) \| < \varepsilon \| x - c \| < \varepsilon \delta$.

and within $K\delta$ of $f(c)$

$\Rightarrow f(x)$ lies in a parallelopiped of volume

$$(2\varepsilon\delta)(2K\delta)^{n-1}$$

Finally: Need to cover crit. locus C w/ countably many balls/cubes s.t. we can use above est. in each one.

Easy: Subdivide cube into N^n small boxes s.t.

diam (small cubes) $< \delta$ i.e., $\sqrt{n}\,\frac{1}{N} < \delta$

$\Rightarrow$ covered critical locus of f w/ boxes of diam $< \delta$.

So,

$$\mathrm{vol}\,(f(\text{small cubes} \cap C)) \leq \left(2\varepsilon \frac{\sqrt{n}}{N}\right)\left(2K \frac{\sqrt{n}}{N}\right)^{n-1} N^n$$

$$= (2\varepsilon\sqrt{n})(2K\sqrt{n})^{n-1} \xrightarrow{\varepsilon \to 0} 0.$$

$\blacksquare$

LECTURE 17

Nov 15th, 2024

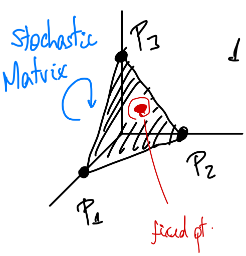
* **APPLICATIONS OF SARD:**

Thm: (Brouwer Fixed Point) Any continuous map $f: B^n \to B^n$, $B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$ has a fixed point.

Manifold w/ boundary

Rmk: Space of prob. distrib. on n outcomes $\{1, 2, \ldots, n\}$ is an n-simplex $\Delta^{n-1} = \{(p_1, \ldots, p_n) : \sum p_i = 1, \ 0 \leq p_i \leq 1\}$

$\cong B^{n-1}$



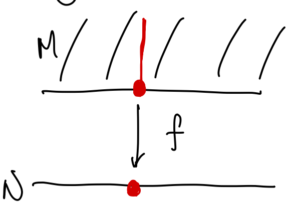
$1 = p_1 + \cdots + p_n$

$0 \leq p_i \leq 1$

Lemma: Let M be a compact mnfd w/ bdry. There is no smooth retract to ∂M; i.e., there are no $f: M \to \partial M$ s.t. $f|_{\partial M} = \mathrm{Id}_{\partial M}$.

Regular Value Thm. for mnfds w/ bdry: $q \in N$ is reg. val. of f and $\partial f \implies f^{-1}(q)$ is a submnfd w/ bdry given by

$$\partial f^{-1}(q) = f^{-1}(q) \cap \partial M$$



Pf: (Lemma) By contradiction assume such f exists.
By Sard's Thm, $\exists$ a reg. value $q \in \partial M$ for f
(since $f|_{\partial M} = Id_{\partial M}$ q is also reg. val. for ∂f)

$\Rightarrow$ $f^{-1}(q)$ is a submnfd w/ bdry of B^n s.t.

single point

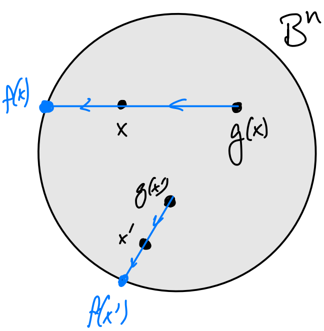
$$\partial(f^{-1}(q)) = f^{-1}(q) \cap \partial B^n = f^{-1}(q) \cap S^{n-1} = \{*\}$$

But $\dim f^{-1}(q) = 1$ $\Rightarrow$ we built a compact 1-dim mnfd w/ bdry $\cong \{q\}$ $\hookrightarrow\!\!/\!\!/\!\!\leftarrow$

$\square$

Smooth Brouwer Fixed Pt. Thm: $\forall g \in C^\infty(B^n, B^n)$, $\exists p$ $g(p) = p$

Pf: Contradiction: suppose g has no fixed points.



Extend the line $g(x) \longmapsto x$ to ∂B^n and obtain

$$f : x \longmapsto \overrightarrow{g(x)\,x} \cap \partial B^n$$

which is a smooth retraction. $\hookrightarrow\!\!/\!\!/\!\!\leftarrow$

$$f(x) = x + tu \quad \text{where } u = \frac{x - g(x)}{\|x - g(x)\|}$$

t = positive solution to a quadratic eq. w/ positive discriminant.

since g is smooth and f is given by this smooth formula f is smooth

$\square$

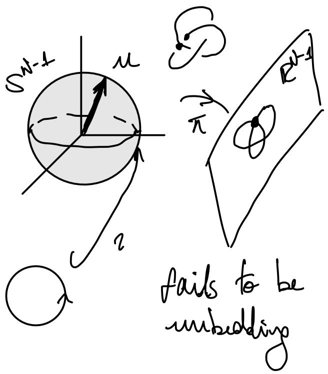
Pf: (BFPT) Weierstrass approx. of continuous $B^n \circlearrowleft$ by smooth maps, scaled slightly to fit in B^n.

$\square$

APPLICATION OF SARD:

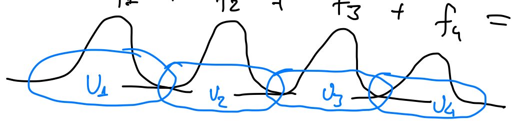
> **Thm:** (Whitney Embedding Theorem) Any n-mnfd can embed in $\mathbb{R}^{2n+1}$.

Idea of Pf: Construct N w/ partition of unity and get the dimensions down by studying when projections fail to give embeddings.



POU:

$$f_1 + f_2 + f_3 + f_4 = 1$$



> **Thm:** (Partions of Unity) Given a regular covering of M $\{(V_i, \varphi_i)\}$, $\exists \{f_i\}$ partition of unity subordinate to this covering s.t. $f_i > 0$ on $V_i \subset \varphi_i^{-1}(B_0(1))$ and $\operatorname{supp} f_i \subset \varphi_i^{-1}(\overline{B_0(2)})$.

Def: A regular covering is a locally finite (i.e., each $x \in M$ intersects finitely many charts) open cover $\{(U_i, \varphi_i)\}$ of M by coord. charts s.t. $\varphi_i(U_i) = B_0(3)$ and $V_i = \varphi_i^{-1}(B_0(1))$ also form a covering.

Def: A POU subordinate to cover $\{(U_i, \varphi_i)\}$ is a collection $\left(f_i : M \xrightarrow{C^\infty} [0,1]\right)$ s.t. (1) $\operatorname{Supp} f_i = \overline{f_i^{-1}(\mathbb{R}\setminus\{0\})} \subset \varphi_i^{-1}(B_0(2))$

only makes sense on the loc. finite case since only finitely many f_i's are $\neq 0$

(2) $\sum_{i=1}^{n} f_i(x) = 1 \quad \forall x \in M$

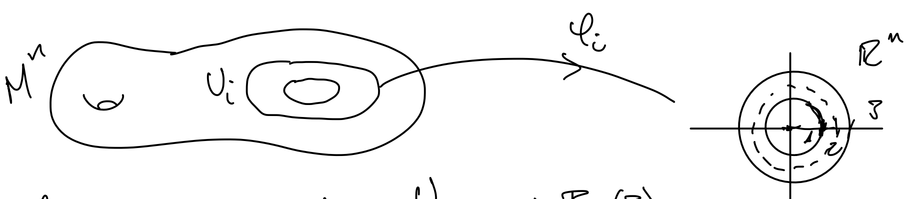
LECTURE 18

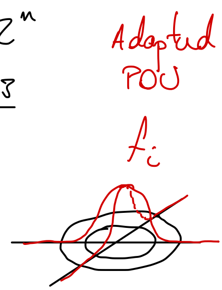
Nov 21st, 2024

same idea for non-compact but more bookkeeping.

* **WHITNEY EMBEDDING**: Let M^n be compact mnfd.

Idea: Start w/ P.O.U.:





Regular covering of M^n:
$$\begin{array}{ccc} U_i & \longrightarrow & B_0(3) \\ \cup & & \cup \\ V_i & \longrightarrow & B_0(1) \end{array}$$

1. Extend charts $U_i \xrightarrow{\varphi_i} \mathbb{R}^n$ to maps $M \to \mathbb{R}^n$ by doing
$U_i \subset M$

$$f_i \varphi_i := \begin{cases} f_i \varphi_i, & \text{for } x \in U_i \\ 0, & \text{else} \end{cases}.$$

This map is going to be an embed. on U_i but not anywhere else.

2. If we exit U_i and enter U_j, pass before to $f_j \varphi_j$.

i.e.: for a covering $\{U_i\}_{i=1}^N$, define

$$\begin{aligned} M &\longrightarrow \mathbb{R}^n_1 \times \mathbb{R}^n_2 \times \cdots \times \mathbb{R}^n_N \\ x &\longmapsto (f_1\varphi_1(x), f_2\varphi_2(x), \ldots, f_N\varphi_N(x)) \end{aligned}$$

3. But to make this into an embedding, we need more information: define $\Phi : M \longrightarrow (\mathbb{R}^n)^N \times \mathbb{R}^N = \mathbb{R}^{N(n+1)}$

$$\Phi(x) := \Big(f_1\varphi_1(x), \ldots, f_N\varphi_N(x), \underbrace{f_1(x), \ldots, f_N(x)}_{\text{Bookkeeping information to track in which open set } x \text{ lives in.}}\Big)$$

Claim: Φ is injective

Pf: If $\Phi(x) = \Phi(y) \Rightarrow$ for some i, $f_i(x) = f_i(y) \neq 0$

$\Rightarrow x, y \in U_i$ and $\cancel{f_i}\varphi_i(x) = \cancel{f_i}\varphi_i(y)$

$\Rightarrow x = y$ b/c φ_i is injective (coord. map) □

Claim: Φ is an immersion

Pf: Check $D\Phi : T_p M \longrightarrow \mathbb{R}^{N(n+1)}$ is injective

$$D_x\Phi(v) = \Big(D_x f_1(v)\,\varphi_1(x) + f_1(x)\, D_x\varphi_1(x),\ \_\_\_,\ D_x f_2(v),\ \_\_\_ \Big)$$

If $D_x\Phi(v) = 0$, must have $D_x f_i(v) = 0 \ \forall i$. Plugging this into the first part $\Rightarrow f_i(x)\, D_x\varphi_i(v) = 0 \ \forall i$

$\Rightarrow$ for some i, $f_i(x) \neq 0 \quad x \in U_i$

$\Rightarrow D_x\varphi_i(v) = 0 \Rightarrow v = 0$

φ_i coörd. charts

$\square$

Claim: Φ embedding

Pf: Φ injective immersion.

Φ homeo. b/c it's injective on a compact Hausdorff space. $\square$

Upshot: Can embed $\mathbb{RP}^2 \xhookrightarrow{\text{emb.}} \mathbb{R}^9 \leftarrow \underbrace{(2+1)}_{\#\text{ of charts on atlas for } \mathbb{RP}^2} \cdot \overbrace{(2+1)}^{\text{as above}}$

Actually can reduce $\mathbb{RP}^2 \xhookrightarrow{\text{emb.}} \mathbb{R}^5$

surgery theory... $\xhookrightarrow{\text{emb.}} \mathbb{R}^4$

$$\boxed{\textbf{Thm}: \text{Any manifold } M \text{ embeds into } \mathbb{R}^{2\dim M + 1} \quad \left(\text{actually } \mathbb{R}^{2\dim M}\right)}$$

Pf: Here, assume M is compact (see notes for generalization)

Strategy: 1. Embed $M \hookrightarrow \mathbb{R}^N$ using previous result

2. Show that, if $N > 2\dim M + 1$, $\exists$ directions $u \in S^{N-1}$ s.t. projections to $\perp u$ is an embedding. Use Sard's Thm to show that the bad directions are measure 0.

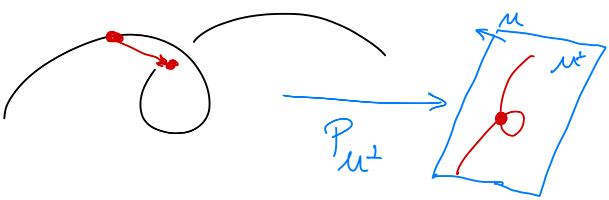
Q: What makes a direction bad?

(i) Failure of injectivity of $P_{u^\perp} \circ \Phi : M \to \mathbb{R}^{N-1}$

(ii) Failure of injectivity of the derivative

WTS: Both these failures only occur on sets of measure 0 in the sphere S^{N-1}.

(i) This failure occurs when $\exists (x,y) \in \overbrace{M \times M \setminus \Delta}^{(2n)\text{-dim mnfd}}$ s.t.



$$\frac{\Phi(y) - \Phi(x)}{\|\Phi(y) - \Phi(x)\|} = u$$

i.e.: $P_{u^\perp} \circ \Phi$ fails injectivity when $u \in S^{N-1}$ is in the image of the map:

2n dim

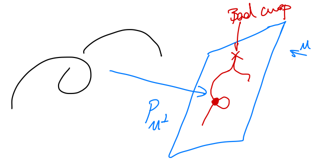
$$M \times M \setminus \Delta \longrightarrow S^{N-1} \quad \leftarrow N-1 \text{ dim}$$

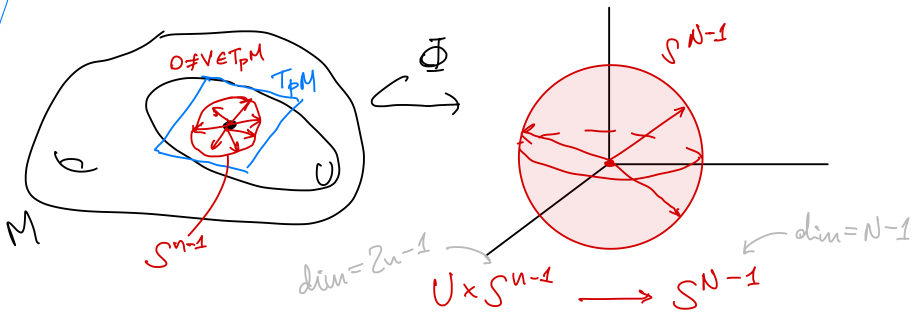
$$(x,y) \longmapsto \frac{\Phi(y) - \Phi(x)}{\|\Phi(y) - \Phi(x)\|}$$

Sard: Image has measure of $2n < N-1$

$\Rightarrow$ Embedding into $N = 2n+1$ is the best we can do here.

(ii) Failure of injectivity of $D(P_{u^\perp} \circ \Phi)$ occurs for $u \in S^{N-1}$ which lie in $T_p\Phi(M)$





$$U \times S^{n-1} \longrightarrow S^{N-1}$$

$$(p, v) \longmapsto \frac{D_p\Phi(v)}{\|D_x\Phi(v)\|}$$

Immersion fails on image of this

Sard: immersion fails on a measure zero set if

$$2n - 1 < N - 1 \;;\; \text{i.e., } 2n < N$$

Upshot: $\begin{cases} \text{Embed into } \mathbb{R}^{2\dim M + 1} \\ \text{Immerse into } \mathbb{R}^{2\dim M} \end{cases}$

$\square$

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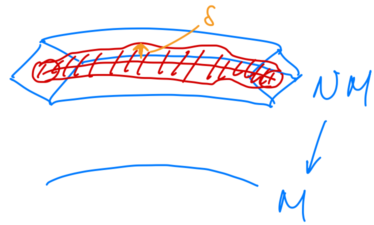
* **TUBULAR NEIGHBORHOODS** (for embeddings $M \hookrightarrow \mathbb{R}^N$)

Thm: Every submanifold M of $\mathbb{R}^N$ has a tubular nbhd.

i.e., U open nbhd of M which is diffeom. to the image of a solid cylinder in NM (normal bundle)

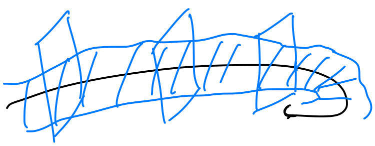
- Each point $x \in M$ has a normal space $N_x = \mathbb{R}^N / T_x M = T_x M^{\perp}$

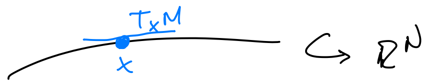
$\Rightarrow$ Form a vector bundle $\begin{array}{c} NM \\ \downarrow \\ M \end{array}$ called the NORMAL BUNDLE



- Solid cylinder: Given $\delta : M \to \mathbb{R}_{>0}$

$$V = \{ (v, y) \in NM : \|v\| < \delta \}$$





Map $V \longrightarrow \mathbb{R}^N$

$(y, v) \longmapsto \Phi(y) + v$

Since $V \cong$ cylinder

$V \xrightarrow[\text{retraction}]{r} M$ $\qquad$ $V \xleftarrow[\text{inclusion}]{\imath} M$

$(y, v) \longmapsto y$ $\qquad$ $(y, 0) \longleftarrow y$

$r \circ \imath = Id_M$

IMPORTANCE OF THIS:

* $f : M \to N$ may not be transverse to $g : K \to N$.

To modify f onto a map which is transverse to g, need to put f in a family of maps

tangent space of $f(M) \subset M$ not enough for $\pitchfork$, but we can add more directions by

$F : M \times S \to N$ s.t. $F|_{M \times \{s_0\}} : M \to N = f$

$s_0 \in S$ and $F \pitchfork g$. $\underset{\text{reg. val. thm.}}{\Longrightarrow}$ $\exists s \in S$ for which $f_s \pitchfork g$.

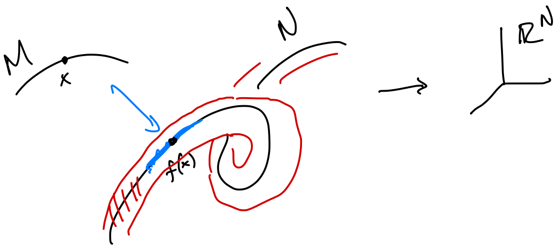
Q: How to deform f in this family $F : M \times S \to N$ to make it transverse to g?

<u>A</u>: Given $f: M \to N$, embed N in $\mathbb{R}^K$

$M \longrightarrow N \hookrightarrow \mathbb{R}^K$

$S = B^K$ open ball in $\mathbb{R}^K$

use this to translate



$F(x,s) = r\big(f(x) + s\big)$

r: retraction of tubular nbhd

LECTURE 19

Nov 22^{nd}, 2024

* **VECTOR BUNDLES** ($\subseteq$ Fiber bundles)

Def: $\begin{array}{c} E \\ \pi\downarrow \\ M \end{array}$ π = bundle projection surjective

s.t. $\forall p \in M$ $\exists$ nbhd U and Φ homeo s.t. the diagram commutes

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\Phi"] \arrow[dr, "\pi"'] & & U \times \mathbb{R}^K \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

<u>Overlap</u>: if U, V are such nbhds w/ Φ_U, Φ_V resp. Then

$$
\begin{tikzcd}
 & \pi^{-1}(U\cap V) \arrow[dl, "\Phi_U"'] \arrow[dr, "\Phi_V"] & \\
(U\cap V)\times\mathbb{R}^k \arrow[rr, "\Phi_V\Phi_U^{-1}"] \arrow[dr] & & (U\cap V)\times\mathbb{R}^k \arrow[dl] \\
 & U\cap V &
\end{tikzcd}
$$

$\Phi_V\Phi_U^{-1}$ = "transition function" : $U\cap V \xrightarrow{C^\infty} GL(k,\mathbb{R})$

Construction: To build a vector bundle over M

1. Build M from opens $\{U_i \subset \mathbb{R}^k\}$, $U_i \supset U_{ij}$, via φ_{ij} gluing maps $\left(U_{ij} \xrightarrow{\varphi_{ij}} U_{ji}\right)$ diffeo.

2. Provide $g_{ij} : U_i \cap U_j \longrightarrow GL(k,\mathbb{R})$ and define

$$E := \bigcup_{i\in I} (U_i \times \mathbb{R}^k) \Big/ \begin{array}{l}(x,u)\sim(y,v)\\ y=\varphi_{ij}(x)\\ v = g_{ij}(x)\,u\end{array}$$

Equiv. relation b/c we have $\varphi_{jk}\varphi_{ij} = \varphi_{ik}$ and

$g_{jk}\,g_{ij} = g_{ik}$

cocycle condition (nonabelian 1-cocycle)

Ex: $S^2 = U_0 \cup U_1$. A vector bundle over S^2 is obtained by choosing

$U_0 \cap U_1 = \mathbb{R}^2\setminus\{0\} = \mathbb{C}^*$

$g_{01} : U_0 \cap U_1 \longrightarrow GL(2,\mathbb{R})$

e.g.: $g_{01}(z\in\mathbb{C}^*) = r^n\begin{pmatrix}\cos n\theta & -\sin n\theta\\ \sin n\theta & \cos n\theta\end{pmatrix} = [z^n]$

$\cap\; GL(2,\mathbb{R})$ $\qquad \cap\; GL(1,\mathbb{C})$

n = # of times this * plane turns as it goes around the intersection before it's identified w/ the bottom plane

$\Rightarrow$ $\mathbb{Z}$ - family of rank-2 vector bundles over S^2.

Rmk: FB vs. VB

- **FB**: fiber type F $\qquad U_i \cap U_j \xrightarrow{g_{ij}} \mathrm{Homeo}(F)$
- **VB**: fiber type $\mathbb{R}^k$ $\qquad U_i \cap U_j \xrightarrow{g_{ij}} GL(k, \mathbb{R})$

Variants: • Affine bundle: fiber type $\mathbb{R}^k$ (as affine space)

$$U_i \cap U_j \xrightarrow{g_{ij}} \mathrm{Aff}(\mathbb{R}^k) = GL(k, \mathbb{R}) \ltimes \mathbb{R}^k$$

• Right Principal G-bundle: fiber type G Lie group viewed as a right G-space

$$U_i \cap U_j \xrightarrow{g_{ij}} G \text{ (acting on the left)}$$

* **OPERATIONS ON VECTOR BUNDLES**: "Any functor involving vec. spaces can be upgraded to $VB \to M$."

e.g.: Duals

V finite-dim. $\mathbb{R}$-vec. space $\qquad V^*$ dual $\qquad V^* = \mathrm{Hom}(V, \mathbb{R})$

$$V \xrightarrow[\text{linear}]{f} W \qquad V^* \xleftarrow{f^*} W^* \qquad \text{(contravariant functor)}$$

If $\begin{array}{c} E \\ \pi\downarrow \\ M \end{array}$ is built from $\begin{array}{c} U_i \times \mathbb{R}^k \\ \downarrow \\ U_i \end{array}$ and $g_{ij} : U_i \cap U_j \longrightarrow GL(k, \mathbb{R})$

Apply functor to

Inverse is a conseq. of the contra-variance of dual functor

$$\begin{array}{ccc} U_i \times (\mathbb{R}^k)^* & \xrightarrow{(g_{ij}^*)^{-1}} & U_j \times (\mathbb{R}^k)^* \\ \downarrow & & \downarrow \\ U_i & & U_j \end{array}$$

get

$$\boxed{\begin{array}{ccc} g_{ij} & \longleftrightarrow & E \\ (g_{ij}^*)^{-1} & \longleftrightarrow & E^* \end{array}}$$ dual VB

e.g.: Sum $V, W \longmapsto V \oplus W$

$f : V \to V'$
$g : W \to W'$ $\rightsquigarrow$ $f \oplus g : V \oplus W \longrightarrow V' \oplus W'$
$(v, w) \longmapsto (f(v), g(w))$

rank-k	rank-ℓ
E	E
$\downarrow$	$\downarrow$
M	M
g_{ij}	h_{ij}
$U_{ij} \to GL(k, \mathbb{R})$	$U_{ij} \to GL(\ell, \mathbb{R})$

Direct sum VB

$E \oplus F$ rank-$(k+\ell)$
$\downarrow$
M

$$G_{ij} = \left(\begin{array}{c|c} \overset{k \times k}{g_{ij}} & 0 \\ \hline 0 & \underset{\ell \times \ell}{h_{ij}} \end{array}\right)$$

e.g.: Tensor product $V, W \longmapsto V \otimes W$

$f: V \to V$ $\dim V = k$ $\qquad f \otimes g : V \otimes W \longrightarrow V \otimes W$

$g: W \to W$ $\dim W = l$

$$\sum_{i,j} v_i \otimes w_j \longmapsto \sum_{i,j} f(v_i) \otimes g(w_j)$$

Bases $(e_i^V), (e_i^W)$ for V, W.

$\downarrow$

as matrices $f \otimes g$ "Kronecker product".

Basis for $V \otimes W$: $(e_1^V \otimes e_1^W, e_1^V \otimes e_2^W, \dots, e_1^V \otimes e_l^W, e_2^V \otimes e_1^W, \dots)$

$\dim V \otimes W = (\dim V)(\dim W) = kl$

$$\overset{f}{\boxed{k \times k}} \otimes \overset{g}{\boxed{l \times l}} \xrightarrow{\text{Kronecker product}} \begin{pmatrix} \overset{l\times l}{\boxed{gf_{11}}} & \boxed{gf_{12}} & \cdots & \boxed{} \\ \boxed{gf_{21}} & \cdots & & \\ \vdots & & & \end{pmatrix}$$

* ASSOCIATED VB TO THE TANGENT BUNDLE

M C^∞ mfd $\Rightarrow$ TM tangent bundle w/ transition fcts

$$U_i \times \mathbb{R}^n \longrightarrow U_j \times \mathbb{R}^n$$
$$(x, u) \longmapsto (\varphi_{ij}(x), D_x \varphi_{ij}(u))$$

TM has many associated VBs:

Ex: COTANGENT BUNDLE T^*M

$$g_{ij} = \left((D_x \varphi_{ij})^*\right)^{-1}$$

Element of T_x^*M is a linear fct on T_xM.

Ex: $\underbrace{TM \otimes \cdots \otimes TM}_{k} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{\ell}$ ← rank $n^{k+\ell}$

$$g_{ij} = (D_x \varphi_{ij}) \otimes \cdots \otimes (D_x \varphi_{ij}) \otimes \left((D_x \varphi_{ij})^*\right)^{-1} \otimes \cdots \otimes \left((D_x \varphi_{ij})^*\right)^{-1}$$

↳ This is the bundle of (k, ℓ)-tensors k – covariant
ℓ – contravariant

Interpretation: $\omega \in T_x^*M \otimes T_x^*M$ is a bilinear fct. on T_xM

$$\| $$

$$\underbrace{\Lambda^2 T_x^*M}_{\text{skew-symmetric bilinear forms}} \oplus \underbrace{\mathrm{Sym}^2 T_x^*M}_{\text{symmetric bilinear forms}}$$

> **Def:** A section of VB $E \xrightarrow{\pi} M$ is a C^∞ map
> $s: M \to E$ s.t. $\pi s = \mathrm{Id}_M$

e.g.: Riem. mnfds – M C^∞ mnfd
– g sym. bilinear form, positive def. on each tangent space varying over M

$g \in C^\infty(M, \mathrm{Sym}^2 T^*M) \longrightarrow g$ is a $(0,2)$-tensor

$\mathrm{Sym}^2 T^*M$ = VB of symmetric 2-tensors

$$\begin{array}{c} \mathrm{Sym}^2 T^*M \\ \cap \\ T^*M \otimes T^*M \end{array}$$

X, Y vec. fields $\Rightarrow g(X,Y) \in C^\infty(M, \mathbb{R})$

$$g(X,Y)_x = g_x(X_x, Y_x).$$

e.g.: Symplectic Manifold

- M C^∞ mnfd
- $\omega \in C^\infty(M, \Lambda^2 T^*M)$ nondegenerate skew-symmetric bilinear form on each tangent space (and need $d\omega = 0$).

only possible when $\dim M$ = even

e.g.: Complex Manifold

- M C^∞ mnfd
- Complex structure on each tangent space. Need to define multiplication by i to define this structure:

$I : T_xM \longrightarrow T_xM$ i.e., $I \in T_x^*M \otimes T_xM$

$I^2 = -\mathrm{id}$ i.e., $I \in C^\infty(M, T^*M \otimes TM)$

s.t. $I^2 = -\mathrm{id}_{TM}$.

only possible when $\dim M$ = even

(also need involutivity needed to get complex coordinates on $\mathbb{C}^n$)

Ex: • Tensor algebra

$V = \mathbb{R}$- vec. space $\dim V = n$

$\bigotimes^\bullet V =$ free algebra generated by V

$= (\mathbb{R} = \otimes^0 V) \oplus V \oplus (V \otimes V) \oplus (V \otimes V \otimes V) \oplus \cdots$

$\dim \bigotimes^\bullet V = \infty$

Non - commutative

Tautological product:

$$(a_1 \otimes \cdots \otimes a_k) \cdot (b_1 \otimes \cdots \otimes b_\ell) = a_1 \otimes \cdots \otimes a_k \otimes b_1 \otimes \cdots \otimes b_\ell$$

• Exterior algebra: $\Lambda^\bullet V = \bigotimes^\bullet V \Big/ \langle x \otimes x : x \in V \rangle$

i.e., expressions w/ $a \otimes a$ always get killed

$$(x+y) \otimes (x+y) = \underbrace{x \otimes x}_{=0} + x \otimes y + y \otimes x + \underbrace{y \otimes y}_{=0}$$

$$\underbrace{(x+y) \otimes (x+y)}_{=0}$$

$$\Rightarrow x \otimes y = -y \otimes x$$

That's why we call $\Lambda^\bullet V$ a graded commutative algebra (b/c it's almost commutative except for a minus sign).

$\dim V = n$

$$\Lambda^\bullet V = (\underset{1}{\mathbb{R}} = \Lambda^0 V) \oplus \underset{\substack{e_1 \\ \vdots \\ e_n}}{V} \oplus \underset{\substack{e_1 \wedge e_2 \\ e_1 \wedge e_3 \\ \vdots}}{\Lambda^2 V} \oplus \underset{\substack{e_i \wedge e_j \wedge e_k \\ i<j<k}}{\Lambda^3 V} \oplus \cdots \oplus \underset{\substack{e_1 \wedge \cdots \wedge e_n \\ 1\text{-dim}}}{\Lambda^n V} \oplus 0 \oplus 0 \oplus \cdots$$

$\vdots$
$e_2 \wedge e_3$
$e_2 \wedge e_4$
$\vdots$
$e_3 \wedge e_4$ $\qquad \dim \Lambda^\bullet V = 2^n$
$\vdots$

$\updownarrow$

$e_i \wedge e_j$
$i<j$

$\mathbb{Z}$ - Graded b/c $\Lambda^\bullet V = \bigoplus_{k \in \mathbb{Z}} \Lambda^k V$ and

commutative algebra

α, β deg $k, l \longrightarrow \deg(k \wedge l) = k + l$

$$\alpha \wedge \beta = (-1)^{(\deg \alpha)(\deg \beta)} \beta \wedge \alpha$$

LECTURE 20

Nov 28th, 2024

* **DIFFERENTIAL FORMS**: M C^∞ mnfd.

diff. k-forms

Differential graded commutative algebra: $\Omega^\bullet(M) = \bigoplus_{k \geq 0} \Omega^k(M)$

non zero only for $k = 0, 1, \dots, \dim M$

where $\Omega^k(M) = C^\infty(M, \Lambda^k T^* M)$.

The product in this algebra is the wedge product: $\alpha \wedge \beta$.

Note: $\alpha \wedge \beta = (-1)^{(\deg \beta)(\deg \alpha)} \beta \wedge \alpha$

* **THE DIFFERENTIAL**: In degree-0, we have

(fcts $M \xrightarrow{C^\infty} \mathbb{R}$)

$$\Omega^0(M) = C^\infty(M, \Lambda^0 T^*M) = C^\infty(M, \mathbb{R})$$

$$\begin{gathered} \Psi \\ f : M \to \mathbb{R} \end{gathered}$$

$$Df : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$$

$$(x, v) \mapsto (f(x), D_x f(v))$$

$v \mapsto D_x f(v)$ is a linear fct. on $T_x M$

Def: $\pi_2 \circ Df =: df$ is a smooth section of T^*M

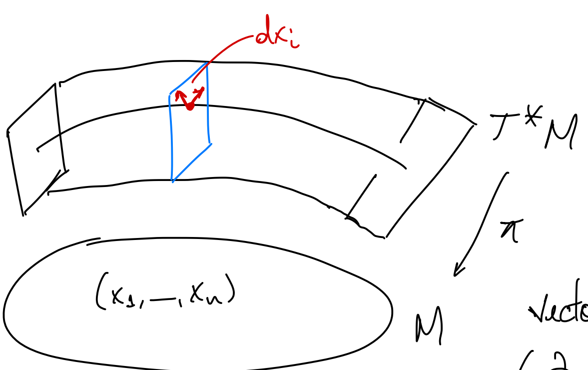
i.e., $df \in \Omega^1(M) = C^\infty(M, \Lambda^1 T^*M)$.

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \quad \text{de Rham operator}$$

(exterior derivative)

In local coordinates: $(x_1, —, x_n)$ coords $(x_i : M \to \mathbb{R})$

have derivatives $(dx_1, —, dx_n)$, $dx_i \in \Omega^1(M)$.



For $f \in \Omega^0(M)$,

$$df = a_1 dx_1 + \cdots + a_n dx_n,$$

$$a_i \in \Omega^0(M).$$

How to find these a_i? Use vector fields! They have a basis $\left(\frac{\partial}{\partial x_1}, —, \frac{\partial}{\partial x_n}\right)$ (sections of TM).

Duality pairing: $df\left(\frac{\partial}{\partial x_k}\right) = a_k$

$$\| \\ \frac{\partial f}{\partial x_k}$$

$\Downarrow$

Upshot: In coordinates, $\boxed{df = \sum_{k=1}^{n} \left(\frac{\partial f}{\partial x_k}\right) dx_k}$

Cor: $df = 0 \implies f$ is constant on each connected component of M.

$\downarrow$ From this, can define the general exterior derivative.

Def: $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is the unique (1) graded derivation of deg $+1$ s.t. df is as above (2) (i.e., $df(X) = X(f)$) for $f \in \Omega^0(M)$; and $d(df) = 0$ (3) $\forall f \in \Omega^0(M)$.

$$0 \xrightarrow{d} \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0$$

Graded Derivation: Leibniz rule

degree $d = +1$ always

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^{|d||\alpha|} \alpha \wedge (d\beta)$$

⚠

To show this exists, we can <u>explicitly compute it in coordinates</u>: let $\rho = \sum_{k=1}^{n} \xi_k \, dx_k \in \Omega^1(M)$, $\xi_k \in \Omega^0(M)$.

Then

$$d\rho = d\left(\sum_{k=1}^{n} \xi_k \, dx_k\right)$$

$$= \sum_{k=1}^{n} \left((d\xi_k) \wedge dx_k + \overset{(-1)^{|d||\xi_k|} = (-1)^{1\cdot 0} = +1}{\xi_k} \, \underbrace{d(dx_k)}_{=0} \right)$$

$$= \sum_{k=1}^{n} \left(\sum_{j=1}^{n} \frac{\partial \xi_k}{\partial x_j} dx_j \right) \wedge dx_k$$

$$\boxed{d\rho = \sum_{j<k} \left(\frac{\partial \xi_k}{\partial x_j} - \frac{\partial \xi_j}{\partial x_k} \right) dx_j \wedge dx_k}$$

Now, take a general k-form $\rho \in \Omega^k(U)$, where the coords. in U are $(x_1, \ldots, x_n)$. Then we can write

$$\rho = \sum_{i_1 < i_2 < \cdots < i_k} \overset{\in \Omega^0(M)}{\rho_{i_1, \ldots, i_k}} \, dx_{i_1} \wedge \cdots \wedge dx_{i_k}$$

(*)
$$\boxed{d\rho = \sum_{l=1}^{n} \left(\sum_{i_1 < \cdots < i_k} \frac{\partial \rho_{i_1, \ldots, i_k}}{\partial x_l} dx_l \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \right) \in \Omega^{k+1}(M)}$$

In order for the def. of $d : \Omega^\bullet(M) \to \Omega^\bullet(M)$ to be well-defined, we need to check that the proposed operator (*) satisfies ① + ② + ③.

② Clear

③ $$d(df) = d\left(\sum_k \frac{\partial f}{\partial x_k} dx_k\right) = \sum_k \left(d\left(\frac{\partial f}{\partial x_k} dx_k\right)\right)$$

$$= \sum_k \left(d\left(\frac{\partial f}{\partial x_k}\right) \wedge dx_k \overset{(-1)^{1\cdot 0} = (+1)}{+} \left(\frac{\partial f}{\partial x_k}\right) \wedge \overset{=0}{\cancel{d(dx_k)}} \right)$$

$$= \sum_k \sum_\ell \frac{\partial^2 f}{\partial x_k \partial x_\ell} dx_\ell \wedge dx_k$$

$$= \sum_{k<\ell} \left(\cancel{\frac{\partial^2 f}{\partial x_k \partial x_\ell}} - \cancel{\frac{\partial^2 f}{\partial x_\ell \partial x_k}} \right) dx_k \wedge dx_\ell = 0 .$$

① For $f, g \in \Omega^0(M)$,

$$d\left(\left(f\, dx_{i_1} \wedge \dots \wedge dx_{i_p}\right) \wedge \left(g\, dx_{j_1} \wedge \dots \wedge dx_{j_q}\right)\right)$$

$$= d\left(fg\, dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \dots dx_{j_q}\right)$$

<u>use fact</u>: $d(fg) = (df)g + f(dg)$ (old Leibniz rule)

$$= f\, dg \wedge dx_{i_1} \wedge \dots \wedge dx_p \wedge dx_{j_1} \wedge \dots \wedge dx_{j_q}$$
$$+ g\, df \wedge dx_{i_1} \wedge \dots \wedge dx_{i_p} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_q}$$

$$= d(f\, dx_{i_1} \cdots dx_{i_p}) \wedge g\, dx_{j_1} \cdots dx_{j_q}$$
$$+ (-1)^p f\, dx_{i_1} \cdots dx_{i_p} \wedge d(g\, dx_{j_1} \cdots dx_{j_q})$$

Thm: $d^2 = 0$.

Pf: d graded derivation $\Rightarrow d \in \mathrm{Der}^1(\Omega^\bullet, \wedge)$.

graded Derivations are closed under graded commutator (i.e., they form a graded Lie algebra).

$$D_1 \in \mathrm{Der}^{d_1}, \quad D_2 \in \mathrm{Der}^{d_2}$$

$$[D_1, D_2] := D_1 \circ D_2 - (-1)^{d_1 d_2} D_2 \circ D_1 \quad \text{(Graded commutator)}$$

Exercise: $[D_1, D_2] \in \mathrm{Der}^{d_1 + d_2}$.

Pf: Take α, β, $[D_1, D_2](\alpha \wedge \beta) = \underbrace{D_1 \underbrace{D_2(\alpha \wedge \beta)}_{\text{Leibniz}}}_{\text{Leibniz etc...}} - (-1)^{d_1 d_2} D_2 D_1 (\alpha \wedge \beta)$ $\square$

$$[d, d] = d \circ d - (-1)^{1 \cdot 1} d \circ d = 2 d^2$$

$\Rightarrow d^2$ is a graded derivation of degree 2.

$\Rightarrow$ Suffices to check $d^2 = 0$ on forms of type $\boxed{\begin{array}{l} f \in \Omega^0 \\ dg,\ g \in \Omega^0 . \end{array}}$

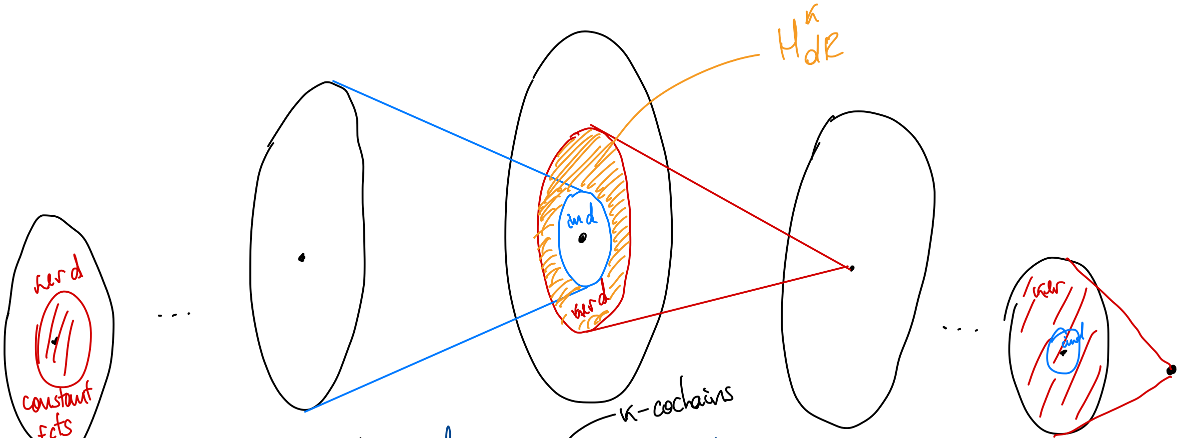
local generators for Ω^0

But $d^2 f = 0$ by def. (*)

$$d^2(dg) = d(d^2 g) = d(0) = 0 .$$

■

CONSEQUENCE OF $d^2 = 0$: im $d \subset$ ker d b/c $d^2 = 0$



k-cochains

$$\Omega^0 \cdots \Omega^{k-1} \xrightarrow{d} \Omega^k \xrightarrow{d} \Omega^{k+1} \cdots \Omega^{\dim M} \to 0$$

$\cup$

$(\ker d) \cap \Omega^k = k$-cocycles

$\cup$

$(\operatorname{im} d) \cap \Omega^k = k$-coboundaries

dual terminology to ∂ boundary for mnfds, and sing. homology

↓

Leads to definition

↓

Def: The de Rham cohomology is the graded algebra given by

$$H^{\bullet}_{dR}(M) := \bigoplus_{k=0}^{\dim M} \underbrace{\left({\ker d \cap \Omega^k} \big/ {\operatorname{im} d \cap \Omega^k} \right)}_{\parallel \atop H^k_{dR}(M)}$$

Topological invariant of spaces.

LECTURE 21

Nov 29th, 2024

__Differential forms__: $\rho \in \Omega^k(M)$ can be written locally as

$$\rho \underset{loc}{=} \sum \rho_{i_1, \ldots, i_k} \, dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

$M \rightsquigarrow$ diff. commutative graded algebra $(\Omega^{\bullet}(M), \wedge, d)$

$M \longrightarrow C^\infty(M, \mathbb{R})$ comm. alg.

$F^*(f) = f \circ F$

__Pull-back__:

$$\begin{tikzcd}
N \arrow[r, mapsto] & C^\infty(N, \mathbb{R}) \arrow[d, "F^*"] \\
M \arrow[u, "F"] \arrow[r, mapsto] & C^\infty(M, \mathbb{R})
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{R} & \mathbb{R} \\
M \arrow[u] \arrow[r, "F"'] & N \arrow[u, "f"'] \\
C^\infty(M,\mathbb{R}) & C^\infty(N,\mathbb{R}) \arrow[l, "F^*"']
\end{tikzcd}$$

$$\begin{tikzcd}
\Lambda^k T^*M \arrow[d] & & \Lambda^k T^*N \arrow[ll] \arrow[d] \\
M \arrow[u, bend left, "F^*\rho"] \arrow[rr, "F"'] & & N \arrow[u, bend right, "\rho \in \Omega^k(N)"']
\end{tikzcd}$$

pullback of ρ

$$TM \xrightarrow{DF} TN$$

$$T^*M \xleftarrow{DF^*} T^*N$$

So, transform mnfds into this algebraic object

$$\begin{tikzcd}
N & & (\Omega^\bullet(N), \wedge, d) \arrow[d, "F^* \text{ morphism of cdga}"] \\
M \arrow[u, "F"] \arrow[r, "\Omega"', "\text{functor}"] & {} & (\Omega(M), \wedge, d)
\end{tikzcd}$$

In local coordinates:

$$M \xrightarrow{F} N \to \mathbb{R} \qquad y \overset{F}{=} y(x)$$

$x_i \qquad y_j \qquad$ basis dy_i for $\Omega^1(N)$

$$F^* dy_i = \sum_{j=1}^{n} \underbrace{\frac{\partial F_i}{\partial x_j}}_{\substack{\text{entries of Jacobian of } F \\ \text{(i.e., of } DF\text{); } F_i = y_i \circ F}} dx_j$$

So, for $\rho \in \Omega^k(N)$,

$$F^*\left(\rho_{j_1\cdots j_k}(y)\, dy_{j_1}\wedge\cdots\wedge dy_{j_k}\right)$$

$$= \left(\rho_{j_1\cdots j_k}\circ F\right)(x)\sum_{i_1}\cdots\sum_{i_k}\left(\frac{\partial y_{j_1}}{\partial x_{i_1}}\cdots\frac{\partial y_{j_k}}{\partial x_{i_k}}\right) dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$

$$dy_{j_1} = \frac{\partial F_{j_1}}{\partial x_1}dx_1+\cdots+\frac{\partial F_{j_1}}{\partial x_n}dx_n$$
$$\vdots$$
$$dy_{j_k} = \frac{\partial F_{j_k}}{\partial x_1}dx_1+\cdots+\frac{\partial F_{j_k}}{\partial x_n}dx_n$$

$$= \sum_{i_1<\cdots<i_k}\begin{pmatrix}k\times k \text{ minor}\\ \text{of } DF\end{pmatrix} dx_{i_1}\wedge\cdots\wedge dx_{i_k}$$

Rmk: If F is a diffeo, DF is $n\times n$, then

top degree

$$F^*(dy_1\wedge\cdots\wedge dy_n) = (\underbrace{\det \overset{= \partial y_i/\partial x_j}{DF}}_{\text{Jacobian determinant}})\, dx_1\wedge\cdots\wedge dx_n .$$

$\Rightarrow$ Integration is well-defined on manifolds ⁉

Recall: Change of Variables Formula

$$\int f(y)\, dy_1\cdots dy_n \overset{y=y(x)}{=} \int f(y(x))\left|\det\frac{\partial y_i}{\partial x_j}\right| dx_1\cdots dx_n .$$

For these 2 formulas to agree, need F to "preserve orientation" i.e., need $\det DF > 0$.

Def: Let M be an n-dim. mnfd.. It is **orientable** when the line bundle $\Lambda^n T^*M \simeq M \times \mathbb{R}$.

An **orientation** is an equivalence class of nonvanishing section $v \in \Omega^n(M)$ where $v' \sim v$ if $v' = e^f v$, for $f \in C^\infty(M, \mathbb{R})$.

positive function

M is oriented when it's endowed w/ orientation $(M, [v])$.

Cor: If M is oriented, we can cover M with charts U_i s.t. the gluing maps φ_{ij} have positive Jac. det..

Pf: For each chart $\varphi_i : U_i \to \mathbb{R}^n$

$\varphi_i^*(dx_1 \wedge \cdots \wedge dx_n)$ — if this is in $[v]$ OK ✓

— if this is in $[-v]$, change the sign of first entry of φ_i:

$$\begin{pmatrix} -1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & \ddots \; 1 \end{pmatrix} \circ \varphi_i .$$

Upshot: For M oriented, have a well-defined INTEGRAL

Def: Let $(M^{[v]})$ be an oriented n-mnfd. The integral is the unique linear map $\int : \Omega^n_c(M) \longrightarrow \mathbb{R}$ s.t. it

← compact supported forms

is compatible with the usual Lebesgue integral in $\mathbb{R}^n$

i.e., if $h : V \underset{\text{open}}{\subseteq} \mathbb{R}^n \xrightarrow{\text{diffeo}} U \underset{\text{open}}{\subseteq} M$ is orientation preserving (i.e., $h^* v \sim [dx_1 \wedge \cdots \wedge dx_n]$) and if $\alpha \in \Omega^n_c(M)$ with support in U, then $\int_M \alpha = \int_{\mathbb{R}^n} h^* \alpha$

* We cannot integrate functions on manifolds. We can only integrate top-degree forms !

Pf: Idea: • Use property to write unique expression for $\int$
• Prove it satisfies claim

If $\alpha \in \Omega^n_c(M)$ choose oriented atlas (U_i, φ_i)
" P.O.U. ψ_i subordinate to (U_i, φ_i)

$$\alpha = \left(\sum_i \psi_i\right)\alpha = \sum_i \psi_i \alpha$$

$$\int \alpha = \sum_i \int \psi_i \alpha = \sum_i \int_{\varphi_i(U_i)} (\varphi_i^{-1})^* \psi_i \alpha .$$

Prove this satisfies the properties.

$\square$

• Each nonvanishing element $v \in \Omega^n(M)$ defines a <u>measure</u> (or <u>volume</u> <u>form</u>) on M

$$\Omega^0(M) \ni f \longmapsto \int_M f v \quad \text{makes sense}$$

e.g.: if a Riemannian metric is chosen on an oriented M $\Rightarrow$ obtain a canonical $v_g \in \Omega^n(M)$ "volume" measure (usually called vol)

STOKES' THEOREM: Let M be an n-mnfd with boundary ∂M. Let $[v]$ be an orientation on M, which induces an orientation on ∂M using outward convention. Let $\rho \in \Omega^{n-1}(M)$, then

Make precise: $\iota_\partial : \partial M \hookrightarrow M$ submnfd

$$\int_M d\rho = \int_{\partial M} \rho = \int_{\partial M} \iota_\partial^* \rho$$

Rmk: This shows the duality between $d \leftrightarrow \partial$.

Cor 1: If $\partial M = \emptyset$, then $\int_M d\rho = 0 \quad \forall \rho \in \Omega^{n-1}(M)$.

Cor 2: If $\partial M = \emptyset$ and v is a volume form, $v \in \Omega^n(M)$, then, using orientation $[v]$, $\int_M v > 0$

$\Rightarrow$ v cannot be in the image of d

$\Rightarrow [v] \neq 0$ in $H^n_{dR}(M)$.

Philosophy: may think of $\Omega^k(M)$ as follows:

Ω^0 evaluate on points (functions on pts)

Ω^1 function on 1-dimension embedded submnfds

e.g., 1-form on M can be pulled back to, say, a circle and we can integrate it there b/c, there, it's a top form $\quad S^1 \hookrightarrow M$

$\vdots$

Ω^k functions on maps from k-dim. domains

$\vdots$

Ω^n

* MAYER - VIÉTORIS: Method for computing $H^\bullet_{dR}(M)$ (local-to-global)

Main Input
- Partition of unity
- Poincaré Lemma: $H^k_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k=0 \\ 0, & \text{else} \end{cases}$

$k=0$, $H^0_{dR}(\mathbb{R}^n)$ is just const. fcts. & $\mathbb{R}^n$ is connec.

Lemma: If $f, g : M \to N$ are homotopic i.e., $\exists H : [0,1] \times M \xrightarrow{C^\infty} N$, $H(0,-) = f$, $H(1,-) = g$, then the pullback morphisms f^*, g^* are chain homotopic i.e., $\exists$ homotopy K between them

$$\begin{tikzcd}
\Omega^{k-1}(M) \arrow[r, "d"] & \Omega^{k}(M) \arrow[r, "d"] & \Omega^{k+1}(M) \\
\Omega^{k-1}(N) \arrow[r, "d"'] \arrow[u, shift left, "g^*"'] \arrow[u, shift right, "f^*"] & \Omega^{k}(N) \arrow[r, "d"'] \arrow[u, shift left, "g^*"'] \arrow[u, shift right, "f^*"] \arrow[ul, "K"] & \Omega^{k+1}(N) \arrow[u, shift left, "g^*"'] \arrow[u, shift right, "f^*"] \arrow[ul, "K"]
\end{tikzcd}$$

i.e., $K : \Omega^k(N) \to \Omega^{k-1}(M)$ (degree $K = -1$) s.t.

$$\boxed{f^* - g^* = Kd + dK} = [K, d]$$

e.g.: if $\alpha \in \Omega^k(N)$, $d\alpha = 0$, then $f^*\alpha, g^*\alpha$ are closed $d(f^*\alpha) = d(g^*\alpha) = 0$. But note:

$$(f^* - g^*) = (Kd + dK)\alpha = d(K\alpha)$$

$\Rightarrow f^* - g^*$ is exact

$\Rightarrow [f^*\alpha] = [g^*\alpha]$ in $H^k_{dR}(M)$

Cor: If f, g are homotopic $\Longrightarrow$ induced maps f^*, g^* on $H^\bullet_{dR}$ agree

Cor: If M, N are homotopic $\Longrightarrow H^\bullet_{dR}(M) \cong H^\bullet_{dR}(N)$.

$$\underset{\text{mnfds}}{M} \overset{\text{functor}}{\longmapsto} \underset{\text{cdga}}{(\Omega^\bullet(M), \wedge, d)} \overset{\text{functor}}{\longmapsto} \underset{\text{cga}}{(H^\bullet_{dR}(M), \wedge)}$$

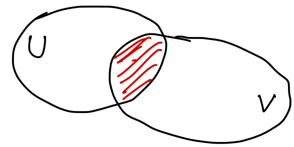
$d = 0$ here...

Mayer - Viétoris Sequence:

Suppose $M = U \cup V$ $\left(\text{e.g.: } S^n = U_N \cup U_S\right)$

open sets

$$\begin{tikzcd}
U & U \cap V \arrow[l, "\partial_U"'] \arrow[d, "\partial_V"] \\
M & V \arrow[l, "\iota_V"]
\arrow[from=1-1, to=2-1, "\iota_U"']
\end{tikzcd}$$



Apply $\Omega^\bullet$

$$\begin{tikzcd}
\Omega^\bullet(U) \arrow[r, "\partial_U^*"] & \Omega^\bullet(U \cap V) \\
\Omega^\bullet(M) \arrow[u, "\iota_U^*"] \arrow[r, "\iota_V^*"'] & \Omega^\bullet(V) \arrow[u, "\partial_V^*"']
\end{tikzcd}$$

$$0 \xrightarrow{0} \overset{1}{\Omega^\bullet(M)} \xrightarrow{(\iota_U^*, \iota_V^*)} \overset{2}{\Omega^\bullet(U) \oplus \Omega^\bullet(V)} \xrightarrow{\partial_V^* - \partial_U^*} \overset{3}{\Omega^\bullet(U \cap V)} \xrightarrow{0} 0$$

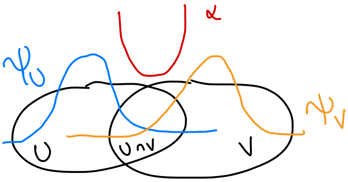
Note: 0. complex $(d^2 = 0)$

1. (ι_U^*, ι_V^*) is injective (ker = im at 1) so we say the seq. is exact at 1

2. $\ker(\partial_V^* - \partial_U^*) = \operatorname{im}(\iota_V^*, \iota_U^*)$ b/c if they agree on $U \cap V$ they glue to a form of $U \cup V$. $\Rightarrow$ Exact at 2

3. Seq. is exact at 3 i.e., $\partial_V^* - \partial_U^*$ is surjective.

Not trivial to show $\longrightarrow$ if $\alpha \in \Omega^k(U \cap V)$ use P.O.U.



$$\psi_U + \psi_V = 1$$

$$\alpha = \underbrace{\psi_V \alpha}_{\substack{\text{well-def.} \\ \text{in } U}} - \underbrace{(-\psi_U \alpha)}_{\text{well-def. in } V}$$

□

Def: Such a 3-term exact sequence

$$0 \longrightarrow (A^\bullet, d_A) \longrightarrow (B^\bullet, d_B) \longrightarrow (C^\bullet, d_C) \longrightarrow 0$$

is called a short exact sequence of complexes.

Upshot: de Rham complexes of M, $U \sqcup V$, $U \cap V$ are related by a short exact sequence.

Thm: Any short exact sequence $0 \to A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{g} C^\bullet \to 0$ induces a long exact sequence on $H^\bullet(A)$, $H^\bullet(B)$, $H^\bullet(C)$:

Mayer-Vietoris sequence

$$\cdots \longrightarrow H^{k-1}(C) \longrightarrow H^k(A) \xrightarrow{f_*} H^k(B) \xrightarrow{g_*} H^k(C) \xrightarrow{\delta} H^{k+1}(A) \longrightarrow H^{k+1}(B) \longrightarrow H^{k+1}(C) \longrightarrow \cdots$$

δ connecting homomorphism