LECTURE 1

Sept 30th, 2025

PHASE SPACE: coordinates that describe "dynamic" variables

Dynamical System $\longleftrightarrow$ law of evolution of the phase space

$$\frac{dx}{dt} = \underbrace{v}(t,x), \qquad x \in \{\text{phase space}\} \subset \mathbb{R}^d.$$

VECTOR FIELD

$$v(t,x) \in \mathbb{R}^d$$

base pt

- Velocity describes the infinitesimal perturbation at x at time t.

- AUTONOMOUS Dynamical System: vector field v is indep. of time
- NON-AUTONOMOUS Dynamical System: vector field varies w/ time.

- Implicitly: $F(x, \dot{x}, \ddot{x}, \ldots, \frac{d^k x}{dt^k}) = 0$

convert to 1^{st} order

$$\left\{\begin{array}{l} x_0 := x \\ x_1 := \frac{dx}{dt} \\ \vdots \\ x_k := \frac{d^{k-1}x}{dt^{k-1}} \end{array}\right. \longrightarrow z := \begin{bmatrix} x_0 \\ \vdots \\ x_{k-1} \end{bmatrix} \longrightarrow \frac{dz}{dt} = v(t,z)$$

Note: if $x \in \mathbb{R}^d$, then $z \in \mathbb{R}^{kd}$.

WLOG: we look at 1^{st} order ODEs $\frac{dx}{dt} = v(t,x)$, $x \in \mathbb{R}^d$

Flow $\equiv$ continuous-time dynamical systems.

ODEs

- Initial Value Problem

single sol.ⁿ $\left[\dfrac{dx}{dt} = v(t,x), \quad x(0) = a \right.$

- Numerical time integration
- Existence and uniqueness
- Regularity

Dynamics

- All possible solutions to an ODE
- Geometry of phase space?
- Volume of initial conditions
- Ergodic theory: relation of long-time evolution of single solution and volumes in phase space
- Perturbations

SOLUTION OF DYNAMICAL SYSTEMS $\leftrightsquigarrow$ TRAJECTORY (or ORBIT) OF THE DYNAMICS

IDEA: Dynamics deals with the qualitative understanding of the orbit structure.

flow

$$x(t) = \varphi(t, x(0))$$

Semigroup Property: $\varphi(t+s, x(0)) = \varphi\big(t, \varphi(s, x(0))\big)$

HOLDS FOR AUTONOMOUS SYSTEMS

i.e., $\varphi^t \circ \varphi^s = \varphi^{t+s} = \varphi^s \circ \varphi^t$.

DISCRETE-TIME DYNAMICS: $\dfrac{dx}{dt} = v(t,x)$ and look at discrete times $x(0), x(\tau), x(2\tau), \dots$ for $\tau \in \mathbb{R}$; i.e., $x(\tau) = \varphi^{\tau}(x(0))$

$$\Downarrow$$

$x_{t+1} = F_t\, x_0,\ t \in \mathbb{Z}$

MAP

AUTONOMOUS DYNAMICS: $x_{t+1} = F(x_t)$,

where $F: \mathbb{R}^d \to \mathbb{R}^d$ is called a self-map

$$x_t = F^t(x_0) = \underbrace{(F \circ \cdots \circ F)}_{t \text{ times}}(x_0)$$

EXAMPLES: • Low-dimensional dynamics (dim = 1, 2, 3, 4)

• Climate dynamics: – very high-dimensional system ($x \in \mathbb{R}^d$, $d = O(10^{11})$)

• Chaos: sensitivity to initial conditions.

• Lorenz Equations ⇝ came from modeling atmospheric convection.

—— // ——

LINEAR DYNAMICS $\frac{dx}{dt} = \underbrace{A}_{\text{linear operator}} x$, $x \in \mathbb{R}^d$, $A \in \mathbb{R}^{d \times d}$.

Solution: $x(t) = e^{tA} x(0)$.

• CASE 1: A is diagonalizable; i.e., $A = P \Lambda P^{-1}$

columns are eigenvectors of A

$\Lambda = \mathrm{diag}(\lambda_1(A), \dots, \lambda_n(A))$

LECTURE 2

LINEAR HOMOGENEOUS ODES

Oct 2nd, 2025

LINEAR HOMOGENEOUS ODES

$$(*) \begin{cases} \dot{x} = A(t)\, x, & x \in \mathbb{R}^n \\ & A(t) \in \mathbb{R}^{n \times n} \text{ continuous in time.} \end{cases}$$

$x \equiv 0$ solves $(*)$.

SUPERPOSITION PRINCIPLE: if $x_1(t)$ and $x_2(t)$ are solutions to (*), then $x(t) := c_1 x_1(t) + c_2 x_2(t)$ is also a solution $\forall c_1, c_2 \in \mathbb{R}$.

NOTE: If we can find n lin. indep. solutions $x_1(t), \dots, x_n(t)$ then the general solution to (*) is

$$x(t) = c_1 x_1(t) + \dots + c_n x_n(t).$$

AUTONOMOUS PROBLEM:

$$\begin{cases} \dot{x} = Ax, & x \in \mathbb{R}^n \text{ and } A \in \mathbb{R}^{n \times n} \\ x(0) = x_0 \in \mathbb{R}^n \end{cases}$$

• APPROACH 1: If A has n distinct eigenvalues $\lambda_1, \dots, \lambda_n$ and n lin. indep. eigenvectors $v_1, \dots, v_n$.

satisfy the characteristic eq. of A

$$0 \overset{!}{=} \det(A - \lambda \mathbb{1})$$

General Solution:

$$x(t) = \sum_{j=1}^{n} c_j e^{\lambda_j t} v_j = \begin{bmatrix} | & & | \\ e^{\lambda_1 t} v_1 & \cdots & e^{\lambda_n t} v_n \\ | & & | \end{bmatrix} \underbrace{\begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}}_{\text{determined by initial condition}}$$

$$x(0) = x_0 = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}^{-1} x_0$$

invertible b/c eigenvectors v_j are lin. indep

NOTATION: $$\Phi(t) := \begin{bmatrix} | & & | \\ e^{\lambda_1 t} v_1 & \cdots & e^{\lambda_n t} v_n \\ | & & | \end{bmatrix}$$ (FUNDAMENTAL MATRIX SOLUTION)

This solves $\frac{d}{dt}\Phi(t) = A\,\Phi(t)$. CONVENTION: $\Phi(0) = \mathbb{1}$

UPSHOT: $\dot{x} = Ax$, $x(0) = x_0$ is solved by $x = \Phi(t)\,x_0$.

ASIDE:
- If some λ_j is complex, $\lambda_j = \alpha + i\beta$, then v_j is also complex. Note that $\bar{\lambda}_j = \alpha - i\beta$ is also an eigenvalue with eigenvector $\bar{v}$.

  In this case, the real solution involving λ is

  $$x(t) = c\,e^{\lambda_j t}\,v_j + \bar{c}\,e^{\bar{\lambda}_j t}\,\bar{v}_j, \quad c \in \mathbb{C}.$$

- If A is defective; i.e., doesn't have n lin. indep. eigenvectors (repeated eigenvalues). Look at generalized eigenvectors.

  Ex: λ is an eigenvector w/ multiplicity 2 and its eigenvector v spans a 1-dim subspace $\leftarrow (A - \lambda\mathbb{1})v = 0$.

  Generalized eigenvector w $\leftarrow$ $\begin{cases} (A - \lambda\mathbb{1})^2 w = 0 \\ \& \ (A - \lambda\mathbb{1})\,w = 0 \end{cases}$

  One possibility is $(A - \lambda\mathbb{1})\,w = v \iff Aw = \lambda w + v$.

  Solution: $x_1(t) = e^{\lambda t}v$ and $x_2(t) = e^{\lambda t}(tv + w)$ are lin. indep. solutions.

- APPROACH 2: Solution to $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases}$ is $x(t) = e^{tA}x_0$.

Recall: $e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!}$.

Check: $x(0) = e^{0A} x_0 = x_0$

$$\dot{x} = \lim_{h\to 0} \frac{x(t+h) - x(t)}{h} = \lim_{h\to 0} \frac{e^{(t+h)A}x_0 - e^{tA}x_0}{h} = \lim_{h\to 0} \frac{e^{hA} - Id}{h} e^{tA}x_0$$

$$= \underbrace{\left(\lim_{h\to 0} \frac{e^{hA} - Id}{h}\right)}_{=A} x(t) = Ax.$$

since autonomous, they commute. False if non-autonomous.

• CASE 1: Diagonalizable $A \rightsquigarrow A = P\Lambda P^{-1}$, where

$\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$, $P = \begin{bmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{bmatrix}$

Then $x(t) = e^{tA} x_0 = P \underbrace{e^{t\Lambda}} P^{-1} x_0$

$e^{t\Lambda} = \mathrm{diag}(e^{t\lambda_1}, \dots, e^{t\lambda_n})$

• CASE 2: Defective A. Recall the Jordan decomposition of a complex matrix $A = S + N$ where $\underbrace{[S, N] = 0}$.

$S = PDP^{-1}$ semisimple (diagonalizable)

nilpotent

S and N commute

Solution: $x(t) = e^{tA} x_0 = e^{t(S+N)} x_0 = e^{tS} e^{tN} x_0$

$$= \underbrace{P e^{tD} P^{-1}}_{S \text{ diagonalizable}} \underbrace{\left(\sum_{k=0}^{n-1} \frac{(tN)^k}{k!} \right)}_{N \text{ nilpotent w/ nilpotency } n} x_0 .$$

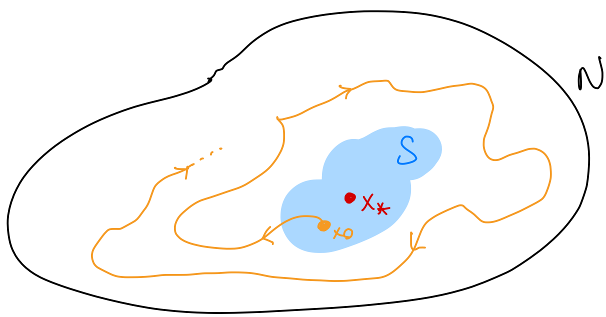
LECTURE 3

Oct 7th, 2025

Motivation: Q2 of HW04 asks about the stability of solution $x=0$ to $\dot{x} = A(t)x$.

DEF: We say a pt. x_* is an equilibrium for the system $\dot{x} = V(x)$ iff $V(x_*) = 0$.

DEF: We say an equilibrium pt. x_* (e.g., $x_* = 0$) is Lyapunov stable iff for every neighborhood $N \ni x_*$, there exists a neighborhood $S \subset N$ such that, for all $x(0) \in S$, we have $x(t) \in N$ $\forall t \geq 0$ (i.e., we never escape).



- x_* is UNSTABLE $\iff$ x_* is not Lyapunov stable.

- x_* is ASYMPTOTICALLY STABLE $\iff$ x_* is Lyapunov stable and $\lim_{t\to\infty} x(t) = x_*$ for $x(t) \in N$

REMARK: For $\dot{x} = Ax$, $x = 0$ is asymptotically stable if all eigenvalues $\{\lambda_j\}$ of A have $\mathrm{Re}(\lambda) < 0$. If $\mathrm{Re}(\lambda) > 0$ for any eigenvalue, then $x = 0$ is unstable

FLOQUET THEORY & MONODROMY MATRIX

Consider $\begin{cases} \dot{x} = A(t)\,x & \text{s.t.} \quad A(t+T) = A(t) \quad \forall t \geq 0. \\ x(0) = x_0 \end{cases}$

The fundamental matrix solution $\Phi(t)$ is s.t.

$$\frac{d\Phi}{dt} = A(t)\,\Phi \quad \text{and} \quad \Phi(0) = \mathbb{1}.$$

($\Phi \approx$ propagator)

Thus, we have $x(t) = \Phi(t)\,x(0)$.

DEF: The monodromy matrix is defined as $\underbrace{M := \Phi(T)}$

! Very useful to know the eigenvals. of M

Say M has eigenvalues $\{\mu_j\}$ called FLOQUET MULTIPLIERS.

REMARK: $x_1 = \Phi(T)\,x_0 = M x_0$

$\vdots$

$$\boxed{x(nT) = M^n x_0}$$

Strobe light analogy

if $|\mu_j| > 1$, then $|\mu_j|^n \xrightarrow{n\to\infty} \infty$ but if $|\mu_j| < 1$ then $|\mu_j|^n \xrightarrow{n\to\infty} 0$

* If $|\mu_j| < 1 \ \forall j$, then $x=0$ is asymptotically stable
* If any μ_j has $|\mu_j| > 1$, then $x = 0$ is unstable

EXISTENCE & UNIQUENESS FOR ODEs (Ch 3)

Consider the (IVP) $\begin{cases} \dot{x} = V(x), & V: \mathbb{R}^n \to \mathbb{R}^n \\ x(0) = x_0, & x_0 \in \mathbb{R}^n \end{cases}$.

no uniqueness...

THM: (Local Picard) If $V \in C^1(\mathbb{R}^n, \mathbb{R}^n)$ then there is an interval of time containing zero $(-\varepsilon, \varepsilon)$ such that a solution to (IVP) exists and is given by

$$\dot{x}(t) = x_0 + \int_0^t V(x(s))\, ds, \quad t \in (-\varepsilon, \varepsilon).$$

WARNING: This doesn't imply global existence of solution! Solution can diverge in finite time

E.g.: $\begin{cases} \dot{x} = x^2 \\ x(0) = 1 \end{cases}$ $\quad \frac{dx}{dt} = x^2 \rightsquigarrow \frac{dx}{x^2} = dt \rightsquigarrow \int_{x(0)}^{x(t)} \frac{dx}{x^2} = \int_0^t dt'$

$\rightsquigarrow \left[-\frac{1}{x} \right]_1^{x(t)} = t \rightsquigarrow x(t) = \frac{1}{1-t}$, which as $t \to 1$, gives $x \to +\infty$.

$\Rightarrow$ Solution only exists for $t < 1$.

THM: Let $a, b > 0$. If $V: B_b(x_0) \to \mathbb{R}^n$ is K-Lipschitz, then (IVP) has a unique solution $x(t)$ for

$\{x \in \mathbb{R}^n : \|x - x_0\| < b\}$

$$t \in J := [-a, a] := \left[-\frac{b}{M}, \frac{b}{M} \right], \quad M := \max_{x \in B_b(x_0)} |V(x)|$$

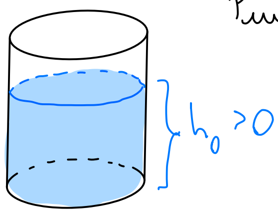
Ensures $x(t)$ doesn't escape $B_b(x_0)$ in time $t \in J$

"max speed"

RECALL: $V : B_b(x_0) \to \mathbb{R}^n$ is K-Lipschitz iff

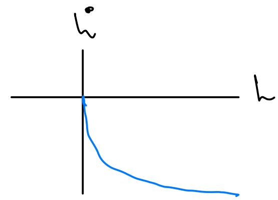
$$\| V(x) - V(y) \| \leq K \|x - y\| \quad \forall x, y \in B_b(x_0).$$

EXAMPLE: (Non-uniqueness in physics)



Punch hole in container at $t=0$. From fluid mechanics

$$\dot{h} = \begin{cases} -\sqrt{h}, & h > 0 \\ 0, & h = 0 \end{cases}$$



❗ NOT LIPSCHITZ at zero

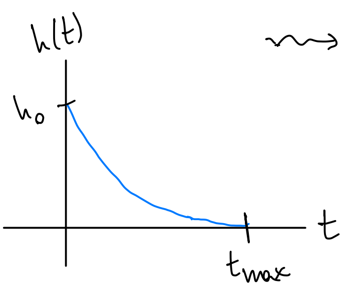
Now, solve for $h_0 > 0$ mindlessly:

$$\frac{dh}{dt} = -\sqrt{h} \rightsquigarrow -\frac{dh}{\sqrt{h}} = dt \rightsquigarrow -\int_{h(0)}^{h(t)} \frac{dh}{\sqrt{h}} = \int_0^t ds$$

$$\rightsquigarrow -2\sqrt{h(t)} + 2\sqrt{h_0} = t, \quad t < 2\sqrt{h_0} =: t_{max}$$

$$\rightsquigarrow h(t) = \begin{cases} \left(\sqrt{h_0} - t/2\right)^2, & t < 2\sqrt{h_0} \\ 0, & t \geq 2\sqrt{h_0} \end{cases}.$$

how long it takes to fully drain



❗ But if you choose your initial condition that gives $h_0 = 0$ (where it's not Lipschitz), there are ∞-many solutions.

LECTURE 4

Oct 9th, 2025

! **STUDY FOR QUALS.** (proof of existence & uniqueness w/ fixed pts & iterations and Grönwall's inequality)

* TOWARDS THE PROOF OF PICARD'S THM

Rewrite (IVP) as $x(t) = x_0 + \int_0^t V(x(s))\,ds$] Equivalent to (IVP)

and the Contraction Mapping Thm.

DEF: (PICARD'S MAP) $T(u(t)) := x_0 + \int_0^t V(u(s))\,ds$

! REMARK: A fixed pt. of T (i.e., x s.t. $Tx = x$) is a solution of (IVP) b/c

$$x(t) = T(x(t)) = x_0 + \int_0^t V(x(s))\,ds.$$

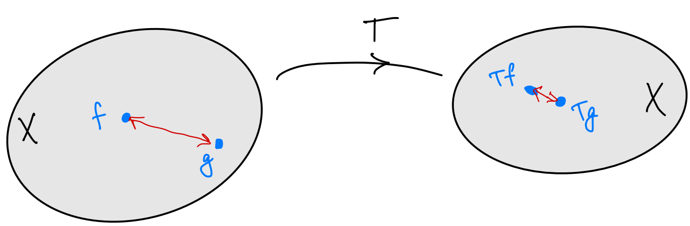
PICARD ITERATION: $u_0(t) \xrightarrow{T} u_1(t) \xrightarrow{T} u_2(t) \xrightarrow{T} \cdots$

and hope it converges.

CONTRACTION MAPS

Let X be a Banach space. A contraction map T looks like:

$f, g \in X$



$Tf, Tg \in X$

THM: Let X be a Banach space ← i.e., complete; i.e., Cauchy seqs converge and let $T: X \to X$. If T is a contraction (i.e., $\exists c \in (0,1)$ such that

d = metric on X → $d(T(f), T(g)) \leq c\, d(f,g) \quad \forall f,g \in X$).

Then, T has a unique fixed point $x_* \in X$; i.e., $Tx_* = x_*$.

PICARD'S MAP: Take $X = C^0(J, B_b(x_0))$ with

↓ Time interval in Picard's Thm

$$d(u(t), v(t)) := \sup_{t \in J} |u(t) - v(t)|.$$

Set $T: X \to X$ given by

(Note: $T \in Aut(X)$ b/c $\int \in Aut$ (cont. fcts.).)

$$T(u(t)) = x_0 + \int_0^t V(u(s))\, ds.$$

CLAIM 1: $T: X \to X$ is a contraction.

PF: Let $u_1, u_2 \in X = C^0(J, B_b(x_0))$. Then

$$|Tu_1 - Tu_2| = \left| x_0 + \int_0^t V(u_1(s))\, ds - x_0 - \int_0^t V(u_2(s))\, ds \right|$$

$$= \left| \int_0^t [V(u_1(s)) - V(u_2(s))]\, ds \right|$$

$$\leq \int_0^t |V(u_1(s)) - V(u_2(s))|\, ds$$

V is K-Lipschitz

$$\leq \int_0^t K \, |u_1(s) - u_2(s)| \, ds$$

Bound the integral

$$\leq K \cdot a \cdot \sup_{t \in J} |u_1(t) - u_2(t)|$$

$$= \underbrace{Ka}_{} \, d(u_1, u_2) .$$

Take a s.t. $2Ka < 1 \iff a < 1/K$

■

Pf: (PICARD) By Claim 1 above and Banach's fixed pt. thm, we're done. ■

THM: (Global Existence of Solutions) If $V: \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz and bounded, then the solutions to

$$(\text{IVP}) \begin{cases} \dot{x} = V(x) \\ x(0) = x_0 \end{cases}$$

exist for all $t \in \mathbb{R}$.

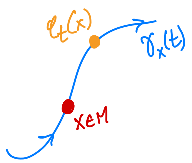
FLOW DYNAMICS (Ch 4)

DEF: A complete (exists $\forall t\in\mathbb{R}$) flow $\varphi_t(x)$ is a 1-parameter (i.e., $t\in\mathbb{R}$) mapping $\varphi: \mathbb{R}\times M \to M$ with the following properties

(i) $\varphi_0(x) = x \quad \forall x\in M$; i.e., $\varphi_0 = id_M$

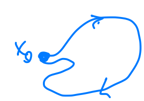
(Groupoid Property) (ii) $\varphi_t\circ\varphi_s = \varphi_{t+s} = \varphi_s\circ\varphi_t \quad \Rightarrow \varphi_t$ is invertible: $\varphi_t^{-1} = \varphi_{-t}$.

REMARK: $\varphi_t(x)$ defines a curve $\gamma_x(t)$ in M as t varies; i.e., $\gamma_x(t) = \left(t\longmapsto\varphi_t(x)\right)$. This $\gamma_x(t)$ is called an <u>ORBIT</u> or <u>TRAJECTORY</u> through x.



EXAMPLE: (1) $\varphi_t(x_0) = x_0 \quad \forall t\in\mathbb{R} \Rightarrow x_0$ is a fixed pt.

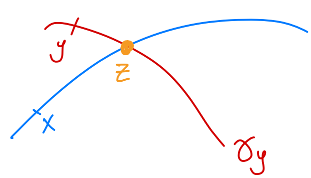
(2) If $\exists\, T>0$ s.t. $\varphi_T(x_0) = x_0$ and $\varphi_t(x_0)\neq x_0 \quad \forall t\in(0,T)$, then we have a closed orbit w/ period T



<u>Obs</u>: any pt. in this closed orbit is also T-periodic b/c $\forall\, t_*\in(0,T)$

$$\varphi_T\left(\varphi_{t_*}(x_0)\right) = \varphi_{t_*}\left(\varphi_T(x_0)\right) = \varphi_{t_*}(x_0).$$

(3) The groupoid property implies that two distinct orbits <u>cannot</u> intersect.



<u>Pf</u>: Assume, by contradiction, that they do cross at z. Then

$\varphi_s(y) = z = \varphi_t(x)$ for some t, s. But then

$$\varphi_r(z) = \varphi_{r+t}(x) = \varphi_{r+s}(y) \quad \longleftrightarrow // \longleftrightarrow$$

we assumed $\gamma_x \neq \gamma_y$. $\square$

REMARK: (Connection w/ ODEs) Flows are differentiable mappings (more precisely, local diffeomorphisms) so we can associate a vector field $V \in \mathfrak{X}(M)$ with them via:

$$V(x) = \lim_{\varepsilon \to 0} \frac{\varphi_t(x) - X}{t} = \frac{d}{dt}\left(\varphi_t(x)\right)\Big|_{t=0}$$

Thus, $\varphi_t(x)$ solves the following (IVP)

$$\frac{d}{dt}\left(\varphi_t(x_0)\right) = V\left(\varphi_t(x_0)\right), \quad \varphi_0(x_0) = x_0 .$$

LECTURE 5

Oct 14^{th}, 2025

STABILITY OF LINEAR SYSTEMS

Recall: $T : (X, d) \to (X, d)$ is a contraction if $\exists q \in (0,1)$ s.t.

$$d(T(x), T(y)) \leq q \, d(x,y) \quad \forall x, y \in X .$$

NOTE:
$$d(T^k(x), T^k(y)) \leq q^k d(x,y) .$$

Since $0 < q < 1$, as $k \to \infty$, the orbits converge exponentially fast to a fixed pt. of T.

$\hookrightarrow \{T^k(x)\}_{k \in \mathbb{N}}$ is an orbit of x.

$\boxed{\text{THM: (Banach Contraction Fixed Pt.) Any contraction in a Banach space has a unique fixed point.}}$

PF: Since (X,d) is Banach (i.e., complete; i.e., every Cauchy sequence converges), it suffices to show that orbits of any $x \in X$ are Cauchy sequences.

Indeed, let $m \geq n$,

$$d\left(T^m(x), T^n(x)\right) \leq \sum_{i=n}^{m-1} d\left(T^{i+1}(x), T^i(x)\right)$$

Triangle Inequality

T is a q-contraction

$$\leq \sum_{i=n}^{m-1} q^i d\left(T(x), x\right)$$

$$\leq \underbrace{\frac{q^n}{1-q}}_{\substack{\text{can make this} \\ \text{arbitrarily small} \\ \text{b/c } q \in (0,1)}} d\left(T(x), x\right)$$

$\Rightarrow \{T^n(x)\}_{n \in \mathbb{N}}$ is Cauchy, hence converges to the fixed point x_* of T.

Take $m \to +\infty$, then

$$d(x_*, T^n(x)) \leq q^n C, \quad C \in \mathbb{R} \text{ constant}$$

The orbit of any x approaches x_* exponentially fast!

■

REMARK: If $F: X \to X$ is s.t. $\|\nabla F(x)\| < 1$ and $\sup_{x \in X} \|\nabla F(x)\| < 1$, then, by MVT, F is a contraction

$$\|Fx - Fy\| \leq \|\nabla f(\xi)\| \, \|x - y\|.$$

REMARK: Take $A \in \mathbb{R}^{n \times n}$, $x \in \mathbb{R}^n$, $x_{t+1} = A x_t$. Then

$$F_A(x) := Ax$$

is a contraction if $|\lambda_j| < 1$ $\forall \lambda_j \in \mathrm{Spec}(A)$. Then the fixed pt. is $x = 0$.

REMARK: For $\dot{x} = Ax$, write the eigendecomposition of A (either diagonalize or Jordan form)

$$x(t) = e^{tA} x_0 = P^{-1} e^{t\Lambda} P^{-1} x_0$$

Thus, A is a contraction if all $\lambda_j \in \mathrm{Spec}(A)$ are s.t. $\mathrm{Re}(\lambda_j) < 0$.

LINEAR STABILITY ANALYSIS

Idea: study orbits infinitesimally close to fixed pts.

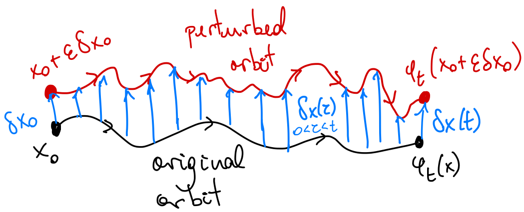
Consider problems of the form

$$\frac{dx}{dt} = V(x) \quad \longleftrightarrow \quad \frac{d\varphi_t(x)}{dt} = V(\varphi_t(x)), \quad t \in \text{interval of existence}$$

x is initial condition; i.e., $\varphi_0(x) = x$.

Let δx_0 be a perturbation to initial condition.

Then



$$\delta x(t) = \lim_{\varepsilon \to 0} \frac{\varphi_t(x_0 + \varepsilon\delta x_0) - \varphi_t(x_0)}{\varepsilon}$$

Now,

$$\frac{d\,\delta x(t)}{dt} = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\left(\frac{d\varphi_t}{dt}(x_0+\varepsilon\delta x_0) - \frac{d\varphi_t(x_0)}{dt}\right)$$

$$\overset{(\text{ODE})}{=} \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(V(\varphi_t(x_0+\varepsilon\delta x_0)) - V(\varphi_t(x_0))\Big)$$

Assume $V \in C^1$

$\Downarrow$

we can Taylor expand:

$\varphi_t(x_0+\varepsilon\delta x_0) = \varphi_t(x_0) + \varepsilon\,\delta x(t) + O(\varepsilon^2)$

$$= \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\Big(V(\varphi_t(x_0)) + \varepsilon\,\nabla V(\varphi_t(x_0))\,\delta x(t) - V(\varphi_t(x_0))\Big)$$

Upshot: FUNDAMENTAL EQUATION OF LINEAR STABILITY ANALYSIS

$$\boxed{\frac{d\,\delta x(t)}{dt} = \nabla V(\varphi_t(x_0))\,\delta x(t)}$$

REMARKS: (1) The orbit $\varphi_t(x_0)$ is fixed

(2) The ODE that describes how the infinitesimal perturbation $\delta x(t)$ changes over time is LINEAR. That is,

$$\frac{dy}{dt} = Y(t,y).$$

(3) The linear stability ODE is not autonomous because the Jacobian $\nabla V(\varphi_t(x_0)) \in \mathbb{R}^{n\times n}$ is changing with time.

(4) If x_0 is a FIXED POINT, then the ODE is autonomous because x_0 is fixed !

* LINEAR STABILITY ANALYSIS AROUND FIXED POINTS

Fixed point $\leftrightarrow$ x_* s.t. $x_* = \varphi_t(x_*)$ $\forall t$.

In this case, the perturbation ODE from above is

$$\frac{d\,\delta x(t)}{dt} = \nabla V(x_*)\,\delta x(t)$$

Upshot: If $Re(\lambda) < 0$ $\forall \lambda \in Spec(\nabla V(x_*))$, then

$$\delta x(t) \xrightarrow{t\to+\infty} 0 \quad \Rightarrow \boxed{x_* \text{ is stable}}$$

since $\delta x(t) = e^{t\nabla V(x_*)}\,\delta x_0$

⚠ WARNING: This only holds locally around x_* where Taylor expansion is valid.

LECTURE 6

Oct 16th 2025

INFINITESIMAL PERTURBATIONS: Recall $u(t) \in \mathbb{R}^n$ is the infinitesimal perturbation to $\varphi_t(x)$ satisfies

$$\frac{du(t)}{dt} = \nabla V(\varphi_t(x))\, u(t)$$

Linear

Flow satisfies $\dfrac{d\varphi_t(x)}{dt} = V(\varphi_t(x))$, $\varphi_t(x) \in \mathbb{R}^n$.

NOTE: $\{\varphi_t(x)\}_{t \in \mathbb{R}_{>0}}$ is an ORBIT of x.

SPECIALIZED EQUATION TO FIXED POINTS: if $\varphi_t(x_*) = x_*$ $\forall t \geq 0$, we say x_* is a fixed point of V. So, perturbations of fixed points satisfy the following AUTONOMOUS linear system

$$\frac{du(t)}{dt} = \nabla V(\varphi_t(x_*))\, u(t) = \nabla V(x_*)\, u(t)$$

$$\text{i.e.,} \quad u(t) = e^{t \nabla V(x_*)}\, u(0),$$

$\nabla V(x_*) \equiv$ Jacobian of matrix $V: \mathbb{R}^n \to \mathbb{R}^n$ at fixed point x_*.

Jacobian

Remark: • if, $\forall \lambda \in \mathrm{Spec}(\nabla V(x_*))$, we have $\mathrm{Re}(\lambda) < 0$, then x_* is asymptotically stable since

$$\|u(t)\| \leq C e^{-\alpha t} \|u(0)\|$$

for some $\alpha, C > 0$.

• if at least one $\lambda \in \mathrm{Spec}(\nabla V(x_*))$ have $\mathrm{Re}(\lambda) > 0$, then, for almost every $u(0)$,

Jacobian

$$\|u(t)\| \xrightarrow{t \to \infty} \infty .$$

$$u(t) = e^{t \nabla V(x_*)} u(0) = P^{-1} e^{t \Lambda} P u(0)$$

DEF: A fixed point x_* is called <u>hyperbolic fixed point</u> if $\mathrm{Re}(\lambda) \neq 0 \quad \forall \lambda \in \mathrm{Spec}(\nabla V(x_*))$.

Jacobian

RECALL: (Aside on maps) $x_{t+1} = F(x_t)$ (map)

$x_{t+1} = DF(x_t)$ (linearization)

$$u_t = (DF(x_*))^t u_0$$

• $\mathrm{Re}(\lambda) > 1$ even for one eigenvalue λ of $DF(x_*)$

$$\Downarrow$$

<u>almost all</u> solutions u_t blow up!

If v_1 is eigenvec. w/ eigenval. λ_1, $\mathrm{Re}(\lambda_1) < 1$, then $v_1 \to 0$, but choosing v_1 w/ $P = 0$.

REMARK: The behavior of the nonlinear system is described "locally" by its linearization around fixed point.

* ASYMPTOTIC STABILITY: We say that a fixed point x_* $(\varphi_t(x_*) = x_*)$ is asymptotically stable iff there exists a nbhd U of x_* s.t. $\varphi_t(x) \xrightarrow{t\to\infty} x_*$ $\forall x \in U$.

* LINEAR STABILITY: $\mathrm{Re}(\lambda) < 0$ $\forall \lambda \in \mathrm{Spec}(DV(x_*))$

THM: If $V \in C^1$ and $\exists x_*$ s.t. $V(x_*) = 0$, then Linear stability $\Rightarrow$ Asymptotic stability.

Pf: Given $\dfrac{d\varphi_t(x)}{dt} = V(\varphi_t(x))$. Say x_* is a fixed pt. of φ_t; i.e., $\varphi_t(x_*) = x_*$ $\forall t$. Let $y(t) := \varphi_t(x) - x_*$.

Note
$$\frac{dy}{dt} = V(\varphi_t(x)) = V(y(t) + x_*)$$

Taylor Expansion

b/c x_* is fixed pt

Nonlinear terms in y.

$$= \cancel{V(x_*)}^{=0} + DV(x_*)\,y(t) + \overbrace{\underbrace{N(y(t))}}$$

Assumption, take $N(y(t)) = V(y(t) + x_*) - DV(x_*)\,y(t)$

$$\Rightarrow \frac{dy}{dt} = DV(x_*)\,y(t) + N(y(t))$$

If $V \in C^1$, then $N(y(t)) = o(y(t))$

$$\frac{N(y(t))}{y(t)} \leq c\, y(t)$$

!! IMPORTANT !

<u>DUHAMEL's FORMULA</u>: The solution to an equation of the form

$$\frac{dy}{dt} = A y(t) + N(y(t))$$

is

$$y(t) = e^{tA} y(0) + \int_0^t e^{(t-\tau)A} N(y(\tau))\, d\tau .$$

By assumption of linear stability, $\mathrm{Re}(\lambda) < 0 \quad \forall \lambda \in \mathrm{Spec}(\underbrace{DV(x_*)}_{\overset{!!}{A}})$; i.e., $\|e^{tA} y(0)\| \leq C e^{-\alpha t} \|y(0)\|$. So, taking norms in Duhamel's formula,

$$\|y(t)\| = \left\| e^{tA} y(0) + \int_0^t e^{(t-\tau)A} N(y(\tau))\, d\tau \right\|$$

$$\leq \|e^{tA} y(0)\| + \left\| \int_0^t e^{(t-\tau)A} N(y(\tau))\, d\tau \right\|$$

$$\leq C e^{-\alpha t} \|y(0)\| + \int_0^t \|e^{(t-\tau)A} N(y(\tau))\|\, d\tau$$

$$\leq C e^{-\alpha t} \|y(0)\| + \int_0^t e^{(t-\tau)A} \|N(y(\tau))\|\, d\tau$$

Now, by differentiability of V, and continuity of $t \mapsto y(t)$ $\forall \kappa > 0$, $\exists \delta > 0$ s.t. ← means κ and δ don't depend on τ

$$\|N(y(\tau))\| \le \kappa \|y(\tau)\| \text{ whenever } \|y(\tau)\| < \delta.$$

Then

$$\|y(t)\| \overset{(*)}{\le} C e^{-\alpha t} \|y(0)\| + C\kappa \int_0^t e^{-\alpha(t-\tau)} \|y(\tau)\| \, d\tau$$

! IMPORTANT: MEMORIZE FOR QUAL !

GRÖNWALL'S INEQUALITY !

If $\beta \in C^1([0,T])$ and

$$\beta(t) \le C + \int_0^t \gamma(\tau)\beta(\tau)\, d\tau, \quad \text{where } \gamma(\tau) > 0$$

then

$$\beta(t) \le C e^{\int_0^t \gamma(\tau)\, d\tau} \quad \forall t \in [0,T].$$

EXERCISE: PROVE THIS !

So, multiplying $e^{\alpha t}$ on both sides of $(*)$, we get

$$\underbrace{e^{\alpha t}\|y(t)\|}_{=\beta(t)} \le \underbrace{C\|y(0)\|}_{\text{constant}} + \int_0^t \underbrace{C\kappa}_{\gamma(\tau)} \underbrace{e^{\alpha\tau}\|y(\tau)\|}_{\beta(\tau)} \, d\tau.$$

So, by Grönwall's inequality,

$$e^{\alpha t}\|y(t)\| \le C e^{C\kappa t} \|y(0)\|$$

SIGN ERROR SMWH

$$\|y(t)\| \leq C e^{t(CK-\alpha)} \|y(0)\|$$

$\Rightarrow \quad CK - \alpha < 0$ if $K < \alpha/C$. Then $\|y(t)\| \xrightarrow{t\to\infty} 0$.

$\|\varphi_t(x) - x_*\| \to 0$ as $t \to \infty$

$\Rightarrow \quad \alpha < -\mathrm{Re}(\lambda) \quad \forall \lambda \in \mathrm{Spec}(A)$. ■

THM: (HARTMAN-GROBMAN) Let x_* be a hyperbolic equilibrium of a C^1 vector field V with flows $\varphi_t(x)$. Then, $\exists$ nhbh $N \ni x_*$ s.t. φ is topologically conjugate to its linearization on N.

LECTURE 7

Oct 21st 2025

STABILITY:

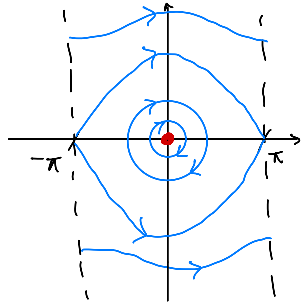
(1) x_* is Lyapunov stable $\iff$ $\forall$ nhbh $N \ni x_*$, $\exists M \subset N$ s.t. $\varphi_t(x) \in N$ $\forall t \geq 0$, $\forall x \in M$.

(2) x_* is asymptotically stable $\iff$ x_* is Lyapunov stable and $\varphi_t(x) \xrightarrow{t\to\infty} x_*$ $\forall x$ close enough to x_*.

EXAMPLES: 1. (IDEAL PENDULUM) $\ddot{\theta} = -\sin\theta$

Convert it into a 2×2 system: $\begin{cases} \dot{\theta} = y \\ \dot{y} = -\sin\theta \end{cases}$

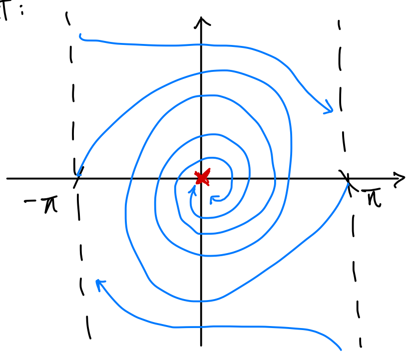
Phase Portrait:



$\theta_* = 0$ is Lyapunov stable (but not asymptotic)

2. (DAMPED PENDULUM) $\ddot{\theta} = -\sin\theta - \beta\dot{\theta}$ $(\beta > 0)$ $\Rightarrow \begin{cases} \dot{\theta} = y \\ \dot{y} = -\sin\theta - \beta y \end{cases}$

Phase Portrait:



$\theta_* = 0$ is both Lyapunov and asymptotically stable

REMARK: Linearizing is useful when the eigenvalues of the system have non-zero real part.

PROBLEM: Center in the ideal pendulum has both eigenvals. w/ zero real part.

solution

Lyapunov Functions $\simeq$ energy function whose level sets are the orbits on the phase portrait.

Ex: (Undamped Pendulum) $E(\theta, y) = \frac{1}{2}y^2 - \cos\theta$

$$\frac{dE}{dt} = y\dot{y} + \sin\theta\, y = -y\sin\theta + \sin\theta\, y = 0$$

Energy is always conserved !

(Damped Pendulum) Same energy $E(\theta, y) = \frac{1}{2}y^2 - \cos\theta$ works.

$$\frac{dE}{dt} = y\dot{y} + \sin\theta\, y = y(-\sin\theta - \beta y) + \sin\theta\, y = -\beta y^2 < 0$$

Energy is always lost !

DEF: A continuous function $L: \mathbb{R}^n \to \mathbb{R}^n$ is a (strong) Lyapunov function for x_* if $\exists$ open set $U \ni x_*$ where

1) $L(x_*) = 0$

2) $L(x) > 0 \quad \forall x \in U \setminus \{x_*\}$

3) $L(\varphi_t(x)) < L(x) \quad \forall x \in U \setminus \{x_*\}$ and all $t \geq 0$

If "$\leq$", we call L a weak Lyapunov fct.

THM: Let x_* be an equilibrium of φ_t.

1. If L is a weak Lyapunov function, then x_* is Lyapunov stable.

2. If L is a strong Lyapunov function, then x_* is asymptotically stable.

❗ CONVERSE is NOT TRUE: Damped pendulum energy $E(\theta, y)+1$ is weak Lyapunov even though we know $\theta_* = 0$ is asymptotically stable

Doesn't decay instantaneously along the $y=0$ axis.

Pf: WLOG, $x_* = 0$. Assume L is weak-Lyapunov.

Fix $\varepsilon > 0$ and set $m := \min\{L(x) : |x| \leq \varepsilon\} > 0$.

Exists b/c the set is closed & bdd

By continuity, $L(x) \xrightarrow{x \to 0} 0$. So, we can find $\delta > 0$ s.t. $L(x) < m$ $\forall x \in B_\delta(0)$. Since $L(\varphi_t(x)) \leq L(x) < m$ $\forall x \in B_\delta(0)$, $\varphi_t(x)$ cannot cross $L = m$. Thus, x_* is Lyapunov stable.

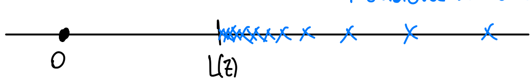
Assume L is strong-Lyapunov. By the above, $\forall x \in B_\delta(0)$, $\varphi_t(x) \in B_\varepsilon(0)$ $\forall t \geq 0$.

CLOSED + BOUNDED $\underset{\text{Bolzano Weierstrass}}{\Rightarrow}$ $\exists t_n$ s.t. $t_n \to \infty$, $\varphi_{t_n}(x)$ converges ~~to~~ smth as $n \to \infty$.

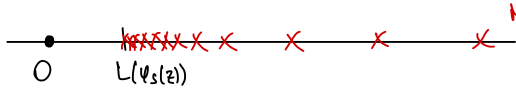
By contradiction, suppose that $\varphi_{t_n}(x) \xrightarrow{n \to \infty} z \neq x_* = 0$.

Passing to L, $L(\varphi_t(x))$ is monotone decreasing and has limit $L(z)$ (by continuity).

Monotone decreasing seq



Monotone decreasing $L(\varphi_{t_n+s}(z))$



We get a new seq. $L(\varphi_{s+t_n}(z))$ that is non-decreasing and has limit $L(\varphi_s(z))$. So, have $t_m > t_n + s$, we get

$$L(z) < L(\varphi_{t_m}(x)) < L(\varphi_{t_n+s}(x)) < L(z)$$

$$\longrightarrow\!/\!/\!\longleftarrow$$

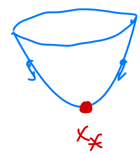
$\blacksquare$

← Usually really hard to find...

EASY LYAPUNOV FUNCTIONS:

1. $\dot{x} = -\nabla F(x)$. If x_* is a local minimum of $F(x)$, then it is stable and F is its Lyapunov function.

$$\frac{d}{dt} F(x(t)) = \nabla F(x(t))\, \dot{x}(t) = -|\nabla F(x)|^2$$



2. $\ddot{x} = -\nabla F(x)$. Can always rewrite this as

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\nabla F \end{cases}$$

$E(x,y) = \frac{1}{2}|y|^2 + F(x)$ is the Lyapunov function.

3. (HAMILTONIAN SYSTEM) $\begin{cases} \dot{x} = \dfrac{\partial H}{\partial y} \\ \dot{y} = -\dfrac{\partial H}{\partial x} \end{cases}$ Hamiltonian is conserved

$$\frac{dH}{dt} = \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial y}\dot{y} = 0$$

$\Rightarrow$ Hamiltonian H is Lyapunov fct. near equilibrium of H.

THM: (LASALLE's INVARIANCE PRINCIPLE) Suppose x_* is an equilibrium for $\dot{x} = f(x)$ and L is weak-Lyapunov on a compact, forward-invariant set $U \ni x_*$.
Let $Z = \{ x \in U : \frac{dL}{dt} = 0 \}$. Then, if x_* is the largest forward invariant subset of Z, x_* is asymptotically stable.

LEMMA: An asymptotically stable linear system has a strong Lyapunov function of the form

$$L = x^T S x$$

for some symmetric positive-definite S. $\left(\begin{matrix} S^T = S \\ S \succ 0 \end{matrix} \right)$

Pf: $\dot{x} = Ax$. We want the decreasing property of L. So,

$$\frac{dL}{dt} < 0 \rightsquigarrow \frac{dL}{dt} = \dot{x}^T S x + x^T S \dot{x}$$

$$= x^T A^T S x + x^T S A x$$

$$= x^T (A^T S + SA) x$$

Would be done if $A^T S + SA = -\mathbb{1}$

$$\Leftrightarrow e^{A^T t} (A^T S + SA) e^{At} = -e^{A^T t} e^{At}$$

$$\Longleftrightarrow \quad \underbrace{A^T e^{A^T t}} S e^{At} + e^{A^T t} S \underbrace{A e^{At}} = - e^{A^T t} e^{At}$$

$$\Longleftrightarrow \quad \frac{d}{dt}\left(e^{A^T t} S e^{At}\right) = - e^{A^T t} e^{At}$$

Integrate

$$\Longleftrightarrow \quad e^{A^T t} S e^{At} = C - \int_0^t e^{A^T \tau} e^{A\tau} \, d\tau$$

Exists b/c we know A is stable

$$C = \int_0^\infty e^{A^T t} e^{At} dt$$

$$= \int_0^\infty e^{A^T t} e^{At} dt - \int_0^t e^{A^T \tau} e^{A\tau} d\tau$$

$$= \int_t^\infty e^{A^T t} e^{At} dt$$

Then $S = \int_0^\infty e^{A^T \tau} e^{A\tau} d\tau$ is symm. positive def.

$$S^T = \left(\int_0^\infty e^{A^T \tau} e^{A\tau} d\tau\right)^T = \int_0^\infty \left(e^{A^T \tau} e^{A\tau}\right)^T d\tau = S.$$

Let $x \in \mathbb{R}^n$, $x^T S x = \int_0^\infty x^T \underbrace{e^{A^T \tau} e^{A\tau}}_{\geq 0} x \, d\tau \geq 0.$

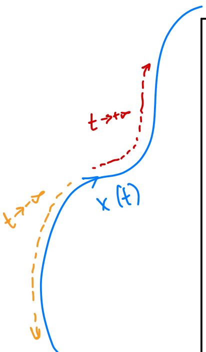
$\blacksquare$

LECTURE 8

Oct 23rd, 2025

LONG-TIME BEHAVIOR OF SOLUTIONS (α- and ω-sets)

* α- AND ω- LIMIT SETS : Suppose $\dot{x} = V(x)$.



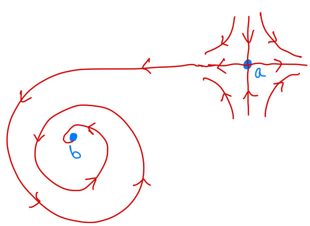
DEF: (ω-limit set)

$$\omega(x) := \{ y \in \operatorname{dom} V : \exists t_j \nearrow +\infty \text{ s.t. } \lim_{j \to \infty} x(t_j) = y \}$$

DEF: (α-limit set)

$$\alpha := \{ y \in \operatorname{dom} V : \exists t_j \searrow -\infty \text{ s.t. } \lim_{j \to \infty} x(t_j) = y \}$$

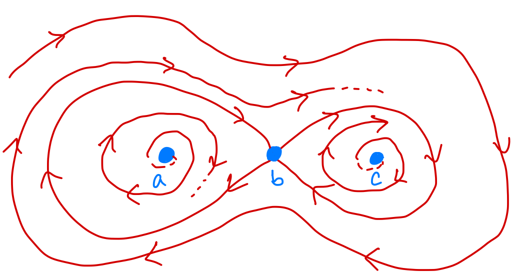
EXAMPLES: 1. Saddle and stable spiral



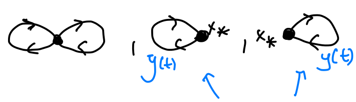
$$\alpha(x) = \{a\}$$

$$\omega(x) = \{b\}$$

2.



The ω-limit sets are



homoclinic

$\omega(y) = \alpha(y) = x_*$

DEF: Solutions connecting different fixed points are called <u>HETEROCLINIC</u> orbits and solutions connecting the same point are called <u>HOMOCLINIC</u> orbits.

!

<u>PROPERTIES OF LIMIT SETS</u>

1. (Existence) ω-limit set of a bounded orbit is always non-empty. (Heine-Borel)
2. (Closedness) ω-limit sets are closed.
3. (Invariance) If $x \in \omega(x_0)$, then $\varphi_t(x) \in \omega(x_0)$ $\forall t \geq 0$.
4. (Connectedness) ω-limit set of bounded orbit is connected
5. (Transitivity) If $z \in \omega(y)$ and $y \in \omega(x)$, then $z \in \omega(x)$.

<u>THM</u>: (POINCARÉ - BENDIXSON) For planar systems

$$\dot{x} = V(x), \; x \in \mathbb{R}^2, \; V \text{ is a vector field in } \mathbb{R}^2,$$

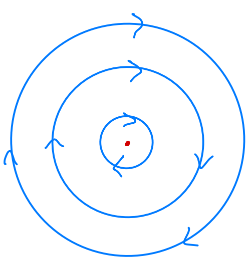
that generate $\varphi_t(x)$, if fixed points are isolated and $\varphi_t(x)$ is bounded. Then, either

1. $\omega(x)$ is an equilibrium, or 

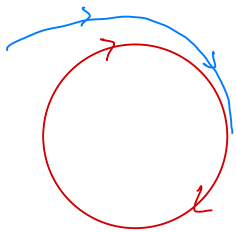
2. $\omega(x)$ is a periodic orbit, or 
3. for each $u \in \omega(x)$, we have that $\alpha(u)$ and $\omega(u)$ are equilibria.

* LIMIT CYCLES

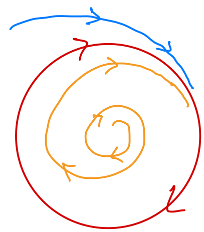
- A periodic orbit γ that is the ω-limit or α-limit set of a point $x \notin \gamma$.



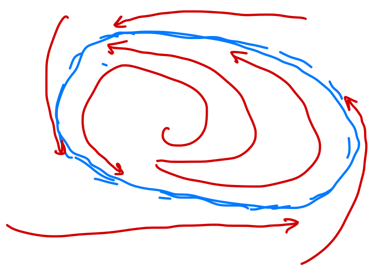
Periodic orbit but no limit cycles



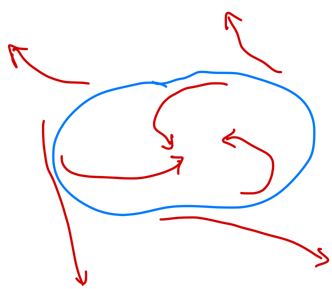
limit cycles are isolate



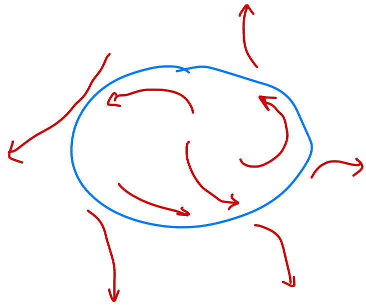
Stable limit cycle →



Unstable limit cycle →



Half-stable limit cycle →



REMARK: Two approaches for linear stability of limit cycle $\gamma(t)$:

LINEARIZE AROUND $\gamma(t)$ vs POINCARÉ RETURN MAP

(∗) (∗)

periodic orbit

(∗) LINEARIZE AROUND $\gamma(t)$:

$$\begin{cases} \dot{x} = V(x), \quad x \in \mathbb{R}^n, \quad n \geq 2 \\ \qquad\qquad V \in C^k(\mathbb{R}^n), \quad k \geq 1. \end{cases}$$

let $x(t) := \gamma(t) + \underbrace{y(t)}_{\text{small perturbation}}$

So, using the ODE,

$V(\gamma(t)) = \dot{\gamma}$ since $\gamma(t)$ solves the ODE

$$\dot{x}(t) = \dot{\gamma}(t) + \dot{y}(t) = V(x(t)) = V(\gamma(t) + y(t))$$

Taylor Expand

$$= V(\gamma(t)) + \underbrace{D_x V(\gamma(t))}_{\text{Jacobian of } V} y + \cancel{h.o.t.}$$

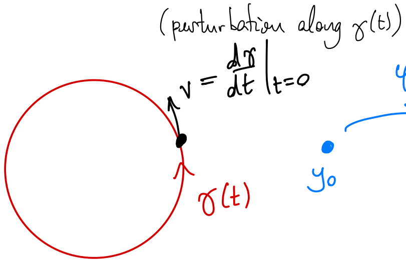
linearize

NOTE: $\dot{y} = \underbrace{D_x V(\gamma(t))}_{\substack{\| \\ A(t) = A(t+T)}} y$

Floquet theory: fundamental matrix solution $\Phi(t)$ is

$$\frac{d\Phi}{dt} = A(t)\,\Phi(t), \quad \Phi(0) = \mathbb{1}$$

- If $y(0) = y_0$, then $y(t) = \Phi(t)\, y_0$ and $y(T) = \underbrace{\Phi(T)}_{\substack{\text{Monodromy Matrix} \\ \text{!!} \\ M}} y_0$



$$y_0 \xrightarrow{\varphi_T} My_0 \xrightarrow{\varphi_T} M^2 y_0 \xrightarrow{\varphi_T} M^3 y_0 \ \cdots$$

$$\Downarrow$$

Examine the eigenvalues of M to determine stability

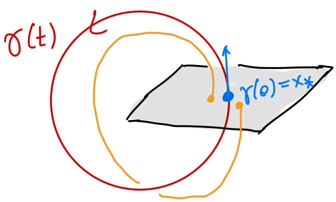
- if $\exists \mu$ with $|\mu| > 1 \Rightarrow$ instability
- if all $\underbrace{\text{but one } \mu}_{Mv = v}$ have $|\mu| < 1 \Rightarrow$ stability

$Mv = v$ (v = perturbation along γ)

$\Downarrow$

$\mu = 1$

(*) Poincaré Return Map



$\Sigma :=$ $(n-1)$-dim. surface transverse to $\gamma(t)$ at $\gamma(0)$ $\Big\}$ $\Sigma \pitchfork \gamma(t)_{\gamma(0)}$

Define the map $P: \Sigma_0 \to \Sigma$ $\qquad P = \varphi_{\tau(x)}(x)$,

$P(x_*) = x_*$ $\qquad \tau(x) = 1^{st}$ return time to Σ

LECTURE 9

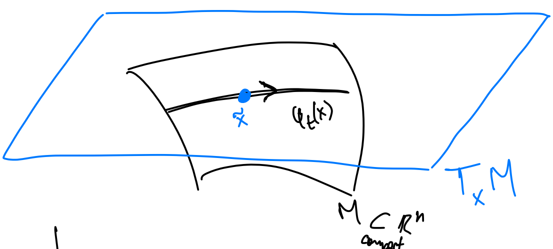
STABLE & UNSTABLE MANIFOLDS

Oct 28th 2025

Recall:
$$\frac{d\varphi_t(x)}{dt} = V(\varphi_t(x)) \quad , \quad \frac{dx(t)}{dt} = V(x(t)) .$$

REMARK: Linear non-autonomous equations that describe the stability (or lack thereof) of infinitesimal perturbations to the flow:
$$\frac{du(t)}{dt} = DV(\varphi_t(x))\, u(t)$$
$u(t) \equiv$ infinitesimal perturbation to $\varphi_t(x)$ on some fixed orbit $\{\varphi_t(x)\}_{t\geq 0}$.

! HYPERBOLIC SYSTEMS: $\frac{d\varphi_t(x)}{dt} = V(\varphi_t(x))$ is a uniformly hyperbolic flow if $\exists\, C, \alpha > 0$ such that, at every x, there is a direct sum decomposition



$$\mathbb{R}^n = E^s(x) \oplus E^u(x) \oplus E^0(x)$$
not necessarily orthogonal

where

(STABLE SUBBUNDLE) $E^s(x) := \{ u(0) \in T_x\mathbb{R}^n : \|u(t)\| \leq Ce^{-\alpha t}\|u(0)\| \,,\, C, \alpha > 0 \}$

(UNSTABLE SUBBUNDLE) $E^u(x) := \{ u(0) \in T_x\mathbb{R}^n : \|u(t)\| \leq Ce^{-\alpha|t|}\|u(0)\| \text{ as } t \to -\infty \}$

REMARK: $E^s \cup E^0 \longleftrightarrow$ perturbations that don't blow up in forward time

(CENTER SUBBUNDLE) $E^0(x) := \operatorname{span} V(x)$

PROBLEM FOR MIDTERM: Show that changing $u(t)$ to $V(\varphi_t(x))$ does not show exponential growth or decay:

$$\frac{du(t)}{dt} = DV(\varphi_t(x))\, u(t).$$

- Fixed point x_* such that $V(x_*) = 0 \iff \varphi_t(x_*) = x_* \ \forall t$.

Thus $\underline{x_* \text{ is hyperbolic}}$ whenever $\mathrm{Re}(\lambda) \neq 0 \ \forall \lambda \in \mathrm{Spec}(DV(x_*))$.

$\iff \exists \alpha > 0$ such that

$$\mathrm{Re}(\lambda) > \alpha \ \text{ or } \ \mathrm{Re}(\lambda) < -\alpha \quad \forall \lambda \in \mathrm{Spec}(DV(x_*))$$

NOTE: Perturbation system is now, for hyperbolic equilibria

$$\frac{du(t)}{dt} = DV(\varphi_t(x_*))\, u(t) = \underbrace{DV(x_*)\, u(t)}_{\text{linear autonomous system}}$$

$$\Rightarrow \quad u(t) = e^{tA} u(0)$$

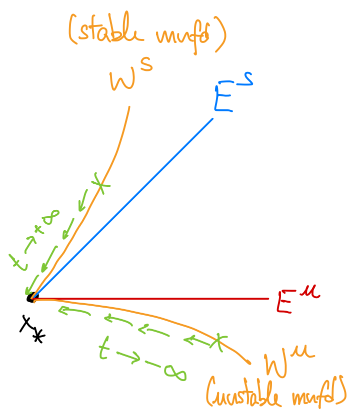
Thus: $\forall u(0) \in E^s$, $\quad \|u(t)\| \leq C e^{-\alpha t} \|u(0)\| \quad$ as $t \to \infty$

$\forall u(0) \in E^u$, $\quad \|u(t)\| \leq C e^{-\alpha |t|} \|u(0)\| \quad$ as $t \to -\infty$

For fixed point: $E^0 = \{0\}$.

REMARK: $y(t) := \varphi_t(x) - x_*$. Thus, as before,

$$\frac{dy}{dt} = A y(t) + N(y(t)) \qquad V(x_*) = 0, \quad A := DV(x_*)$$



DEF: Global stable and unstable mfds:

(GLOBAL STABLE MANIFOLD) $W^s(x_*) := \left\{ x \in \mathbb{R}^n : \| \varphi_t(x) - x_* \| \xrightarrow{t \to +\infty} 0 \right\}$

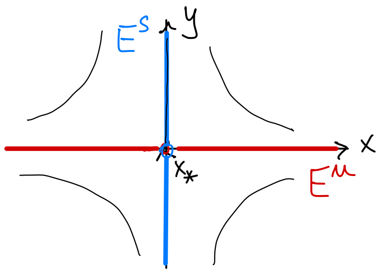
(GLOBAL UNSTABLE MANIFOLD) $W^u(x_*) := \left\{ x \in \mathbb{R}^n : \| \varphi_t(x) - x_* \| \xrightarrow{t \to -\infty} 0 \right\}$

EXAMPLE:

$$F(a,b) := \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \xrightarrow[F^n \text{ ntimes}]{\text{composition}} \begin{bmatrix} 2^n & 0 \\ 0 & \left(\frac{1}{3}\right)^n \end{bmatrix}$$

blows up $n \to \infty$

0 as $t \to +\infty$



$x_* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is fixed pt.

LECTURE 10

Oct 30^{th}, 2025

- Hyperbolic Fixed Points: $\varphi_t(x_*) = x_* \Leftrightarrow V(x_*) = 0$

  $\mathrm{Re}(\lambda) \neq 0 \quad \forall \lambda \in \mathrm{Spec}(DV(x_*))$.

- Uniformly Hyperbolic Fixed Point: pressence of E^s, E^u, E^c exponential subspaces of $\mathbb{R}^n$ at every x with uniform (in x) rates of exponential expansion / contraction.

EXAMPLE: $F: \mathbb{R}^2 \to \mathbb{R}^2$, $F(x_*) = x_*$. Suppose $\mathrm{Re}(\lambda) \neq 0$ for all $\lambda \in \mathrm{Spec}\, DF(x_*)$.

$$F\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a \\ b/3 \end{pmatrix} \rightsquigarrow F \doteq \begin{bmatrix} 2 & 0 \\ 0 & 1/3 \end{bmatrix}$$

$$F^n \doteq \begin{bmatrix} 2^n & 0 \\ 0 & (1/3)^n \end{bmatrix}$$. So, the flow here is

$$V(x) = \begin{bmatrix} \overset{>0}{\log 2} & 0 \\ 0 & \underset{<0}{\log \frac{1}{3}} \end{bmatrix} x$$

(LINEAR HYPERBOLIC DYNAMICS)

- $\log 2 > 0 \Rightarrow$ exponential growth in x_1
- $\log \frac{1}{3} < 0 \Rightarrow$ exponential decay in x_2

DEF: The unstable and stable manifolds for a fixed point (can also be defined similarly for general points or even sets)

(STABLE MANIFOLD) $W^s(x_*) := \{ x \in \mathbb{R}^n : \| \varphi_t(x) - \underbrace{\varphi_t(x_*)}_{= x_* \text{ (fixed pt.)}} \| \xrightarrow{t \to +\infty} 0 \}$

(UNSTABLE MANIFOLD) $W^u(x_*) := \{ x \in \mathbb{R}^n : \| \varphi_t(x) - \underbrace{\varphi_t(x_*)}_{= x_* \text{ (fixed pt.)}} \| \xrightarrow{t \to -\infty} 0 \}$.

We can define the LOCAL stable and unstable mnfds for a hyperbolic fixed point as

(LOCAL STABLE MANIFOLD) $W_\delta^{loc,s}(x_*) := \{ x \in W^s(x_*) : \| \varphi_t(x) - x_* \| < \delta \quad \forall t \geq 0 \}$.

(LOCAL UNSTABLE MANIFOLD) $W_\delta^{loc,u}(x_*) := \{ x \in W^u(x_*) : \| \varphi_t(x) - x_* \| < \delta \quad \forall t \leq 0 \}$.

* REMARK: These manifolds are invariant

$$\varphi_t(W^s(x_*)) \subset W^s(x_*) \qquad \forall t \geq 0$$

$$\varphi_t(W^u(x_*)) \subset W^u(x_*) \qquad \forall t \leq 0$$

* LINEARIZATION: $\dfrac{du(t)}{dt} = DV(x_*)u(t) \rightsquigarrow u(t) = e^{tDV(x_*)}u(0)$

$v \in E^s(x_*) \Rightarrow \| e^{tDV(x_*)} v \| \leq C e^{-\alpha t} \|v\|$ as $t \to +\infty$

$v \in E^u(x_*) \Rightarrow \| e^{tDV(x_*)} v \| \leq C e^{-\alpha |t|} \|v\|$ as $t \to -\infty$.

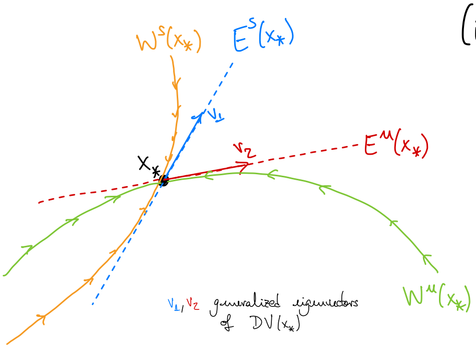
Ex: (Same as before) $(0,0)$ is hyperbolic fixed pt.

$$F\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \xrightarrow{\text{Linear map}} DF(x_1,x_2) = \begin{pmatrix} 2 & 0 \\ 0 & 1/3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Exponential subspaces:	Manifolds:
$E^s(x_*) = x_2$-axis	$W^s(x_*) = x_2$-axis
$E^u(x_*) = x_1$-axis	$W^u(x_*) = x_1$-axis

In this class, we only do this then for hyperbolic fixed points

THM: (STABLE / UNSTABLE MANIFOLD THM) If V is a C^r vector field with a hyperbolic fixed point x_* $(V(x_*)=0)$, then



(i) $W^s(x_*)$ is a C^r manifold and its tangent space at x_* has the same dimension as the stable space $E^s(x_*)$ of x_*

(ii) $W^u(x_*)$ is a C^r manifold and its tangent space at x_* has the same dimension as the unstable space of x_*

(iii) $W^{loc,s}(x_*)$ is a Lipschitz graph $(x, L(x))$ over $E^s(x_*)$

LECTURE 11

Nov 4th 2025

OSELEDETS THEOREM AND LYAPUNOV EXPONENTS

STUDY THE PROOF OF STABLE / UNSTABLE MANIFOLD THEOREM FOR MIDTERM

Highlights of proof: x_* hyperbolic fixed pt. $\Rightarrow \mathbb{R}^n = E^s(x_*) \oplus E^u(x_*)$

Set $y(t) := \varphi_t(x) - x_*$, $x \in B_{\delta_0}(x_*)$.

$$\frac{\partial y(t)}{\partial t} = \frac{\partial \varphi_t(x)}{\partial t} = V(\varphi_t(x)) = V(y(t) + x_*)$$

$$= \underbrace{DV(x_*)\, y(t)}_{\text{linear portion}} + \underbrace{N(y(t))}_{\text{nonlinear portion}}$$

Here, y is a function $[0,\infty) \to \mathbb{R}^n$. If $\|y(t)\|$ is bounded for all $t \geq 0$, then $y(t)$ satisfies

$$\frac{dy(t)}{dt} = DV(x_*)\, y(t) + N(y(t)) \qquad (1)$$

provided that

$A := DV(x_*)$

$$y(t) = e^{tA}(y(0))_s + \int_0^t e^{(t-z)A} (N(y(z)))_s \, dz - \int_t^\infty e^{(t-z)A} (N(y(z)))_u \, dz \qquad (2)$$

For all $v \in \mathbb{R}^n$, since $\mathbb{R}^n = E^s(x_*) \oplus E^u(x_*)$, can write $v = v_s + v_u$, $v_s \in E^s(x_*)$ and $v_u \in E^u(x_*)$.

EXERCISE: Show that $y(t)$ satisfies (1).

Why is $y(t)$ bounded $\forall t \geq 0$?

Can $y(t) = \varphi_t(x) - x_*$ be unbounded? Yes!

If $y(t)$ is bounded, is the solution unique? Yes!

Claim: If $y(t)$ is bounded, the solution is unique.

Pf: (Contraction Mapping Theorem) Define

$$T : C^0([0,\infty), \mathbb{R}^n) \longrightarrow C^0([0,\infty), \mathbb{R}^n)$$

$$T[y](t) := e^{tA}(y(0))_s + \int_0^t e^{(t-\tau)A}(N(y(\tau)))_s \, d\tau - \int_t^\infty e^{(t-\tau)A}(N(y(\tau)))_u \, d\tau$$

Sub Claim 1: T is well-defined (i.e., it maps C^0 fcts. to C^0).

Sub Claim 2: Show that T is a contraction on the subspace

$$V(\delta) := \{ t \mapsto y(t) : \|y\| \leq \delta \},$$

w.r.t. $\|y\| := \sup_{t \in [0,\infty)} |y(t)|$.

NTS: (i) $T : V(\delta) \to V(\delta)$

(ii) T contraction

If you're trying to prove uniqueness, there's usually a hidden contraction somewhere.

Pf: (i) Note that

$$\|T[y](t)\| \leq \|e^{tA}(y(0))_s\| + \int_0^t \|e^{(t-\tau)A} N(y(\tau))_s\| \, d\tau$$

Hyperbolicity and definition of stable/unstable vector

$$+ \int_t^\infty \| e^{(t-z)A} N(y(z))_u \| \, dz$$

$$\leq C e^{-\alpha t} \|y(0)\| + C \int_0^t e^{-\alpha(t-z)} \|N(y(z))_s\| \, dz$$

$$+ C \int_\infty^t e^{-\alpha(t-z)} \|N(y(z))_u\| \, dz$$

Since $y \in V(\delta)$, we have $\|y\| \leq \delta \ \forall t$. Now, $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $\|N(v)\| < \varepsilon \|v\|$ whenever $\|v\| < \delta$. Thus,

$$\|T[y](t)\| \leq C e^{-\alpha t} \|y(0)_s\| + C\varepsilon \int_0^t e^{-\alpha(t-z)} \|y(z)\| \, dz$$

$$+ C\varepsilon \int_\infty^t e^{-\alpha(t-z)} \|y(z)\| \, dz$$

$$\leq \underbrace{C e^{-\alpha t} \|y(0)_s\|}_{\overset{!}{<} \frac{\delta}{2}} + \underbrace{\frac{\tilde{C}\varepsilon\delta}{\alpha}}_{\overset{!}{<} \frac{\delta}{2}} \rightsquigarrow \varepsilon < \frac{\alpha}{2\tilde{C}}$$

So, take it to be

$$C e^{-\alpha t} \|y(0)_s\| < \delta/2 .$$

Then $T : V(\delta) \to V(\delta)$, as desired. □

Pf: (ii) T is a contraction b/c

$$\|T[y] - T[\tilde{y}]\| \leq e^{tA} \|y(0)_s - \tilde{y}(0)_s\| + (\cdots) + (\cdots)$$

$\leq$ Triangle Inequality

$$\cdots < \frac{1}{2} \|y - \tilde{y}\|$$ using same argument as above

□

So, (i) + (ii) $\Rightarrow$ T is a contraction on a complete space

$\Rightarrow$ T has a unique fixed point in $V(\delta)$.

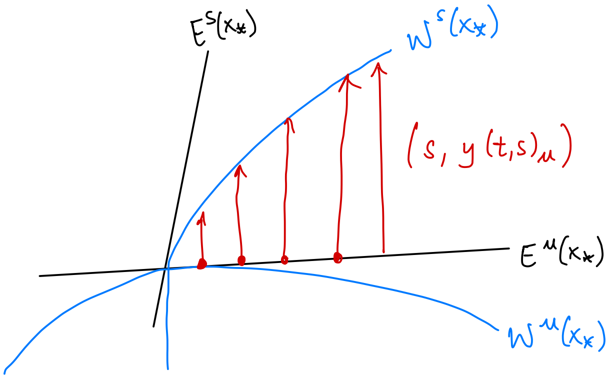
Upshot: If y is bounded $\forall t \geq 0$ and solves (1), then y is unique.

Lastly: Use Generalized Grönwall to get that

$$\|y(t)\| \leq \tilde{C} e^{-\tilde{\alpha} t} \|y(0)_s\|.$$

Remark: Setting $s := y((0))_s$

$$\|y(t,s) - y(t,\tilde{s})\| \lesssim K \|s - \tilde{s}\| \Rightarrow$$ y is a Lipschitz graph over $E^s(x_*)$.



INTUITION: Stable / Unstable Mnfd Thm gives us a new coordinate system to write vectors in

$n_s := \dim E^s(x_*) = \dim W^s_{loc}(x_*)$

$n_u := \dim E^u(x_*) = \dim W^u_{loc}(x_*)$

$n = n_s + n_u$

$$\mathbb{R}^n = E^s(x_*) \oplus E^u(x_*).$$

Namely: $(\underset{\substack{\uparrow \\ n_s \text{ entries}}}{s}, \underbrace{y(t,s)_u}_{n_u \text{ entries}})$

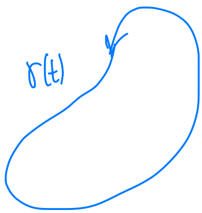
$n_s + n_u = n$

∎

Lecture 12

Nov 9, 2025

Stability Properties of Cycles:



$$\dot{x} = V(x, \underset{\text{parameter } c \in \mathbb{R}}{c}), \quad x \in \mathbb{R}^n$$

$$\dot{\gamma}(t) = V(\gamma(t))$$

$$\gamma(t) = \gamma(t+T)$$

Linear Stability of $\gamma(t)$: $x(t) = \gamma(t) + y(t)$

$$\Rightarrow \dot{\gamma} + \dot{y} = V(\gamma + y) = V(\gamma(t)) + \underbrace{DV(\gamma(t))}_{A(t)} y + \text{h.o.t.}$$

Jacobian

Linearized Problem: $\dot{y} = A(t)\, y$

Note: $\gamma(t+T) = \gamma(t) \Rightarrow A(t+T) = A(t)$.

Fundamental Matrix Solution: $\dot{\Phi}(t) = A(t)\,\Phi_0$, $\quad \Phi_0 = \mathbb{1}$.

"Propogator"

If $y(0) = y_0$, then $y(t) = \Phi(t)\, y_0$

Monodromy Matrix: $\Phi(T) := M$

$$y_0 \xrightarrow{M} y(T) \xrightarrow{M} y(2T) \longrightarrow \cdots$$

$$y(T) = M y_0, \qquad y(2T) = M^2 y_0$$

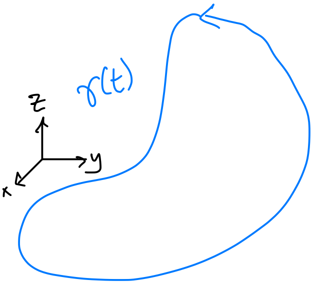
→ Eigenvalues of M:

$$\mu_1 = 1, \mu_2, \ldots, \mu_n.$$

← If $|\mu_i| > 1$ for some $i \geq 2$, then perturbations actually GROW

(HW03)

<u>EXAMPLE</u>: (ROESSLER ATTRACTOR) $a = b = 0.2$ and $c = 2.5$.



$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x-c) \end{cases}$$

$$DV(x,y,z) = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x-c \end{bmatrix}$$

Consider the 6-dim system

$$\begin{cases} \dot{x} = -y - z \\ \dot{y} = x + ay \\ \dot{z} = b + z(x-c) \\ \delta\dot{x} = -\delta y - \delta z \\ \delta\dot{y} = \delta x + a\,\delta y \\ \delta\dot{z} = z\,\delta x + \delta z(x-c) \end{cases}$$

<u>WANT</u>: Monodromy Matrix $\Phi(T) =: M$

Initial conditions

$(x_0, y_0, z_0, 1, 0, 0)$

$(x_0, y_0, z_0, 0, 1, 0)$

$(x_0, y_0, z_0, 0, 0, 1)$

<u>GOAL</u>: Choose x_0, y_0, z_0 so that they are <u>on</u> the limit cycle!

Flow them by T:

Need to choose x_0, y_0, z_0 on the cycle!?

$$\varphi_T(x_0, y_0, z_0, 1, 0, 0) = (\underbrace{x_0, y_0, z_0}, a, b, c)$$
$$\varphi_T(x_0, y_0, z_0, 0, 1, 0) = (x_0, y_0, z_0, d, e, f)$$
$$\varphi_T(x_0, y_0, z_0, 0, 0, 1) = (x_0, y_0, z_0, g, h, k)$$

$$\Rightarrow \boxed{M = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}}$$

Verify this is a good M:

(i) Check that one of the eigenvalues is ≈ 1.

(ii) Check that the eigenvector is tangent to the limit cycle.

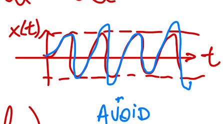
(iii) <u>ABEL'S IDENTITY</u>:

$$\det(\Phi(t)) = \exp\left[\int_0^t \operatorname{tr}(A(s))\, ds\right]$$

$$\rightsquigarrow \det(M) = \exp\left[\int_0^T (\alpha + x(s) - c)\, ds\right] \overset{!}{=} \mu_2 \mu_3 \cdot \underbrace{1}_{\mu_1} \overset{!}{>} 0$$

Verify that this is the case

<u>FIND T</u>: Times the time series crosses zero

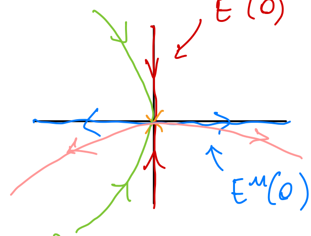


Make sure we <u>HAVE A SINGLE PEANUT</u> (aka 1-cycle)

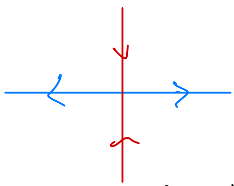
CHECK BY COUNTING <u>PEAKS</u> IN TIME SERIES

* APPROXIMATING STABLE / UNSTABLE MANIFOLDS

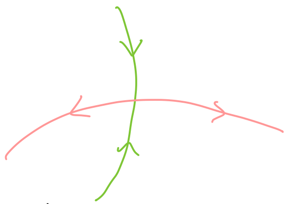
Ex: $\begin{cases} \dot{x} = 2x + y^2 \\ \dot{y} = -2y + x + y^2 \end{cases}$ $\underset{\text{about origin}}{\overset{\text{linearize}}{\rightsquigarrow}}$ $\underset{\ddot{A}}{\underset{\shortparallel}{DV(0)(x,y)}} = \begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$



(HW03 → find $W^u_{loc}(0)$)



(linearization)



(nonlinear)

Use Stable/Unstable Mnfd Thm to find $W^s_{loc}(0)$:

– Recall $W^s_{loc}(0)$ is a Lipschitz graph over $E^s(0)$; i.e.,

$$W^s(0) = \{ (x,y) = \underbrace{(h_s(y), y)} : |y| \le \delta \}$$

(1) $h_s(0) = 0 \Leftarrow$ W^s_{loc} is a graph over E^s

(2) $h_s'(0) = 0 \Leftarrow$ and it's tangent to E^s at (0,0)

(3) If $(x_0, y_0) = (h_s(x_0), y_0)$ is the initial condition, then $\Leftarrow$ and it's invariant under the flow

$$(x(t), y(t)) = (h_s(y(t)), y(t))$$

So $x(t) = h_s(y(t)) \rightsquigarrow \dot{x} = h_s'(y(t))\,\dot{y}$

$$\overset{\text{(Using the equations)}}{\rightsquigarrow} \quad (2x + y^2)\Big|_{x=h_s} = h_s'(y(t))\,(-2y + x + y^2)\Big|_{x=h_s}$$

$$\rightsquigarrow \begin{cases} (2h_s + y^2) = h_s'(-2y + h_s + y^2) \\ h_s'(0) = h_s(0) \end{cases}$$

↖ valid for $|y|$ suff. small

GOAL: Approximate h_s using a power series

$$h_s(y) = \alpha_1 + \alpha_1 y + \alpha_2 y^2 + \alpha_3 y^3 + \dots$$

Substitute this into the eq.:

$$2\alpha_2 y^2 + 2\alpha_3 y^3 + \dots + y^2 = (2\alpha_2 y + 3\alpha_3 y^2 + \dots)$$

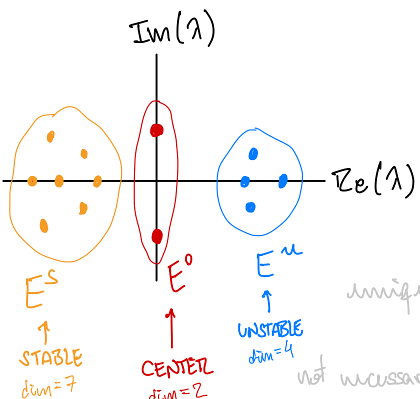
$$\cdot \left(-2y + \alpha_2 y^2 + \alpha_3 y^3 + \dots + y^2\right)$$

Take terms up to $O(y^2)$: $2\alpha_2 + 1 = -4\alpha_2 \Rightarrow \alpha_2 = -\frac{1}{6}$.

NON-HYPERBOLIC CASE: CENTER MANIFOLD

$$\begin{cases} \dot{x} = V(x), \quad V \in C^k, \ k \geq 1. \\ V(0) = 0 \quad \leftarrow \text{WLOG} \end{cases}$$

Let $\lambda :=$ eigenvalues of $DV(0)$



- There is a neighborhood of the origin where $\exists$ locally C^k invariant manifolds:

unique
W^s_{loc} = tangent to E^s, $|x(t)| \to 0$ as $t \to +\infty$

W^u_{loc} = tangent to E^u, $|x(t)| \to 0$ as $t \to -\infty$

not necessarily unique
W^0_{loc} = tangent to E^0

LECTURE 13

CHAOTIC SYSTEMS

Nov 11, 2025

OSELEDETS MULTIPLICATIVE ERGODIC THEOREM (MET)

* LINEARIZATION EXPLAINING BEHAVIOR OF FULL NONLINEAR SYSTEM

(Discrete Time)

MAP: $F: M \subset \mathbb{R}^n \to M \subset \mathbb{R}^n$, M compact.

LINEARIZED DYNAMICS: $DF(x): \mathbb{R}^n \to \mathbb{R}^n$. Fix an orbit $\{x_0, x_1, \dots\}$ s.t. $x_i = F(x_{i-1})$. Then

linear in v, nonautonomous $\longrightarrow$ $v_t = DF(x_{t-1})\, v_{t-1}$,

$v_0 \equiv$ infinitesimal perturbation at x_0

$v_t \equiv$ infinitesimal perturbation at x_t.

- if x is a fixed point $\Rightarrow$ can use Floquet theory as before to study the dynamics.

- if x is not a fixed point: $v_t \in \mathbb{R}^n$, $v_t = A_{t-1} v_{t-1}$

$A_t := DF(x_t)$

$$\Longrightarrow \quad v_t = \underbrace{A_{t-1} A_{t-2} \cdots A_1 A_0}_{B(t)} v_0$$

ASSUME: $A_t \in \mathbb{R}^{n\times n}$ $\forall t$ is nonsingular and $\underbrace{\frac{1}{t}\log\|A_t\| \xrightarrow{t\to\infty} 0}_{\text{should be subexponential}}$.

LYAPUNOV EXPONENT: $\mathbb{R} \ni \lambda(x_0, v) := \lim_{t\to\infty} \frac{1}{t} \log \|B(t)v\|$.

existence/uniqueness $\Rightarrow$ fixing IC x_0 fixes the orbit and $B(t) = A_{t-1} \cdots A_0$, $A_0 = DF(x_0)$

→ The rate of the exponential growth of $\|B(t)v\|$ as $t \to +\infty$

$\Longleftrightarrow$ $\lambda(x_0, v)$ is a scalar that gives the rate of asymptotic exponential growth / decay of infinitesimal linear perturbations along a reference orbit of x_0 in the direction of v at x_0

$$\|v_t\| = \|B_t(x_0)\| \approx O\left(e^{\lambda(x_0,v)t} v\right) \quad \text{as } t \to \infty$$

OSSELEDETS THM: (1) The Lyapunov exponents $\lambda(x_0, v)$ only take a finite number $p \in \mathbb{N}$ of distinct values $p \leq n$ for any v.

Let $\{\lambda_i\}_{i=1}^{p}$, $p \leq n$, be the Lyapunov exponents ordered as

$$\lambda_1 > \lambda_2 > \cdots > \lambda_p$$

and multiplicity k_i of λ_i, $i = 1, \ldots, p$. Then,

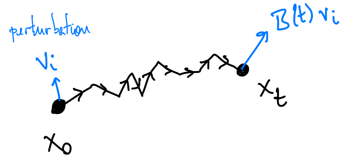
$$\sum_{i=1}^{p} k_i \lambda_i = \lim_{t\to\infty} \frac{1}{t} \log |\det B(t)|.$$

(2) $$\lim_{t\to\infty} \left[(B(t))^T (B(t)) \right]^{1/2t} = \operatorname{diag}\left(e^{\lambda_1}, \ldots, e^{\lambda_p}\right)_{n\times n}$$

multiplicity...

NOTE: $B(t) = U(t)\,\Sigma(t)\,V(t)^T$ (SVD)

$$\Rightarrow B(t)\, \underbrace{v_i}_{\text{right sing. vec.}} = \sigma_i \underbrace{u_i}_{\text{left sing. vec.}}$$



$$\Rightarrow \mathrm{vol}_t(k) = \mathrm{vol}_0(k) \cdot \left(\sum_{i=1}^{k} \sigma_i\right)^t \quad \text{(discrete time)}$$

$$\mathrm{vol}_t(k) = \mathrm{vol}_0(k) \cdot \exp\left[\sum_{i=1}^{k} \sigma_i\right] \quad \text{(continuous time)}$$

EXAMPLE: $V_0 := \{0\}$, then $B(t)V_0 = \{0\}$.

So $\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \log \|B(t)v\| = \lim_{t\to\infty} \frac{1}{t} \log(0) = -\infty$.

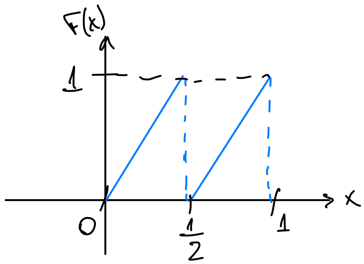
$V_1 := \{v \in \mathbb{R}^n : \lambda(v) \le \lambda_1\} \cong \mathbb{R}^n$ b/c λ_1 is the largest Lyapunov exponent

$V_2 := \{v \in \mathbb{R}^n : \lambda(v) \le \lambda_2\}$ is $(n - k_1)$-dimensional

$$\mathbb{R}^n \cong V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_p \supset \underbrace{V_{p+1}}_{:=\{0\}} \cdots$$

DEF: In a chaotic dynamical system, there is at least one positive Lyapunov exponent $\lambda_1 > 0$. $\Rightarrow V_i \neq \emptyset$

EXAMPLE: (1-dim chaos) For $x \in [0,1]$,



$$F(x) := \begin{cases} 2x, & x < 1/2 \\ 2x-1, & x > 1/2 \end{cases}, \quad x_t = F(x_{t-1})$$

$x_* = 0$ is a fixed point

$\Downarrow$

$$A := DF(x_*) = 2$$

Then, infinitesimal perturbation system $v_{t+1} = DF(x_*)\, v_t$

$$v_{t+1} = 2 v_t, \quad v_0 \neq 0, \ \|v_0\| = \varepsilon$$

Then, the Lyapunov exponent of v is

$$\lambda(v) = \lim_{t\to\infty} \frac{1}{t} \log \|\underbrace{B(t)\, v}_{2^t v_0}\| = \log 2 > 0 \implies \text{CHAOS}$$

Orbits: $x = 0 \rightarrow$ fixed pt.

$x = 0.1 \rightarrow 0.1, 0.2, 0.4, 0.8, 0.6, 0.2, 0.4, \dots$ (periodic)

$x = 0.1 + \varepsilon \rightarrow 0.1 + \varepsilon, 0.2 + \varepsilon', 0.4 + \varepsilon'', \dots$ (decorelation although they exist on $[0,1]$)

LECTURE 14

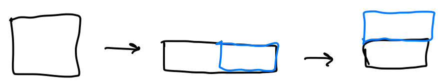
Nov 13^{th}, 2025

CHAOS $\longleftrightarrow$ sensitivity to ICs & loss of memory of IC.

Ex: Smale Horseshoe] Chaos coming from deterministic systems
Baker unfolded map

EXAMPLE: (Baker Unfolded Map)

$$F(x,y) = \begin{pmatrix} 2x - \lfloor 2x \rfloor \\ y/2 - \lfloor 2x \rfloor /2 \end{pmatrix}$$



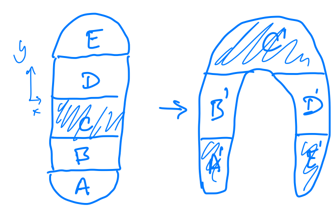
SYMBOLIC DYNAMICS: $x \in [0,1]$ in binary: $0.01\ldots$

times 2 $\Longleftrightarrow$ $0.1\ldots$ (left shift)

divide by 2 $\Longleftrightarrow$ $0.001\ldots$ (right shift)

NOTE: Smale Horseshoe dynamics on

$$\Lambda := \{(x,y) \in B \cup D : F_{SH}^n(x,y) \in B \cup D\}$$



Removing middle thirds

Attractor of the Smale Horseshoe F_{SH} map is $\simeq C \times C$, C = middle third Cantor set

OSELEDETS THEOREM : Deals w/ behavior of linearizations

Fix orbit $\{\varphi_t(x)\}_{t\geq 0}$, and

$$\frac{d\varphi_t(x)}{dt} = V(\varphi_t(x)) \rightsquigarrow \frac{du}{dt} = DV(\varphi_t(x))\, u(t)$$

Suppose we have a discrete time map F for the dynamics.

$F^t(x) := \underbrace{F\circ\cdots\circ F}_{t \text{ times}}(x)$

$M(0) \in \mathbb{R}^{n\times n}$, $\quad \dfrac{dM(t)}{dt} = DV(\varphi_t(x))\, M(t)$

$u_{t+1} = DF(x_t)\, u_t$
$=: A_t u_t$

$$M(t) = \underbrace{F(t)}\, M(0)$$

$A(t) := A_{t-1} \cdots A_0$

tangent propagator $\leftrightarrow$ fundamental matrix solution

$\underset{\mathbb{R}^{n\times n}}{u(t) = F(t)\, u(0)}$. Take SVD of $F(t)$:

$$F(t) = U(t)\, \Sigma(t)\, V(t)^T$$

$$F(t)\, v_i(t) = \sigma_i(t)\, u_i(t)$$

$$O(t) := \lim_{t\to\infty} \left(F(t)^T F(t)\right)^{1/2t} = \lim_{t\to\infty} \left(V(t)\, \Sigma(t)^2\, V(t)^T\right)^{1/2t}$$

$$= \lim_{t\to\infty} V(t)\, \Sigma(t)^{1/t}\, V(t)^T$$

$\left[V(t) \xrightarrow{t\to\infty} \text{matrix of eigenvectors of } O(t) \right]$

$$= \operatorname{diag}\left(e^{\lambda_1}, \_\_, e^{\lambda_p}\right)_{n\times n}$$

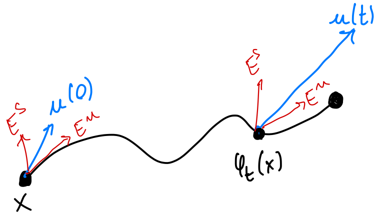
exp. of Lyapunov exponent

multiplicity of Lyapunov exponents

$\lambda_1 > \cdots > \lambda_p$ w/ multiplicity

NOTE: $\lambda_i(t) = \frac{1}{t} \log \sigma_i(t)$

OMET: as $t \to \infty$, $\frac{1}{t} \log(\sigma_i(t))$ converges as $t \to \infty$



$u(0) \sim$ randomly chosen in $\mathbb{R}^n$

$\|u(t)\| \sim O(e^{\lambda_1 t})$

$\boxed{\text{i.e., } \lambda_1 > 0 \Rightarrow \varphi_t \text{ IS CHAOTIC}}$

$$\frac{du(t)}{dt} = DV(\varphi_t(x))\, u(t)$$

COMPUTE λ_1: $\frac{du(t)}{dt} = A(t)\, u(t)$. Thus λ_1 is approx. the rate of growth $\|u(t)\| \sim O(e^{\lambda_1 t})$ for each initial condition. That is,

$$\boxed{\lambda_1 \approx \frac{1}{t} \log \frac{\|u(t)\|}{\|u(0)\|}}\ .$$

NUMERICAL ERROR: be careful with taking logs / dividing by, e.g., e^{100} b/c that's basically too.

HW04: $\mathbb{R}^6 = E^u \oplus E^s \oplus E^0$

positive Lyapunov exps $=$ dim E^u

There's always a zero Lyapunov exponent b/c V itself satisfies the linearized equation $\Rightarrow V \in E^0$

LECTURE 15

BIFURCATION THEORY

Nov 20, 2025

POINCARÉ 1885: Topological change in the structure of a dynamical system when a parameter crosses a bifurcation value.

GOAL: LOCAL BIFURCATIONS $\longrightarrow$ $\dot{x} = V(x; \lambda)$, $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^k$

C^r, $r \geq 1$

parameters

local to a non-hyperbolic equilibrium x_0 and occurring at a value λ_0

REMARK: If x_0 is hyperbolic, then small changes of λ do not change $\dim E^s(x_0)$ or $\dim E^u(x_0)$

$\Rightarrow$ Stability doesn't change (equilibrium x_0 cannot be destroyed).

$\Rightarrow$ Hyperbolic equilibria are ROBUST. $\Leftrightarrow$ structurally stable

$\Rightarrow$ Need to study non-hyperbolic equilibria for bifurcation

DEF: STEADY STATE BIFURCATION $\leadsto$ creation or destruction of equilibria.

IMPLICIT FUNCTION THEOREM: If $DV(x_0, \lambda_0)$ is invertible (i.e., $\det DV(x_0,\lambda_0) \neq 0$), then $\exists \underset{\mathbb{R}^n}{V} \ni x_0$, $\underset{\mathbb{R}^n}{W} \ni \lambda_0$, and a unique C^r function $\xi(\lambda): W \to V$ for which

$$V(\xi(\lambda); \lambda) = 0, \quad \text{and} \quad x_0 = \xi(\lambda_0)$$

If $\det D_x V(x_0, \lambda_0) \neq 0$, then we can use linear approximation to approximate $\xi(\lambda)$:

$$0 = V(x, \lambda) = V(x_0, \lambda_0) + D_x V(x_0, \lambda_0)(x - x_0) + D_\lambda V(x_0, \lambda_0)(\lambda - \lambda_0) + \cdots$$

$$\Rightarrow \boxed{x \approx x_0 - [D_x V(x_0, \lambda_0)]^{-1} D_\lambda V(\lambda_0, x_0)(\lambda - \lambda_0)}$$

NON-HYPERBOLIC HARTMAN-GROBMAN: Let

$$(*) \begin{cases} \dot{x} = Cx + F(x,y,z) & \text{(center directions)} \\ \dot{y} = Sy + G(x,y,z) & \text{(stable directions)} \\ \dot{z} = Uz + H(x,y,z) & \text{(unstable directions)} \end{cases}$$

F, G, H are $o(x,y,z)$. Then, $\exists N \ni 0$ s.t. ← WLOG via translations

$$W^c_{loc}(0,0,0) = \{(x, g(x), h(x)) : x \in E^c\} \cap N,$$

and the dynamics of $(*)$ are topologically conjugate to

$$\dot{x} = Cx + F(x, g(x), h(x)), \quad \dot{y} = Sy, \quad \dot{z} = Uz$$

! Dimension reduction: can just ignore the "nice" directions that are hyperbolic (E^u, E^s) and just focus on the problematic non-hyperbolic directions (E^c).

BIFURCATION PROBLEM: Consider the extended 2-dim. center mnfd (2-dim b/c $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$).

$$\begin{cases} \dot{x} = 0 \cdot x + \underbrace{F(x, \lambda, g(x,\lambda), h(x,\lambda))} =: f(x,\lambda) \\ \dot{\lambda} = 0 \cdot \lambda \quad \leftarrow \lambda \text{ is a parameter (const.)} \end{cases}$$

[: Don't need the $\dot{y}, \dot{z}$ b/c of non-hyperbolic HG]

Thus: $\dot{x} = f(x, \lambda)$, $x \in \mathbb{R}$, $\lambda \in \mathbb{R}$.

Assume equilibrium at $x = 0$ and bifurcation at $\lambda = 0$.

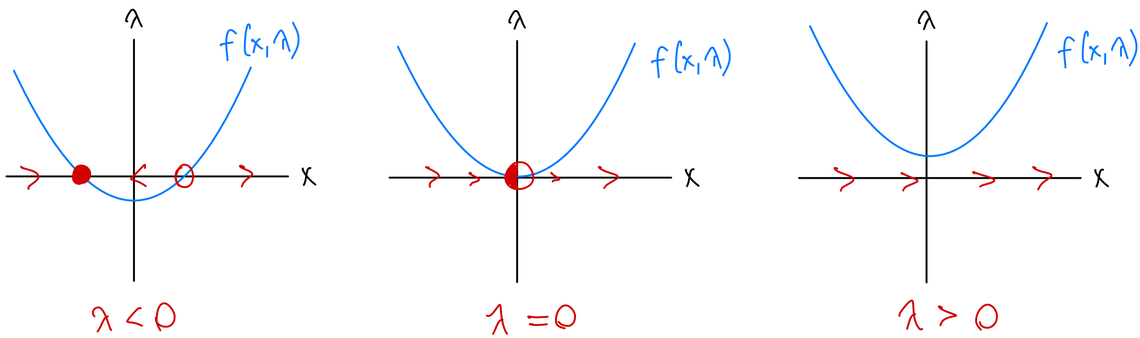
$$\left.\begin{aligned} &f(0,0) = 0 \\ &\frac{\partial f}{\partial x}(0,0) = \mu = 0 \end{aligned}\right\} \text{DEFINING PROPERTIES OF STEADY-STATE BIFURCATION}$$

Taylor Expand in (x, λ):

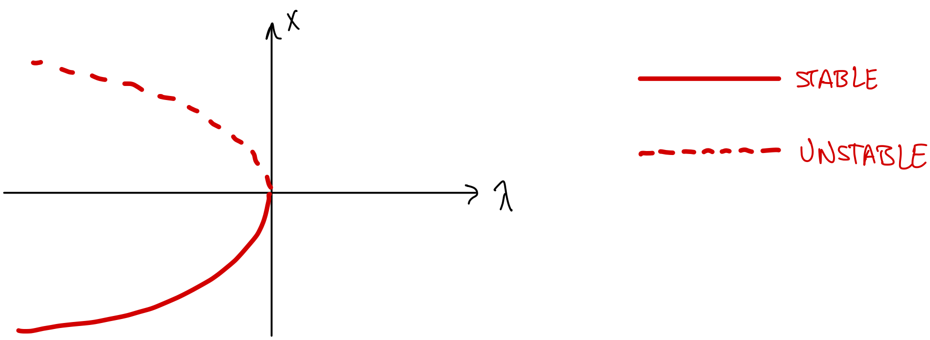
$$f(x,\lambda) = \cancel{f(0,0)}^{=0} + \cancel{\frac{\partial f}{\partial x}(0,0)}^{=0} x + \overbrace{\frac{\partial f}{\partial \lambda}(0,0)}^{\text{assume} \neq 0} \lambda + \overbrace{\frac{1}{2}\frac{\partial^2 f}{\partial x^2}(0,0)}^{\text{assume} \neq 0} x^2$$

$$+ \frac{\partial^2 f}{\partial x \partial \lambda}(0,0)\, x\lambda + \frac{1}{2}\frac{\partial^2 f}{\partial \lambda^2}(0,0)\, \lambda^2$$

SADDLE - NODE BIFURCATION: Equilibria appear / disappear in pairs



$\boxed{\text{SADDLE - NODE BIFURCATION}}$

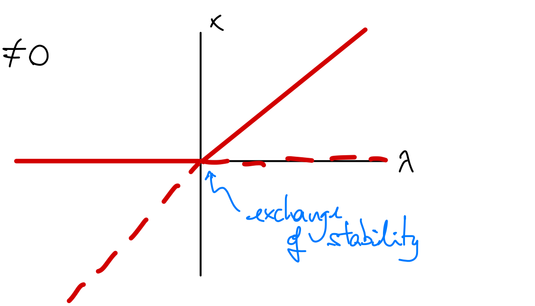


TRANSCRITICAL BIFURCATION: Always 2 equilibria that change stability at the bifurcation value (one of the eq. is always at 0)

$$\frac{\partial f}{\partial \lambda}(0,0) = 0 \,, \quad \frac{\partial^2 f}{\partial x^2}(0,0) \neq 0 \,, \quad \frac{\partial^2 f}{\partial x \partial \lambda}(0,0) \neq 0$$

$$\boxed{\dot{x} = f(x,\lambda) = x\, g(x,\lambda)}$$

Ex: Logistic $\dot{x} = x(\lambda - x)$



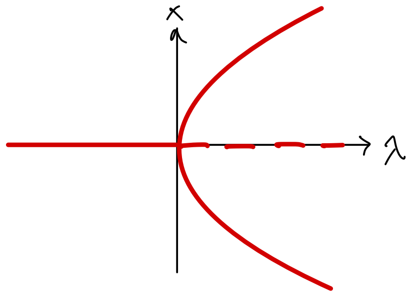
PITCHFORK BIFURCATION: when we have $x \mapsto -x$ symmetry;

i.e., $x = \tilde{x}(t)$ solution to $\dot{x} = f(x) \Rightarrow x = -\tilde{x}(t)$ also solution

$\Rightarrow f(\tilde{x}) = -f(-\tilde{x})$ (f ODD in x)

1) SUPERCRITICAL PITCHFORK

$$\dot{x} = \lambda x - x^3$$



2) SUBCRITICAL PITCHFORK

$$\dot{x} = \lambda x + x^3$$