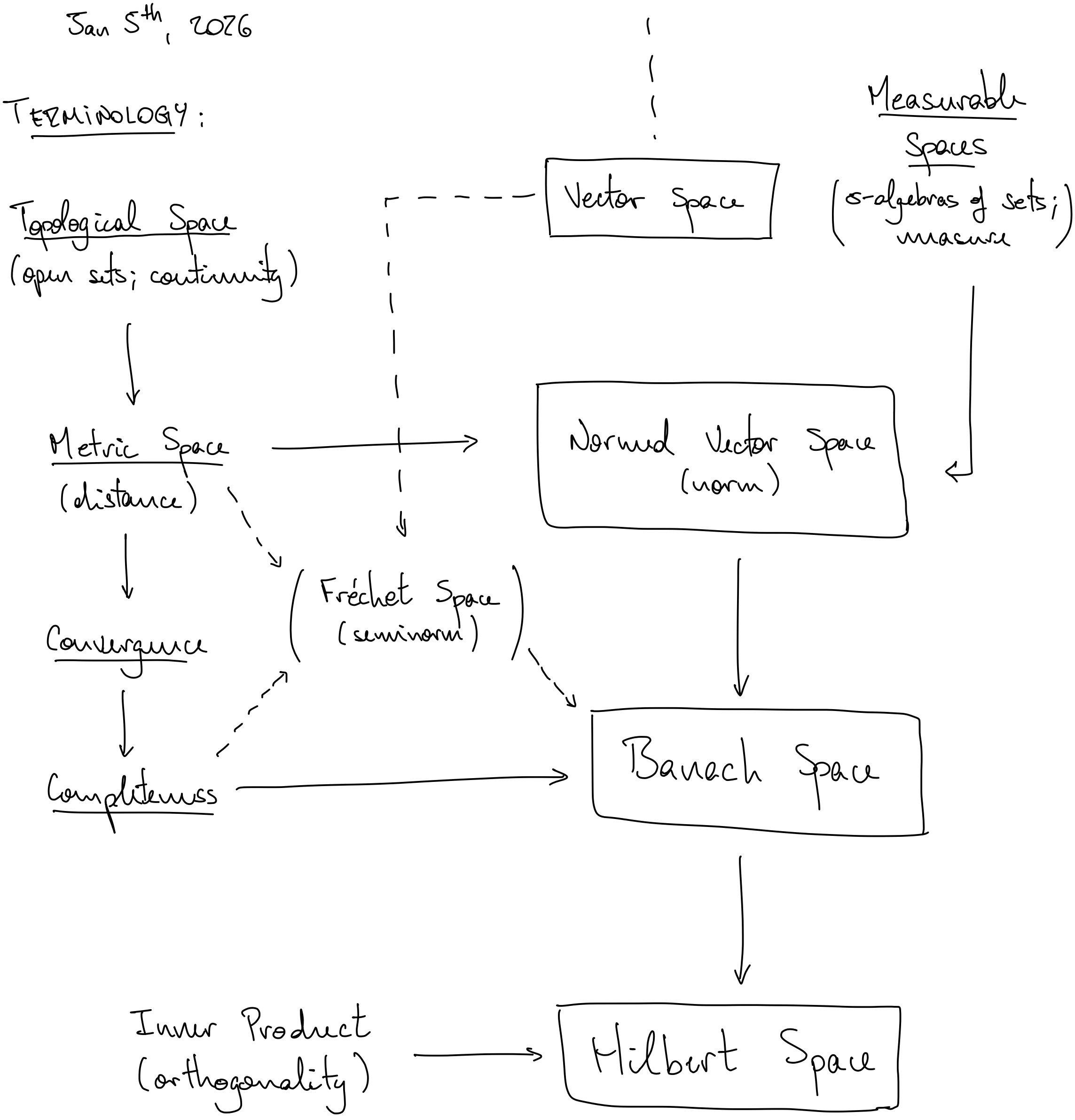
LECTURE 1

(Ch 1–4)

Jan 5th, 2026

TERMINOLOGY:



TOPOLOGICAL SPACES (Ch 4)

DEF: A topology is a collection $\mathcal{T}$ of open subsets of X s.t.

(1) $\emptyset$ and X are open

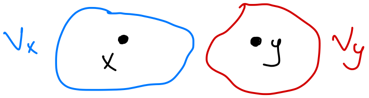
(2) Family of open sets closed under

- (2.1) arbitrary unions
- (2.2) finite intersections

$(X, \mathcal{T}) \rightsquigarrow$ TOPOLOGICAL SPACE $\qquad$ $A \subset X$ closed $\Leftrightarrow$ $A^c := X \setminus A$ open

DEF: $V \subset X$ is a neighborhood of $x \in X$ if $x \in \underset{\substack{\uparrow \\ \text{open}}}{G} \subset V \subset X$ for some G open.

DEF: $\mathcal{T}$ is Hausdorff if for every $x, y \in X$ distinct, there are nbhds $V_x \ni x$ and $V_y \ni y$ s.t. $V_x \cap V_y = \emptyset$



$\hookrightarrow$ Non-examples: long-line; line with two origins;

$(X, \mathcal{T} := \{X, \emptyset\})$ and $\# X \geq 2$ $\qquad$ $(\mathbb{R}_1 \sqcup \mathbb{R}_2)/\sim$, $(x,1) \sim (x,2)$ $\forall x \neq 0$

DEF: • Discrete topology $\equiv$ every point is open

$\mathcal{P}(X) = 2^X$ power set (too big)

- Indiscrete topology $\equiv$ ϕ and X are the only open sets
(too trivial)

- Generated topology by subsets $\mathcal{T}_0 \equiv$ intersection of all topologies $\mathcal{T}$ such that $\mathcal{T}_0 \subset \mathcal{T}$. (Always defined since discrete top. is defined)

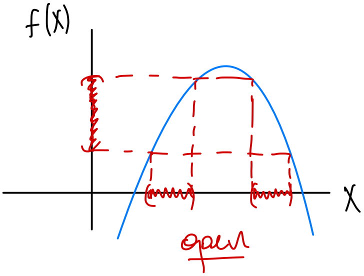
DEF: Convergence $\equiv$ $x_n \longrightarrow x$ in X if for all nbhd $V_x \ni x$, $x_n \in V_x$ for n large enough.

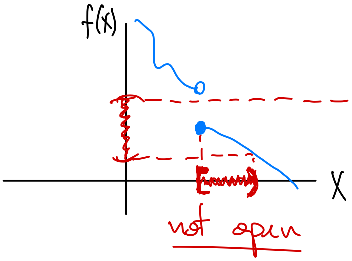
DEF: Continuity $\equiv$ $f: (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$ is continuous at $x \in X$ if for all nhbd $W_{f(x)} \subset Y$ of $f(x)$ there is a nhbd $V_x \subset X$ of x s.t. $f(V_x) \subset W_{f(x)}$.

- $f: X \to Y$ is continuous if continuous at every $x \in X$.

THM: $f: (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T}_Y)$ continuous iff $f^{-1}(G) \in \mathcal{T}_X \quad \forall G \in \mathcal{T}_Y$.

f^{-1}(open) is open





DEF: $f: X \to Y$ is a homeomorphism if it is continuous with continuous inverse. (i.e., f bijection s.t. f, f^{-1} cont.)

- If there is a homeomorphism between X and Y then X and Y are homeomorphic (i.e., indistinguishable as topological spaces).

DEF: Compactness $\equiv$ $K \subset X$ is compact if every open cover of K admits a finite subcover

NOTE: On $\mathbb{R}$, topology generated by open intervals

$$(a,b) := \{ x \in \mathbb{R} : a < x < b \}.$$

- Then $(0,1)$ is not compact. Covered by $\bigcup_{n \geq 1} \left(\frac{1}{n}, 1 - \frac{1}{n} \right)$
- $[0,1]$ is compact (Heine-Borel)

REMARK: It is "easier" to be compact when the topology has "fewer" open sets.

- Compactness is one of the main tools providing existence of limits (nets have convergent subnets or sequences w/ convergent subsequences).

METRIC SPACES

Include geometry via notion of distance between pts in X.

non-empty set

DEF: A metric on X is a function $d : X \times X \to \mathbb{R}$ s.t.

(i) $d(x,y) = d(y,x)$ (symmetry)

(ii) $d(x,y) \leq d(x,z) + d(z,y)$ (triangle inequality)

(iii) $d(x,y) \geq 0$ (non negativity)

(iv) $d(x,y) = 0 \iff x = y$ (definiteness)

$(X, d) \rightsquigarrow$ METRIC SPACE

- On $\mathbb{R}$, $d(x,y) := |x-y|$.
- On X set of n-letter words in a k-alphabet,

$$d(x,y) = \#\{i : x_i \neq y_i\} \quad \text{(Hamming distance)}$$

- On Cartesian product $X \times Y$

$$d_{X\times Y}\big((x_1,y_1),(x_2,y_2)\big) = d_X(x_1, x_1) + d_Y(y_1,y_2)$$

gives distances in $\mathbb{R}^n$

- Natural Topology $B_\varepsilon(x) := \{y \in X : d(x,y) < \varepsilon\}$ (open ball)

Then topology on $\mathcal{C}$ on X is generated by all open balls $B_\varepsilon(x)$, $x \in X$, $\varepsilon > 0$.

Closed ball : $\overline{B_\varepsilon(x)} = \{ y \in X : d(x,y) \le \varepsilon \}$.

- On $\mathbb{R}$, topology generated by open intervals (a,b).

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VECTOR SPACE V over field $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ when

- $(V, +)$ Abelian group $\quad + : V \times V \to V \quad f + g \in V$
- $\mathbb{F}$ acts on V : $\mathbb{F}$-module $\quad \cdot : V \times \mathbb{F} \to V \quad \lambda f \in V$

EXAMPLES :

- $\mathbb{R}^N \quad x = (x_1, —, x_N)$
- $f : (0,1) \to \mathbb{R}$, $L^2(0,1) \equiv \int_0^1 |f(x)|^2 dx < +\infty$

$f \in L^2(0,1)$, $g \in L^2(0,1) \Rightarrow f + g \in L^2(0,1)$

- $C^0([0,1])$ continuous functions on $[0,1]$

- Unit Sphere $\equiv \left\{ x \in \mathbb{R}^n : |x| = \left(\sum_{i=1}^n x_i^2 \right)^{1/2} = 1 \right\}$ is NOT a vector space (e.g., not closed under $+$). It is a metric space (for natural metric) and hence a topological space (for the associated topology).

DEF: NORM $\|\cdot\| : V \to \mathbb{R}$ s.t.

$$\text{norm}\left\{\begin{array}{l} \text{semi-norm}\left\{\begin{array}{ll} (i) & \|x\| \geq 0 \\ (ii) & \|\lambda x\| = |\lambda| \, \|x\| , \quad \lambda \in \mathbb{F} \\ (iii) & \|x+y\| \leq \|x\| + \|y\| \quad \text{(Triangle Inequality)} \end{array}\right. \\ \\ \quad (iv) \quad \|x\| = 0 \iff x = 0 \end{array}\right.$$

$(V, \|\cdot\|) \rightsquigarrow$ NORMED VECTOR SPACE

- On $(V, \|\cdot\|)$, $d(x,y) := \|x-y\|$ makes (V, d) into a metric space
- For V finite dimensional, all norms are equivalent.

  Not true in infinite dimensions, although a natural one makes the space complete.
- $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent if $\exists\, c, C > 0$ s.t.

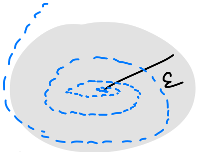
$$c\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1 .$$

- On $\mathbb{R}^n$, $\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$ $\quad 1 \leq p < \infty$

$$\|x\|_\infty := \sup_{1 \leq i \leq n} |x_i|$$

CONVERGENCE

DEF: $\{x_n\}_{n\in\mathbb{N}} \subset X$ converges to $x \in X$ iff $\forall \varepsilon > 0 \ \exists N_\varepsilon > 0$ s.t. $\forall n \geq N_\varepsilon$, $d(x_n, x) < \varepsilon$.



- conv. in metric sense $\Longrightarrow$ conv. in topological sense.

↑ converse is not true

DEF: CAUCHY SEQUENCES $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy if $\forall \varepsilon > 0$, $\exists N > 0$ s.t. $\forall n, m \geq N$, $d(x_n, x_m) < \varepsilon$.

- No notion of limit here

$x_n \to x \Longrightarrow x_n$ Cauchy

$d(x_n, x_m) \leq d(x_n, x) + d(x_m, x)$

$< \varepsilon/2 + \varepsilon/2$.

- In $\mathbb{Q}$, $x_n = \frac{p_n}{q_n} \longrightarrow \sqrt{2} \notin \mathbb{Q}$

x_n Cauchy not converging in $\mathbb{Q}$

$\mathbb{Q}$ completed $\longrightarrow \mathbb{R}$.

DEF: (X, d) is complete if every Cauchy sequence converges in X.

- $\boxed{\text{BANACH SPACE}} \equiv$ complete normed vector space

- $X \subset \mathbb{R}^n$. Then $C^0(X)$; $C^{k,\alpha}(X)$; $L^p(X)$; $W^{m,p}$ are all Banach spaces.

- $C^\infty(X) = \bigcap_{k \geq 1} C^k(X)$; $S(\mathbb{R}^n)$ (Schwarz); $S'(\mathbb{R}^n)$ are Fréchet spaces (not Banach)

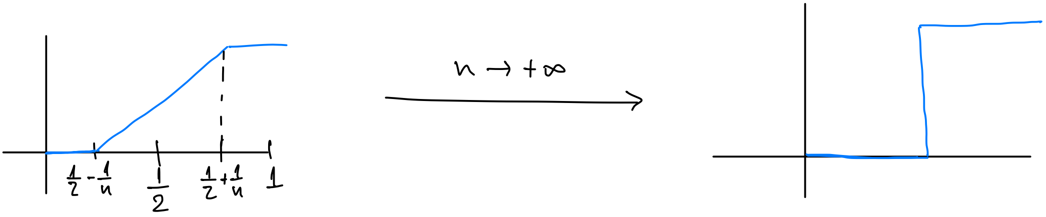
THM: Every metric space (X, d) has a completion $(\tilde{X}, \tilde{d})$

- $X \subset \tilde{X}$; $\tilde{d}(x,y) = d(x,y)$ $\forall x, y \in X$; X dense in $\tilde{X}$.

$\forall \tilde{x} \in \tilde{X}, \forall \varepsilon > 0, \exists x \in X$ s.t. $d(x, \tilde{x}) < \varepsilon$.

EXAMPLE: $C^0([0,1])$ continuous functions

$$\|u\|_2 = \left(\int_0^1 |u(x)|^2 \, dx \right)^{1/2}.$$



Cauchy sequence for $\|\cdot\|_2$
but limit is not in C^0

- Completion of $(C^0([0,1]), \|\cdot\|_2)$ $\underline{\underline{\text{is}}}$ $(L^2(0,1), \|\cdot\|_2)$

space of square-integrable classes of equivalence of fcts defined up to measure zero sets.

- Completion of $(C^0, \|\cdot\|_p)$ is $(L^p, \|\cdot\|_p)$ for $1 \leq p < \infty$.

- $(C^0([0,1]), \|\cdot\|_\infty)$ is Banach with uniform norm.

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CONTINUITY IN METRIC SPACES $(X, d_X), (Y, d_Y)$

DEF: $f : X \to Y$ continuous at $x_0 \in X$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $d_X(x, x_0) < \delta$ then $d_Y(f(x_0), f(x)) < \varepsilon$.

DEF: $f : X \to Y$ is continuous if continuous at every $x \in X$.

- Above $\delta = \delta(\varepsilon; x_0)$ depends on x_0.
  When $\delta(\varepsilon)$ is independent of x_0, f is uniformly cont.

- Sequential Continuity: f is seq. cont. at x if
  $(x_n \to x) \Rightarrow (f(x_n) \to f(x))$.

PROP: $F \subset X$ closed iff
$$\left\{ \begin{array}{l} (x_n \to x) \Rightarrow x \in F \\ x_n \in F, \ x \in X \end{array} \right\}.$$

- Closure $\overline{A}$ of A in X ≡ smallest closed set that contains A.

$$\overline{A} = \{ x \in X : \exists\, x_n \in A,\ x_n \longrightarrow x \}.$$

- Density $A \subset X$ is dense in X when $\overline{A} = X$ (e.g. $\overline{\mathbb{Q}} = \mathbb{R}$)

(X, d) dense in its completion $(\tilde{X}, \tilde{d})$ (for metric $\tilde{d}$!).

- (X, d) is separable if it has a countable dense subset.

Ex: $\mathbb{R} = \overline{\mathbb{Q}}$, $\#\mathbb{Q} = \#\mathbb{N}$.

COMPACTNESS

DEF: $K \subset X$ is sequentially compact if every sequence in K admits a converging subsequence in K.

$\mathbb{N}^* := \mathbb{N} \cup \{+\infty\}$

- Subsequence: $\varphi : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ s.t. $\varphi(n+1) \geq \varphi(n) + 1$.

- $x_n = (-1)^n$ does not converge but x_{2n} and x_{2n+1} are converging subsequences.

- A is precompact if $\overline{A}$ is compact.

THM: (Heine–Borel) Subsets of $\mathbb{R}^n$ are compact if and only if they are closed and bounded.

THM: (Bolzano–Weierstrass) Every bounded sequence in $\mathbb{R}^n$ has a converging subsequence.

EXERCISE: Prove that (i) K compact $\Rightarrow$ K closed & bounded.

(ii) Bolzano-Weierstrass $\Rightarrow$ Heine-Borel

- Heine-Borel FALSE in infinite dimensions: in

$$\ell^2 := \left\{ x = (x_n)_{n\in\mathbb{N}^*} : \left(\sum_{n=1}^{\infty} x_n^2 \right)^{1/2} < \infty \right\}$$

construct a "basis" $e^k \in \ell^2$ such that $e^k_j = \delta_{jk}$. Then $e_k \in \overline{B}$ closed unit ball. Choose $x_k = e^k$ for $k \geq 1$. Then $x_k \in \overline{B}$ is bounded. But $\|e^k - e^l\|_2^2 = 2$ when $k \neq l$. So, for any subsequence $\|x_{\varphi(k)} - x_{\varphi(l)}\|_2^2 = 2 \Rightarrow$ not Cauchy so not convergent.

◢

* The following holds for *any* compact subsets of metric spaces:

DEF: $\{G_\alpha : \alpha \in \mathcal{A}\}$ *cover* of A if $A \subset \bigcup_{\alpha \in \mathcal{A}} G_\alpha$.

DEF: $\{x_\alpha : \alpha \in \mathcal{A}\}$ is an *ε-net* of A if $A \subset \bigcup_{\alpha \in \mathcal{A}} B_\varepsilon(x_\alpha)$.

DEF: $A \subset X$ is *totally bounded* if it has a *FINITE* ε-net for all $\varepsilon > 0$.

THM: $A \subset X$ is *sequentially compact* *iff* it is *complete* and *totally bounded*.

↓

LECTURE 2

Jan 6th 2025

CONTINUOUS FUNCTIONS

THM: $f: K \to Y$ continuous and K compact, then $f(K)$ compact

(Continuous fcts map compact to compact)

THM: K metric compact and $f: K \to \mathbb{R}$ continuous, then f attains its minimum and maximum.

Pf: $f(K)^{\subset \mathbb{R}}$ cpt $\Rightarrow$ admits minimizer $m \leq f(x_n) \leq m + \frac{1}{n}$

$x_n \in K$

K compact $\Rightarrow$ $x_{\varphi(n)} \longrightarrow x \in K$

f continuous $\Rightarrow$ $f(x_{\varphi(n)}) \longrightarrow f(x) = m$

■

NOTE: $f_n(x) = x^n$ on $[0,1]$

$$f_n(x) \xrightarrow{n \to \infty} \begin{cases} f(x) = 0 & 0 \leq x < 1 \\ f(x) = 1 & x = 1 \end{cases} \Rightarrow \text{not continuous}$$

Upshot: $(C^0, \|\cdot\|_2)$ not complete so can't compare in L^2 sense

Need: Uniform norm $\|f\|_\infty := \sup_{x \in X} |f(x)|$

DEF: $f_n \to f$ uniformly iff $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$

THM: $\|\cdot\|_\infty$ preserves continuity

i.e., f_n bdd, continuous, and $\|f_n - f\|_\infty \to 0 \Rightarrow f$ continuous.

Pf:
$$|f(x) - f(y)| \le \underbrace{|f(x) - f_n(x)| + |f(y) - f_n(y)|} \le \frac{2\varepsilon}{3} \text{ for } n \text{ large b/c } \|f_n - f\|_\infty \to 0$$
$$+ \underbrace{|f_n(x) - f_n(y)|}_{\text{continuity of } f_n \text{ to get } \delta(\frac{\varepsilon}{3})}$$

■

- K compact metric space and $f : K \to \mathbb{R}$ continuous $\Rightarrow f$ bounded

So $\|f\|_\infty := \sup_{x \in X} |f(x)|$ defined (i.e., $< +\infty$)

- $(C^0(K), \|\cdot\|_\infty)$ is a normed vector space.

THM: K compact metric space. Then $(C^0(K), \|\cdot\|_\infty)$ is complete.

$$\boxed{(C^0(K), \|\cdot\|_\infty) \text{ is Banach space}} .$$

Pf: Let f_n be Cauchy $\Rightarrow$ for each $x \in K$, $f_n(x)$ is Cauchy

Since K compact, $f_n(x) \to y =: f(x)$.

$f_n \to f$. Now,

$$\|f_n - f\|_\infty = \sup_{x \in K} |f_n(x) - f(x)|$$

$$= \sup_{x \in K} \liminf_{m \to \infty} |f_n(x) - f_m(x)|$$

$$\leq \liminf_{m \to \infty} \sup_{x \in K} |f_n(x) - f_m(x)|$$

$$= \liminf_{m \to \infty} \| f_n - f_m \|_\infty$$

$$\xrightarrow{n \to \infty} 0 \text{ b/c Cauchy.}$$

Thus, f is continuous (by previous results on $\|\cdot\|_\infty$ preserving continuity).

∎

DEF: Support of f $\operatorname{supp} f := \overline{\{x \in X : f(x) \neq 0\}}$.

DEF: $C_c(X) := \{$ continuous compactly supported fcts on $X \}$.

DEF: $C_0(X) := \overline{C_c(X)}$

Example: $C_c(\mathbb{R}) \not\ni e^{-x^2} \in C_0(\mathbb{R})$.

e^{-x^2} approximated by compactly supported functions.

$C_0(\mathbb{R}) = \{$ fcts $\to 0$ at $\infty \}$.

DEF: $C_b(X) := \{$ bounded continuous functions on $X \}$.

$$C_c(X) \subset C_0(X) \subset C_b(X) \subset C(X)$$

↑ not closed; not Banach

↑ Banach ↑

↑ $\|\cdot\|_\infty$ not defined when X is unbounded for d

THM: (Weierstrass) $X \subset \mathbb{R}$. Polynomials are dense in $C^0([a,b], \|\cdot\|_\infty)$.

⚠ MIDTERM

EXERCISE: $\mathcal{F} \subset K$ equicontinuous $\Rightarrow$ $\mathcal{F}$ is uniformly continuous

↑ compact

(i.e., $\forall \varepsilon > 0 \ \exists \delta(\varepsilon)$ s.t. $\forall x, f$ the ε-δ rule holds)

COMPACTNESS IN ∞-DIMS.

- Heine-Borel does not apply.
- Total boundedness is difficult to check.

DEF: A family $\mathcal{F}$ of continuous fcts $(X, d_X) \to (Y, d_Y)$ is equicontinuous if $\delta = \delta(\varepsilon, x)$ is independent of $f \in \mathcal{F}$:

$$\forall x \in X,\ \forall \varepsilon > 0\ \exists \delta = \delta(\varepsilon, x) > 0 \text{ s.t.}$$

$$d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon,\ \forall f \in \mathcal{F}.$$

⚠

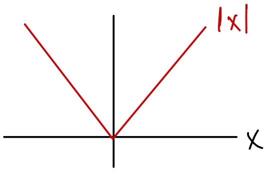
THM: (Arzelà – Ascoli) Let K be compact metric space. A subset of $(C(K), \|\cdot\|_\infty)$ is compact iff it is closed, bounded, and equicontinuous.

DEF: $f : X \to \mathbb{R}$ is Lipschitz continuous on X if $\exists L > 0$ s.t.

$$|f(x) - f(y)| \leq L\, d_X(x,y).$$

DEF: $$\mathrm{Lip}(f) := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_X(x,y)} < +\infty$$ (LIPSCHITZ CONSTANT)

EXAMPLE:



$$\mathrm{Lip}(|x|) = 1.$$

δ doesn't depend on x
↑

PROPOSITION: $\mathcal{F}_M := \{ f \text{ Lipschitz with } \mathrm{Lip}(f) \leq M \}$

is equicontinuous (actually, even uniformly continuous)

↳ not necessarily uniformly bounded... take $f_n = n$.

Pf: $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. $d_X(x,y) < \delta \Rightarrow |f(x) - f(y)| < \varepsilon$

choose $\delta(\varepsilon) = \varepsilon / M$. Then $|f(x) - f(y)| \leq M \, d_X(x,y)$

$$< M \, \varepsilon/M = \varepsilon .$$

⚠ $\mathcal{F}_M$ not bounded.

EXERCISE

$\mathcal{F}_{M,N} := \{ f \in C^0(K, \|\cdot\|_\infty) : \|f\|_\infty \leq N \text{ and } \mathrm{Lip}(f) \leq M \}$.

bounded, closed, equicontinuous $\Rightarrow \mathcal{F}_{M,N}$ compact family/subset

Exercise

- $\mathcal{G}_\alpha := \{ f \in C(K, \|\cdot\|) : \|f\|_\infty \leq N, \ |f(x) - f(y)| \leq M \, d_X(x,y)^\alpha \}$

$0 < \alpha \leq 1$

Hölder-continuous functions

$\mathcal{G}_\alpha$ is closed, bounded, Exercise

and equicontinuous

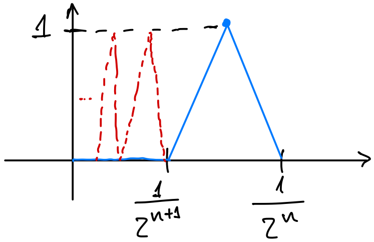
(Take: $\delta = (\varepsilon/M)^{1/\alpha}$)

EXERCISE: and find counterexamples

IMPORTANT: CHECK (1), (2), (3)

f_n continuous

COUNTEREXAMPLE: $\mathcal{F} := \{f_n\}_{n\in\mathbb{N}^*}$ where



(1) $f_n(x) \longrightarrow 0 \quad \forall x \in [0,1]$

(2) $f_{\varphi(n)}$ is not Cauchy: $\|f_n - f_m\|_\infty = 1 \quad \forall n \neq m$

$\Rightarrow$ No convergent subsequence!

(3) $\mathcal{F}$ is not equicontinuous (slope $\to +\infty$ around zero as $n \to \infty$)

Upshot: $B_1(0)$ (or any other ball containing zero) NOT compact in $(C^0([0,1]), \|\cdot\|_\infty)$. Above $\mathcal{F} \subset B_1(0)$.

GENERALIZATION: (K, d_X) metric compact;
(Y, d_Y) complete metric space.

- $C^0(K, Y) \equiv$ space of continuous fcts $K \to Y$ with $d(f,g) := \sup_{x\in K} d_Y(f(x), f(y))$
- Then $(C(K), d)$ is complete.
- Subsets are compact iff bounded, closed, equicontinuous.

CONTRACTION MAPPING (Ch 3)

<u>DEF</u>: (X,d) Banach (i.e., complete). $T : X \to X$ is a contraction if $\exists\, c \in (0,1)$ such that

$$d(Tx, Ty) \le c\, d(x,y) \quad \forall x,y \in X.$$

- T may be nonlinear.
- X not necessarily a vector space (often a ball in vec. space).

⚠

THM: (Banach's Contraction Mapping) If $T : X \to X$ is a contraction, then $\exists!\, x_* \in X$ such that $T x_* = x_*$.

EXAMPLE: 2^{nd} order Fredholm integral equation

$$f(x) = g(x) + \underbrace{\int_a^b k(x,y)\, f(y)\, dy}_{=:\, Kf} \quad \text{for } g \in C^0([a,b]), \; k \in C^0([a,b]^2)$$

- Find $f \in C^0([a,b])$ that satisfies eq. above.
- Rewrite eq. as $f = g + Kf$ $\qquad f \overset{?}{=} (1-K)^{-1} g$
- Want to write f as the sol^{n} of $f = T(f)$.
- Thus, let $T(f) = Tf =: g + Kf$.

- T maps $C^0([a,b])$ to $C^0([a,b])$ with $d(f,h) = \|f-h\|_\infty$ complete. Is T a contraction?

- $$d(Tf, Th) = \|Tf - Th\|_\infty$$

$$= \sup_{x\in[a,b]} \left| \int_a^b K(x,y)\,[f(y) - h(y)]\,dy \right|$$

$$\le \|f-h\|_\infty \underbrace{\sup_{x\in[a,b]} \int_a^b |K(x,y)|\,dy}_{!!}$$

T contraction $\Leftrightarrow c < 1$.

- When $c < 1$, above eq. admits a unique solution

$$f = g + Kf \qquad (1-K)f = g \qquad f = (1-K)^{-1} g = \underbrace{\sum_{n=0}^{\infty} K^n g}$$

$\|K^n g\|_\infty \le \|K\|^n \|g\|_\infty = c^n \|g\|_\infty \Longrightarrow$ converges in $\|\cdot\|_\infty$ norm b/c $c < 1$.

$$(1-K)^{-1} g = \sum_{n=0}^{\infty} K^n g$$

NEUMANN SERIES EXPANSION

LECTURE 3

Jan 12th 2026

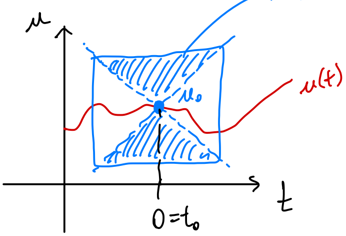
PEANO CONSTRUCTION

$$\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(0) = u_0 \end{cases}$$

$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ continuous, $t \in I \ni 0$.

$|f(t,u)| \leq M$

By the bound $|f(t,u)| \leq M$ we get a bound in the velocity of u. So,



u doesn't go there by the velocity bound

GOAL: Construct $\underline{\underline{\text{a}}}$ sol$^{\underline{n}}$.

Possibly non unique

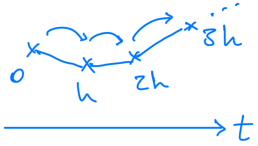
1) Construct an approximation by evolving in time.

2) Prove they are equicontinuous to get compactness (b/c it's already bounded and closed).

3) Prove limit solves the ODE.

1) for $0 \ll h < 1$ in the limit $h \to 0$,

$$\frac{u(t+h) - u(t)}{h} \simeq f(t, u(t)).$$



(Forward Euler)

Take $t = kh \Rightarrow u_{h,k} = u_{h,k-1} - h f(h(k-1), u_{h,k-1})$

$u_h(t) :=$ linear interpolation

2) • $u_h(t)$ are bounded in the box

• $u_h(t)$ are equicontinuous in the box b/c they're Lipschitz w/ $\mathrm{Lip}(u_h) \le M$.

Arzelà-Ascoli $\Rightarrow u_{\varphi(h)}(t) \xrightarrow{(C, \|\cdot\|_\infty)} u(t)$, $h = \frac{1}{n}$ $\quad \Longleftrightarrow \|u_{\varphi(h)}(t) - u(t)\|_\infty \to 0$

3) <u>Claim</u>: This limit $u(t)$ solves ODE.

Pf: Rewrite ODE in integral form

b/c we don't know what $\dot u$ is...

$$u(t) = u_0 + \int_0^t f(s, u(s))\,ds$$

$\Rightarrow \dot u$ is continuous

$$\begin{tikzcd}
u_h(t) \arrow[ddd, "h\to 0"'] & = u_h(0) + \int_0^t \dot u_h(s)\,ds & & \\
 & = u_h(0) \arrow[dd] & + \int_0^t f(s, u_h(s))\,ds \arrow[dd, "f \text{ continuous ptwise}"] & + \int_0^t \underbrace{\left[\dot u_h(s) - f(s, u_h(s))\right]} ds \\
 & & & r_h(s) \arrow[d, "h\to 0"] \\
u(t) & u(0) & + \int_0^t f(s, u(s))\,ds & + 0
\end{tikzcd}$$

$$\Rightarrow u(t) = u_0 + \int_0^t f(s, u(s))\,ds \quad ✓$$

■

LECTURE 4

CONTRACTION MAPPING THM

Jan 14th 2026

(Ch 5)

DEF: (X, d) complete metric space. $T : X \to X$ is a contraction if $\exists\, c \in [0,1)$ s.t. $d(Tx, Ty) \le c\, d(x,y) \quad \forall x, y \in X$.

THM: (Contraction Mapping) $T : X \to X$ contraction on complete (X,d) then $\exists!\, x_* \in X$ s.t. $Tx_* = x_*$.

Pf: Keep applying T to a pt. to get a Cauchy seq. that converges on (X,d).

$x_1 = Tx_0$

$x_2 = Tx_1$

$\vdots$

$x_{n+1} = Tx_n$

$$\begin{aligned} d(x_{n+m}, x_n) &\overset{\text{Triangle Ineq}}{\le} d(x_{n+m}, x_{n+m-1}) + \dots + d(x_{n+1}, x_n) \\ &\le c\, d(x_{n+m-1}, x_{n+m-2}) + \dots \\ &\le c^{n+m-1} d(x_1, x_0) + \dots + c^n d(x_1, x_0) \\ &\le c^n (1 + c + c^2 + \dots)\, d(x_1, x_0) \\ &\le \frac{c^n}{1-c}\, d(x_1, x_0) \xrightarrow{n \to \infty} 0 \end{aligned}$$

$$\Rightarrow x_n \text{ Cauchy} \overset{\text{completeness}}{\Rightarrow} x_n \xrightarrow{n \to \infty} x_* \in X$$

T is c-Lipschitz, hence continuous $\Rightarrow$

$$\begin{tikzcd} Tx_n \arrow[r] \arrow[ddr, equal] & Tx_* \arrow[d, equal] \\ & x_* \\ & x_{n+1} \arrow[u] \end{tikzcd}$$

$x, y \in X$, $Tx = x$, $Ty = y$ $\rightsquigarrow$ $\underset{\overset{\|}{d(x,y)}}{d(Tx, Ty)} \le c\, d(x,y) \Rightarrow d(x,y) = 0$

∎

EXAMPLE: Picard's Thm for ODEs $\begin{cases} \dot{u}(t) = f(t, u(t)) \\ u(t_0) = u_0 \end{cases}$ f Lipschitz in u.

$f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$u : \mathbb{R} \to \mathbb{R}^n$

$$T[f](t) := u(t_0) + \int_{t_0}^{t} f(s, u(s))\, ds$$

Two ways of proving this

1) Show T is a contraction on $C([t_0 - \delta, t_0 + \delta])$ δ small

2) Show T^n is a contraction on $C([t_0 - T_0, t_0 + T_0])$ T_0 large

↘ $t > s_1 > s_2 > \dots > s_n > 0$ $\quad t^n / \boxed{n!}$ = vol. of tetrahedron in n-dim.

gives contraction

◢

BANACH SPACES (Ch 5)

Not all norms are equivalent in ∞-dim.

DEF: $(X, \|\cdot\|)$ Banach $\iff$ complete normed space w.r.t. $\|\cdot\|$

Ex:
- $C(K, \|\cdot\|_\infty)$ K compact
- $C^r(K, \|\cdot\|_{r,\infty})$ r-times continuously differentiable w.r.t

$$\|f\|_{r,\infty} = \sum_{j=0}^{r} \|f^{(j)}\|_\infty .$$

Equivalent norms $\boxed{\equiv} \|f\|_\infty + \|f^{(r)}\|_\infty$

- $C^\infty(K) = \bigcap_{r \geq 0} C^r(K)$ with $d(f,g) = \sum_{r=0}^{\infty} \dfrac{2^{-r} \|f^{(r)} - g^{(r)}\|_\infty}{1 + \|f^{(r)} - g^{(r)}\|_\infty}$

(FRÉCHET SPACE) semi-norm, complete

- L^p spaces, $W^{m,p}$ Sobolev spaces (m "derivatives" in L^p).
- $\ell_p(\mathbb{N})$ spaces of infinite seqs. $x = (x_n)_{n \geq 1}$, $x_n \in \mathbb{C}$,

with
$$\begin{cases} \|x\|_p := \left(\sum_{n=1}^{\infty} |x_n|^p \right)^{1/p} & 1 \leq p < \infty \\ \|x\|_\infty := \sup_{n \geq 1} |x_n| \end{cases}$$

⚠ **IMPORTANT INEQUALITIES (MEMORIZE!)**

Proposition: $(\ell_p(\mathbb{N}), \|\cdot\|_p)$ is a Banach for $1 \leq p \leq \infty$.

⚠ **Pf:** ($p = \infty$) is very similar to functions.

($1 \leq p < \infty$) **Claim 1:** $\|\cdot\|_p$ is a norm on $\ell_p(\mathbb{N})$

Pf (Claim 1): All easy except triangle inequality for $p \neq 1$.

- $\|x+y\|_p \leq \|x\|_p + \|y\|_p$ (**Minkowski's Inequality**)

Define $q := \frac{p}{p-1} \iff \frac{1}{p} + \frac{1}{q} = 1$ for $1 \leq p \leq \infty$

- **Young's Inequality** for $1 < p < \infty$

$\frac{1}{p} + \frac{1}{q} = 1$ and $a, b \geq 0 \implies \boxed{ab \leq \frac{a^p}{p} + \frac{b^q}{q}}$

Since log is concave,

$$\log\left(\frac{a^p}{p} + \frac{b^q}{q}\right) \geq \frac{1}{p}\log(a^p) + \frac{1}{q}\log(b^q) = \log(ab)$$

Exp and exp(·) is increasing. □

- **HÖLDER'S INEQUALITY** $x \in \ell_p,\ y \in \ell_q,\ \frac{1}{p} + \frac{1}{q} = 1$, then

$$\|xy\|_1 \le \|x\|_p \|y\|_q$$

TRICK: $ab = (\lambda a)\left(\frac{b}{\lambda}\right)$

Pf: $\sum_{n\ge1} x_n y_n = \sum_{n\ge1} (x_n \lambda^{1/p})(y_n \lambda^{1/p})$

Young's Inequality $\le \sum_{n\ge1} \left(\frac{\lambda x_n^p}{p} + \frac{\lambda^{-p/q} y_n^q}{q} \right)$ Minimize over λ.

$\lambda := \frac{\|y\|_q}{\|x\|_p^{p-1}} \leadsto = \left(\frac{1}{p} + \frac{1}{q}\right) \|x\|_p \|y\|_q$.

$\square$

- **MINKOWSKI'S INEQUALITY** $\|x+y\|_p \le \|x\|_p + \|y\|_p$.

$$\|x+y\|_p^p = \sum_{n\ge1} |x_n + y_n|^p$$

$$\le \sum_{n\ge1} |x_n + y_n|^{p-1} (|x_n| + |y_n|)$$

Hölder ↪ $\le \left(\sum_{n\ge1} |x_n + y_n|^{\overbrace{q(p-1)}^{\overset{p}{\shortparallel}}} \right)^{1/q} \left[\left(\sum_{n\ge1} |x_n|^p \right)^{1/p} + \left(\sum_{n\ge1} |y_n|^p \right)^{1/p} \right]$

$$= \|x+y\|_p^{p-1} (\|x\|_p + \|y\|_p) .$$

■

LECTURE 4

(Ch 5)

Jan 21, 2026

Banach Spaces:
- $(C(K), \|\cdot\|_\infty)$, $(C^r(K), \|\cdot\|_r)$, $(L^p, \|\cdot\|_p)$
- $L^p \rightsquigarrow \ell^p := \left\{ \underset{\sim}{x} \in \mathbb{C} : \left(\sum_{n=1}^{\infty} |x_n|^p\right)^{1/p} < +\infty \right\}$
- $(\ell^p, \|\cdot\|_p)$
- $B(X,Y) := \{$ bdd linear operators $T : X \to Y \}$.

Pf: [$(\ell^p, \|\cdot\|_p)$ is Banach (CTD)] We now claim that every Cauchy sequence converges.

Take $x^{(k)} \in \ell^p$ Cauchy $\iff \sum_{n=1}^{\infty} |x_n^{(k)} - x_n^{(k+l)}|^p \xrightarrow{k\to\infty} 0$

$x^{(k)} \overset{?}{\longrightarrow} y$. So, $\forall n$, $|\underbrace{x_n^{(k)} - x_n^{(k+l)}}_{\in \mathbb{C} \text{ which is complete}}| \xrightarrow{n\to\infty} 0 \rightsquigarrow x_n^{(k)} \xrightarrow{k\to\infty} y_n$.

$y \in \ell^p$? $\|x^{(k)} - y\| \overset{?}{\longrightarrow} 0$

$$\sum_{n=1}^{N} |y_n|^p = \sum_{n=1}^{N} \lim_{k\to\infty} |x_n^{(k)}|^p = \lim_{k\to\infty} \sum_{n=1}^{N} |x_n^{(k)}|^p$$

$$\leq \limsup_{k\to\infty} \underbrace{\sum_{n=1}^{N} |x_n^{(k)}|^p}_{\leq C \text{ since } x_n^{(k)} \text{ is Cauchy}}$$

Thus,

$$\sum_{n=1}^{\infty} |y_n|^p \leq C \Rightarrow y_n \in \ell^p.$$

Now,

$$\|x_n^{(k)} - y_n\|_p^p \overset{\text{! GOOD TRICK}}{\leq} \frac{\varepsilon}{2} + \sum_{n=1}^{N(\varepsilon)} |x_n^{(k)} - y_n|^p$$

$$= \frac{\varepsilon}{2} + \sum_{n=1}^{N(\varepsilon)} \lim_{l \to \infty} |x_n^{(k)} - x_n^{(l)}|^p$$

$$= \frac{\varepsilon}{2} + \lim_{l \to \infty} \sum_{n=1}^{N(\varepsilon)} |x_n^{(k)} - x_n^{(l)}|^p$$

$$\leq \frac{\varepsilon}{2} + \limsup_{l \to \infty} \underbrace{\|x_n^{(k)} - x_n^{(l)}\|_p^p}_{\leq \varepsilon \text{ for large enough } l.}$$

$$\leq \varepsilon.$$

$$\blacksquare$$

———————— // ————————

BOUNDED LINEAR OPERATORS

DEF: $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is linear if

- $T(x+y) = Tx + Ty \quad \forall x, y \in X$
- $T(\lambda x) = \lambda T(x) \quad \forall x \in X, \lambda \in \mathbb{C}$

DEF: T is injective if $Tx = Ty \Rightarrow x = y$.

DEF: T is surjective if $\forall z \in Y \ \exists x \in X$ s.t. $Tx = z$.

DEF: T injective + surjective = bijective.

DEF: $T^{-1} : Y \to X$ and is linear.

DEF: A linear operator $T : (X, \|\cdot\|_X) \to (Y, \|\cdot\|_Y)$ is BOUNDED if $\exists M > 0$ such that

$$\|Tx\|_Y \leq M \|x\|_X \quad \forall x \in X.$$

$$\Leftrightarrow \sup_{\|x\|_X = 1} \|Tx\|_Y < \infty$$

$$\Leftrightarrow \sup_{\|x\|_X \leq 1} \|Tx\|_Y < \infty$$

$$\Leftrightarrow \|T\| := \sup_{X \ni x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} < \infty$$

- **FACT**: Space of bdd linear operators $\mathcal{B}(X,Y)$ is a normed vector space with operator norm:

$$\|T\| := \inf\{M : \|Tx\|_Y \le M \|x\|_X\}$$

$$= \sup_{X \ni x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\| = 1} \|Tx\|_Y \overset{\text{bdd}}{<} +\infty$$

[$\|\cdot\|$ is called OPERATOR or UNIFORM norm]

- **FACT**: $\|Tx\| \le \|T\| \, \|x\|$.

- **ALGEBRAIC PROPERTIES**: L^p not algebra ($\frac{1}{\sqrt{x}}$ near origin...)

  But, if we have $T : X \to Y$ and $S : Y \to Z$ are bounded, then $ST : X \to Z$ is also bounded ($\|ST\| \le \|S\| \, \|T\|$)

Ex: $X = (C([0,1]), \|\cdot\|_\infty)$

$$T : X \to X$$
$$f \mapsto \int_0^x f(t)\,dt$$

$$\|Tf\|_\infty = \sup_{x \in [0,1]} \left| \int_0^x f(t)\,dt \right| \le \sup_{x \in [0,1]} \int_0^x \|f\|_\infty \,dt$$

$$\le \|f\|_\infty$$

$$\Rightarrow \|T\| \le 1 \quad \text{but} \quad \|T\mathbb{1}(x)\| = 1 \quad \Rightarrow \|T\| = 1.$$

- **DERIVATIVE** : On $C^\infty([0,1]) := \bigcap_{r \geq 1} C^r([0,1])$.

Let $X := \left(C^\infty([0,1]), \|\cdot\|_\infty\right)$ ← NOT BANACH ⚠

But still a normed space nonetheless.

$$T: X \to X$$
$$f \mapsto f'$$

← Need C^∞ to ensure that derivative is in the same space

T is unbounded in X b/c : $f_n(x) := e^{nx} \in C^\infty$

$$f_n'(x) = n e^{nx}$$

$$\|f_n\|_\infty = e^n$$
$$\|Tf_n\|_\infty = \|f_n'\|_\infty = ne^n$$
$$\Rightarrow \quad \frac{\|f_n'\|}{\|f_n\|} = n \nearrow \infty \, .$$

- Can make it bounded : $X := \left(C^1[0,1], \|\cdot\|_{1,\infty}\right)$ ← This IS Banach

$$Y := \left(C^0[0,1], \|\cdot\|_\infty\right)$$

Set $T: X \to Y$, $f \mapsto f'$ $\Rightarrow \|T\| \leq 1$.

∎

- NOTE : $A: X \to Y$, $\dim X = n < \infty$, $\dim Y = m < \infty$ $\Rightarrow X \simeq \mathbb{C}^n$, $Y \simeq \mathbb{C}^m$.

In this basis, can represent $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \in \mathbb{C}^{m \times n}$.

THM: $T: X \to Y$ linear. TFAE

(1) T is bounded

(2) T is continuous (Lipschitz, in fact)

(3) T is continuous at 0

Pf: (1) $\Rightarrow$ (2)

$$\|Tx - Ty\| = \|T(x-y)\| \overset{T \text{ bounded}}{\leq} \|T\| \, \|x-y\|$$

$\Rightarrow$ $\|T\|$ - Lipschitz in fact.

(2) $\Rightarrow$ (3) Clear.

(3) $\Rightarrow$ (1) Note $T0 = 0$ by linearity. WTS: $\forall \varepsilon > 0 \; \exists \delta > 0$ s.t. $\|x\| < \delta \Rightarrow \|Tx\| < \varepsilon$.

Say $\varepsilon = 1$ and $z \in X$, $z \neq 0$,

$$\frac{\|\delta z\|}{\|z\|} \leq \delta \Rightarrow \left\| T \frac{\delta z}{\|z\|} \right\| \leq 1 \Rightarrow \frac{\|Tz\|}{\|z\|} \leq \frac{1}{\delta}$$

$\Rightarrow$ T bounded.

∎

THM: (Bounded linear transformation)

$T : \underset{\substack{\uparrow \\ \text{linear subspace}}}{M} \subset X \longrightarrow \underset{\nwarrow \text{Banach space}}{Y}$, $M \subset X$ dense (i.e, $\bar{M} = X$)

Then, if T is bounded, there is a unique extension

$\bar{T} : X \longrightarrow Y$ such that $\bar{T}|_M = T$ and $\|\bar{T}\| = \|T\|$

Pf: Given $x \in X$, take a sequence

$$M \ni x_n \longrightarrow x \in X$$

T bounded $\Longrightarrow$ $\|y_n - y_m\|_Y \leq \|T\| \underbrace{\|x_n - x_m\|_X}_{\text{Cauchy b/c it converges}}$

$\swarrow$ y_n is Cauchy

$\swarrow$

y_n converges b/c
Y is Banach

So, $y_n \to y \in Y$ thus set $\bar{T}x := y$.

Check: $\bar{T}|_M = T$ and $\|\bar{T}\| = \|T\|$.

∎

⚠ IMPORTANT #1

OPEN MAPPING THEOREM: Let X, Y be Banach spaces and let $T \in B(X,Y)$ be surjective. Then T is open.

$T(\text{open})$ is open

COROLLARY: If $T \in B(X,Y)$ is bijective, then T is open; hence continuous (preimages); hence T^{-1} is continuous; hence T^{-1} is bounded.

NOTE: T open $\iff T(B_1(0)) \supset B_r(0)$ for some $r > 0$.

LECTURE 5

(Ch 5)

Jan 23, 2026

COROLLARY 1 (OMT): X Banach for $\|\cdot\|_1, \|\cdot\|_2$ and $\exists C > 0$ s.t. $\|\cdot\|_1 \leq C \|\cdot\|_2$. Then, $\exists C_2 > 0$ s.t. $\|\cdot\|_2 \leq C_2 \|\cdot\|_1$.

Pf: $Id : (X, \|\cdot\|_2) \longrightarrow (X, \|\cdot\|_1)$ is bdd, linear, bijection since b/c $\|\cdot\|_1$ is bdd by $\|\cdot\|_2$. So inverse is also bdd by OMT. □

COROLLARY 2 (OMT): Assume $(X, \|\cdot\|_1)$, $(X, \|\cdot\|_2)$ are Banach with same Cauchy limits. Then $\|\cdot\|_1 \sim \|\cdot\|_2$.

WARNING: Only need 1 norm to be Banach! Usually norms are not equivalent.

CLOSED GRAPH THEOREM: $T: X \to Y$ linear and X, Y Banach. $\mathrm{Grap}(T) := \bigcup_{x \in X} [x, Tx] \subset X \times Y$. We say T is closed if its $\mathrm{Graph}(T) \subset X \times Y$ is closed; i.e.,

$$\{(x_n, Tx_n) \to (x,y)\} \Longrightarrow \{y = Tx\}.$$

$$T \text{ CLOSED} \Longrightarrow T \text{ BOUNDED}$$

Pf: $(X, \|\cdot\|_X + \|T\cdot\|_Y) \equiv$ Graph norm $\square$

REMARK: T bounded $\Rightarrow$ T continuous $\Rightarrow$ T closed.

DEF: $T: X \to Y$ linear $\quad \ker T := \{x \in X : Tx = 0\}$

$$\mathrm{im}\, T := \{y \in Y : \exists x \in X \text{ s.t. } Tx = y\}$$

$$= \bigcup_{x \in X} Tx.$$

THM: $\ker(T) \subset X$ and $\operatorname{im}(T) \subset Y$ are linear subspaces.

- $T \in B(X,Y) \Rightarrow \ker T$ is closed (not true in general for im)
- $T: X \to Y$ injective $\Leftrightarrow \ker T = \{0\}$
  surjective $\Leftrightarrow \operatorname{im} T = Y$

EXAMPLE: $X = (C([0,1]), \|\cdot\|_\infty)$, $K: X \to X$, $f \mapsto Kf(x) := \int_0^x f(t)\,dt$

$\|K\| = 1$. $\operatorname{im} K = \{f \in C^1([0,1]) : f(0) = 0\}$

$$\overline{\operatorname{im} K}^{\|\cdot\|_\infty} = \{f \in C^0([0,1]) : f(0) = 0\} \supsetneq \operatorname{im} K.$$

- $T = I + K : X \to X$, $Tf(x) = f(x) + \int_0^x f(t)\,dt$.

$$Tf = g \Leftrightarrow f(x) = g(x) - \int_0^x e^{-(x-y)} g(y)\,dy =: T^{-1} g(y)$$

$\Rightarrow \operatorname{im} T = X$ (closed). ◢

PROPOSITION: $T: X \to Y$ bounded, X, Y Banach. Then:

$$\{\exists c > 0 \text{ s.t. } c\|x\| \le \|Tx\| \ \forall x \in X\} \Leftrightarrow \{\operatorname{im} T \text{ closed and } \ker T = \{0\}\}.$$

METHOD OF
A PRIORI
ESTIMATES

If we can show Tx bounded below, then $Tx = y$ has a unique result!

//

CONVERGENCE

DEF: $T_n \xrightarrow{n\to\infty} T$ uniformly if $\|T_n - T\| \xrightarrow{n\to+\infty} 0$.

THM: X normed and Y Banach. Then $(\mathcal{B}(X,Y), \|\cdot\|_\infty)$ is Banach.

//

COMPACT OPERATORS

DEF: $T: X \to Y$ is COMPACT if $T(B_1(0)) \subset Y$ is precompact

- T compact maps bounded families to compact families
- T compact $\iff$ for each seq. $(x_n)_{n\in\mathbb{N}} \subset X$ with $\|x_n\|_X \leq C$
  $\exists$ subseq. $x_{\varphi(n)}$ s.t. $T x_{\varphi(n)}$ converges in Y.

- X, Y Banach spaces $(\mathcal{B}(X,Y), \|\cdot\|_\infty)$ is Banach and an algebra.
- Let $K(X,Y)$ be the space of compact operators in $\mathcal{B}(X,Y)$.

Proposition: $K(X,Y) \subset \mathcal{B}(X,Y)$ is closed linear subspace.

Obs: $T_n \xrightarrow{n\to\infty} T$ uniformly and T_n compact $\Rightarrow$ T compact ⚠

↖ **USEFUL !**

- $\dim \operatorname{im} T < \infty \Rightarrow T$ compact (by Heine-Borel)
- $S \in K(\cdot,\cdot)$ and $T \in \mathcal{B}(\cdot,\cdot)$ then ST compact
  TS compact } when defined

⚠ **VERY USEFUL:** T_n compact and $\|T_n - T\|_\infty \to 0 \Rightarrow T$ compact

Def: (STRONG CONVERGENCE) $T_n \in \mathcal{B}(X,Y)$ converges to T strongly iff $\lim_{n\to\infty} T_n x = Tx \quad \forall x \in X$.

i.e., $\|T_n x - Tx\|_Y \to 0 \quad \forall x \in X$ (in strong topology of Y)

THM: $T_n \to T$ uniformly $\Rightarrow$ $T_n \to T$ strongly

Pf: $\|T_n x - Tx\|_Y \leq \|T_n - T\| \, \|x\|_X$.

WARNING!

($\Leftarrow$) NOT TRUE!

Convergence in uniform norm is stronger than strong convergence!

UNIFORM BOUNDEDNESS THEOREM (Banach-Steinhaus Thm)

X, Y Banach, $(T_i)_{i\in I} \subset \mathcal{B}(X,Y)$.

not necessarily countable

Assume $\sup_{i\in I} \|T_i x\|_Y < \infty$ $\forall x \in X$. Then $\exists\, C > 0$ s.t.

$$\|T_i x\|_Y \leq C \|x\|_X \quad \forall x \in X \text{ and } i \in I$$

uniform bound

COROLLARY: X, Y Banach spaces. $T_n \in \mathcal{B}(X,Y)$ and $T_n \to T$ strongly. Then, $\sup_n \|T_n\| < \infty$ and $T \in \mathcal{B}(X,Y)$ with

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\|.$$

- Application of strong convergence : LAX EQUIVALENCE THEOREM.

- $A : X \to Y$ bounded and A_h seq. of approximations $h := \frac{1}{n} \to 0$

- Consider $Au = f$ and $A_h u_h = f_h$.

- Assume $A_h : X \to Y$ bijection, A_h^{-1} bounded.

DEF: Approximations scheme is convergent if $f_h \to f \Rightarrow u_h \to u$

———— consistent if $A_h v \xrightarrow{h \to 0} Av \quad \forall v \in X$.

———— stable if $\| A_h^{-1} \| \leq M \quad \forall 0 < h \leq h_0$

THM: A consistent scheme is convergent iff it is stable

Useful direction

Pf: $(\Leftarrow)$ $u - u_h = A_h^{-1} \left[(A_h - A) u + f - f_h \right]$

So, $\| u - u_h \| \leq \| A_h^{-1} \| \left(\| (A_h - A) u \| + \| f - f_h | \right) \longrightarrow 0$

$\Rightarrow u_h = A_h^{-1} f$. $u_h \to u \Rightarrow A_h^{-1} f$ bounded $\forall f$
for $f_h = f$

Uniform Boundedness $\Rightarrow \| A_h^{-1} \| \leq C$ uniformly in h. ∎

- Applications of convergence to continuous groups.

- $A: X \to Y$ bounded X Banach.

- Define $e^A := \sum_{k=0}^{\infty} \frac{A^k}{k!}$, $\|e^A\| \leq e^{\|A\|} \Rightarrow e^A$ defined in $\mathcal{B}(X)$.

ODEs: $\begin{cases} \dot{x} = Ax \\ x(0) = x_0 \end{cases} \rightsquigarrow x(t) = e^{tA} x_0$. $T(t) := e^{tA}$ solution operator

- $T(t)$ is a uniformly continuous group if

  (i) $T(0) = id$

  (ii) $T(t+s) = T(t)T(s)$ $\forall t, s \in \mathbb{R}$

  (iii) $T(t) \to id$ uniformly as $t \to 0$.

- $T(t) = e^{tA}$ is a uniformly continuous group.

- $T(t)$ is a strongly continuous group when (iii) is replaced by

  (iii'): $T(t) \to id$ strongly as $t \to 0$.

Obs: When A unbounded, this is the best we can hope for.

LECTURE 6

(Ch 5)

Jan 26th, 2026

← strong topology

- Banach $(B, \|\cdot\|)$. $T_n : X \to Y$

  ↳ Convergence (uniform) $T_n \to T$ when $\|T_n - T\| \to 0$.

← weaker than uniform

- Strong convergence: $T_n \xrightarrow{\text{strongly}} T$ when $T_n x \to T \;\; \forall x \in X$

- Uniform $\Rightarrow$ strong but $\not\Leftarrow$

APPLICATION: Continuous Groups

$$\begin{cases} \dot{u} = Au \\ u(0) = u_0 \end{cases} \rightsquigarrow u(t) = e^{tA} u_0$$

- Uniformly Continuous Groups

  (i) $T(t+s) = T(t)\,T(s) \quad \forall\, t, s \in \mathbb{R}$

  (ii) $T(0) = \mathbb{1}$

  (iii) $T(t) \to \mathbb{1}$ uniformly as $t \to 0$

- Strongly Continuous Group: $T(t) \to \mathbb{1}$ strongly as $t \to 0$

- If A bounded, then $T(t) = e^{tA}$ is a unif. cont. group.

Uniform Boundedness Principle:

$T_n \to T$ strongly $\Rightarrow$ $\|T_n\| \le C$ uniformly
and $\|T\| < \infty$.

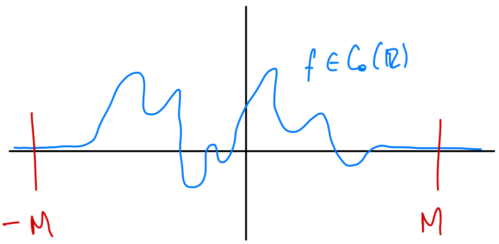
EXAMPLE: SHIFT OPERATORS

- $f \in C_0(\mathbb{R})$
- Shif Operator $\tau_h : C_0(\mathbb{R}) \longrightarrow C_0(\mathbb{R})$

  $f(x) \longmapsto f(x+h) =: (\tau_h f)(x)$
- $\tau_0 = \mathbb{1}$ and $\tau_{h_1+h_2} = \tau_{h_1} \circ \tau_{h_2}$.
- $\tau_h \xrightarrow{h\to 0} \mathbb{1}$ strongly (but not uniformly)
- GOAL: Compute $\sup_{x\in\mathbb{R}} \left|(\tau_h f)(x) - f(x)\right| \overset{?}{\underset{h\to 0}{\longrightarrow}} 0$

$$\shortparallel$$

f is unif. cont. $\Longleftrightarrow$ $\sup_{x\in\mathbb{R}} \left|f(x+h) - f(x)\right|$ ← Property of unif continuity.



Since $f \in C_0(\mathbb{R})$, $\forall \varepsilon > 0$, $\exists M > 1 > h > 0$ s.t. $|f(x)| < \varepsilon$ on $|x| > M$.

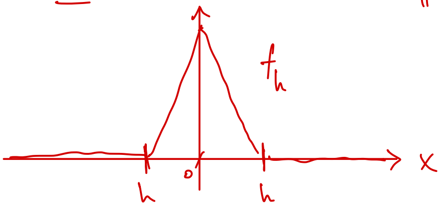
$\Rightarrow |f(x+h) - f(x)| < 2\varepsilon$ on $|x| > 2M$

$\Rightarrow$ On $(-2M, 2M)$, f is unif. continuity ✓

⚠ **WARNING**: τ_h is not unif. convergent to $\mathbb{1}$ because all of the M's, ε's, etc above depend entirely on the function f!

$\Rightarrow$ Not uniform.

Ex:



$$\|(\tau_h f_h)(x) - f_h(x)\|_\infty = 1 .$$

BUT: $\|\tau_h - \mathbb{1}\| \longrightarrow 0$

$$\Rightarrow \|(\tau_h f_h) - f_h\| \not\to 0$$

NOTE: $(\tau_h f)(x) \overset{\text{def}}{=} f(x+h) = e^{h\frac{\partial}{\partial x}}$ $= f(x) + h f'(x) + \dots$ if the derivatives exist.

$$\Rightarrow \tau_h = e^{h\frac{\partial}{\partial x}} \text{ associated to } A = \underbrace{\frac{\partial}{\partial x}}_{\text{unbounded}} .$$

—————————— // ——————————

DUAL SPACES

DEF: Given a $\overset{\wedge}{\text{normed}}$ vector space X. The space of linear continuous (hence bounded) operators $X \to \mathbb{F}$ is the topological dual.

$\uparrow$ $\mathbb{R}$ or $\mathbb{C}$

$$X^* = \mathcal{B}(X, \mathbb{F})$$

→ Banach b/c $\mathbb{R}$ or $\mathbb{C}$ are Banach and the operators are linear!

↳ If nonlinear, this fails...

- Norm on X^*: $\|\varphi\| = \sup_{0 \neq x \in X} \dfrac{|\varphi(x)|}{\|x\|_X}$

DEF: $\varphi \in X^*$ bounded $|\varphi(x)| \leq \|\varphi\| \|x\| < +\infty$.

NOTATION: $\varphi(x) = \langle \varphi, x \rangle_{X^*, X}$ (DUALITY PRODUCT) ← only "inner" of Hilbert

EXAMPLE: $(\mathbb{R}^n)^* \cong \mathbb{R}^n$.

$\varphi \in \mathcal{B}(\mathbb{R}^n, \mathbb{R}) \rightsquigarrow$ Choose a basis $(e_1, \ldots, e_n)$ for $\mathbb{R}^n$. Then

$$\varphi(x) = \varphi\left(\sum_{i=1}^{n} x_i e_i\right) = \sum_{i=1}^{n} \underset{\substack{\uparrow \\ [x_i(x)] \text{ linear form}}}{x_i} \underbrace{\varphi(e_i)}_{\in \mathbb{R}}$$

$\Rightarrow$ The n $[x_i(x)]$ "coord. fcts" form a basis of $(\mathbb{R}^n)^*$

$\Rightarrow (\mathbb{R}^n)^* \cong \mathbb{R}^n$.

∎

EXAMPLES:

1) $(\mathbb{R}^n)^* \cong \mathbb{R}^n$ (also true for separable Hilbert spaces)

2) For $1 < p < \infty$, then $(L^p)^* \cong L^q$, $\frac{1}{p} + \frac{1}{q} = 1$.

$(L^2)^* \cong L^2$ $\qquad$ $(\ell^p)^* \cong (\ell^q)$, $\frac{1}{p} + \frac{1}{q} = 1$.

Q 1: $p \neq q \Rightarrow L^p \neq L^q$ (not trivial)

Note: $L^p(0,1)$ $\quad$ $\mathbb{R} \ni \underset{\displaystyle \langle g,f\rangle_{L^q,L^p}}{\underset{\shortparallel}{\int_0^1 fg\,dx}} \leq \|f\|_p \|g\|_q$ (Hölder)

Now, $\varphi_f(g) := \int fg \in \mathbb{R}$ by Hölder, linear, continuous b/c of

$\leq \|f\|_p \|g\|_q < \infty$ $\quad$ i.e., $|\varphi_f(g)| \leq C\|g\|_q$.

$\Rightarrow \varphi_f \in (L^q)^* \underset{\uparrow}{\cong} L^p$

Riesz Representation

⚠ WARNING: This gives $(L^p)^{**} = L^p$ ← Not TRUE IN GENERAL!
If true, called REFLEXIVE space.

3) $X := C([a,b])$.

p smooth. $\qquad \varphi_p(f) := \int_a^b p(x)\, f(x)\, dx$.

$f \in X$. $\qquad \Longrightarrow \varphi_p \in X^*$.

Now, if p is not smooth, e.g., $p = \delta_{x_0}$ $\left(\delta_{x_0}(f) = f(x_0) \right)$.

i.e., $\int_a^b \delta(x - x_0)\, f(x)\, dx = f(x_0)$

<u>Obs</u>: $|\delta(f)| = |f(x_0)| \leq \|f\|_\infty < \infty$ ✓

$\Longrightarrow \delta_{x_0} \in X^*$.

Bounded measures on $[a,b]$

<u>Claim</u>: $X^* = \mathcal{M}_b([a,b])$ $\qquad$ (MARKOV – RIESZ REPRESENTATION)

<u>HAHN–BANACH</u>: Let X be a $\mathbb{R}$-vector space and $p : X \to \mathbb{R}$ be a <u>sublinear</u> functional on X. Let $\ell : M \subset X \to \mathbb{R}$ be a linear functional such that $\ell(x) \leq p(x)$ $\forall x \in M$. Then $\exists\, \tilde{\ell} : X \to \mathbb{R}$ linear functional on X such that

$p(x+y) \leq p(x) + p(y)$ and $p(\lambda x) = \lambda p(x)$ $\forall \lambda \geq 0$

$$\tilde{\ell}(x) \leq p(x) \quad \forall x \in X \quad \text{and} \quad \tilde{\ell}\big|_M = \ell .$$

↓

VANILLA HAHN-BANACH: Let $Y \subset X$ subspace and $\psi : Y \subset X \to \mathbb{R}$ with $\|\psi\|_{Y^*} = M < \infty$. Then, $\exists \varphi : X \to \mathbb{R}$ bounded linear such that

$$\varphi\big|_Y = \psi \quad \text{and} \quad \|\varphi\|_{X^*} = \|\psi\|_{Y^*} .$$

⚠

IMPORTANT : MEMORIZE PROOF (MIDTERM / QUAL)

COROLLARY 1:

not unique

$$\forall x_0 \in X \text{ then } \exists f_0 \in X^* \text{ such that } \|f_0\|_{X^*} = \|x_0\|_X$$

$$\text{and } \langle f_{x_0} , x_0 \rangle_{X^*, X} = \|x_0\|_X^2$$

Pf: $Y = \mathbb{R} x_0 \subset X$

$$\psi(t x_0) = t \underbrace{\psi(x_0)}_{\text{Need to define}} = t \|x_0\|^2 \Rightarrow \psi \in Y^*$$

By Hahn-Banach, f_0 extends to X^* and $\|\psi\|_{X^*} = \|x_0\|_X$

$$\Longrightarrow \|f_0\|_{X^*} = \|x\|_X \quad \text{and} \quad \langle \underset{\underset{\psi}{\equiv}}{f_0} , x_0 \rangle = \|x_0\|^2$$

∎

COROLLARY 2: Let $x \in X$, then

$$\|x\| = \max_{\substack{f \in X^* \\ \|f\|_{X^*} = 1}} |\langle f, x \rangle| .$$

Pf: Take $x = x_0$ in Cor. 1 and $f_1 = \frac{f_0}{\|x\|}$. Then

$$\|x\| = \langle f_1, x \rangle = \frac{1}{\|x\|} \langle f_0, x \rangle \checkmark$$

$$|\langle f_1, x \rangle| \leq \|f\| \, \|x\| = \|x\|$$

■

COROLLARY 3: Assume $\varphi(x) = \varphi(y)$ $\forall \varphi \in X^*$. Then $x = y$

Pf: $x - y \in X$

$$\|x-y\| = \langle f, x-y \rangle \Rightarrow \|x-y\| = \langle f, x \rangle - \langle f, y \rangle$$

$$= \varphi(x) - \varphi(y)$$

$$= 0 .$$

$$\Rightarrow x = y .$$

■

LECTURE 7

(Ch 5)

Jan 28 2026

DOUBLE DUAL: $x \in X$, set $F_x(\varphi) \equiv x(\varphi) := \varphi(x) \quad \forall \varphi \in X^*$

$$\Rightarrow \quad F_x \in X^{**} \quad \Rightarrow \quad X \subset X^{**}$$

$\supset$ IS FALSE IN GENERAL!

DEF: When all $F \in X^{**}$ can be written as $F(\varphi) = \varphi(x)$ $\forall x \in X$ then $X^{**} \subset X$, and $X \cong X^{**}$ $\Rightarrow$ X is CALLED REFLEXIVE

Ex: For $1 < p < \infty$, $(L^p)^* = L^q$, $\frac{1}{p} + \frac{1}{q} = 1$.

$\Rightarrow (L^p)^{**} = L^p$ i.e., L^p is reflexive for $1 < p < \infty$.

NOTE: $p = 1 \rightsquigarrow (L^1)^* = L^\infty$. However, $(L^\infty)^* \not\supseteq L^1$

$\Rightarrow L^1, L^\infty$ not reflexive.

DEF: $x_n \xrightarrow[\{x_n\} \subset X]{\text{weak}} x$ iff $\forall \varphi \in X^*$ $\varphi(x_n) \longrightarrow \varphi(x)$.

$\downarrow$ weaker

DEF: $x_n \xrightarrow[\{x_n\} \subset X]{S-\text{weak}} x$ iff $\forall \varphi \in \underset{\uparrow \text{subspace}}{S} \subset X^*$ $\varphi(x_n) \longrightarrow \varphi(x)$.

DEF: $\varphi_n \xrightarrow[\{\varphi_n\} \subset X^*]{\text{weak}} \varphi$ iff $\forall F \in X^{**}$ $F(\varphi_n) \longrightarrow F(\varphi)$.

$\downarrow$ weaker

DEF: $\varphi_n \xrightarrow[\{\varphi_n\} \subset X^*]{\text{weak}-*} \varphi$ in X^* iff $\forall x \in X \subset X^{**}$ $\underset{x(\varphi_n) =}{\varphi_n(x)} \longrightarrow \underset{= x(\varphi)}{\varphi(x)}$

DEF: $\{F_n\}_{n\in\mathbb{N}} \subset X^{**}$ converges weak-* $\overset{\text{in } X^{**}}{\vee}$ to $F \in X^{**}$:

$F_n \xrightarrow{w^*} F$ iff $F_n(\varphi) \longrightarrow F_n(\varphi)$ $\forall \varphi \in X^*$.

Weak-* convergence in X^{**} means ptwise convergence on X^*. If the elements in X^{**} come from $X \subset X^{**}$ then this weak-* in X^{**} is the same as weak convergence in X: $x(\varphi_n) = \varphi_n(x) \longrightarrow \varphi(x) = x(\varphi)$.

$X \subset X^{**}$ ⇝ weak on X = weak-* on $\overset{\text{embedding } X \hookrightarrow X^{**}}{J(X)} \subset X^{**}$

X^* vs X ⇝ weak and weak-* never coincide (different spaces)

$X = X^{**}$ ⇝ weak on X = weak-* on $X = X^{**}$

NOTE: Strong $\Rightarrow$ Weak since $|\varphi(x) - \varphi(x_n)| \leq \|\varphi\| \|x - x_n\| \to 0$.

$\cancel{\Leftarrow}$ Not in general (true in finite dims.)

NOTE: (X^*, τ_{w*}) has less opens than (X^*, τ_w); which has less opens than (X^*, τ_s) ⇒ LESS OPENS = EASIER TO BE COMPACT

THM: (BANACH - ALAOGLU) Closed unit ball of X^* is compact in the weak-* topology.

Can promote to "sequence" if X is separable (i.e., has countable dense subset)

BANACH - ALAOGLU: Every net (φ_α) in the closed unit ball

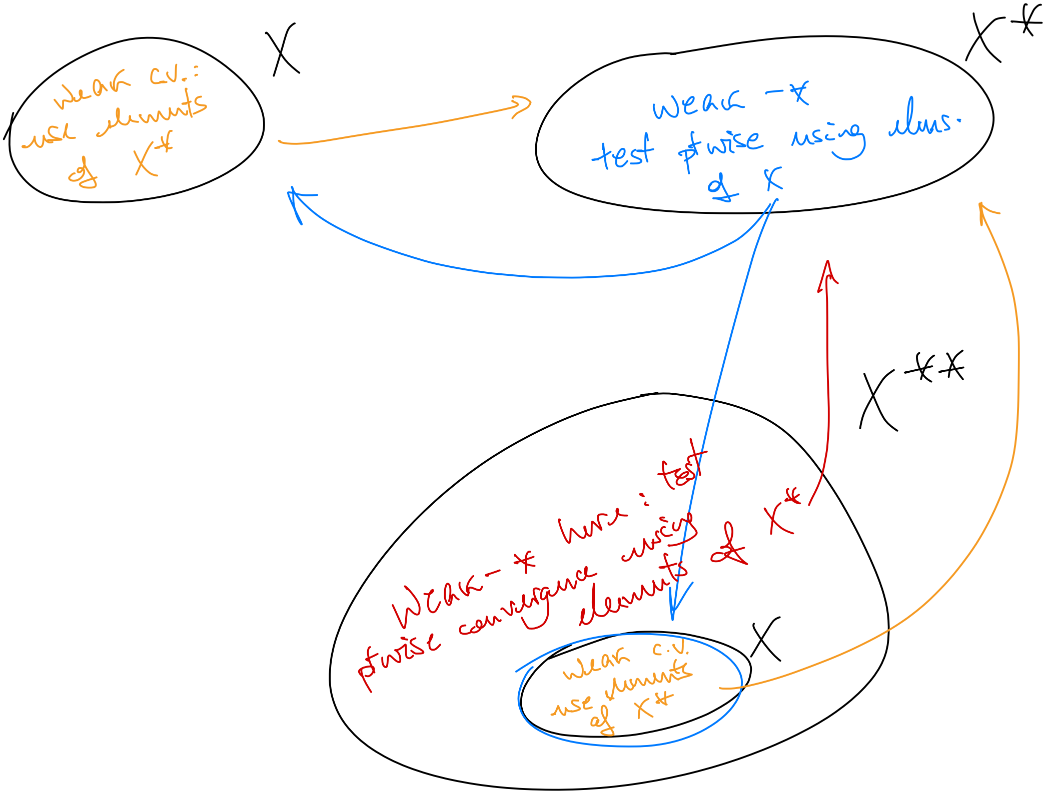
$$\overline{B_{X^*}} = \{ \varphi \in X^* : \|\varphi\| \leq 1 \}$$

has a subnet (φ_{α_β}) for which $\exists\, \varphi \in B_{X^*}$ such that

$$\varphi_{\alpha_\beta}(x) \longrightarrow \varphi(x) \quad \forall x \in X \subset X^{**} \iff \varphi_{\alpha_\beta} \xrightarrow{w^*} \varphi \quad .$$

THM: (Kakutani)

$$X \text{ reflexive } (X^{**} \cong X) \iff \overset{\{x\in X : \|x\| \leq 1\}}{\overset{\shortparallel}{\overline{B_1(0)}}} \text{ is compact for } (X, \mathcal{T}_w) .$$


X
weak cv.: use elements of X^*
X^*
weak -* test ptwise using elms. of X
X^{**}
Weak-* here: test ptwise convergence using elements of X^*
X
weak c.v. use elements of X^*

LECTURE 8

(Ch 12)

Feb 2, 2026

MEASURE THEORY

DEF: (σ-ALGEBRA) Collection of sets $\mathcal{A}$ s.t.

(i) $\emptyset \in \mathcal{A}$

(ii) $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$

(iii) $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{A} \Rightarrow \bigcup_{i \in \mathbb{N}} A_i \in \mathcal{A}$

Ex: (X, τ) topological space $\rightsquigarrow$ σ-algebra generated by open sets of τ is called BOREL σ-ALGEBRA.

DEF: A measure on $(X, \mathcal{A})$ is $\mu : (X, \mathcal{A}) \to [0, \infty]$ such that

(i) $\mu(\emptyset) = 0$

(ii) $\{A_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$, $A_i \cap A_j = \emptyset$ $\forall i \neq j$, then $\mu\left(\bigsqcup_{i \in \mathbb{N}} A_i\right) = \sum_{i \in \mathbb{N}} \mu(A_i)$.

DEF: Measure μ is finite iff $\mu(X) < +\infty$.

σ-finite iff $\exists \{A_i\}_{i \in \mathbb{N}}$, $A_i \cap A_j = \emptyset$ $\forall i \neq j$, s.t. $X = \bigsqcup_{i \in \mathbb{N}} A_i$ and $\mu(A_i) < \infty$ $\forall i$.

DEF: $(X, \mathcal{A}, \mu)$ MEASURE SPACE.

Ex: $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \mathcal{L}eb)$

$\mathcal{L}eb \equiv \lambda$ such that $\lambda\left(\prod_{i=1}^{n}(a_i, b_i)\right) = \prod_{i=1}^{n}(b_i - a_i) \;\forall\, b_i \geq a_i$.

FACT: λ is σ-finite. Moreover, λ indeed exists!

◢

Ex: $(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \delta_{x_0})$ $\forall A \in \mathcal{B}_{\mathbb{R}^n}$,

$$\delta_{x_0}(A) := \begin{cases} 1, & x_0 \in A \\ 0, & x \notin A \end{cases}.$$

◢

THM: (λ is REGULAR) For Lebesgue measure, $\forall A \in \mathcal{B}_{\mathbb{R}^n}$,

$$\lambda(A) = \inf\{\lambda(U) : U \text{ open and } A \subset U\}$$
$$= \sup\{\lambda(K) : K \text{ compact and } K \subset A\}$$

DEF: A measure space is <u>complete</u> if every subset of measure 0 sets are measurable (i.e., in the σ-algebra)

THM: $(X, \mathcal{A}, \mu) \longrightarrow (X, \tilde{\mathcal{A}}, \tilde{\mu})$ can be completed.

Ex: The completion of the Borel σ-algebra is LEBESGUE σ-algebra

$$(\mathbb{R}^n, \mathcal{B}_{\mathbb{R}^n}, \lambda) \xrightarrow{\text{complete}} (\mathbb{R}^n, \mathcal{L}_{\mathbb{R}^n}, \lambda)$$

DEF: (ESSENTIAL SUPREMUM) "A.e. - sup"

$$\operatorname{ess\,sup}\{A\} = \inf\{C : x \leq C,\ \forall x \in A \setminus N \text{ and } \mu(N) = 0\}.$$

e.g.: $\mu = \lambda = \text{Leb} \rightsquigarrow \operatorname{ess\,sup}([0,1] \cup \{2\}) = 1$.

LEBESGUE - STIELTJES: $F: \mathbb{R} \to \mathbb{R}$ càdlàg, then

$$\mu_F((a,b]) := F(b) - F(a)$$

is called the Lebesgue - Stieltjes measure on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$.

e.g.: $\text{Leb} = \mu_F$ with $F(x) = x$

$$\delta_{x_0} = \mu_F \text{ with } F(x) = \begin{cases} 1, & x \geq x_0 \\ 0, & x < x_0 \end{cases}.$$

NOTE: If $\overline{F((-\infty,+\infty))} = [0,1] \Rightarrow \mu_F = \mathbb{P}$ prob. measure.

DEF: (MEASURABLE FUNCTIONS) $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ measurable spaces. Then, $f : (X, \mathcal{A}) \to (Y, \mathcal{B})$ is measurable iff $\underset{\{x \in X:\, f(x) \in B\}}{\underset{\shortparallel}{f^{-1}(B)}} \in \mathcal{A} \quad \forall B \in \mathcal{B}$.

PROPOSITION: $f : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ is measurable on $(X, \mathcal{A})$ iff $\underset{f^{-1}([-\infty, C))}{\underset{\shortparallel}{\{x \in X : x < C\}}} \in \mathcal{A} \quad \forall C \in \overline{\mathbb{R}}$.

DEF: $\{f_n\}_{n \in \mathbb{N}}$, $f_n : X \to \overline{\mathbb{R}}$

- $f_n \to f$ pointwise $\iff f_n(x) \to f(x) \quad \forall x \in X$.
- $f_n \to f$ μ-a.e. $\iff f_n(x) \to f(x) \quad \forall x \in X \setminus N \quad \forall N \in \mathcal{A}$ with $\mu(N) = 0$.

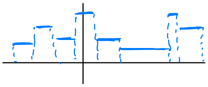
THM: (i) f_n measurable and $f_n \xrightarrow{\text{ptwise}} f$ ptwise $\Rightarrow$ f measurable

(ii) f_n measurable and $f_n \xrightarrow{\text{a.e.}} f$ a.e. AND $(X, \mathcal{A}, \mu)$ IS COMPLETE $\Rightarrow$ f measurable.

INTEGRATION

DEF: (SIMPLE FUNCTION) $\varphi:(X,\mathcal{A})\to\mathbb{R}$ is SIMPLE if, $n<\infty$,

$$\varphi=\sum_{i=1}^{n} c_i\,\chi_{A_i},$$



for disjoint sets $\{A_i\}\subset\mathcal{A}$, $A_i\cap A_j=\emptyset\ \forall i\neq j$, $c_i\in\mathbb{R}$.

$0\leq\varphi_1\leq\varphi_2\leq\varphi_3\leq\cdots\leq f$

THM: If $f:X\to\overline{\mathbb{R}}$ is measurable, then $\exists\{\varphi_n\}_{n\in\mathbb{N}}$, $\varphi_n\geq 0\ \forall n$, increasing sequence of simple functions such that $\varphi_n\nearrow f$ ptwise. [and uniformly on any set in which f is bounded]

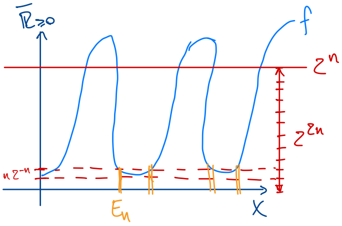
Pf: Decompose $f:X\to\overline{\mathbb{R}}$ as $f=f_+ + f_-$, $f_+:=\max\{f,0\}$

$f_-:=\max\{-f,0\}$.

$$E_n^{\ell}:=f^{-1}\left(\left[\ell\cdot 2^{-n},(\ell+1)2^{-n}\right)\right)$$

Define

$$R_n:=f^{-1}\left([2^n,\infty]\right)$$



Now, define the simple functions as:

$$\phi_n(x):=\sum_{\ell=0}^{2^{2n}-1}\ell\cdot 2^{-n}\chi_{E_n^{\ell}}(x)+2^n\chi_{R_n}(x)$$

□

LECTURE 9

(Ch 12)

Feb 4, 2026

Midterm focus → Ch 5

Will have some measure theory

Also a bit of Ch 1–4 (T/F?)

- Thms/facts
- Problems from book
- Past analysis exams

INTEGRATION

- INTEGRAL OF SIMPLE FUNCTIONS: $\varphi = \sum_{i=1}^{n} c_i \chi_{A_i}$ simple,

define
$$\int \varphi \, d\mu := \sum_{i=1}^{n} c_i \mu(A_i) .$$

- INTEGRAL OF MEASURABLE FUNCTIONS

DEF: $f : X \to [0,\infty]$ and let seq. of simple fcts $\varphi_n \nearrow f$ then
$$[0,\infty] \ni \int f \, d\mu := \lim_{n\to\infty} \int \varphi_n \, d\mu$$

THM: This definition is independent of choice of $\{\varphi_n\}$

DEF: For $f : X \to \overline{\mathbb{R}}$, write $f_+ = \max\{f, 0\}$ and $f_- := \max\{-f, 0\}$,

then
$$\int f \, d\mu := \int f_+ \, d\mu - \int f_- \, d\mu$$

as long as one of the terms is bounded.

NOTE: $L^1(X, \mathcal{A}, \mu) = \{ f \text{ s.t. } \int |f| \, d\mu = \int f_+ \, d\mu + \int f_- \, d\mu < \infty \}$

↖ Not a Banach space (need μ-a.e. equal class of equiv.)

CONVERGENCE THEOREMS

MONOTONE CONVERGENCE THEOREM: $\{f_n\} \subset L^+$ such that $f_n \leq f_{n+1}$ $\forall n$ (i.e., monotonically increasing) and $f_n \nearrow f$ ptwise then

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim f_n \, d\mu = \int f \, d\mu$$

COUNTEREXAMPLES:

$f_n(x) = \mathbb{1}_{[n,n+1]}$ ← Not monotonic & $f_n \to 0$ ptwise

$1 = \int f_n \not\to \int f = 0$

$f_n(x) = n \, \mathbb{1}_{[0,1/n]}$ ← Not monotonic & $f_n \to 0$ ptwise

but $1 = \int f_n \not\to \int f = 0$.

FATOU'S LEMMA: For any $\{f_n\}_{n\in\mathbb{N}} \subset L^+$,

$$\int \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int f_n \, d\mu .$$

LEBESGUE DOMINATED CONVERGENCE THEOREM: Let $\{f_n\}_{n\in\mathbb{N}} \subset L^1$ such that $\exists g \in L^1$ with $|f_n| \leq g$ ptwise $\forall n \in \mathbb{N}$ and $f_n \to f$ ptwise. Then $f \in L^1$ and

$$\lim_{n\to\infty} \int f_n \, d\mu = \int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu .$$

COROLLARY: Let $f : X \times I \to \mathbb{R}$, where I is a space of parameters (e.g., time). Suppose

(1) $f(\cdot, t)$ integrable $\forall t \in I$ and

(2) $f(\cdot, t)$ differentiable for a.e. x

(3) $\left| \frac{\partial f}{\partial t}(x,t) \right| \leq g(x)$ for some $g \in L^1$.

Then,
$$\frac{dJ}{dt}(t) = \int_X \frac{\partial f}{\partial t}(x,t)\, d\mu(x).$$

❗

Pf: Need existence of limit

$$\lim_{n\to\infty} \frac{J(t+1/n) - J(t)}{1/n} = \lim_{n\to\infty} \int \frac{f(x, t+1/n) - f(x,t)}{1/n}\, d\mu(x)$$

$$\text{LDCT} \rightarrow = \int \frac{\partial f}{\partial t}(x,t)\, d\mu(x).$$

□

—— // ——

PRODUCT SPACES

- $(X, \mathcal{A}, \mu)$ and $(Y, \mathcal{B}, \nu)$ measure spaces.

- $(X \times Y, \mathcal{A} \otimes \mathcal{B}, \mu \otimes \nu)$ with $\mathcal{A} \otimes \mathcal{B}$ generated by $A \times B$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$, and

  $$(\mu \otimes \nu)(A \times B) = \mu(A)\, \nu(B)$$

  generalized to $\mathcal{A} \otimes \mathcal{B}$.

FUBINI THEOREM: Let $f : X \times Y \longrightarrow \overline{\mathbb{R}}$ be measurable. If

$$\left(\begin{array}{c} f \text{ is integrable with} \\ \int_{X \times Y} |f| \, d\mu \otimes d\nu \end{array} \right) \iff \left(\begin{array}{l} \text{Either} \\ \int_X \left| \int_Y |f(x,y)| \, d\nu(y) \right| d\mu(x) < \infty \\ \text{or} \\ \int_Y \left| \int_X |f(x,y)| \, d\mu(x) \right| d\nu(y) < \infty \end{array} \right)$$

Then $\int f \, d\mu \otimes d\nu = \int_X \left(\int_Y f \, d\nu \right) d\mu = \int_Y \left(\int_X f \, d\mu \right) d\nu$.

——————— // ———————

DEF: (L^p SPACES) Define, for $1 \leq p < \infty$,

$$L^p(X, \mathcal{A}, \mu) := \left\{ [f] : \|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{1/p} < \infty \right\}$$

Equivalence class of functions $f: X \to \mathbb{R}$ that are equal μ-a.e..

$$L^\infty(X, \mathcal{A}, \mu) := \left\{ [f] : \|f\|_\infty := \underset{x \in X}{\operatorname{esssup}} |f(x)| < \infty \right\}$$

THM: $(X, \mathcal{A}, \mu)$ measure space and $1 \leq p \leq \infty$,

$L^p(X, \mathcal{A}, \mu)$ is BANACH SPACE

Pf: Claim 1: $\|\cdot\|_p$ indeed norm on L^p. Need that $\|f\|_p = 0 \Rightarrow f = 0$ a.e.. Take $\varphi_n \nearrow |f|^p$ and see either $c_i = 0$ or $\mu(A_i)$ $\forall i$ $\Rightarrow f = 0$ μ-a.e.. b/c $|f|^p = 0$ a.e.

Claim 2: L^p is complete.

(All Cauchy seq. converge) $\Longleftrightarrow$ $\left(\sum_{n=0}^{\infty} \|f\|_p < \infty \Rightarrow \sum_{n=0}^{\infty} f \text{ converges in } L^p \right)$

$p=1$: By MCT,

$$\int \sum_{j=1}^{\infty} |f_j| \overset{MCT}{=} \sum_{j=1}^{\infty} \int |f_j| < \infty .$$

So, $$g(x) := \sum_{j=1}^{\infty} |f_j(x)| \in \underbrace{L^1}_{\Rightarrow g \text{ is finite a.e.}}$$

def $$h_n(x) := \sum_{j=1}^{n} f_j(x), \text{ then } |h_n| \le g \text{ a.e.}$$

Thus, by LDCT, $h_n \xrightarrow{L^1} h$.

■

FACTS :

(1) $\forall f \in L^p$, $\exists \varphi_n$ simple s.t. $\forall \varepsilon > 0$, $\|f - \varphi_n\|_p < \varepsilon$

(2) $1 \le p < \infty$ $L^p(\mathbb{R}^n)$ is separable (countable dense subset)

(3) $1 \le p < \infty$ $C_c^\infty(\mathbb{R}^n)$ is dense in L^p

⚠ (4) (JENSEN'S INEQUALITY) φ convex

$$\Rightarrow \varphi\left(\frac{1}{\mu(X)} \int f \, d\mu\right) \le \frac{1}{\mu(X)} \int \varphi \circ f \, d\mu .$$

⚠ (5) (HÖLDER INEQUALITY) $\frac{1}{p} + \frac{1}{q} = 1$, $1 \le p, q \le \infty$,

$\forall f, g$ measurable, $\|fg\|_1 \le \|f\|_p \|g\|_q < \infty$

i.e., $fg \in L^1$.

⚠ (6) (CHEBYSHEV INEQUALITY) $f \in L^p$, $1 \le p < \infty$

$$\mu(\{x \in X\epsilon : |f| > \varepsilon\}) \le \|f\|_p^p / \varepsilon^p .$$

(7) (YOUNG INEQUALITY) $f \in L^p$, $g \in L^q$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r} \Rightarrow \|f * g\|_r \le \|f\|_p \|g\|_q$

LECTURE 10

(Ch 6)

HILBERT SPACES

Feb 11, 2026

DEF: (INNER PRODUCT) $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{F}$ s.t.

(1) $\langle x, \lambda y + \mu z \rangle = \lambda \langle x, y \rangle + \mu \langle x, z \rangle$ (linearity)

(2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (skew-symmetry)

(3) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$ (positive-def)

$(X, \langle \cdot , \cdot \rangle)$ is a pre-Hilbert space.

- $\langle \lambda x + \mu y, z \rangle = \overline{\langle z, \lambda x + \mu y \rangle} = \overline{\lambda} \, \overline{\langle z, x \rangle} + \overline{\mu} \, \overline{\langle z, y \rangle}$
  $= \overline{\lambda} \langle x, z \rangle + \overline{\mu} \langle y, z \rangle$.

- X $\mathbb{R}$-vector space $\Rightarrow \langle \cdot , \cdot \rangle : X \times X \to \mathbb{R}$ BILINEAR FORM
  X $\mathbb{C}$-vector space $\Rightarrow \langle \cdot , \cdot \rangle : X \times X \to \mathbb{C}$ SESQUILINEAR FORM

- $\|x\| := \sqrt{\langle x, x \rangle}$ $\forall x \in X$ is a norm!

$$\Rightarrow \boxed{\left(X, \|\cdot\| := \sqrt{\langle \cdot , \cdot \rangle}\right) \text{ is A HILBERT SPACE}}$$

i.e., a complete pre-Hilbert space.

Ex: • $\mathbb{C}^n$ with $\langle x,y\rangle = \sum_{i=1}^{n} \overline{x_i}\, y_i$

• $C^0([a,b])$, $\langle f,g\rangle := \int_a^b \overline{f(x)}\, g(x)\, dx$

$(C^0([a,b]), \langle\cdot,\cdot\rangle)$ is pre-Hilbert

$$\Big\downarrow \text{ Completion}$$

$L^2([a,b])$

• $C^k([a,b])$, $\langle f,g\rangle_k := \sum_{j=0}^{k} \int_a^b \overline{f^{(j)}(x)}\, g^{(j)}(x)\, dx$

$(C^k, \langle\cdot,\cdot\rangle_k)$ is pre-Hilbert

$$\Big\downarrow \text{ Completion}$$

$H^k(a,b) \equiv W^{k,2}(a,b)$

THM: (Cauchy-Schwarz) $x,y \in X \Rightarrow |\langle x,y\rangle| \le \|x\|\,\|y\|$.

Pf: X $\mathbb{R}$-v.s., $\lambda \in \mathbb{R}$, $0 \le \|x-\lambda y\|^2 = \langle x-\lambda y, x-\lambda y\rangle$

$$= \|x\|^2 + \lambda^2\|y\|^2 - 2\lambda\langle x,y\rangle$$

So, using $\pm\lambda \in \mathbb{R}$, get $2|\lambda|\,|\langle x,y\rangle| \le \|x\|^2 + \lambda^2\|y\|^2$.

Clear for $y=0$ or $x=0$. For $\begin{matrix} y\neq 0 \\ x\neq 0\end{matrix}$ take $\lambda = \frac{\|x\|}{\|y\|}$ or $\frac{\|y\|}{\|x\|}$ to get $2|\langle x,y\rangle| \le 2\|x\|\,\|y\|$.

$\square$

COROLLARY: $\|x+y\| \leq \|x\| + \|y\|$. ($\Longrightarrow \|\cdot\| = \sqrt{\langle \cdot,\cdot \rangle}$ indeed a norm)

MUST KNOW HOW TO PROVE: FINAL & QUAL

$\downarrow$

THM: (PARALLELOGRAM LAW)

$(X, \|\cdot\|)$ is an inner product space $\Longleftrightarrow$ $\forall x,y \in X$, $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$

Pf: ($\Longrightarrow$) clear

($\Longleftarrow$) NOT EASY: define

$$\langle x,y \rangle = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2\right)$$

and show inn. prod. $\square$

- PRODUCT SPACE: on $X \times Y$, define

$$\left\langle (x_1, y_1), (x_2, y_2) \right\rangle_{X \times Y} := \langle x_1, x_2 \rangle_X + \langle y_1, y_2 \rangle_Y .$$

$\Longrightarrow \left(X \times Y, \langle \cdot,\cdot \rangle_{X \times Y} \right)$ pre-Hilbert.

THM: $\langle \cdot , \cdot \rangle : X \times X \to \mathbb{C}$ is continuous (for $(X, \|\cdot\|)$).

* ORTHOGONALITY in HILBERT SPACES $\mathcal{H}$

DEF: $x \perp y \iff \langle x, y \rangle = 0$

$A \perp B \iff x \perp y \quad \forall x \in A, y \in B.$

$A^{\perp} := \{ x \in \mathcal{H} : \langle x, a \rangle = 0 \quad \forall a \in A \}.$ (ORTHOGONAL COMPLEMENT)

THM: $A^{\perp}$ is a closed linear subspace of $\mathcal{H}$.

EXERCISE

← PROVE

⚠

THM: (Projection) Let $M \subset \mathcal{H}$ be a closed linear subspace.

(1) $\forall x \in \mathcal{H}$, $\exists !\, y \in M$ such that $\|x - y\| = \min\limits_{z \in M} \|x - z\|$.

(2) $y \in M$, closest pt. to x, is the unique pt. s.t. $x - y \in M^{\perp}$.

- Let $M, N \subset \mathcal{H}$ closed linear subspaces with $M \perp N$.

DEF: $M \oplus N := \{ x+y : x \in M, y \in N \}$. (DIRECT SUM)

$$\boxed{\text{COROLLARY: } M \subset \mathcal{H} \text{ closed linear subspace} \Rightarrow \mathcal{H} = M \oplus M^\perp}$$

Pf: $x \in \mathcal{H} \Rightarrow x = \underset{\substack{\cap \\ M}}{y} + \underset{\substack{\cap \\ M^\perp}}{\underbrace{x-y}}$

□

- If M is not closed, $\boxed{\mathcal{H} = \overline{M} \oplus M^\perp}$

↑ ALWAYS AVAILABLE FOR HILBERT SPACES

* ORTHOGONAL BASES

DEF: A set $U \subset \mathcal{H}$ is orthogonal if $u \perp v$ $\forall u, v \in U$, $u \neq v$.
An orthogonal set $U \subset \mathcal{H}$ is orthonormal if $\|u\| = 1$ $\forall u \in U$.

- Let I be a linear index (possibly uncountable), and let $J \subset I$ be a finite subset

DEF: Unordered sum $S_J := \sum_{\alpha \in J} x_\alpha$, $x_\alpha \in \mathcal{H}$.

DEF: (1) An unordered sum converges unconditionally to $x = \sum_{\alpha \in I} x_\alpha$ if $\forall \varepsilon > 0 \ \exists J^\varepsilon \subset I$ finite s.t. $\| S_J - x \| < \varepsilon$

⇒ Convergence is then independent of reordering of the terms.

(2) $\sum_{\alpha \in I} x_\alpha$ converges absolutely if $\sum_{\alpha \in I} \| x_\alpha \|$ converges unconditionally.

NOTE: Absolute convergence $\Longrightarrow$ unconditional convergence

$\not\Longleftarrow$ FALSE: $\sum_{n=1}^{\infty} \frac{1}{n} e_n$ $\{e_n\}$ o.n.

$$\Big\| \sum_{n=1}^{N} \frac{1}{n} e_n \Big\|^2 = \sum_{n=1}^{N} \frac{1}{n^2} < \infty \text{ but}$$

$$\sum_{n=1}^{N} \frac{1}{n} \sim \log N \nearrow \infty .$$

LEMMA: $U := \{u_\alpha : \alpha \in I\}$ orthonormal set in $\mathcal{H}$.
Then $\sum_{\alpha \in I} u_\alpha$ converges unconditionally
iff $\sum_{\alpha \in I} \|u_\alpha\|^2 < \infty$.

⚠

THM: (BESSEL'S INEQUALITIES) $U = \{u_\alpha : \alpha \in I\}$ orthonormal in $\mathcal{H}$
$x \in \mathcal{H}$, then

(1) $\sum_{\alpha \in I} |\langle u_\alpha, x \rangle|^2 \leq \|x\|^2$

(2) $x_U := \sum_{\alpha \in I} \langle u_\alpha, x \rangle u_\alpha$ is convergent

(3) $x - x_U \in U^\perp$.

DEF: (Span) U orthonormal

$$\operatorname{span} U := \left\{ \sum_{x \in U} c_u u \ : \ c_u \in \mathbb{C}, \ \sum_u |c_u|^2 < \infty \right\}$$

(smallest closed subspace of $\mathcal{H}$ containing U)

⚠ **IMPORTANT** ⚠

THM: $U = \{u_\alpha : \alpha \in I\}$ orthonormal set in $\mathcal{H}$. TFAE:

(1) $(u_\alpha, x) = 0 \quad \forall \alpha \in I \implies x = 0$

(2) $x = \sum_{\alpha \in I} \langle u_\alpha, x \rangle u_\alpha \quad \forall x \in \mathcal{H}$

(3) $\|x\|^2 = \sum_{\alpha \in I} |\langle u_\alpha, x \rangle|^2 \quad \forall x \in \mathcal{H}$ (PARSEVAL)

(4) $\operatorname{span} U = \mathcal{H}$.

DEF: $U \subset \mathcal{H}$ orthonormal set is COMPLETE if $\operatorname{span} U = \mathcal{H}$

Then U is called an orthonormal basis

THM: U o.n.b. for $\mathcal{H}$, $x = \sum_\alpha a_u u_\alpha$

$$y = \sum_\alpha b_\alpha u_\alpha$$

Then $\langle x, y \rangle = \sum_\alpha \bar{a}_\alpha b_\alpha$

THM: Every Hilbert space admits an o.n.b. (Zorn's Lemma...)

COROLLARY: $\mathcal{H}$ separable $\Rightarrow$ $\mathcal{H} \cong \ell^2(\mathbb{N})$.

(Identify basis elements)

$\hookrightarrow$ Usually not very useful, but tells us there is really only one separable Hilbert space.

- Can use Gram-Schmidt to get o.n.b.

LECTURE 11

(Ch 7)

FOURIER SERIES

Feb 13, 2026

- $f : \mathbb{R} \to \mathbb{C}$ 2π-periodic $\longleftrightarrow$ $f : \overset{\mathbb{T}}{\overset{\shortparallel}{\mathbb{R}/2\pi\mathbb{Z}}} \longrightarrow \mathbb{C}$
- In $C(\mathbb{T})$, define $\|f\| := \left(\int_0^{2\pi} |f(x)|^2 \, dx \right)^{1/2}$ and completion of $C(\mathbb{T})$ is $L^2(\mathbb{T})$.
- Inner product: $\langle f, g \rangle := \int_0^{2\pi} \overline{f(x)} g(x) \, dx \rightsquigarrow \underset{\text{HILBERT}}{\left(L^2(\mathbb{T}), \langle \cdot, \cdot \rangle \right)}$

THM: Let $c_n(x) := \frac{1}{\sqrt{2\pi}} e^{inx} = \frac{1}{\sqrt{2\pi}} (\cos nx + i \sin nx)$, $n \in \mathbb{Z}$,

so $c_n \in C^\infty(\mathbb{T}) \subset C(\mathbb{T})$ and $\langle c_n, c_m \rangle = \delta_{nm}$.

Then $U := \{ c_n : n \in \mathbb{Z} \}$ is an o.n.b. for $L^2(\mathbb{T})$

and so $f(x) = \sum_{n \in \mathbb{Z}} \langle c_n, f \rangle c_n(x)$ $\forall f \in L^2(\mathbb{T})$.

$f_n := \langle c_n, f \rangle =$ FOURIER COEFFICIENTS OF $f(x)$.

<u>OBS</u>: $\varphi_n(x) = c_n (1 + \cos x)^n \geq 0$ w/ c_n s.t. $\int_{-\pi}^{\pi} \varphi_n(x) \, dx = 1$

Thus $\int_{\delta < |x| < \pi} \varphi_n(x) \, dx \xrightarrow{n \to \infty} 0$ and $\varphi_n(x) = \sum_{k=-n}^{n} 2^{-n} \binom{2n}{n+k} e^{ikx}$

$$\Rightarrow (\varphi_n * f)(x) = \int_{-\pi}^{\pi} \varphi_n(x-y) f(y) \, dy = \sum_{k=-n}^{n} 2^{-n} \binom{2n}{n+k} \left[\int_{-\pi}^{\pi} e^{-iky} f(y) \, dy \right] e^{ikx}$$

LECTURE 12

(Ch 8) **BOUNDED OPERATORS ON $\mathcal{H}$**

Feb 16 2026

PROJECTION : $P : \mathcal{H} \hookrightarrow M$, $P^2 = P$ (orthogonal iff $P^* = P$)

↖ doesn't need to be bdd a priori...

THM: If we can write $X = M \oplus N$, $\exists!\, P$ s.t. $\operatorname{im} P = M$, $\ker P = N$.

THM: If P is a projector, then $X = (\operatorname{im} P) \oplus (\ker P)$.

- IN HILBERT SPACES CAN SAY MORE: with $\langle \cdot , \cdot \rangle$,

THM: For $M \subset \mathcal{H}$ closed, let $P : \mathcal{H} \hookrightarrow M$, then

$$\mathcal{H} = (\operatorname{im} P) \oplus \underbrace{(\ker P)}_{= (\operatorname{im} P)^{\perp},\ b/c\ P^* = P.}.$$

DEF: T on Hilbert space is SYMMETRIC iff $\forall x, y \in \operatorname{dom} T$,

$$\langle x, Ty \rangle = \langle Tx, y \rangle.$$

DEF: The ADJOINT of T is $\langle T^* x, y \rangle = \langle x, Ty \rangle$

$\forall x \in \operatorname{dom} T^* := \{ x \in \mathcal{H} : |\langle x, Ty \rangle| \le C_x \|y\| \}$

$\forall y \in \operatorname{dom} T$

Def: T is self-adjoint iff $T^* = T$.

$$\boxed{T \text{ self-adjoint}} \Longrightarrow \boxed{T \text{ symmetric}} \qquad \left(\begin{array}{c}\text{converse} \\ \underline{\text{not}} \text{ true}\end{array}\right)$$

$$\boxed{\begin{array}{l} T \text{ bounded} \\ \underline{\text{and}} \text{ symmetric} \end{array}} \Longrightarrow \boxed{T \text{ self-adjoint}}$$

Proposition: $P^2 = P$, then $\|P\| = 1$ unless $P \equiv 0$.

Pf: $\|P\| \geq 1$ clear

$$Py = y \rightsquigarrow \|Px\| = \frac{\langle Px, Px \rangle}{\|Px\|} = \frac{\langle x, P^2 x \rangle}{\|Px\|}$$

$$\leq \frac{\|x\| \, \cancel{\|Px\|}}{\cancel{\|Px\|}} = \|x\| . \qquad \square$$

Thm: (1) P orthogonal projection. Then $\operatorname{im} P$ is <u>closed</u> and $\mathcal{H} = \underset{\overset{\|}{(\ker P)^\perp}}{(\operatorname{im} P)} \oplus \underset{\overset{\|}{(\operatorname{im} P)^\perp}}{(\ker P)}$.

(2) If $\mathcal{H} = \overline{M} \oplus M^\perp$, then $\exists!\, P$ orthogonal projection s.t. $\overline{M} = \operatorname{im} P$.

Pf: (1) P projection $\rightsquigarrow \mathcal{H} = (\operatorname{im} P) \oplus (\ker P)$

P orthogonal $\rightsquigarrow \ker P \perp \operatorname{im} P$

$$x = \underline{Px} + \underline{(x - Px)}$$

im P ker P

$$\Rightarrow \text{im } P = (\ker P)^{\perp} \quad \text{hence closed}$$

ORTHOGONAL COMPLEMENTS ARE ALWAYS CLOSED

$\square$

EXAMPLE : (1) Even part : $(Pf)(x) := \dfrac{f(x) + f(-x)}{2}$

CHECK [orthogonal projection onto even functions $f \in L^2(\mathbb{R})$

(2) $A \subset \mathbb{R}$ measurable, then

$$(P_A f)(x) = \chi_A f$$

is orthogonal projection for $f \in L^2(\mathbb{R})$.

(3) Rank-1 projection : $P_u(x) := \langle u, x \rangle u, \quad u \in V$.

NOTATION : $P_u = u \otimes u = u u^T$.

(4) $\mathcal{H} = L^2(\mathbb{T}), \quad \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$

$$Pf := \langle f \rangle = \frac{1}{2\pi} \int_T f(x) \, dx$$ } Proj. onto the first Fourier basis element

is an orthogonal projection

* DUAL SPACES : $\mathcal{H}^* = \mathcal{B}(\mathcal{H}, \mathbb{F})$

- Define the functional for $y \in \mathcal{H}$, ← bounded $\|\varphi_y\|_{\mathcal{H}^*} = \|y\|_{\mathcal{H}}$

$$\varphi_y(x) := \langle y, x \rangle, \quad x \in \mathcal{H}$$

QUESTION: Is this all? Namely, is any $\varphi \in \mathcal{H}^*$ of the form $\varphi = \varphi_y$?

IMPORTANT

RIESZ REPRESENTATION: Let $\varphi \in \mathcal{H}^*$ bounded and linear. Then $\exists!\, y \in \mathcal{H}$ such that

$$\varphi(x) = \varphi_y(x) \overset{def}{=} \langle y, x \rangle \quad \forall x \in \mathcal{H}$$

NOTE: $J : \mathcal{H} \to \mathcal{H}^*$
$y \mapsto \varphi_y$

ISOMETRY between $\mathcal{H}$ and $\mathcal{H}^*$.

$$\Downarrow$$

$$\boxed{\mathcal{H} \overset{J}{\cong} \mathcal{H}^*}$$

↑ inn. prod. here inherited from $\mathcal{H}$ by parallelogram law

Pf: (Riesz Representation)

If $\varphi \equiv 0$, then $y = 0$.

If $\ker \varphi \neq \mathcal{H}$, then $\exists z \in \mathcal{H}$ s.t. $\varphi(z) \neq 0$.

Write $\mathcal{H} = (\ker \varphi) \oplus \underbrace{(\ker \varphi)^{\perp}}_{\substack{\text{not empty b/c } \ker \varphi \neq \mathcal{H} \\ \Rightarrow \text{ Take } 0 \neq z \in (\ker \varphi)^{\perp}.}}$

Define: $\boxed{\begin{array}{l} P : \mathcal{H} \to \mathcal{H} \\ \qquad x \longmapsto \dfrac{\varphi(x)}{\varphi(z)}\, z \end{array}}\ \boxed{0 \neq z \in (\ker \varphi)^{\perp}}$

Claim: $P^2 = P$ $\longrightarrow$

$$P^2 x = \frac{\varphi(Px)}{\varphi(z)}\, z = \frac{\varphi\left(\frac{\varphi(x)}{\varphi(z)} z\right)}{\varphi(z)}\, z$$

$$= \frac{\varphi(x)\,\frac{\varphi(z)}{\varphi(z)}}{\varphi(z)}\, z = Px.$$

- P projection $\Longrightarrow$ $\mathcal{H} = (\operatorname{im} P) \oplus (\ker P)$
- $Px = 0 \Leftrightarrow \varphi(x) = 0$, i.e., $\ker P = \ker \varphi$.
- $\operatorname{im} P = \mathbb{C} z$.

Thus, since $z \in (\ker \varphi)^{\perp}$ $\Rightarrow$ $z \in (\ker P)^{\perp}$

$\Rightarrow$ $\operatorname{im} P \perp \ker P$

$\Rightarrow$ P orthogonal projection.

$$\Rightarrow \boxed{\mathcal{H} = \mathbb{C} z \oplus \ker P}$$

$$\mathcal{H} \ni x = \alpha z + n, \quad \langle z, n \rangle = 0, \quad \text{where } \alpha = \frac{\langle z, x \rangle}{\|z\|^2} \in \mathbb{C}.$$

$$\Rightarrow \varphi(x) = \varphi(\alpha z + n) = \alpha\, \varphi(z) + 0$$

$$= \frac{\langle z, x\rangle}{\|z\|^2}\, \varphi(z)$$

Linearity $\longrightarrow$ $$= \left\langle \frac{z\, \varphi(z)}{\|z\|^2} , x \right\rangle$$

$$=: y$$

Uniqueness : $\varphi_{y_1}(x) = \varphi_{y_2}(x) \Rightarrow \langle y_1 - y_2, x \rangle = 0 \quad \forall x$

$$\Rightarrow y_1 = y_2 .$$

$\square$

* ADJOINT OPERATORS

DEF: The ADJOINT of an operator T is defined via

$$\langle T^* x, y \rangle = \langle x, Ty \rangle$$

$\forall x \in \operatorname{dom} T^* := \{ x \in \mathcal{H} : |\langle x, Ty\rangle| \le C_x \|y\|_{\mathcal{H}} \ \forall y \in \mathcal{H} \}$

$\forall y \in \operatorname{dom} T$.

- CONSTRUCT THE ADJOINT : Fix $x \in \mathcal{H}$ and $A \in \mathcal{B}(\mathcal{H}, \mathcal{H})$.

$\langle x, Ay \rangle = \varphi_x(y)$ is a bdd lin. functional

Riesz

$\Rightarrow$ $\varphi_x(y)$ has a unique representation:

$\exists!\ z \in \mathcal{H}$ s.t. $\varphi_x(y) = \langle z, y \rangle$

defined via $z = A^* x$.

A^* conjugate transpose

PROPERTIES : $A^{**} = A$, $(AB)^* = B^* A^*$.

Ex : On $L^2([0,1])$, $(Kf)(x) = \int_0^1 k(x,y)\, f(y)\, dy$ ⚠

Then $K^* f(x) = \int_0^1 \overline{k(y,x)}\, f(y)\, dy$. CHECK

THM : $A : \mathcal{H} \to \mathcal{H}$ bounded. Then ⚠

(i) $\overline{\operatorname{ran} A} = (\ker A^*)^\perp$

(ii) $\ker A = (\operatorname{im} A^*)^\perp$

$$\left[\Rightarrow \mathcal{H} = \overline{\operatorname{im} A} \oplus \ker A^* = \overline{\operatorname{im} A^*} \oplus \ker A \right]$$

Pf : (i) $(\subset)$ Let $x \in \operatorname{im} A$, $Ay = x$ and $z \in \ker A^*$

$$\langle x, z \rangle = \langle Ay, z \rangle = \langle y, A^* z \rangle = 0.$$

$$\Rightarrow \operatorname{im} A \subset \underbrace{(\ker A^*)^\perp}_{\text{closed}}.$$

closed $\Rightarrow \overline{\operatorname{im} A} \subset (\ker A^*)^\perp$ ✓

(⊃) $X^\perp \subset Y^\perp \Rightarrow Y \subset X$.

Let $x \in (\mathrm{im}\,A)^\perp \Rightarrow 0 = \langle Ay, x \rangle = \langle y, A^* x \rangle \quad \forall y$

$$\Rightarrow A^* x = 0$$

$$\Rightarrow (\mathrm{im}\,A)^\perp \subset \ker A^*$$

$$\Rightarrow (\ker A^*)^\perp \subset \underset{\substack{\| \\ \text{A \underline{bounded}} \rightarrow \ \mathrm{im}\,A}}{(\mathrm{im}\,A)^{\perp\perp}}$$

(ii) Use (i) and that $A^{**} = A$.

□

LECTURE 13

Feb 18, 2026

<u>NOTE</u>: Since $\mathcal{H} = \overline{\mathrm{im}\,A^*} \oplus \ker A$, $\overset{\substack{\text{big if} \\ \downarrow}}{\underline{\text{if}}}$ im A is closed, then $Ax = y$ solvable if $y \in (\ker A^*)^\perp$; then x is unique up to an element in $\ker A$.

<u>COROLLARY</u>: $\overset{\text{BOUNDED BELOW}}{\left\{ \begin{array}{c} \|Ax\| \geq C\|x\|, \\ C > 0 \end{array} \right\}} \iff \left\{ \begin{array}{c} \mathrm{im}\,A \text{ closed}, \\ \ker A = \{0\} \end{array} \right\}$

→ If $\|Ax\| \geq C\|x\|$ then $Ax = y$ uniquely solvable for $y \in \mathrm{im}\,A$.

FREDHOLM ALTERNATIVE : EITHER

(A) $\dim \ker A = \dim \ker A^* = 0$, and, hence, $Ax = y$ and $A^*x = y$ are uniquely solvable

OR

(B) $\dim \ker A = \dim \ker A^* > 0$, hence $Ax = y$ is solvable iff $y \in (\ker A^*)^\perp$ and solution is up to an element of $\ker A$ (same for $A \rightsquigarrow A^*$)

Ex:
- $A: \mathbb{C}^n \to \mathbb{C}^n$ always Fredholm
- In ∞-dim, A might not be Fredholm
  - im A not closed
  - $\dim \ker A \neq \dim \ker A^*$ (e.g., integration)

- $M_x : L^2([0,1]) \to L^2([0,1])$

  $(M_x f)(x) := x f(x)$ $\Rightarrow$ **CHECK** :

  $M^* = M$

  $\text{im}\, M \neq \overline{\text{im}\, M} = L^2([0,1])$

  ↑ i.e.: im M not closed

  $xf =: g$

  $f = \cancel{\frac{g}{x}}$ bad near zero...

But $M_{x+1} f(x) := (x+1) f(x)$ on $L^2([0,1])$ is Fredholm

↑ but not on $L^2([-1,1])$...

- CLOSED IMAGE BUT dim KERNELS DON'T MATCH. (SHIFT OPERATOR)

$$S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$$

$$S(x_1, x_2, \_\_) := (0, x_1, \_\_)$$

$$\Rightarrow \quad \operatorname{im} S = \{ x \in \ell^2(\mathbb{N}) : x_1 = 0 \} \subsetneq \ell^2(\mathbb{N})$$

↖ closed in $\ell^2(\mathbb{N})$

$\mathbb{N} = [1 \ 2 \ 3 \dots)$

↑ closed ↑ open

This ex. is not applicable if you replace $\mathbb{N}$ w/ $\mathbb{Z}$

ADJOINT: $S^*(x_1, x_2, \_\_) = (x_2, x_3, \_\_)$

$$\Rightarrow \quad \ker S = \{0\} \quad \text{but} \quad \ker S^* = \mathbb{C}\{1, 0, 0, \_\}$$

$\Rightarrow$ Not Fredholm ◢

DEF: A bounded operator A is Fredholm iff

(1) $\operatorname{im} A$ closed

(2) $\ker A$ and $\ker A^*$ finite-dimensional

$$(\operatorname{index} A := \dim \ker A - \dim \ker A^*)$$

THM: Fredholm Alternative holds for F
iff F Fredholm w/ index $F = 0$

NOTE: A Fredholm and K compact
$\Rightarrow$ $A+K$ Fredholm and $\operatorname{index}(A+K) = \operatorname{index}(A)$.

EXERCISE: $\mathbb{1} + K$, K compact has closed image
i.e., $\mathbb{1}+K$ is Fredholm

SELF-ADJOINT & UNITARY OPERATORS

DEF: $A : \mathcal{H} \to \mathcal{H}$ self-adjoint iff $A^* = A$.

DEF: Let $A \in \mathcal{B}(\mathcal{H})$, then we have a sesquilinear form $a(x,y) := \langle x, Ay \rangle$

$$A = A^* \Rightarrow a(x,y) = \overline{a(y,x)}$$ (Hermitian symmetric)

DEF: Given $A : \mathcal{H} \to \mathcal{H}$ bounded linear then define the quadratic form $q(x) := a(x,x) = \langle x, Ax \rangle$.

- A nonnegative definite if $A = A^*$ and $\langle x, Ax \rangle \geq 0 \ \forall x \in \mathcal{H}$
- A positive definite if $A = A^*$ and $\langle x, Ax \rangle > 0 \ \forall x \in \mathcal{H}$

In this case, $a(x,y) = \langle x, Ay \rangle$, A positive definite, is an inner product on $\mathcal{H}$.

Ex: If A is bounded below, then the norm of $(\cdot,\cdot)$ is equivalent to $\langle \cdot, A \cdot \rangle$.

NOTE: PARTIAL ORDERING $A \geq B \iff \langle x, Ax \rangle \geq \langle x, Bx \rangle \ \forall x \in \mathcal{H}$

LEMMA: $A^* = A \Rightarrow \|A\| = \sup_{\|x\|=1} |\langle x, Ax \rangle|$.

Pf: Set $\alpha := \sup_{\|x\|=1} |\langle x, Ax \rangle|$ (assuming $\mathbb{F} = \mathbb{R}$)

(Hahn-Banach) (Taking coords. of Ax by inn. prod.)

$$\|A\| \overset{\text{def}}{=} \sup_{\|x\|=1} \|Ax\| = \sup_{\|x\|=\|y\|=1} |\langle y, Ax \rangle| \geq \alpha.$$

$A^* = A \Rightarrow$ Polarization identity

$$4\langle y, Ax\rangle = \langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle$$

Abs. val:

$$4|\langle y, Ax\rangle| \leq \alpha\left(\|x+y\|^2 + \|x-y\|^2\right)$$

Parallelogram Law → $\leq 2\alpha\left(\|x\|^2 + \|y\|^2\right)$

Take $\|x\| = \|y\| = 1 \Rightarrow |\langle y, Ax\rangle| < \alpha$.

$$\Rightarrow \|A\| < \alpha.$$

$\square$

COROLLARY: $\|A^*A\| = \|A\|^2$ ⚠

Pf: $(A^*A)^* = A^*A \Rightarrow A^*A$ self-adjoint

$(AA^*)^* = AA^* \Rightarrow AA^*$ self-adjoint

So, by the Lemma

$$\|A^*A\| = \sup_{\|x\|=1} |\langle x, A^*Ax\rangle| = \sup_{\|x\|=1} |\langle Ax, Ax\rangle|$$

$$= \|A\|^2.$$

$\square$

Note: $S^* = -S$ ⇝ skew-self-adjoint (multiply S by i to get a self-adjoint)

DEF: $U : \mathcal{H}_1 \to \mathcal{H}_2$ is **UNITARY** iff U is invertible and $\langle Ux, Uy \rangle_{\mathcal{H}_2} = \langle x, y \rangle_{\mathcal{H}_1}$.

In particular, $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$,

$U : \mathcal{H} \to \mathcal{H}$ unitary $\iff U^* U = U U^* = \mathbb{1}_{\mathcal{H}}$

DEF: $T : \mathcal{H} \to \mathcal{H}$ NORMAL $TT^* = T^* T$

//

WEAK CONVERGENCE IN $\mathcal{H}$

⚠ **IMPORTANT** By Riesz: $\mathcal{H}^* \cong \mathcal{H} \Rightarrow$ weak and weak-$*$ are equivalent

DEF: $x_n \xrightarrow{w} x \iff \langle x_n, y \rangle \longrightarrow \langle x, y \rangle \ \forall y \in \mathcal{H}$.

UNIFORM BOUNDEDNESS PRINCIPLE: $\varphi_n : X \to \mathbb{C}$, X Banach, linear forms and $|\varphi_n(x)| \le C_x \ \forall n$, then $\exists \tilde{C}$ (indep. of n, x) such that $\|\varphi_n\| < \tilde{C}$.

COROLLARY: If $x_n \xrightarrow{w} x$, then we know the sequence x_n is uniformly bounded.

THM: Let $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ and $D \subset \mathcal{H}$ dense and linear. Then

$$\boxed{x_n \xrightarrow{w} x}$$

Trivial by UBP

$\langle x_n, y\rangle \to \langle x, y\rangle \Rightarrow |\langle x_n, y\rangle| \leq C_y$

$\text{UBP} \Rightarrow \exists \tilde{C} \ \|x_n\| < \tilde{C}$

$\Longrightarrow$

iif

$\Longleftarrow$

$$\boxed{\begin{array}{l} (1)\ \|x_n\| \leq M \\ (2)\ \langle x_n, y\rangle \longrightarrow \langle x, y\rangle \\ \qquad \forall y \in D \end{array}}$$

⚠ **IMPORTANT:** Suffices to prove bounded + convergence coordinate-wise when you have a countable basis...

LECTURE 14

(ch 9)

Feb 23 2026

WEAK CONVERGENCE : $\mathcal{H}^* \cong \mathcal{H}$, $\xrightarrow{w} \iff \xrightarrow{w^*}$

$$x_n \xrightarrow{\text{weakly}} x \iff \langle x_n, y\rangle \to \langle x, y\rangle , \quad \forall y \in \mathcal{H}$$

UBP: $x_n \xrightarrow{\text{weakly}} x \Rightarrow \exists C > 0$ s.t. $\|x_n\| \le C < \infty \quad \forall n.$

THM: $\{e_\alpha\}_{\alpha\in\mathbb{N}}$ countable orthonormal basis

$$\left\{ x_n \xrightarrow{\text{weakly}} x \right\} \iff \left\{ \begin{array}{l} \|x_n\| \le C < \infty \\ \langle e_\alpha, x_n\rangle \to \langle e_\alpha, x\rangle , \quad \forall \alpha\in\mathbb{N} \end{array} \right\}$$

Parseval / Bessel : $y = \sum_{m=1}^{M} \langle e_m, y\rangle e_m$, $M<\infty$ is dense in $\mathcal{H}$

Ex: $\mathcal{H} \cong L^2([0,1])$

NOTE : $f_n^2 \xrightarrow{\text{weakly}} \frac{1}{2}$

$$f_n(x) := \sin(n\pi x) \Rightarrow f_n \xrightarrow{\text{weakly}} 0 .$$

Let $\boxed{e_m := e^{2\pi i m x}}$ ← Basis of $L^2([0,1])$

$$\langle e_m, f_n\rangle = \int \overline{e_m} f_n = \int e^{-2\pi i m x} \left(\frac{e^{\pi i n x} - e^{-\pi i n x}}{2i} \right) dx$$

$$= 0 \text{ for large enough } m.$$

PROPOSITION :

(i) $x_n \xrightarrow{\text{weakly}} x \implies \|x\| \le \liminf_{n\to\infty} \underbrace{\|x_n\|}_{\text{bdd by UBP}}$

(ii) $x_n \xrightarrow{\text{weakly}} x$ and $\|x_n\| \longrightarrow \|x\|$, then $x_n \xrightarrow{\text{strongly}} x$.

Pf: Say $\mathcal{H}$ is $\mathbb{R}$ - space.

$$\|x_n - x\|^2 = \|x\|^2 + \|x_n\|^2 - 2\underbrace{\langle x, x_n\rangle} \longrightarrow \langle x, x\rangle \text{ by assumption}$$

$$0 \le \liminf_n \|x_n - x\|^2 = \|x\|^2 + \liminf_n \|x_n\|^2 - 2\underline{\|x\|}^2$$

$$= \liminf_n \left(\|x_n\|^2 - \|x\|^2\right)$$

(i) Add $\|x\|^2$ to both sides

(ii) RHS $= 0$ when $\|x_n\|^2 \to \|x\|^2$.

■

BANACH – ALAOGLU$^{(\mathcal{H} \cong \mathcal{H})}$: The closed unit ball is weakly sequentially compact.

↳ If $\|x_n\| \leq 1 \Rightarrow x_{n_j} \xrightarrow{\text{weakly}} x$ in $\mathcal{H}$.

Pf: ($\mathcal{H}$ separable) Take o.n.b. $\{e_m\}$.

$|\langle x_n, e_m \rangle| \leq \|x_n\| \|e_m\| \leq 1 \Rightarrow$ Have a subseq.

$\langle x_{\varphi_1(n)}, e_1 \rangle \xrightarrow{n \to \infty} c_1$

by Heine-Borel for $\mathbb{C}$.

Iterate: $\langle x_{\varphi_k \circ \cdots \circ \varphi_1(n)}, e_k \rangle \xrightarrow{n \to \infty} c_k$

Set

$$\mathcal{H}^* \ni \phi(y) := \lim_{k \to \infty} \langle x_{\varphi_k \circ \cdots \circ \varphi_1(k)}, y \rangle .$$

↑ well-defined for

$$y = \sum_{m=1}^{M} \langle e_m, y \rangle e_m, \quad M < \infty .$$

So, $\phi(y)$ is well-defined on a dense linear set in $\mathcal{H}$ and is bounded because the limits converge as before $\Longrightarrow$ ϕ extends to $\mathcal{H}$ by Hahn-Banach with same norm.

Note: $\|\phi\| \leq 1 \Rightarrow$ Riesz: $\exists!\, x \in \mathcal{H}$ s.t. $\phi(y) = \langle x, y \rangle$ $\forall y \in \mathcal{H}$.

$\Rightarrow$ Subsequence $\langle x_{\varphi_k \circ \cdots \circ \varphi_1(k)}, y \rangle \longrightarrow \langle x, y \rangle \quad \forall y \in \mathcal{H}$

$\Rightarrow$ Closed unit ball is sequentially compact. ■

//

SPECTRAL THEORY FOR BOUNDED LINEAR OPERATORS (Ch 9)

- Assume $A u_k = \lambda_k u_k$, $\lambda_k \in \mathbb{C}$ and $\{u_k\}_{k=1}^{n}$ form an o.n.b. for $\mathbb{C}^n$.

  $\sigma(A) := \{\lambda_k\}_{k=1}^{n}$ SPECTRUM of A (i.e., where $A - \lambda \mathbb{1}$ not invertible)

  Set

  $$U := \begin{bmatrix} | & & | \\ u_1 & \cdots & u_n \\ | & & | \end{bmatrix} \in \mathbb{C}^{n \times n}, \quad U e_k = u_k$$

  and $U^{-1} = U^*$ (unitary) b/c u_k o.n.b.

- $AU = UD$, $D = \operatorname{diag}(\lambda_1, -, \lambda_n)$

  $\Rightarrow \boxed{A = UDU^*}$

  Note: $AA^* = UDU^*UD^*U$
  $= U|D|^2U$
  $= A^*A$

  $A^* = UD^*U^*$

  $\Rightarrow$ A normal.

THM: ($\mathbb{C}^n$ – SPECTRAL THEOREM)

A is normal iff $\mathbb{C}^n$ has an o.n.b. of ev's of A

DEF: (RESOLVENT SET) The resolvent set of $A \in \mathcal{B}(\mathcal{H})$ is:

$$\rho(A) := \mathbb{C} \setminus \sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda \mathbb{1} \text{ is } \underline{\text{invertible}}\} \subset \mathbb{C}$$

i.e., 1-1 and onto

DEF: (SPECTRUM) The spectrum of $A \in \mathcal{B}(\mathcal{H})$ is

$$\sigma(A) := \mathbb{C} \setminus \rho(A) = \{\lambda \in \mathbb{C} : A - \lambda \mathbb{1} \text{ } \underline{\text{not}} \text{ invertible}\} \subset \mathbb{C}.$$

Note: Open Mapping Thm gives that $(A - \lambda \mathbb{1})^{-1}$ bounded on the resolvent set $\rho(A)$.

Note: $Au = \lambda u$ for $0 \neq u \in \mathcal{H} \Rightarrow \underset{\text{not } 1-1}{(A - \lambda \mathbb{1})} \Rightarrow \lambda \in \sigma(A)$.

⚠ IMPORTANT TECHNIQUE TO SHOW $\lambda \in \sigma(A)$:

FACT: By Open Mapping Thm, for $\underline{T \in \mathcal{B}(\mathcal{H})}$,

T invertible $\iff$ T bounded below ← $(\exists C > 0 : \|Tx\| \geq C\|x\| \ \forall x \in \mathcal{H})$

and dense image

Can contradict this by constructing sequence (x_n) s.t. $\|x_n\|_{\mathcal{H}} = 1 \ \forall n$ but $\|Tx_n\| \longrightarrow 0$ as $n \nearrow \infty$.

⚠ Useful for showing $A - \lambda \mathbb{1}$ not invertible which gives $\lambda \in \sigma(A)$

show $A - \lambda \mathbb{1}$ is not bdd below

⚠

DEF: (1) POINT - SPECTRUM

$$\sigma_p(A) := \{ \lambda \in \sigma(A) : A - \lambda \mathbb{1} \text{ is not } 1-1 \}$$

(2) CONTINUOUS SPECTRUM

$$\sigma_c(A) := \left\{ \lambda \in \sigma(A) : \begin{array}{l} A - \lambda \mathbb{1} \text{ is } 1-1 \text{ but not} \\ \text{onto and image is dense} \end{array} \right\}$$

i.e., $\operatorname{im}(A-\lambda\mathbb{1}) \subsetneq \overline{\operatorname{im}(A-\lambda\mathbb{1})} = \mathcal{H}$

(3) RESIDUAL SPECTRUM

$$\sigma_r(A) := \left\{ \lambda \in \sigma(A) : \begin{array}{l} A - \lambda \mathbb{1} \text{ is } 1-1 \text{ but} \\ \text{not onto and image is} \\ \text{not dense} \end{array} \right\}$$

i.e., $\overline{\operatorname{im} A-\lambda\mathbb{1}} \subsetneq \mathcal{H}$

$$\sigma(A) = \sigma_p(A) \sqcup \sigma_c(A) \sqcup \sigma_r(A).$$

Ex: $\mathcal{H} = L^2([0,1])$, $\quad Mf(x) := x\, f(x)$, $\quad M := M_x \in \mathcal{B}(\mathcal{H})$.

$(M - x_0)\, f(x) = (x - x_0)\, f(x)$ $\qquad$ Inverse: $(M - x_0)^{-1} = M_{1/x-x_0}$

- if $x_0 \in \mathbb{C} \setminus [0,1]$, then $\frac{1}{x-x_0}$ is bounded
- if $x_0 \in [0,1]$, then problem: $f \in L^2 \Rightarrow \frac{f}{x-x_0} \notin L^2$

$\Rightarrow$ Prove: $\sigma(M_x) = \sigma_c(M_x)$

Obs: $x_0 \in [0,1]$, $\quad (x - x_0) f(x) = 0$

$$(x - x_0)\, \delta_{x_0}(x) = 0$$

$$\int g(x)(x - x_0)\, \delta_{x_0}(x)\, dx = g(x_0)(x_0 - x_0) = 0$$

$\delta_{x_0} \notin \mathcal{H}$ → generalized ev's.

Cl: image is dense: $f(x)\, \chi_{\{|x-x_0| \geq \delta\}} \xrightarrow{L^2} f$ as $\delta \to 0$.

$$\frac{f(x)\, \chi_{\{|x-x_0| \geq \delta\}}}{x - x_0} \in L^2$$

$$\Downarrow$$

M has dense image in L^2.

◢

DEF. (RESOLVENT OPERATOR) For $\lambda \in \rho(A)$, the resolvent operator is defined as

$$R_\lambda := (\lambda \mathbb{1} - A)^{-1} = \frac{1}{\lambda}\left(\mathbb{1} - \frac{A}{\lambda}\right)^{-1}$$

$\in \mathcal{B}(\mathcal{H})$ by OMT

PROPOSITION: (1) $\{\lambda \in \mathbb{C},\ |\lambda| > \|A\|\} \subset \rho(A)$

(2) R_λ is analytic on $\rho(A)$

(3) $\rho(A)$ is open in $\mathbb{C}$ $\Leftrightarrow$ $\sigma(A)$ is closed

Pf: $\lambda_0 \in \rho(A)$, $\lambda = (\lambda - \lambda_0) + \lambda_0$

$$\lambda - A = (\lambda_0 - A)\Big[\mathbb{1} - \underbrace{(\lambda_0 - \lambda)\underbrace{(\lambda_0 - A)^{-1}}_{\text{Bdd}}}_{=:\, K_\lambda}\Big]$$

$$= (\lambda_0 - A)(\mathbb{1} - K_\lambda)$$

$$|\lambda - \lambda_0|\ \|(\lambda_0 - A)^{-1}\| < 1 \implies \|K_\lambda\| \leq 1$$

$$\Rightarrow (\lambda - A)^{-1} (1 - K_\lambda)^{-1} (\lambda_0 - A)^{-1}$$

$$R_\lambda = \sum_{k=0}^{\infty} (K_\lambda)^k (\lambda_0 - A)^{-1} \quad \Rightarrow \quad R_\lambda \text{ analytic.}$$

$$R_\lambda = \frac{1}{\lambda}\left(1 - \frac{A}{\lambda}\right)^{-1} . \quad |\lambda| \geq \|A\| \Rightarrow \left\| \frac{A}{\lambda} \right\| < 1$$

$$\Rightarrow \quad R_\lambda = \frac{1}{\lambda} \sum_{k=0}^{\infty} \frac{A^k}{\lambda^k} \quad \text{defined \& bdd.}$$

□

LECTURE 15

(Ch 9)

Feb 25 2026

SPECTRAL THEORY (ctd)

PROPOSITION: $A \in \mathcal{B}(\mathcal{H})$, $\sigma(A) \neq \emptyset$ ⚠

Pf: $\lambda \mapsto R_\lambda$ is analytic hence entire if $\sigma(A) = \emptyset$

$$\Rightarrow \lambda \mapsto \langle R_\lambda x, y \rangle \text{ entire \& bounded} \xrightarrow{\text{Liouville}} \text{constant}$$

$$\Rightarrow R_\lambda \equiv 0 \quad \hookrightarrow /\!/ \hookleftarrow$$

□

LEMMA: $A = A^*$, $\lambda \in \sigma_p(A) \Rightarrow \lambda = \bar{\lambda}$

Pf:

$$\begin{array}{ccc} \langle x, Ax \rangle & = & \langle Ax, x \rangle \\ \| & & \| \\ \langle x, \lambda x \rangle & & \langle \lambda x, x \rangle \\ \| & & \| \\ \lambda \|x\|^2 & & \bar{\lambda} \|x\|^2 \end{array}$$

□

LEMMA: $A = A^*$

If $Ax = \lambda x$ and $Ay = \mu y$, $\lambda \neq \mu$, then $x \perp y$; i.e., $\langle x, y \rangle = 0$.

Pf:

$$\begin{array}{ccccc} \langle Ax, y \rangle & = & \langle x, Ay \rangle & = & \mu \langle x, y \rangle \\ \| & & & & \\ \lambda \langle x, y \rangle & & & & \end{array}$$

$A^* = A$

$$\Rightarrow (\lambda - \mu) \langle x, y \rangle = 0.$$

□

DEF: $M \subset \mathcal{H}$ closed is *invariant subspace* of $A(M) \subset M$.

LEMMA: $A = A^*$. $A(M) \subset M \Longrightarrow A(M^\perp) \subset M^\perp$.

Pf: $x \in M^\perp$, $y \in M$.

$$\langle Ax, y\rangle \overset{?}{=} 0$$
$$\parallel$$
$$(x, \underbrace{Ay}_{\overset{\cap}{M}}) = 0 \checkmark .$$

$\square$

NOTE:

$$\begin{array}{c|c:c|}
 & M & M^\perp \\
M & A|_M & 0 \\
\hdashline
M^\perp & 0 & A|_{M^\perp}
\end{array} .$$

Divide and conquer

PROPOSITION: $A \in \mathcal{B}(\mathcal{H})$, $\lambda \in \sigma_r(A) \Longrightarrow \bar{\lambda} \in \sigma_p(A^*)$.

Pf: $\mathcal{H} = \overline{\mathrm{im}\,(A - \lambda \mathbb{1})} \oplus \underset{\neq\{0\}}{M^\perp}$

$$= \overline{\mathrm{im}\,(A - \lambda \mathbb{1})} \oplus \underbrace{\ker\left[(A - \lambda \mathbb{1})^*\right]}_{= \ker(A^* - \bar{\lambda}\mathbb{1})} \neq \{0\} \Longrightarrow \begin{array}{l} \exists x \neq 0 \text{ s.t.} \\ A^* x = \bar{\lambda} x . \ \square \end{array}$$

COROLLARY: $A \in \mathcal{B}(\mathcal{H})$, $A = A^* \Rightarrow \sigma_r(A) = \emptyset$.

LEMMA: $A \in \mathcal{B}(\mathcal{H})$, $A = A^*$ then

⚠ $\sigma(A) \subset [-\|A\|, \|A\|] \subset \mathbb{R}$.

Pf: By contradiction, $\lambda = a + ib \in \sigma(A)$, $b \neq 0$.

$$\|(A - \lambda \mathbb{1})x\|^2 = \|(A-a)x - ibx\|^2 = \|(A-a)x\|^2 + \|bx\|^2$$

$\uparrow$ "Pitagore"

$$\geq b^2 \|x\|^2.$$

$$\Rightarrow \|(A - \lambda \mathbb{1})x\| \geq |b| \, \|x\|, \quad |b| > 0.$$

By a priori estimates $\Rightarrow$ im $A - \lambda \mathbb{1}$ is closed.

$$\Rightarrow \lambda \notin \sigma_c(A)$$

$\lambda \notin \sigma_p(A)$ & $\sigma_r(A) = \emptyset \quad \Rightarrow \lambda \notin \sigma(A) \quad \longrightarrow\!/\!/\!\longleftarrow$

$\uparrow$

b/c $\bar{\lambda} \neq \lambda$

$\square$

PROPOSITION: $A \in \mathcal{B}(\mathcal{H})$ compact, ⚠

(1) $\lambda \in \sigma_p(A)$, $\lambda \neq 0$ $\Longrightarrow$ λ finite multiplicity.

(2) $\{\lambda_n\}_{n \in \mathbb{N}} \subset \sigma_p(A)$, $\lambda_n \neq 0$, admits 0 as the only accumulation point.

Pf: Say $Au_k = \lambda u_k$, u_k o.n.b. of ker $(A - \lambda \mathbb{1})$.

Compactness $\rightsquigarrow$ $Au_{\varphi(k)} \longrightarrow y$ $= \lambda u_{\varphi(y)}$ impossible. b/c of orthogonality!

$\{\lambda_n\}$ bounded and $0 \neq \lambda_{\varphi(n)} \longrightarrow \lambda \neq 0$.

$Ae_{\varphi(n)} = \lambda_{\varphi(n)} e_{\varphi(n)}$; $f_{\varphi(n)} := \dfrac{e_{\varphi(n)}}{\lambda_{\varphi(n)}} \cdot \left| \dfrac{1}{\lambda_{\varphi(n)}} \right| \leq M$.

$Af_{\psi \circ \varphi(n)} \longrightarrow y$

$e_{\varphi(n)}$ not Cauchy

} Impossible b/c e's are orthogonal to each other

$\Longrightarrow$ M does not exist $\Rightarrow$ $\dfrac{1}{\lambda_{\varphi(n)}} \longrightarrow \infty$ $\Rightarrow$ $\lambda_{\varphi(n)} \longrightarrow 0$.

□

SPECTRAL THEOREM (Compact + Bounded + Self-Adjoint) :

$A \in \mathcal{B}(\mathcal{H})$, $A = A^*$, A compact. Then, there exists an o.n.b. for $\mathcal{H}$ of eigenvectors of A (i.e., $\sigma(A) = \sigma_p(A)$)

- Non-zero eigenvalues form a (at most) countably infinite set $\{\lambda_k\}_{k\in\mathbb{N}}$, $\lambda_k \in \mathbb{R}$ $\forall k$, such that

$$A = \sum_{k\in\mathbb{N}} \lambda_k P_k \quad , \quad P_k \text{ orthogonal projectors}$$

NOTE : $A u_k = \lambda_k u_k$, $\|u_k\| = 1$, write

$$P_k := u_k \otimes u_k \iff P_k x := \langle u_k , x \rangle u_k .$$

If we have multiplicity, $\lambda_k = \dots = \lambda_{k+m} =: \lambda \neq 0$, m finite, then

$$P_\lambda = \sum_{l=0}^{m} P_{k+l} = \sum_{l=0}^{m} u_{k+l} \otimes u_{k+l}$$

$$\Rightarrow \boxed{A = \sum_{\substack{\lambda\in\sigma(A) \\ \| \\ \sigma_p(A)}} \lambda P_\lambda}$$

VERY USEFUL

$$\to \quad A^n = \sum_{\lambda\in\sigma(A)} \lambda^n P_\lambda$$

$$f(A) = \sum_{\lambda\in\sigma(A)} f(A) P_\lambda$$

NOTE: $A^0 = \mathbb{1} = \sum_{\lambda\in\sigma(A)} P_\lambda$ $\Leftarrow$ projectors add up to "whole space".

Pf: (1) Either $\|A\|$ or $-\|A\| \in \sigma_p(A)$.

$$\|A\| \neq 0, \quad \|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle| = \lim_{\substack{n\to\infty \\ \|x_n\|=1}} |\underbrace{\langle x_n, Ax_n\rangle}_{\in \mathbb{R} \text{ b/c } A = A^*}|$$

x_n bdd $\Rightarrow$ Up to $\overset{\wedge}{\text{taking}}$ subsequence, $\langle x_n, Ax_n\rangle \to \lambda$.

$\uparrow$ b/c $\langle x_n, Ax_n\rangle$ could be pos. or neg. so might need to pick subseq.

$\Rightarrow |\lambda| = \|A\|$.

Now, A compact $\Rightarrow Ax_{\varphi(n)} \to y \in \mathcal{H}, \; y \neq 0$.

$$Ay \overset{?}{=} \lambda y.$$

$$\|(A - \lambda\mathbb{1})y\|^2 = \lim_{n\to\infty} \|(A-\lambda)Ax_{\varphi(n)}\|^2$$

$$\leq \|A\|^2 \liminf_{n\to\infty} \|(A-\lambda)x_{\varphi(n)}\|^2$$

$A = A^* \rightsquigarrow$
$$= \|A|^2 \liminf_{n\to\infty}\Big(\overset{\to \|A\|^2}{\|Ax_{\varphi(n)}\|^2} + \lambda^2 \underbrace{\|x_{\varphi(n)}\|^2}_{=1} - 2\lambda\underbrace{\langle x_{\varphi(n)}, Ax_{\varphi(n)}\rangle}_{\downarrow \lambda}\Big)$$

$$\leq \|A\|^2\left(\|A\|^2 - \lambda^2\right)$$

$|\lambda| = \|A\| \rightsquigarrow = 0$.

(2) $N_1 = \mathcal{H}$, $A_1 = A$

$$A_1 = \left[\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & A_2 \end{array} \right]$$

$A_1 e_1 = \lambda_1 e_1 \in N_1$

$M_2 := \mathbb{C} e_1$ and $N_1 = M_2 \oplus \underbrace{N_2}_{= M_2^\perp}$

$\Rightarrow M_2^\perp = N_2$ is invariant for A because self-adjoint.

Define: $A_2 := A_1 \big|_{N_2} \Rightarrow \| A_2 \| \leq \| A_1 \|$.

Repeat : $M_n := \text{span} \{ e_1, \ldots, e_n \}$

$N_n := M_n^\perp$ $\quad \forall n \in \mathbb{N}$.

$A_n := A \big|_{N_n} = A_{n-1} \big|_{N_n}$

(3) If $A_{n+1} = 0 \underset{\substack{\uparrow \\ \text{finite dimensions}}}{\Rightarrow} A = \sum_{k=1}^{n} \lambda_k e_k \otimes e_k = \sum_{k=1}^{n} \lambda_k P_k$

(TO BE CONTINUED)

LECTURE 16

(Ch 9 ctd)

Mar 2^{nd} 2026

FINAL FOCUS: Ch 6, 7, 8, 9, 10

Recall: Spectral Theorem for bounded compact self-adjoint operators.

← $\mathcal{H}$ separable but could be uncountable...

- $\exists$ o.n.b $\{e_n\}_{n\in\mathbb{N}}$ of eigenfunctions of A $(Ae_k = \lambda_k e_k,$ $\lambda_k \in \mathbb{R})$ and

$$A = \sum_{k\in\mathbb{N}} \lambda_k\, e_k \otimes e_k = \sum_{k\in\mathbb{N}} \lambda_k P_k = \sum_{\lambda\in\sigma(A)} \lambda P_\lambda$$

↖ $\sigma(A) = \sigma_p(A)$

i.e., $Ax = \sum_{k\in\mathbb{N}} \lambda_k \langle x, e_k\rangle e_k$.

Pf: (Spectral Thm – Continued) Either $\|A\|$ or $-\|A\|$ is in $\sigma(A)$.

$$A \rightsquigarrow \left[\begin{array}{c|c} \begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix} & 0 \\ \hline 0 & A\big|_{N_{n+1}} =: A_{n+1}\end{array}\right] \begin{matrix} M_{n+1} \\ \oplus \\ N_{n+1}\end{matrix}$$

If $A_{n+1} = 0 \Rightarrow A = \sum_{n=1}^{N} \lambda_n P_n$ and $\mathcal{H} = \operatorname{span}\{e_k\} \oplus \underset{\operatorname{span}\{f_1,\dots\}}{\underline{N_{n+1}}}$

Done if we have finitely many dimensions, we're done

If $A_{n+1} \neq 0$, then $\|A_n\| \xrightarrow{n\to\infty} 0$ by compactness (accumulation of eigenvalues can only happen at zero).

$$\Rightarrow \quad \underset{\substack{\downarrow \\ 0}}{\|A_{n+1}\|} = |\lambda_{n+1}| \quad \Rightarrow \quad |\lambda_n| \to 0 .$$

i.e., $\lambda_n \to 0$.

Have basis $\{e_n\}$ with $A_n e_n = \lambda_n e_n$.

$$\Rightarrow \quad A = \sum_{n=1}^{N} \lambda_n P_n + \underbrace{A_{n+1}} \quad \text{converges uniformly}$$

$\|A_{n+1}\| \to 0 \Rightarrow$ converges uniformly

$$\Rightarrow \quad \boxed{A := \sum_{n=1}^{\infty} \lambda_n P_n}$$

Obs: λ_n goes to zero but are not exactly 0.

NOTE: If $\dim[\operatorname{im}(\text{compact})] = +\infty \Rightarrow$ image not closed.
(since $\lambda_n \to 0$ but never actually zero)

Cl: $\operatorname{im} A \subset \operatorname{span}\{e_n\}$.

$$x = \sum_{k=1}^{\infty} b_k e_k \ ; \quad \|x\|^2 = \sum_{k=1}^{\infty} |b_k|^2 \quad \text{by Parseval.}$$

PiCARD's CRiTERiON

$$Ax = \sum_{k=1}^{\infty} \underbrace{\lambda_k b_k}_{c_k} e_k \quad \rightsquigarrow \quad \operatorname{im} A = \left\{ \sum_{k=1}^{\infty} c_k e_k : \sum_{k=1}^{\infty} \left| \frac{c_k}{\lambda_k} \right|^2 < \infty \right\}$$

Note: $\operatorname{im} A \subsetneq \overline{\operatorname{im} A} = \left\{ \sum_{k=1}^{\infty} b_k e_k : \sum_{k=1}^{\infty} |b_k|^2 < \infty \right\}$.

Thus: $\overline{\operatorname{im} A} = \operatorname{span}\{e_n\} =: M \rightsquigarrow$ closed.

$$\Rightarrow \quad \mathcal{H} = M \oplus M^{\perp}$$

$$A|_{M^{\perp}} = 0 \quad \text{and} \quad M^{\perp} = \operatorname{span}\{f_n\} = \ker A.$$

i.e., $\mathcal{H} = \operatorname{span}\{e_n\} \oplus \operatorname{span}\{f_n\} \rightsquigarrow A f_n = 0$. □

UPSHOT : $\quad A = \sum_{\lambda \in \sigma(A)} \lambda P_\lambda .$

$$f(A) = \sum_{\lambda \in \sigma(A)} f(\lambda) P_\lambda .$$

e.g.: $\quad (A - z\mathbb{1})^{-1} = \sum_{\lambda \in \sigma(A)} \frac{1}{\lambda - z} P_\lambda \quad \Big\}$ Always defined if $\operatorname{Im} z \neq 0$.

NOTE : Given $\lambda_n \neq 0$, $\ker A = \{0\} \Rightarrow A$ not invertible as bounded operator.

But it is as an unbounded operator $A^{-1} = \sum_{n \geq 1} \frac{1}{\lambda_n} P_{\lambda_n}$.

—————————— // ——————————

EXAMPLES OF COMPACT OPERATORS

⚠

THM : (COMPACTNESS CRITERION) $E \subset \mathcal{H}$. ← separable

(1) If E is precompact, then for all o.n.b.'s $\{e_n\}$ and all $\varepsilon > 0$, $\underline{\exists N = N(\varepsilon) \in \mathbb{N}}$, s.t. ← N independent of x

$$\|(\mathbb{1} - P_N)x\|^2 = \sum_{n=N+1}^{\infty} |\langle e_n, x\rangle|^2 < \varepsilon \quad \underline{\forall x \in E} .$$

(2) If E is bounded and $\exists$ o.n.b. $\{e_n\}$ s.t. $\forall \varepsilon > 0$ $\exists N = N(\varepsilon) \in \mathbb{N}$ s.t.

⚠

$$\sum_{n=N+1}^{\infty} |\langle e_n, x\rangle|^2 < \varepsilon \quad \forall x \in E,$$

then E is precompact.

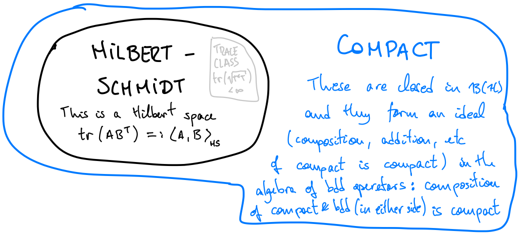
EXAMPLE : $A : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$

$$A(x_1, x_2, x_3, \dots) = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \dots)$$

A compact $\Longleftrightarrow$ $\lambda_n \to 0$ as $n \to \infty$.

//

HILBERT - SCHMIDT OPERATORS :



This is a Hilbert space
$\mathrm{tr}(AB^T) =: \langle A, B\rangle_{HS}$

These are closed in $\mathcal{B}(\mathcal{H})$ and they form an ideal (composition, addition, etc of compact is compact) in the algebra of bdd operators: composition of compact & bdd (in either side) is compact

DEF : $A \in \mathcal{B}(\mathcal{H})$ is Hilbert - Schmidt if $\exists \{e_n\}_{n\in\mathbb{N}}$ o.n.b. of $\mathcal{H}$ such that

$$\|A\|_{HS}^2 := \sum_{n\in\mathbb{N}} \|Ae_n\|^2 < \infty .$$

THM : $A \in \mathcal{B}(\mathcal{H})$ is Hilbert-Schmidt $\Rightarrow$ A compact.

EXAMPLE : (Typical HS operator)

$$K : L^2([0,1]) \longrightarrow L^2([0,1])$$

$$Kf(x) := \int_0^1 \kappa(x,y)\, f(y)\, dy$$

CLAIM : Kernel $\kappa \in L^2([0,1]^2) \iff K$ is Hilbert-Schmidt.

Pf : ($\Rightarrow$) Say $\iint_{[0,1]^2} |\kappa(x,y)|^2 \, dy\, dx < \infty$.

Take Fourier basis $\{e_n\}$ o.n.b. $L^2([0,1])$:

$$\kappa(x,y) = \sum_{i,j=1}^{\infty} \kappa_{ij}\, e_i(x)\, e_j(y) .$$

$\Rightarrow$ Parseval : $\iint_{[0,1]^2} |\kappa(x,y)| \, dy\, dx = \sum_{i,j=1}^{\infty} |\kappa_{ij}|^2$

$$= \sum_{\ell=1}^{\infty} \| K e_\ell \|^2 < \infty$$

$$\Rightarrow \boxed{\| K \|_{HS}^2 = \iint_{[0,1]^2} |\kappa(x,y)|^2 \, dy\, dx < \infty .}$$

$\square$

NOTE : A HS $\Rightarrow A^* A = \sum_n \mu_n P_n$, $\mu_n \geq 0$

$\lambda_n = \sqrt{\mu_n}$ singular values of A.

$\Rightarrow \| A \|_{HS}^2 = \sum_{n=1}^{\infty} \lambda_n^2 < \infty .$

!

(ℓ^∞)	$\lambda_n \searrow 0$	COMPACT
(ℓ^2)	$\sum_n \lambda_n^2 < \infty$	HILBERT - SCHMIDT
(ℓ^1)	$\sum_n \lvert\lambda_n\rvert < \infty$	TRACE CLASS

! THM: $A \in \mathcal{B}(\mathcal{H})$, IMPORTANT ! ⚠

$$A \text{ compact} \iff \left\{ x_n \xrightarrow{\text{weakly}} x \iff Ax_n \to Ax \right\}$$

i.e., compact operators are defined, morally, as operators that replace weak convergence w/ strong convergence

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LECTURE 17

(Ch 10)

Mar 4 2026

* UNBOUNDED OPERATORS

LAPLACE

SCHRÖDINGER

Hamiltonian

- Typical operator: $\begin{cases} -u'' = \lambda u \\ u(0) = u(1) = 0 \end{cases}$, $\begin{cases} i\partial_t u = Hu \\ u(0) = u_0 \end{cases}$

$$u(t) = \underbrace{e^{-itH}}_{\text{unitary}} u_0$$

$$\Updownarrow$$

$$H = H^*$$

$\mathcal{H} := L^2([0,1])$.

Q: Can define u'' in L^2?

- $\overset{\text{linear}}{A}$ unbounded $\rightarrow$ $D(A) \overset{\text{linear}}{\subset} \mathcal{H}$ with $\overline{D(A)} = \mathcal{H}$ where $A : D(A) \to \mathcal{H}$ is well-defined.

$$A = \int \lambda \, dP(\lambda)$$

$$e^{-itA} = \int e^{-it\lambda} \, dP(\lambda)$$

DEF: $D(A) := \{ f \in \mathcal{H} : Af \in \mathcal{H} \}$ (DOMAIN OF A)

DEF: $\tilde{A}$ is an extension of A if $D(A) \subset D(\tilde{A})$ and $\tilde{A}\big|_{D(A)} = A$.

EXAMPLE: Let $A_k u = u''$, $\mathcal{H} = L^2([0,1])$.
different domains as below

$$D(A_1) = \{ u \in C^2([0,1]) : u(0) = u(1) = 0 \}$$ (Dirichlet Boundary)

$A_1 \subsetneq \overline{A_1} = A_3 = \overline{A_3}$, $A_2 \subsetneq \overline{A_2} = A_4 = \overline{A_4}$

$$D(A_2) = C^2([0,1])$$

A_1 essentially self-adjoint (not A_2, not symm.)

$$D(A_3) = \{ u \in \underbrace{H^2([0,1])} : u(0) = u(1) = 1 \}$$

are domain of Δ in $L^2([0,1])$: $D(\Delta) = \{ f \in L^2([0,1]) : \Delta f \in L^2([0,1]) \} =: H^2([0,1])$

$$D(A_4) = H^2([0,1])$$

$$D(A_5) = \{ u \in H^2([0,1]) : u(0) = u(1) = u'(0) = u'(1) = 0 \}$$

$\Rightarrow A_1^* = A_3 = A_3^*$; A_1 symmetric ; $A_2^* \overset{\overset{C^2 \text{ dense in } H^2}{\downarrow}}{=} A_4^* = A_5$

A_2, A_4 not symmetric; A_5 symmetric but not self-adjoint;

NOTE: • If $u \in L^2([0,1])$, CANNOT say $u(0) = 0$ b/c class of equiv.

• If $u \in H^{\overset{!}{①}}$, $u(0) = u(x) - \int_0^x u'(y)\,dy$

$\Rightarrow |u(0)| \leq C \|u\|_{H^1([0,1])}$

L^2 doesn't have evaluations, H^1 does, but H^2 doesn't

DEF: (ADJOINT) The adjoint of operator A is denoted A^* and satisfies, $A^* : D(A^*) \longrightarrow \mathcal{H}$,

$$\langle A^* \psi, \phi \rangle_{\mathcal{H}} = \langle \psi, A\phi \rangle_{\mathcal{H}}$$

$\forall \psi \in D(A^*)$ and $\forall \phi \in D(A)$, where

$$D(A^*) := \{ \psi \in \mathcal{H} : |\langle \psi, A\phi \rangle_{\mathcal{H}}| \leq C_\psi \|\phi\|_{\mathcal{H}} \} .$$

- NOTE: $y \in \mathcal{H}$, $\varphi_y(x) = \langle y, Ax \rangle : D(A) \to \mathbb{C}$

Riesz $\Rightarrow$ $D(A^*)$ is defined where φ_y is bounded on $D(A)$

NOTE:
- If A is bounded, then $D(A^*) = \mathcal{H}$
- A bounded $\Leftrightarrow$ $D(A) = \mathcal{H}$

DEF: An operator A on $\mathcal{H}$ is SYMMETRIC if $\forall x, y \in D(A)$,

$$\langle Ax, y \rangle = \langle x, Ay \rangle .$$

(i.e., if A^* is an extension of A)

DEF: An operator A on $\mathcal{H}$ is SELF-ADJOINT iff $A = A^*$ <u>and</u> $D(A) = D(A^*)$.

PROPERTIES:

A BOUNDED AND SYMMETRIC $\Longrightarrow$ A SELF-ADJOINT

A SELF-ADJOINT $\Longrightarrow$ A SYMMETRIC

↑ converse not true in general

DEF: $A : D(A) \to \mathcal{H}$ is closed iff $\mathrm{Graph}(A) \subset \mathcal{H} \times \mathcal{H}$ is closed;

i.e., $\forall x_n \in D(A)$ with $x_n \longrightarrow x$ and $Ax_n \longrightarrow y$, then A is closed whenever $x \in D(A)$ <u>and</u> $y = Ax$.

i.e., $\mathrm{Graph}(A) := \{ (x,y) : x \in D(A) \text{ and } y = Ax \}$.

A closed $\Longleftrightarrow$ $\mathrm{Graph}(A)$ closed in $\mathcal{H} \times \mathcal{H}$.

DEF: A is closable if $\forall x_n \in D(A)$ with $x_n \to 0$ and $Ax_n \to y$ then $y = 0$.

- THEN: the closure $\bar{A}$ of A is an extension of A with $D(\bar{A})$ is closed: $D(\bar{A}) = \{x : D(A) \ni x_n \to x \text{ and } Ax_n \to y\}$.
- THEN: $\text{Graph}(\bar{A})$ is the closure of $\text{Graph}(A)$.
- i.e., $\bar{A}$ is the "smallest" closed extension of A.

CLAIM: $\boxed{A \text{ symmetric}} \Rightarrow \boxed{A \text{ closable}}$

DEF: A is essentially self-adjoint when its closure is self-adjoint.

DEF: If $A : D(A) \to \mathcal{H}$ is 1-1 and onto, define the inverse $A^{-1} : \mathcal{H} \to \mathcal{H}$, $A^{-1}y = x \iff y = Ax$.

CLAIM: A closed and invertible $\Rightarrow$ A^{-1} closed.

$$A^{-1} : \mathcal{H} \to \mathcal{H} \text{ closed} \xRightarrow{\text{CLOSED GRAPH THM}} A^{-1} \text{ bounded}$$

Obs: Inverse of compact needs to be unbounded: integration ⟷ regularization (compact)
differentiation ⟷ losing info (unbounded)

PROPOSITION: $(A^*)^{-1} = (A^{-1})^*$

NOTE: Unbounded operators may have empty spectrum (e.g., $\partial_y \pm y$)

* WEAK DERIVATIVE

$$H^1([0,1]) := \{ u \in L^2([0,1]) : u' \in L^2([0,1]) \}.$$

↑ Completion of C^1 (or C^∞) for

$$\langle f, g \rangle_{H^1} := \int_0^1 \bar{f} g + \bar{f}' g' \, dx$$

TEST FUNCTION: $\varphi : \mathbb{R} \to \mathbb{C}$ if $\varphi \in C_c^\infty(\mathbb{R})$ (smooth & compact support); e.g., $\varphi(x) = \mathbb{1}_{\{|x| \le 1\}} e^{-1/1-x^2} \in C_c^\infty(\mathbb{R})$.

Define: $\varphi_\varepsilon(x) := \frac{1}{\varepsilon} \varphi\left(\frac{x}{\varepsilon}\right)$, φ test function.

ε^d if $\mathbb{R}^d$.

$$C^\infty(\mathbb{R}) \ni (\varphi_\varepsilon * u)(x) \xrightarrow{L^2} u(x) \Rightarrow \underline{C^\infty \text{ dense in } L^2}.$$

- If $u \in C^1$, $\varphi \in C_c^\infty(\mathbb{R})$

$$\Rightarrow v := u' \Rightarrow \forall \varphi \in C_c^\infty(\mathbb{R}), \int_{\mathbb{R}} v \varphi \, dx \overset{\substack{\text{by parts} \\ \downarrow}}{=} - \int_{\mathbb{R}} u \varphi' \, dx$$

(boundaries → 0 b/c cpt supp)

DEF: $u \in L^2(\mathbb{R})$ has weak derivative $v \in L^2(\mathbb{R})$ iff

$$\forall \varphi \in C_c^\infty(\mathbb{R}), \; \langle v, \varphi \rangle = -\langle u, \varphi' \rangle .$$

DEF: $H^k([0,1]) := \{ u \in L^2([0,1]) : u, u', u'', \ldots, u^{(k)} \in L^2([0,1]) \}$

EXAMPLE: $\frac{d}{dx} : D\left(\frac{d}{dx}\right) = H^1(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ closed

Pf: $u_n \to u$, $u_n' \to v$, $u' = v$ and $u \in H^1$?

$\langle v, u \rangle \leftarrow \langle u_n', \varphi \rangle = -\langle u_n, \varphi' \rangle \to -\langle u, \varphi' \rangle$

Claim: H^k HILBERT

Claim: C_c^∞ dense in H^k

Corollary: $u, v \in H^1(\mathbb{R}) \Rightarrow \int uv' + u'v = 0$

THM: $A = i\frac{d}{dx} : H^1(\mathbb{R}) \to L^2(\mathbb{R})$ is self-adjoint.

THM: A closed and symmetric on $\mathcal{H}$. TFAE:

(1) A self-adjoint

(2) $\ker(A^* \pm i\mathbb{1}) = \{0\}$

(3) $\operatorname{im}(A \pm i\mathbb{1}) = \mathcal{H}$