LECTURE 1 — INTRODUCTION

09/01/2024

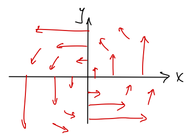
READ: First section of Arnold.

VECTOR FIELDS AS DYNAMICAL SYSTEMS

In a coord. system $(x^1, \dots, x^n)$, a (smooth) vector field is

$$V = \sum_i \underbrace{v_i(x^1, \dots, x^n)}_{\text{smooth fcts}} \frac{\partial}{\partial x^i}$$

e.g.: $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$



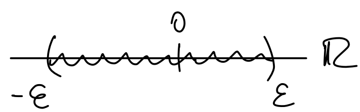
CORRESPONDING DYNAMICAL SYSTEM

Flow of V at $t=0$ at an initial pt. $p = (x^1(0), \dots, x^n(0))$ and then evolve the system as a fct. of $t \in \mathbb{R}$ into a path

$$x(t) = (x^1(t), \dots, x^n(t)).$$

This path $x: \mathbb{R} \xrightarrow{C^\infty} \mathbb{R}^n$

$$t \longmapsto (x^1, \dots, x^n)$$



This path is defined by

$$\boxed{\underbrace{\frac{d}{dt}x(t)}_{\dot{x}(t)} = V(x(t))} \quad (*)$$

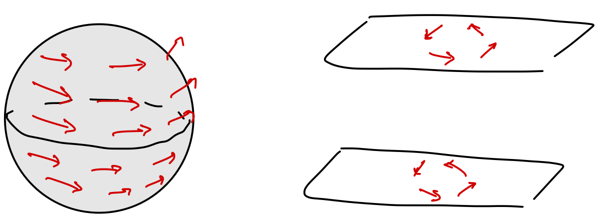
$(*) \iff \boxed{\dfrac{d}{dt} x^i(t) = V^i\left(x^1(t), \ldots, x^n(t)\right)}$ $i = 1, \ldots, n$

System of n 1^{st} order coupled (nonlinear) ODEs.

<u>Moving along charts</u>: $V = V^i \dfrac{\partial}{\partial x^i} = \tilde{V}^i \underbrace{\dfrac{\partial}{\partial \tilde{x}^i}}_{= \sum_j \frac{\partial x^j}{\partial \tilde{x}^i} \frac{\partial}{\partial x^j}}$

Obs: The definition in $(*)$ also allows us to define these dynamical systems on topologically nontrivial spaces (e.g., curved).

e.g., on S^2



$V = V^1 \dfrac{\partial}{\partial x} + V^2 \dfrac{\partial}{\partial y}$

$\tilde{V} = \tilde{V}^1 \dfrac{\partial}{\partial \tilde{x}} + \tilde{V}^2 \dfrac{\partial}{\partial \tilde{y}}$

In this way, systems of n coupled 1^{st} order nonlinear ODEs generalize to vector fields in n-dimensional manifolds

"INTEGRATING" VECTOR FIELDS $\iff$ Solving the ODE; i.e., finding, for an initial condition x_0, the maximal interval $I \subset \mathbb{R}$ and integral curve $x(t)$ satisfying

$$V \in \mathfrak{X}(M) \rightsquigarrow \begin{cases} \dot{x}(t) = V(x(t)) \\ x(0) = x_0 \end{cases}$$

Thm: (Picard) $\exists!$ solution to this initial value problem and it is C^∞ in the initial conditions.

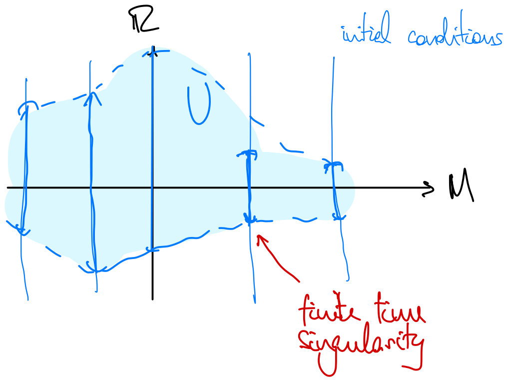
More precisely, $\exists!$ open set $U \subset M \times \mathbb{R}$ and $\exists!$ C^∞ map

initial condition

time

$\phi : U \longrightarrow M$

called the FLOW of V



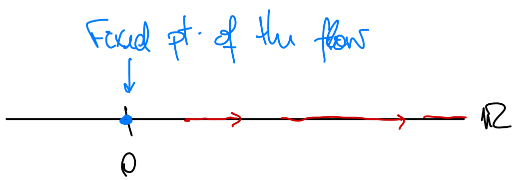
such that

$$\begin{cases} \phi^V(x,0) = x \\ \dfrac{\partial}{\partial t}\phi^V(x,t) = V(\phi^V(x,t)) \end{cases}$$

Def: If $U = M \times \mathbb{R}$, then V is called COMPLETE.

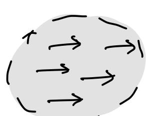
If M is compact, any V is complete.

e.g.: Incomplete vector field on $M = \mathbb{R}$. Consider $V = x^2 \frac{\partial}{\partial x}$



If we start at any $x_0 \neq 0$, then we reach infinity in finite time. (finite time singularity)

MAJOR ENTERPRISE: Classification of vector fields (locally)

0) NONSINGULAR: non-vanishing  $\subset \mathbb{R}^n$

If $V(0) \neq 0$ then V is non-vanishing on the neighborhood

Thm: (Frobenius) If V nonsingular, $\exists$ coörds. $(x^1, \_, x^n)$ s.t. in a neighborhood, $V = \frac{\partial}{\partial x^1}$.

i.e., $V \cong \frac{\partial}{\partial x^1}$ and flow $\phi(x,t) \simeq (x^1(0)+t, x^2(0))$.

1) Near a singular pt (i.e., $V(0) = 0$), classical results focus on the LINEARIZATION of V at 0:

$$V^i(x^1, \_, x^n) = \sum_j c^j x^j + \text{h.o.t.}$$

which gives the linearized vector field:

$$V^{lin} := \sum_{i,j} c^i_j \, x^j \frac{\partial}{\partial x^i}.$$

Q: When is V linearizable? (i.e., does there exist a coörd. change s.t. $V \cong V^{lin}$?)

E.g.: If $c^i_j = \delta_{ij}$, then V is linearizable.

In general, this depends on the matrix c^i_j.

1.1: If c^i_j is invertible, then the eigenvalues give whether it is linearizable or not.

Obs: Vector fields have 2 dramatically different behavior types:

1) ERGODIC (deterministic chaos)

no stable fixed pts / limit cycles; aperiodic; sensitive to initial conditions

2) INTEGRABLE (sort of nice ☺)

LECTURE 2

11/10/2024

GEODESIC FLOWS

↳ THE model of classical mechanics

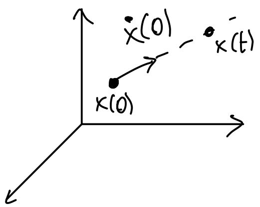
Ingredients: Riemannian manifold (M, g).

e.g.: $M = \mathbb{R}^3$ w/ $g(x,y) = x_1 y_1 + x_2 y_2 + x_3 y_3$

induces $\rightsquigarrow \mathrm{dist}_g(x,y) = \sqrt{(y_1 - x_1)^2 + (y_2 - x_2)^2 + (y_3 - x_3)^2}$

GEODESICS:

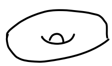
Initial data $(x(0), \dot{x}(0))$. In $\mathbb{R}^3$, we can parametrize geodesics as

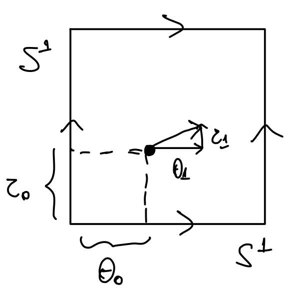


$$\boxed{x(t) = x(0) + t\,\dot{x}(0)}$$

"straight lines"

GEODESIC FLOW:

e.g.: $S^1 \times S^1 =$ 



Geodesic on the product:

$$(x(t), y(t)) = \left(e^{i(\theta_0 + \theta_1 t)},\ e^{i(\tau_0 + \tau_1 t)}\right).$$

We could have non-periodic trajectories.

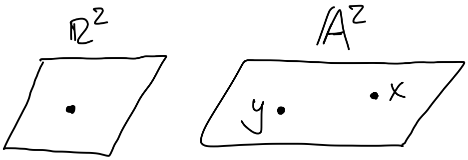
LECTURE 3

16/01/2024

GALILEAN SPACETIME & 1D PHASE PORTRAITS

SPACETIME: Affine space $\mathbb{A}^4$ for $\mathbb{R}^4$: $\mathbb{R}^4 \circlearrowright \mathbb{A}^4$ = GALILEAN SPACETIME

∃! arrow that takes you to any other pt. } freely, transitively



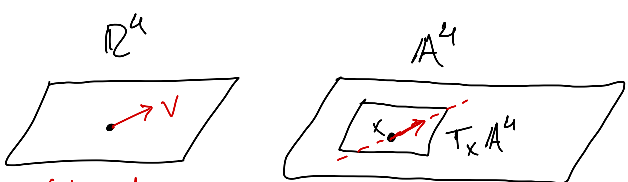
$y - x =$ unique $v \in \mathbb{R}^4$ s.t. $v + x = y$.

NOTE: A choice of "base point" $x_0 \in \mathbb{A}^4$ immediately identifies

$$\mathbb{A}^4 \xrightarrow{\simeq} \mathbb{R}^4$$
$$x_0 + v \longleftarrow\!\shortmid\; v$$

SPACETIME = $\mathbb{A}^4$ (an affine space for $\mathbb{R}^4$)

Obs: for all $x \in \mathbb{A}^4$, the tangent space $T_x\mathbb{A}^4$ is isomorphic to the space that acts on the affine space (namely, $\mathbb{R}^4$)



$\gamma(t) = tv$

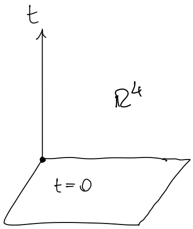
* This $\mathbb{R}^4$ is equipped with a natural linear map:

$$t : \mathbb{R}^4 \longrightarrow \mathbb{R} \quad (\underbrace{\text{time coördinate}})$$

$\Rightarrow$ Well-defined notion of time interval between events

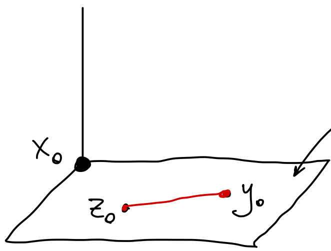
Cannot have space coords. b/c that means we need to pick an origin

* On $\ker t$, we have a positive-definite inner product $\langle\cdot,\cdot\rangle$ (i.e., the Euclidean inner product)



> **Upshot:** The mathematical model of classical mechanics is:
>
> $$\Big(\underbrace{(\mathbb{A}^4, \mathbb{R}^4)}_{\text{affine space}},\ \underbrace{t : \mathbb{R}^4 \to \mathbb{R}}_{\text{time}},\ \langle\cdot,\cdot\rangle \Big)$$

* **CONSEQUENCES**: If we fix an event $x_0 \in \mathbb{A}^4$, then we can identify $\mathbb{R}^4 \xrightarrow[x_0]{} \mathbb{A}^4$.



3d space of events simultaneous with x_0

Then, we have a notion of distance (only!) between simultaneous events:

$$d(y_0, z_0) = \sqrt{\langle y_0 - z_0, y_0 - z_0\rangle}$$

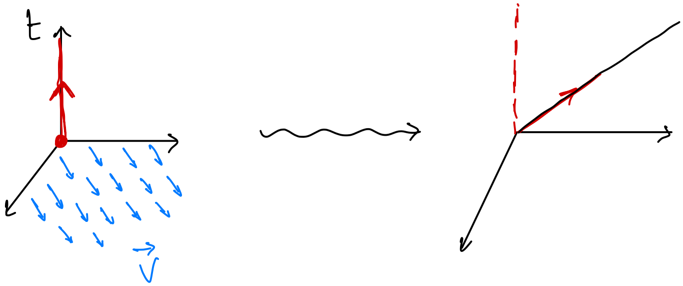
GALILEAN SYMMETRIES:

(i) $(\vec{x}, t) \longmapsto (\vec{x} + \vec{s}, t + s)$ where $(\vec{s}, s) \in \mathbb{R}^4$

(ii) $(\vec{x}, t) \longmapsto (A\vec{x}, t)$, $A \in O(3)$ Group of Orthogonal Matrices ($A^T A = \mathrm{Id}$)

SO(3) (rotations $\det = +1$, reflections $\det = -1$)

(iii) Uniform rectilinear motion $(\vec{x}, t) \longmapsto (\vec{x} + t\vec{v},\ t)$



The compositions of these symmetries form the GALILEAN GROUP, denoted GAL, of symmetries of spacetime.

CONVENIENT NOTATION:

$$\begin{pmatrix} t' \\ \vec{x}' \end{pmatrix} \longmapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{v} & A \end{array}\right) \begin{pmatrix} t \\ \vec{x} \end{pmatrix} + \begin{pmatrix} s \\ \vec{s} \end{pmatrix} = \begin{pmatrix} ct + s \\ A\vec{x} + t\vec{v} + \vec{s} \end{pmatrix}$$

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1 dimension of space $\rightsquigarrow \mathbb{R}^2 \curvearrowright \mathbb{A}^2 \rightsquigarrow$ Symmetries are: $\begin{pmatrix} t' \\ x' \end{pmatrix} = \left(\begin{array}{c|c} 1 & 0 \\ v & a \end{array}\right)\begin{pmatrix} t \\ x \end{pmatrix} + \begin{pmatrix} t_0 \\ x_0 \end{pmatrix}$

1D SYSTEMS: States = information that determines the motion.

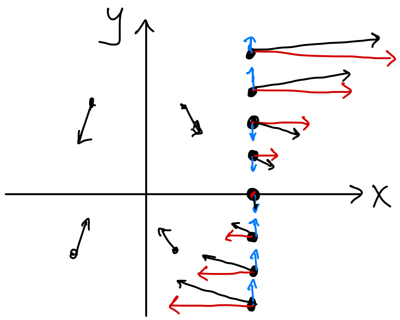
Galileo, Newton $\Longrightarrow \begin{pmatrix} x & \text{position} \\ \dot{x} & \text{velocity} \end{pmatrix}$ at a moment in time

NEWTON'S MODEL: $\boxed{\ddot{x}(t) = F(x, \dot{x}, t)}$ 2^{nd} order ODE

$m = 1$

This ODE can be written as a system of 1^{st} order ODEs: $\begin{cases} x = x \\ y = \dot{x} \end{cases}$. Then:

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ F(x,y,t) \end{pmatrix}$$



PHASE SPACE: $\{(x,y)\}$ = space of states (positions + velocities)

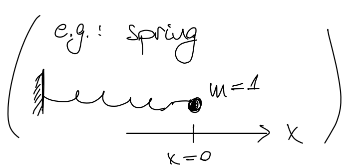
TX, $X = \mathbb{R}$ → spatial slice.

In this space, (x,y) denotes the tangent vector $y\frac{\partial}{\partial x}$ at the point x.

* Newton's law is a vector field in phase space. In there, integral curves of this vector field are "PHASE TRAJECTORIES"

EXAMPLES:

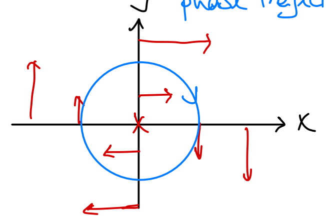
1) $\boxed{\ddot{x} = -x}$



$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

Phase Portrait:

Periodic (!) phase trajectories



Integral Curve through initial state $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$ is:

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = e^{t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}}\begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x(0)\cos t + y(0)\sin t \\ x(0)\sin t + y(0)\cos t \end{pmatrix}$$

Upshot: the motion of this particle in space is given by

$$\boxed{x(t) = x(0)\cos t + \dot{x}(0)\sin t}$$

Obs: Frequency of motion is independent from initial conditions!

LECTURE 4

18/01/2024

1D SYSTEMS

Recall: Phase space $= \underbrace{T\mathbb{A}^1_{space}}_{\text{Tangent bundle of spatial slices}} = \{(x,\dot{x}) : x \in \mathbb{A}^1_{space},\ \dot{x} \in T_x\mathbb{A}^1_{space}\}$

1d systems

1D-systems: We first study 1d systems $\boxed{\ddot{x} = f(x)}$ EQUATION OF MOTION

Rmk: The force here does not depend on $\dot{x}, t$.

This produces very special systems: CONSERVATIVE.

That means we can define an energy function on phase space

$$\boxed{E = \underbrace{\frac{1}{2}\dot{x}^2}_{\text{kinetic}} + \underbrace{U(x)}_{\text{potential}}, \quad U(x) = -\int_{x_0}^{x} f(\tilde{x})\, d\tilde{x}}$$

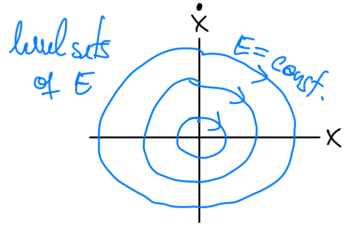
Thm: E is preserved by evolution, i.e., phase flow.

Pf: Consider $E(x(t))$ where $x(t)$ satisfies the equations of motion.

Then:

$$\dot{E} = \dot{x}\ddot{x} + \frac{\partial U}{\partial x}\dot{x} = \dot{x}\left[\ddot{x} - f(x)\right] = 0 .$$

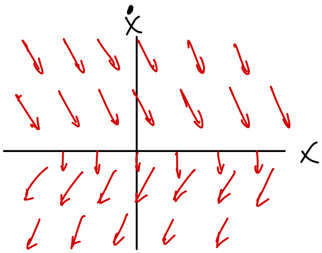
Cor: A trajectory (i.e., a solution to the eq. of motion) must remain on the level sets of $E(x, \dot{x})$.



e.g.: $f(x) = -x$.

Thus $U(x) = +\int_{x_0}^{x} \tilde{x}\, d\tilde{x} = \frac{1}{2}x^2 + C$.

EXAMPLE: $f(x) = -g$, $g = 9.8\,m/s^2$ $(m=1)$. Thus the vec. field in phase space is given by $\boxed{V = y\frac{\partial}{\partial x} - g\frac{\partial}{\partial x}}$.



We can also write it as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\dot{v} = Av + b$$

$\Rightarrow$ Solution: $\boxed{v(t) = e^{tA}\left(\int_0^t e^{-sA} b\, ds + v(0)\right)}$

LECTURE 5

1D & 2D SYSTEMS

23/01/2024

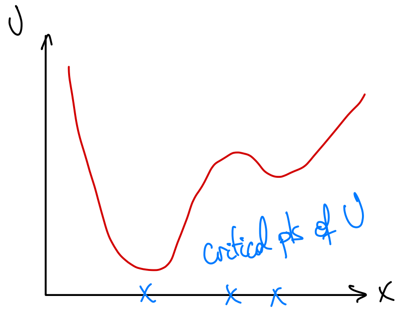
Recall: 1d systems $\boxed{\ddot{x} = f(x)}$ $\rightsquigarrow$ Conservative

KINETIC ENERGY $= \boxed{T := \frac{1}{2}\dot{x}^2}$

POTENTIAL ENERGY $= \boxed{U := -\int_{x_0}^{x} f(t)\, dt}$

$\Big\}$ $\boxed{E = T + U}$ is <u>conserved</u>!

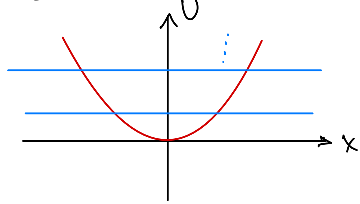
$\Rightarrow$ phase trajectories must remain on the level sets of the energy

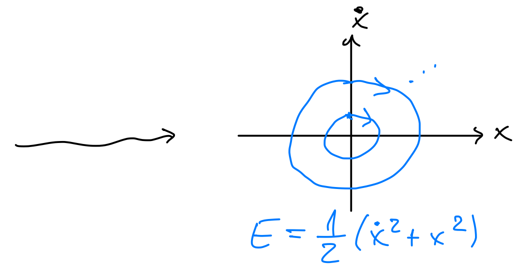


- $E = \text{const.} \Rightarrow$ upper bound on U
- If $U \nearrow \infty$ as $|x| \nearrow \infty$, then the state must remain in a compact region in x (confined space)

<u>e.g.:</u> $U(x) = \frac{1}{2}x^2$

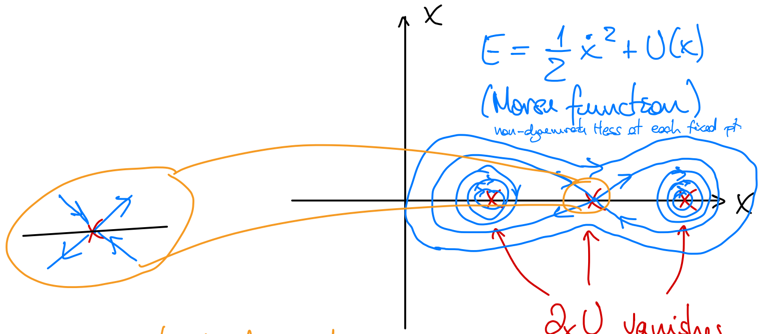
E levels





<u>How to find phase trajectories from potential energy $U(x)$?</u>

$$V = \dot{x}\frac{\partial}{\partial x} - \partial_x U \frac{\partial}{\partial \dot{x}}$$

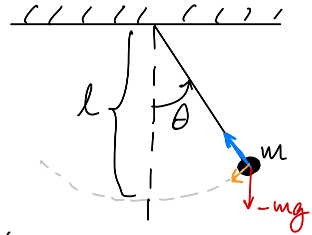


Here, the trajectories that approach this fixed pt take ∞-long to reach it: the flow is $e^{-t}x(0) \Rightarrow$ takes ∞ amount of time to reach that point

HISTORICAL EXAMPLE: $\ddot{x} = -\sin x$

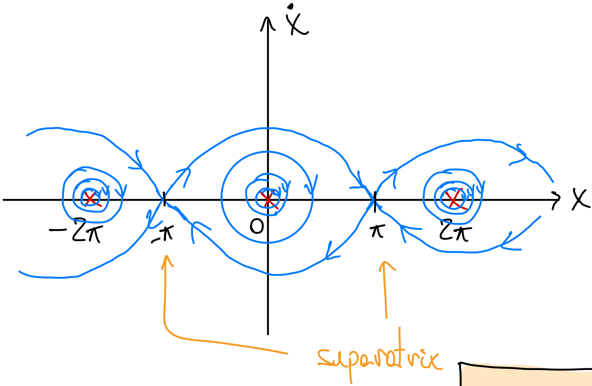
$$U(x) = \int_0^x \sin t \, dt = 1 - \cos x$$

simplify



Actual system

$$\ell \ddot{\theta} = f(\theta) = -mg \sin\theta$$



$\underline{\underline{Q}}$: Fixing E, how long does it take to make 1 full revolution

!!

PERIOD

$$\boxed{\Delta t = T = \int_{x_1}^{x_2} \frac{dx}{\sqrt{2(E-U)}}}$$

(Elliptic Integral)

In this case ($\ddot{x} = -\sin x$) Elliptic Integral of 1st kind

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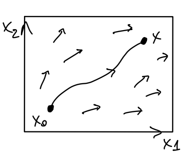
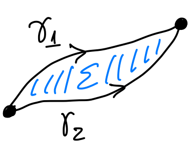
2-DIMENSIONAL SYSTEMS: Now, $x \in A^2$ and the system is "still"

$\boxed{\ddot{x} = f(x)}$ and, as before, we study systems s.t. the force f does <u>not</u> depend on $\dot{x}$ (or t).

MAIN DIFFERENCE 1d $\longrightarrow$ 2d: If we write $U(x) = -\int_{x_0}^{x} \vec{f}(x) \cdot d\vec{s}$, then the integral is path dependent $\Longrightarrow$ $U(x)$ is not well-defined.





$$\int_{\gamma_1} \vec{f} \cdot d\vec{s} - \int_{\gamma_2} \vec{f} \cdot d\vec{s} = \iint_{\Sigma} \underbrace{(\nabla \times f)}_{\text{Needs this to be zero!}} \cdot dA$$

* But we need to find such Σ before doing anything

Upshot: If $\nabla \times f = 0$, then we can* define $U(x)$ and then

$$\boxed{f = -\nabla U.}$$

But this is only possible if $\exists$ a surface Σ s.t. $\partial\Sigma = \gamma_1 \sqcup \gamma_2$.

(e.g.: if $\pi_1(X) = 1$, then this is always possible)

$\Leftrightarrow$ γ_1 homotopic to γ_2

Def: A system $\ddot{x} = f(x)$ is conservative whenever $\exists U(x)$ s.t.

$$f(x) = -\nabla U(x).$$

Thm: Energy is conserved in a conservative system

$$E = \frac{1}{2}\langle \dot{x}, \dot{x}\rangle + U(x)$$

Pf: $\dot{E} = \langle \dot{x}, \ddot{x}\rangle + \langle \dot{x}, \dot{U}(x)\rangle = \langle \dot{x}, \underbrace{\ddot{x} + \nabla U}_{=0}\rangle = 0.$ $\square$

Rmks: Configuration space $X = \mathbb{A}^2$ has a metric

$$g : TX \to T^*X$$

$$\underset{\text{vector field}}{v} \mapsto \underset{\text{1-form}}{\langle v, \bullet\rangle}$$

(force field) $\vec{f} \longmapsto \langle \vec{f}, \bullet\rangle =: g(\vec{f}) \in \Omega^1(X)$

$$\Omega^0(X) \xrightarrow{d} \Omega^1(X) \xrightarrow{d} \Omega^2(X) \to \dots \text{ De Rham}$$

$$\underset{U}{}$$

$g(\vec{f})$ and $\boxed{\nabla \times \vec{f} = 0} \iff \boxed{d(g(f)) = 0}$

This defines a class in the 1^{st} De Rham Cohomology

$$[g(f)] \in H^1_{dR}(X)$$

Want: $g(f) = -dU \iff \boxed{[g(f)] = 0}$ Conservative

e.g.: $H^1_{dR}(S^1) \simeq \mathbb{R}$

$H^1_{dR}(S^1 \times S^1) \simeq \mathbb{R}^2$

$H^1_{dR}(S^2) \simeq 0$

Obs:

$\pi_1(X) = \{1\} \Rightarrow H^1_{dR}(X) = 0$

EXAMPLE: $U(x) = \frac{1}{2}(x_1^2 + x_2^2) \rightsquigarrow -\nabla U = \left(-x_1 \frac{\partial}{\partial x_1}, -x_2 \frac{\partial}{\partial x_2}\right)$

$=: \vec{f}$

Equations of Motion: $\boxed{\frac{d^2}{dt^2}(x_1, x_2) = (-x_1, -x_2)}$

$$\Updownarrow \quad \begin{cases} \ddot{x}_1 = -x_1 \\ \ddot{x}_2 = -x_2 \end{cases}$$

Then $E = \frac{1}{2}(\dot{x}_1^2 + \dot{x}_2^2) + \frac{1}{2}(x_1^2 + x_2^2) = \frac{1}{2}(x_1^2 + \dot{x}_1^2 + x_2 + \dot{x}_2^2)$

Cartesian product: $\underset{x_1}{\overset{\cup}{X_1}} \times \underset{x_2}{\overset{\cup}{X_2}}$ of configuration spaces

$$\begin{tikzcd} & X_1 \times X_2 \arrow[r] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \mathbb{R} \\ \mathbb{R} & X_1 \arrow[l] & X_2 \arrow[r] & \mathbb{R} \end{tikzcd}$$

Phase space $= A^4 \simeq \mathbb{R}^4$

$\Rightarrow$ Energy level sets are $\simeq S^3 \subset \mathbb{R}^4$

$\Longrightarrow$ No longer have the case $E=0$ defining trajectories.

↳ In this example, there are more fundamental quantities

$$E_1 := \frac{1}{2}(x_1^2 + \dot{x}_1^2) \quad \text{and} \quad E_2 := \frac{1}{2}(x_2^2 + \dot{x}_2^2)$$

s.t. E_1, E_2 are conserved separately

level sets

$\Longrightarrow$ Motion is on $E_1^{-1}(e_1) \cap E_2^{-1}(e_2)$

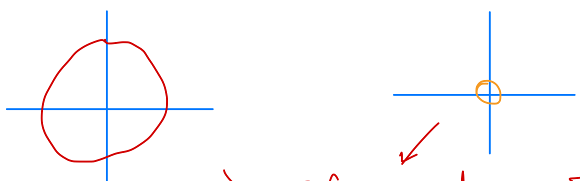
on a 2d surface in phase space

$$\Sigma^2 \subset S^3_{E_1+E_2} \subset \mathbb{A}^4$$

$$z_1 := x_1 + i\dot{x}_1$$

$$z_2 := x_2 + i\dot{x}_2$$

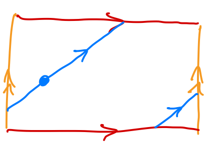
Fix energies $E_1 = \frac{1}{2} z_1 \overline{z_1}$, $E_2 = \frac{1}{2} z_2 \overline{z_2}$



↳ Defines a torus $T^2 = \Sigma^2 \subset \mathbb{A}^4$

$\Rightarrow$ Motion on a torus!

$\Sigma^2 = T^2$



Trajectory on the torus

LECTURE 6

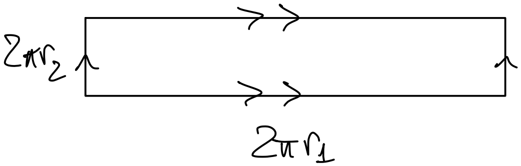
25/01/2024

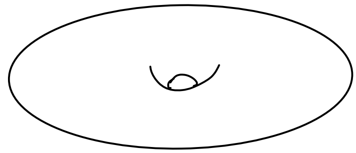
2D - SYSTEMS

Recall: $U(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$ ← 2 Harmonic oscillators

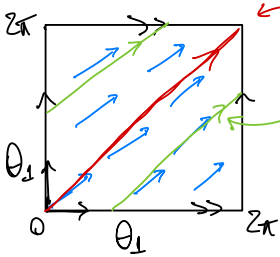
2 conserved quantities: $E_1 = \frac{1}{2}(\dot{x}_1^2 + x_1^2)$, $E_2 = \frac{1}{2}(\dot{x}_2^2 + x_2^2)$

Total energy: $E := E_1 + E_2$. $\Rightarrow$ 2d common level sets $S^1_{r_1} \times S^1_{r_2}$





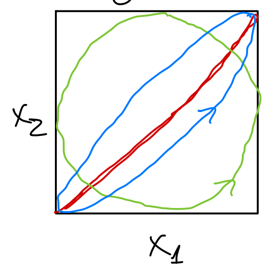
In angular coordinates



trajectories are IN PHASE

out of phase

Lissajous Figures



So, we can write $x_1(t) = c_1 \cos t + c_2 \sin t$

$\dot{x}_1(t) = \ldots$

$\vdots$

Write $z_1 := x_1 + i\dot{x}_1$

$z_2 := x_2 + i\dot{x}_2$ $\rightsquigarrow$

Fully determine the trajectories

$$\begin{cases} z_1(t) = (c_1 + ic_2)\, e^{-it} \\ z_2(t) = (c_3 + ic_4)\, e^{-it} \end{cases}$$

Obs: Invariant under scaling by same phase constant:

$(z_1, z_2) \longmapsto (e^{i\delta} z_1, e^{i\delta} z_2)$ $\leftarrow$ Same trajectory, different initial condition

Upshot: To know the (unparametrized) phase curve on a constant energy surface $E=\frac{1}{2}$ is to know

$$(\underbrace{c_1+ic_2}_{=z_1(0)},\ \underbrace{c_3+ic_4}_{=z_2(0)}) \in \mathbb{C}^2 \quad \text{s.t.} \quad |c_1|^2+|c_2|^2+|c_3|^2+|c_4|^2=1$$

i.e., a point in $S^3 \subset \mathbb{R}^4$ up to a phase change $(z_1(0), z_2(0))e^{i\delta}$.

This defines a $U(1)$ action on S^3:

$$S^3 \xrightarrow{\ \pi\ } S^3/U(1) = \underset{\cup}{\mathbb{C}P^1} = S^2$$

$$(z_1(0), z_2(0)) \longmapsto [z_1(0), z_2(0)]$$

Hopf Fibration

* **PERTURBATION OF THE PREVIOUS SYSTEM**: Consider a system w/ the following potential:

$$U(x_1,x_2) = \frac{1}{2}x_1^2 + \frac{1}{2}\omega^2 x_2^2$$

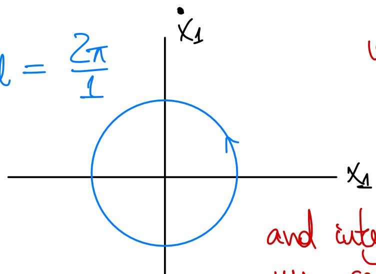
SHO(1) SHO(ω)

$$\ddot{x}_1 = -x_1 \qquad \ddot{x}_2 = -\omega^2 x_2$$

$$z_1(t) = z_1(0)e^{-it} \qquad z_2(t) = z_2(0)e^{-it}$$

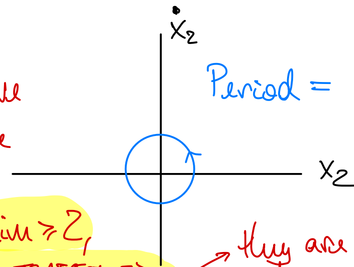
Take ω close to 1:

Period $= \frac{2\pi}{1}$



No periodic motion anymore (Lissajous figures evolve in time). Still have 2 conserved qts. and integrability. But in dim ≥ 2, we can have **NON-CLOSED TRAJECTORIES** $\to$ they are dense!

Period $= \frac{2\pi}{\omega}$



LECTURE 7

CENTRAL FORCES

30/01/2024

2D SYSTEMS: Conservative $\Longleftrightarrow \begin{cases} \nabla \times f = 0 \\ H^1_{dR}(X) = 0 \end{cases} \Longleftrightarrow \exists U$ s.t. $f = -\nabla U$

$$\Downarrow$$

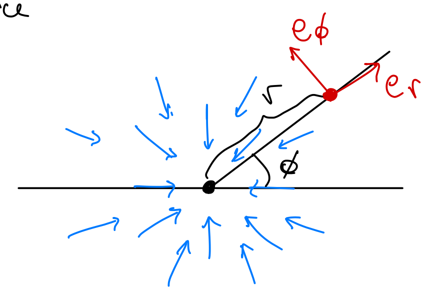
$$E = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle + U(x)$$

is conserved

In the plane

CENTRAL FORCES: Magnitude of the force only depends on r (not on ϕ) and the direction of the force is $\vec{f} \parallel \vec{r}$.

e.g.: Gravity & celestial motion



$$\boxed{\begin{aligned} \dot{e}_r &= (-\sin\phi\, \hat{x} + \cos\phi\, \hat{y})\,\dot{\phi} = \dot{\phi}\, e_\phi \\ \dot{e}_\phi &= -\dot{\phi}\, e_r \end{aligned}}$$

Differentiate $\longleftarrow$ $\begin{cases} e_r = \cos\phi\, \hat{x} + \sin\phi\, \hat{y} \\ e_\phi = -\sin\phi\, \hat{x} + \cos\phi\, \hat{y} \end{cases}$

Obs: Central forces $\rightsquigarrow$ Time independent $\Rightarrow$ E is conserved

Rotational symmetry $\Rightarrow$ Angular mom. conserved

SO(2) symmetry

$\phi \longmapsto \phi + C$

Def: (ANGULAR MOMENTUM) $\vec{M} := [\vec{r}, \dot{\vec{r}}]$ $\left(= \vec{r} \times \dot{\vec{r}} \text{ in } 2d \right)$

example of a Lie bracket, hence the notation

Note that for $\dot{\vec{r}} = \dot{r} e_r + r\dot{\phi} e_\phi$, we have

$$\vec{M} = \vec{r} \times \dot{\vec{r}} = (r e_r) \times (\dot{r} e_r + r\dot{\phi} e_\phi)$$

$$= r^2 \dot{\phi} \text{ in the direction coming out of the page}$$

or: $[x e_x + y e_y, \dot{x} e_x + \dot{y} e_y] = (x\dot{y} - \dot{x}y)$ → Function of 2 out of 4 variables on phase space

Cl: $\vec{M}$ is conserved in central forces.

Pf: $$\dot{\vec{M}} = \frac{d}{dt}[r, \dot{r}] = \underbrace{[\dot{r}, \dot{r}]}_{=0 \text{ b/c } [\cdot,\cdot] \text{ is skew-symmetric}} + [r, \ddot{r}]$$

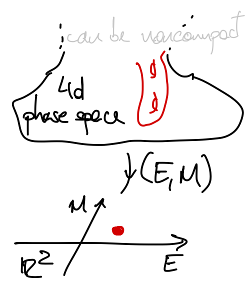
Equation of motion $\ddot{r} = f$ ⟶ $= [r, \ddot{r}] = [r, \vec{f}\,] = 0$ ← Central force ⟺ $\vec{r} \parallel \vec{f}$.

□

Upshot: Reduce phase flow from 4d to 2d by fixing E, M.

⇓

$$\boxed{\text{A priori, this is an effective 1d system! (phase space is 2d)}}$$



REDUCTION OF CENTRAL FORCE TO 1D SYSTEM. Consider

$$\ddot{r} = -\nabla U, \quad U = U(r) \text{ and } \dot{\phi} r^2 = M \text{ is constant}$$

Thm!: This reduces to an effective 1d problem $V(r) = U(r) + M^2/2r^2$.

Rmk: $\partial_r U \to 0$ as $r \nearrow \infty$ (approx.) $\Rightarrow$ $M^2/2r^2$ wins as $r \to 0$

Allows solution to $r(t)$ $\leadsto$ Use $M = \dot\phi r^2 \Rightarrow \phi = \int_{t_1}^{t_2} M/r^2 \, dt$.

Pf: $\vec r = r e_r$

$$\dot{\vec r} = \dot r e_r + r\dot\phi e_\phi$$

Equation of Motion

$$\ddot{\vec r} = \ddot r e_r + \dot r \dot\phi e_\phi + (\dot r\dot\phi + r\ddot\phi) e_\phi - r\dot\phi^2 e_r = -\partial_r U e_r.$$

$$2\dot r\dot\phi + r\ddot\phi = 0$$

$$\ddot r - r\dot\phi^2 = -\partial_r U \longrightarrow \text{Standard form } \boxed{\ddot r = -\partial_r V}.$$

Need $V \overset{!}{=} U + M^2/2r^2$. So,

$$\boxed{\ddot r = -\partial_r \left(U + \frac{M^2}{2r^2}\right)}$$ 1d SYSTEM!

□

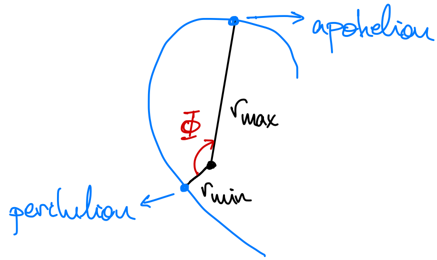
Rmk: $E = \frac{1}{2} m \dot r^2 + V(r)$ is conserved

$$\begin{cases} \dot r = \sqrt{2(E - V(r))} \\ \dot\phi = M/r^2 \end{cases}$$ $\leadsto$ Define vector field on surface in phase space

So, if we only need unparametrized orbits in phase space, we can eliminate t and write $\phi(r)$:

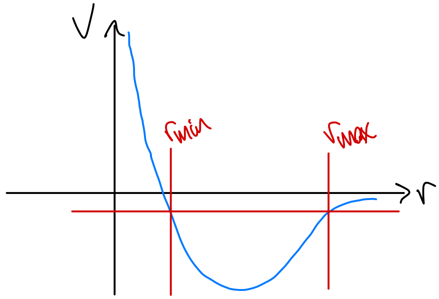
$$\frac{d\phi}{dr} = \frac{M/r^2}{\sqrt{2(E - V(r))}} \leadsto \boxed{\Phi = \int_{r_{min}}^{r_{max}} \frac{M/r^2}{\sqrt{2(E-V(r))}} \, dr}$$

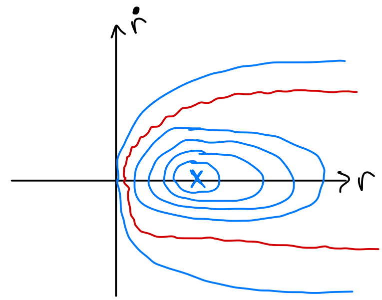
Obs: If $\Phi \in 2\pi\mathbb{Q}$, then the orbit is CLOSED!



EXAMPLE: $\boxed{V(r) = -\frac{\kappa}{r} + \frac{M^2}{2r^2}}$ $\leftarrow$ $U(r) = -\frac{\kappa}{r}$, $\kappa > 0$.

Configuration space: $r \in (0, \infty)$.

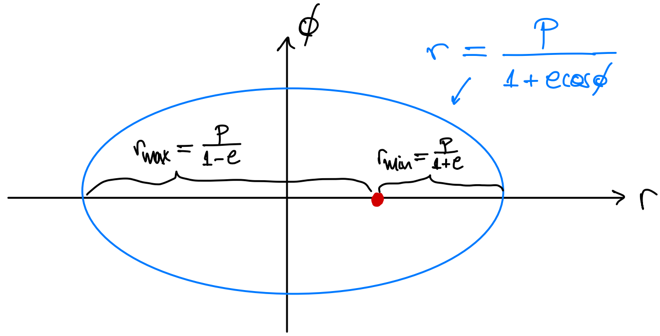




$$\phi = \int \frac{M/r^2 \; dr}{\sqrt{2\left[E - \left(-\kappa/r + M^2/2r^2\right)\right]}} = \arccos\left(\frac{\frac{M}{r} - \frac{\kappa}{M}}{\sqrt{2E + \kappa^2/M}}\right)$$

Define

$$\begin{cases} p := M^2/\kappa & \text{"parameter"} \\ e := \sqrt{1 + \frac{2EM^2}{\kappa^2}} & \text{"eccentricity"} \end{cases} \Longrightarrow \boxed{r = \frac{p}{1 + e\cos\phi} \quad \text{Ellipse in polar coordinates}}$$



Remarks on Central Forces:

1) The only central forces giving periodic trajectories (i.e., $\Phi = 2\pi Q$) are $U(r) = -\frac{\kappa}{r}$ and $U(r) = ar^2$. Others (e.g., r^α, $\alpha \in \mathbb{R}$) will not!

2) Actual planetary orbits do precess (e.g.: Mercury = 5.75 arcsec/year).

2 main sources for precession of orbits: other planets and GR.

(A) (B)

Q: Is there an "effective" change to $V(r)$ coming from (A) and (B)

GAUSS (A): Model each planet averaged out in time (since they move relatively slowly to each other). So planets become "rings" of mass.

After lots of computations, 5.32 out of 5.75 deviation of Mercury is explained by Gauss.

↪ 0.41 due to Einstein

EINSTEIN (B): $$F = -\frac{GM}{r^2} - \frac{3GM\,h^2}{c^2 r^4}$$

const. of motion

This gives the other 0.41.

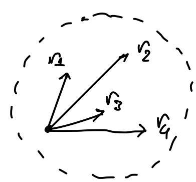
LECTURE 8

MULTIPARTICLE SYSTEMS

01/02/2024

3d MULTIPARTICLE SYSTEM: Configuration space is just the product of n Euclidean spaces

$$\underbrace{\mathbb{R}^3 \times \mathbb{R}^3 \times \cdots \times \mathbb{R}^3}_{n \text{ times}} = \mathbb{R}^{3n}$$



$$r := (r_1, r_2, \underline{\qquad}, r_n)$$

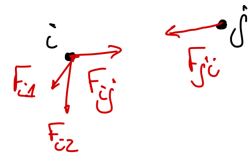
$$\text{masses} = m_1, m_2, \underline{\qquad}, m_n$$

Phase Space: $(r, \dot{r}) \in T\mathbb{R}^{3n}$ ⟵ $\dim T\mathbb{R}^{3n} = 6n$.

Forces:
- External forces F_i'.
- Interaction forces: force of ith body on jth body $:= F_{ij}$.

$F_{ij} = -F_{ji}$, direction is parallel to $r_j - r_i$

skew $n \times n$ matrix of vectors



Note: $F_i = \sum_{j \neq i} F_{ij}$ force acting on i. This defines n 3d vectors or a single vector field $F = (F_1, \ldots, F_n)$ on $\mathbb{R}^{3n}$.

EoM: $m_i \ddot{r}_i = F_i \rightsquigarrow \begin{pmatrix} m_1 & & \\ & \ddots & \\ & & m_n \end{pmatrix} \ddot{r} = F$

Upshot: EoM defines a vector field on $T\mathbb{R}^{3n}$ (a $6n$-dimensional space).

$$F := \sum_{i \neq j} F_{ij} + F_i'.$$

NOTATION: If we only have interaction forces, we call the system "CLOSED"

Def: The LINEAR MOMENTUM of the system is $\vec{p} := \sum_i m_i \dot{r}_i$.

Consequence: $\dfrac{d\vec{p}}{dt} = \sum_i m_i \ddot{r}_i = \sum_i F_i$ Thus, rate of change of total momentum = sum of external forces

$\Rightarrow$ If the system is CLOSED, then $\vec{p}$ is conserved. $\Rightarrow$ 3 fcts on phase space are conserved

Obs: If $\sum F_i' \neq 0$, then only the components of $\vec{p}$ perpendicular to $\vec{F}$ will be conserved.

Terrestrial gravity

e.g.: in $F_i' = m_i g_i e_z$, only the components p_x and p_y are conserved.

Rmk: If we define $r_{CM} := \dfrac{\sum_i m_i r_i}{M := \sum_i m_i}$ $\overset{\text{EoM}}{\rightsquigarrow}$ $\boxed{M \ddot{r}_{CM} = F_{ext} = \sum F_i'}$

CENTER OF MASS

CLOSED SYSTEM $\Rightarrow F_{ext} = 0$

$\Downarrow$

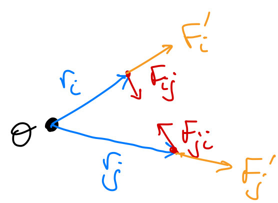
Center of mass travel in uniform motion along a straight line

Def: Total ANGULAR MOMENTUM relative to $O \in \mathbb{R}^3$ is defined as

$$M := \sum_i [r_i, m_i \dot{r}_i]$$ → "Cross product"

Rmk:
$$\frac{dM}{dt} = \sum_i \cancelto{=0}{[\dot{r}_i, m_i \dot{r}_i]} + [r_i, m_i \ddot{r}_i]$$

$$\overset{\text{EoM}}{=} \sum_{i=1}^{n} \Big[r_i, \sum_{j \neq i} F_{ij} + F_i' \Big]$$



All internal forces cancel out b/c they are all pairwise colinear!

$$= \dots = \sum_i \underbrace{[r_i, F_i']}$$

TORQUE of force F_i' exerted on r_i relative to O

proved

Thm: The rate of change of total angular momentum is the sum of the torques of external forces.

Def: Kinetic Energy is $T := \sum_{i=1}^{n} \frac{1}{2} m_i \langle \dot{r}_i, \dot{r}_i \rangle$

Thm: Change in KE ΔT = work done by forces; i.e.,

$$T(t_1) - T(t_0) = \sum_{i=1}^{n} \underbrace{\int_{t_0}^{t_1} \langle F_i, \dot{r}_i \rangle \, dt}_{\text{work done by these forces.}}$$

<u>SPECIAL</u> <u>CASE</u>: Interaction forces are conserved!

$$F = -\nabla(U(r_1, \ldots, r_n)) \ ; \quad U = \sum_{i<j} U_{ij}(|r_i - r_j|).$$

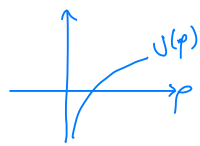
Generalizes the central force result from before

In a closed system w/ n identical bodies, the configuration space is

$$\mathbb{R}^3 \times \cdots \times \mathbb{R}^3 = \mathbb{R}^{3n}$$

$$F = -\nabla U, \quad U = \sum_{i<j} \underbrace{U_{ij}(|r_i - r_j|)}_{\text{function of 1 single variable}}$$

$$\mathbb{R}^3 \times \cdots \times \mathbb{R}^3 \xrightarrow[\binom{n}{2} \text{ such maps}]{i<j,\ r_i - r_j} \mathbb{R}^3 \xrightarrow{|\cdot| \text{ norm}} \mathbb{R}$$



LECTURE 9

LAGRANGIAN MECHANICS

06/02/2024

X = configuration space mnfd w/ coordinates $(x^1, \ldots, x^n)$

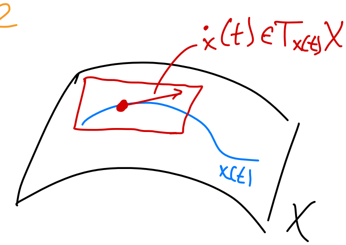
$$g = g_{ij}(x^1, \ldots, x^n) \underbrace{dx^i \otimes dx^j}_{\text{dual basis vector}} = \text{Riem. metric}$$

$$\vec{F} = F^i(x^1, \ldots, x^n) \underbrace{\frac{\partial}{\partial x^i}}_{\text{basis for tangent space (in that chart)}} = \text{vector field on } X$$

Dynamics: "$\vec{F} = m\vec{a}$" $\longleftrightarrow$ "$\ddot{x}(t) = m\vec{F}(x(t))$", $x(t) = (x^1(t), \dots, x^n(t))$

constant $\in \mathbb{R}$

To make sense of $\frac{d}{dt}(\dot{x}(t))$, we need the notion of parallel transport as an identification between tangent spaces along a path.



"Fundamental Theorem of Riem. geom" = $\exists! \, \nabla$ Levi-Civita connection

∇ allow differentiation in <u>any</u> direction

$\nabla_X Y$ = directional derivative of vec. field Y along the direction of vec. field X = vector field

output of $\nabla_X Y$

Christoffel Symbols: $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \Gamma^k_{ij}(x^1, \dots, x^n) \frac{\partial}{\partial x^k}$.

Compute Γ^k_{ij} using $\Gamma^k_{ij} = \frac{1}{2} g^{kp} \left(\frac{\partial g_{pi}}{\partial x^j} + \frac{\partial g_{pj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^p} \right)$

Upshot. "$\ddot{x}(t) = m\vec{F}(x(t))$" $\xrightarrow[\text{MEANS}]{\text{ACTUALLY}}$ $\boxed{\nabla_{\dot{x}(t)} \dot{x}(t) = m\vec{F}(x(t))}$

(THIS is NEWTON's 2^{nd} LAW!

In coordinates

$$\ddot{x}^k(t) + \dot{x}^a \dot{x}^b \Gamma^k_{ab}(x^1(t), \dots, x^n(t)) = m F^k(x^1(t), \dots, x^n(t)).$$

If conservative

$$= -m \nabla U$$

$$= -m g^{-1} dU$$

$$= -m g^{kj} \frac{\partial U}{\partial x^j}.$$

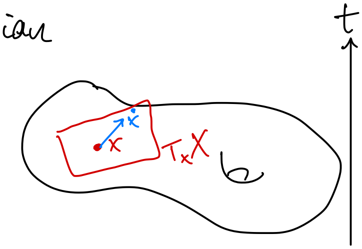
NEWTONIAN MECHANICS (modernized a bit...)

UPDATE

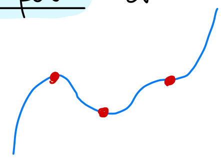
LAGRANGIAN MECHANICS:
$$\begin{cases} X = \text{configuration mnfd} \\ \mathcal{L}(x,\dot{x},t) = \text{Lagrangian} \end{cases}$$

The Lagrangian $\mathcal{L}$ is defined on $TX \oplus \mathbb{R}$

$$\mathcal{L} : TX \oplus \mathbb{R} \longrightarrow \mathbb{R} \qquad (x,\dot{x},t)$$



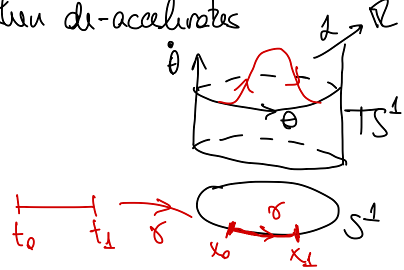
VARIATIONAL PRINCIPLE: the solution curve $x(t)$ is a critical point or "extremum" of the action $S : P(X, x_0, x_1) \longrightarrow \mathbb{R}$,



where $P(X,x_0,x_1) := \left\{ \begin{matrix} \gamma : [t_0,t_1] \to X \\ \text{smooth} \end{matrix} : \begin{matrix} \gamma(t_0) = x_0 \\ \gamma(t_1) = x_1 \end{matrix} \right\}$
given by

$$S[\gamma] := \int_{t_0}^{t_1} \mathcal{L}(x(t),\dot{x}(t),t)\, dt$$

e.g.: S^1 path accelerates and then de-accelerates



LECTURE 10

08/02/2024

LAGRANGIAN MECHANICS

X = config. space (mnfd) e.g. particle in $\mathbb{R}^3 \rightsquigarrow X = \mathbb{R}^3$
n particles in $\mathbb{R}^3 \rightsquigarrow X = \mathbb{R}^{3n}$
rigid body (fixed CM) $\rightsquigarrow X = SO(3)$
triple pendulum $\rightsquigarrow X = S^1 \times S^1 \times S^1$

spherical pendulum $\rightsquigarrow X = S^2$

ball rolling inside a sphere $\longrightarrow X = S^2 \times SO(3)$

Lagrangian $\mathcal{L} : TX \oplus \mathbb{R} \longrightarrow \mathbb{R}$

$$(q, \dot{q}, t) \longmapsto \mathcal{L}(q, \dot{q}, t)$$

generalized position

generalized velocity

These are the only ones chosen

$q = (q^1, \dots, q^n)$ coordinate system on X

$\dot{q} = (\dot{q}^1, \dots, \dot{q}^n)$ induced extended coord. in TX

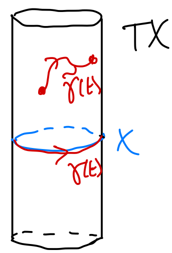
HAMILTON'S PRINCIPLE OF STATIONARY ACTION: The true path is a critical pt. of the action $S : P(X, x_0, x_1) \longrightarrow \mathbb{R}$

$P(X, x_0, x_1) := \left\{ \gamma : [t_0, t_1] \to \mathbb{R} \text{ smooth} : \begin{array}{l} \gamma(t_0) = x_0 \\ \gamma(t_1) = x_1 \end{array} \right\}$

$$S[\gamma] := \int_{t_0}^{t_1} \mathcal{L}(\tilde{\gamma}(t), t)\, dt$$

where $\gamma : [t_0, t_1] \to X$ is a C^∞ path. This path induces a lifted path $\tilde{\gamma} : [t_0, t_1] \longrightarrow TX$

$$\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t))$$



DERIVATION OF THE EQUATIONS OF MOTION: Assume linear variation of the path $\gamma_u(t) = \gamma(t) + u\, v(t)$, $v(t_0) = 0$, $v(t_1) = 0$.

Need $\dfrac{dS(\gamma_u)}{du} = 0 \quad \forall v$. Then

$$\frac{dS(\gamma_u)}{du} = \int_{t_0}^{t_1} \left(\frac{\partial \mathcal{L}}{\partial q} v(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}} \dot{v}(t) \right) dt$$

Fundamental Lemma of Calculus of Variations

$= 0$ EULER LAGRANGE EQUATIONS

$$= \int_{t_0}^{t_1} \left(\boxed{\frac{\partial \mathcal{L}}{\partial q}(\gamma, \dot{\gamma}, t) - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}(\gamma, \dot{\gamma}, t) \right)} \right) v(t)\, dt \overset{!}{=} 0 \quad \forall v(t)$$

LECTURE 11

LAGRANGIAN MECHANICS (ctd)

13/02/2024

Momentum p^i

Implicit system of 1^{st} order nonlinear ODEs for path in TX

EULER LAGRANGE EQUATIONS:
$$\boxed{\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) = \frac{\partial \mathcal{L}}{\partial x^i}} \qquad i = 1, \dots, n$$

Only requires a choice of basis on X (any basis?)

extrema

$\mathcal{L} : TX \oplus \mathbb{R} \to \mathbb{R}$,

Action $S(\gamma) = \int_{t_0}^{t_1} \mathcal{L}(\tilde{\gamma}(t), \dot{\gamma}(t), t)\, dt$ over curves $\gamma : [t_0, t_1] \to X$ w/ fixed endpoints.

$$\begin{tikzcd} & TX \arrow[d, "\pi"] \\ {[t_0,t_1]} \arrow[ur, "\tilde{\gamma}"] \arrow[r, "\gamma"'] & X \end{tikzcd}$$

lift to TX

$$\tilde{\gamma}(t) = \left(\gamma(t), \left.\frac{d}{dt}\right|_t \gamma(t)\right) = \left(x^i(t), \dot{x}^i(t)\right)$$

Rmk: Expand (EL) equations

$$\frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial \dot{x}^j}\ddot{x}^j + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial x^j}\dot{x}^j + \frac{\partial^2 \mathcal{L}}{\partial \dot{x}^i \partial t} = \frac{\partial \mathcal{L}}{\partial x^i}$$

Potentially non-linear coefficients.

unknown $= x^j(t)$, $j = 1, \dots, n$ System of nonlinear 2^{nd} order ODEs

EXAMPLE 1: $X = \mathbb{R}^2_{(x,y)}$, $TX = \mathbb{R}^2 \times \mathbb{R}^2_{(x,y,\dot{x},\dot{y})}$

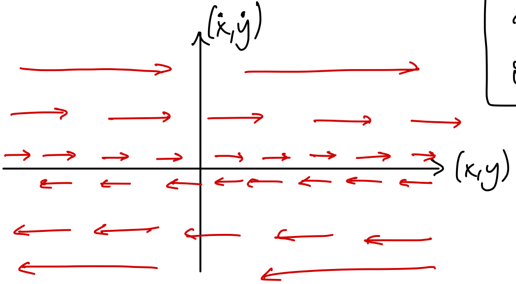
$\mathcal{L}(x, y, \dot{x}, \dot{y}) = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ ($= T$ kinetic energy)

$$\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}} = \dot{x}, \quad \frac{\partial \mathcal{L}}{\partial \dot{y}} = \dot{y}}_{\text{momenta}}, \quad \underbrace{\frac{\partial \mathcal{L}}{\partial x} = 0, \quad \frac{\partial \mathcal{L}}{\partial y} = 0}_{\text{"forces"}}$$

symmetries under translations

(EL): $\frac{d}{dt}\dot{x} = 0, \ \frac{d}{dt}\dot{y} = 0 \Rightarrow$ $\boxed{(x,y) = (x(0), y(0)) + t(\dot{x}(0), \dot{y}(0))}$

i.e. a straight line (geodesic) parametrized in a linear fashion.



Def: When $\mathcal{L}$ does not depend on x^i, this x^i is called a CYCLIC COORD.
(translational symmetry of $\mathcal{L} \Rightarrow$ Conservation of linear $\underbrace{\text{momenta}}_{\frac{\partial \mathcal{L}}{\partial \dot{x}}}$ in those directions)

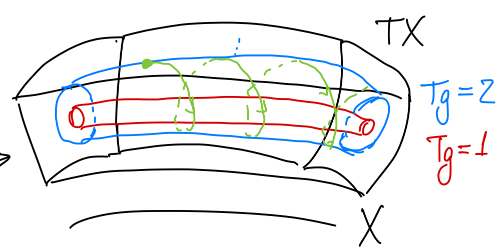
EXAMPLE 2: (Generalize ex 1) Let (X, g) be a Riem. mnfd. and take coord. $(x^1, \dots, x^n)$ s.t. $g = g_{ij}(x)\, dx^i dx^j$, where $g_{ij}(x) = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$.
Use g to define a single $\mathbb{R}$-valued fct.

$$T_g : TX \longrightarrow \mathbb{R}$$

$$(x^1, \dots, x^n, \dot{x}^1, \dots, \dot{x}^n) \longmapsto \frac{1}{2}\, g\left(\dot{x}^1 \frac{\partial}{\partial x^1} + \cdots + \dot{x}^n \frac{\partial}{\partial x^n}, \ \dot{x}^1 \frac{\partial}{\partial x^1} + \cdots + \dot{x}^n \frac{\partial}{\partial x^n}\right)$$

i.e., $$\boxed{T_g(x) = \frac{1}{2}\, g_{ij}(x)\, \dot{x}^i \dot{x}^j}$$

(n−1)-sphere bundles over X



Upshot: (EL) then gives $\mathcal{L} = T_g$ so $\frac{d}{dt}\left(\frac{\partial T_g}{\partial \dot{x}^i}\right) = \frac{\partial T_g}{\partial x^i}$.

That is $\mathcal{L} = \frac{1}{2} g_{ij}(x)\, \dot{x}^i \dot{x}^j$. Then $\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{1}{2} g_{ij}\, \dot{x}^j + \frac{1}{2} \overset{= g_{ij}}{g_{ji}}\, \dot{x}^j$

momenta: $\boxed{\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = g_{ij}(x)\dot{x}^j}$ ← sum...

forces: $\boxed{\frac{\partial \mathcal{L}}{\partial x^i} = \frac{1}{2}\frac{\partial g_{pq}}{\partial x^i}\dot{x}^p\dot{x}^q}$

(EL)

$$\boxed{\frac{d}{dt}\left(g_{ij}(x)\dot{x}^j\right) = \frac{1}{2}\frac{\partial g_{pq}}{\partial x^i}\dot{x}^p\dot{x}^q}$$

HAMILTON-JACOBI EQUATION

(particular case of Euler-Lagrange)

Rmk: Rewrite Hamilton-Jacobi

$$\boxed{\ddot{x}^r = \frac{1}{2}g^{ri}\partial_i g_{pq}\dot{x}^p\dot{x}^q - g^{ri}\partial_k g_{ij}\dot{x}^k\dot{x}^j}$$

GEODESIC EQUATION OF FREE PARTICLE ON X

Using Hamilton-Jacobi, we get this without ever mentioning connections / Christoffel symbols

PROBLEMATIC EXAMPLE: $\mathcal{L}(x,y,\dot{x},\dot{y}) = \sqrt{\dot{x}^2+\dot{y}^2}$.

Note: $S(\gamma) = \int_{t_0}^{t_1}\sqrt{\dot{x}^2+\dot{y}^2}\,dt$ = arclength of γ

(EL): $\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow x,y$ cyclic

$$\frac{\partial \mathcal{L}}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} \qquad \frac{\partial \mathcal{L}}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}$$

$\rightsquigarrow$ (EOM) $\begin{cases} \frac{d}{dt}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0 \\ \frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = 0 \end{cases}$

$\Updownarrow$

$$\begin{cases} \dot{y}^2\ddot{x} - \dot{x}\dot{y}\ddot{y} = 0 \\ -\dot{x}\dot{y}\ddot{x} + \dot{x}^2\ddot{y} = 0 \end{cases} \Longleftrightarrow \begin{pmatrix} \dot{y}^2 & -\dot{x}\dot{y} \\ -\dot{x}\dot{y} & \dot{x}^2 \end{pmatrix}\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = 0$$

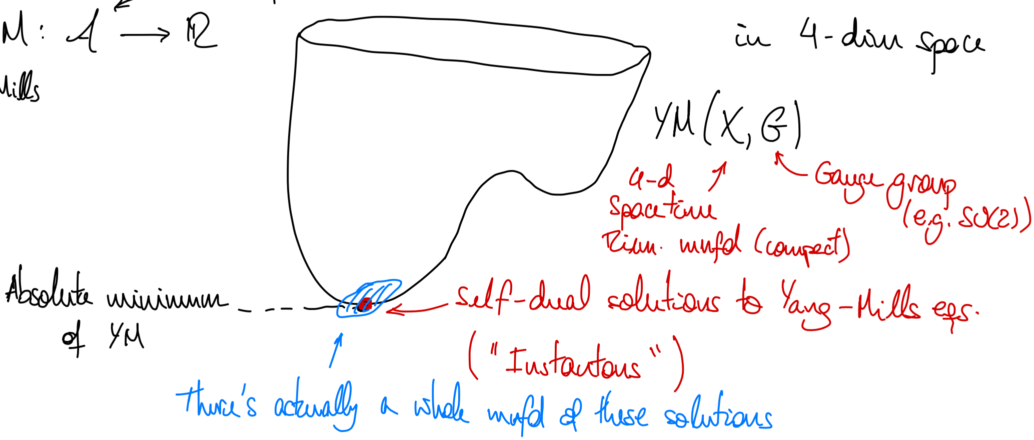
$\ddot{x} = \ddot{y} = 0$ is still a solution (geodesics) but the matrix has a kernel at each pt. of TX.
$\Rightarrow \exists$ solutions w/ $\ddot{x} \neq 0$.

So, take $\dfrac{d}{dt}\left(\dfrac{\dot{x}}{\dot{y}}\right) = 0 \rightsquigarrow \dot{x} = c_1 \dot{y} + c_0 \Rightarrow$ solutions don't come with parametrization!

That is b/c length functional is invariant under reparametrizations $\Rightarrow$ critical pts of length don't come w/ parametrizations!

<u>ASIDE</u>: $YM : \mathcal{A} \longrightarrow \mathbb{R}$ (∞-dim space of connections on vector bundle) in 4-dim space

Yang-Mills



LECTURE 12

15/02/2024

LEGENDRE TRANSFORM

Idea: Instead of working w/ $\mathcal{L} : TX \oplus \mathbb{R} \to \mathbb{R}$, we work in the cotangent bundle for a simpler (yet equivalent) description of the physics.

In $T^*X \oplus \mathbb{R}$, we can easily generalize the theory to simplectic mnfds

$$T^*X \oplus \mathbb{R} \rightsquigarrow \boxed{(M, \omega)} \longrightarrow \text{No configuration space}$$

Means of transforming:

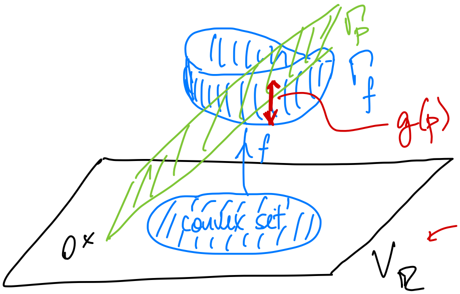
$$\left(TX \oplus \mathbb{R} \xrightarrow{\mathcal{L}} \mathbb{R}\right) \xleftrightarrow{\text{LEGENDRE TRANSFORMATION}} \left(T^*X \oplus \mathbb{R} \xrightarrow{H} \mathbb{R}\right)$$

Lagrangian $(q, \dot{q}, t)$ — Hamiltonian (q, p, t)

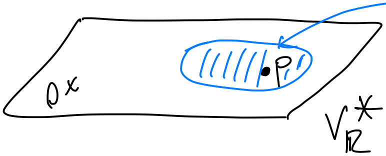
LEGENDRE TRANSFORM: operation in convex geometry. Let $f: U \to \mathbb{R}$ be a convex fct.; i.e., $\forall t \in [0,1]$,

$$f(tx + (1-t)y) \leq t f(x) + (1-t) f(y)$$

$\forall x, y \in U$.



More generally, could be real affine space



Convex set of covectors s.t. $\Gamma_{x \mapsto p(x)} \geq \Gamma_f$ at some pts in U

LEGENDRE TRANSFORM OF f

Define $g(p) := \sup_{x \in U} \left(\langle p, x \rangle - f(x) \right)$. If f is smooth, this sup occurs (if at all) at the point x s.t. $\frac{\partial}{\partial x}\left(\langle p, x\rangle - f(x)\right) = 0$.

MAIN ASSUMPTION: $\mathcal{L}$ is convex in the $\dot{q}$ direction

Actually nondegenerate

$$\boxed{\begin{aligned} \mathcal{L} &= T_g - U, \\ T_g &= g_{ij}\, \dot{q}^i \dot{q}^j \end{aligned}}$$

Coordinates: $X = \operatorname{span}\{x^i\}$

$$TX = \operatorname{span}\left\{\frac{\partial}{\partial x^i}\right\}$$

dual space

$$T^*X = \operatorname{span}\{dx^i\}$$

LEGENDRE TRANSFORM OF $\mathcal{L}$ IS: $H(x,p,t) = \sup_{\dot{x}} \left(\langle p, \dot{x} \rangle - \mathcal{L} \right)$.

This sup occurs at $\dot{x}$ s.t. $\frac{\partial}{\partial \dot{x}} \left(\langle p, \dot{x} \rangle - \mathcal{L} \right) = 0$

i.e., $p - \boxed{\frac{\partial \mathcal{L}}{\partial \dot{x}}} \Big|_{\dot{x}} = 0$

Generalized momentum

HAMILTONIAN: $H(x,p,t) = \langle p, \dot{x}(x,p,t) \rangle - \mathcal{L}(x, \dot{x}(x,p,t), t)$

on $T^*X \oplus \mathbb{R}$ where $\dot{x}$ is a solution to $\frac{\partial \mathcal{L}}{\partial \dot{x}} = p$ with x, p fixed.

Thm: The dynamical system in (x,p) variables is:

$$\boxed{\dot{p}_i = -\frac{\partial H}{\partial q^i} \quad , \quad \dot{q}^i = \frac{\partial H}{\partial p_i}}$$

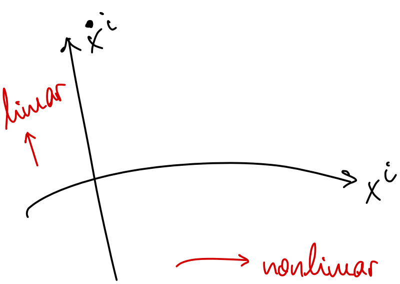
HAMILTON'S EQUATIONS

LECTURE 13

27/02/2023

MORE ON LAGRANGIANS

Note that, in each time slice, $\mathcal{L}_t : TX \to \mathbb{R}$. Then we can Taylor expand $\mathcal{L}_t$ in the coordinates x^i.



$\mathcal{L}_t$'s degree in $\dot{x}^i$ is DEPENDENT on the coordinates x^i!

deg 0: $V(x^1, \ldots, x^n)$ i.e., constant in the tangent directions ("potential energy")

V is a function on configuration space.

e.g.: electric force $\vec{F} = q\vec{E}$ $(= q\vec{\nabla}V)$ ← electric potential.

eg 1: $f_A := A_i(x^1, \dots, x^n)\,\dot{x}^i$ i.e., linear in tangent directions

A is a covector field (i.e., 1-form) given by $A := A_i(x^1, \dots, x^n)\,dx^i$.

Euler Lagrange:

$$\frac{d}{dt}\Big(\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{x}}}_{\text{gen. mom.}}\Big) = \underbrace{\frac{\partial \mathcal{L}}{\partial x}}_{\text{gen. force}} \longrightarrow \text{“}\frac{\partial f_A}{\partial x^i} = \underbrace{(\partial_i A_j)}_{\text{fct of } x}\,\dot{x}^j\text{”}$$

← linear in velocity!

Lorentz force $F = q\,\vec{v} \times \vec{B}$ is an example of this. In this case, such A is called vector potential.

eg 2: $\frac{1}{2} g_{ij}(x^1, \dots, x^n)\,\dot{x}^i \dot{x}^j$ (kinetic energy) i.e, quadratic in $\dot{x}$'s.

Symmetric 2-tensor $g := g_{ij}(x^1, \dots, x^n)\,\underbrace{dx^i \otimes dx^j}_{\text{basis for } (T^*M)\otimes(T^*M)\dots}$ (Riem. metric)

Obs: $T_g := \frac{1}{2} g_{ij}(x^1, \dots, x^n)\,\dot{x}^i \dot{x}^j$ describes free particle motion.

DEPENDENCE OF THE LAGRANGIAN ON TIME & FICTITIOUS FORCES

Suppose we have a Lagrangian system $\mathcal{L}(x, \dot{x}, t)$. Choose new coordinates which are time-dependent:

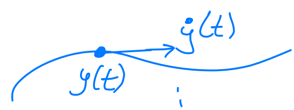
$$y = \phi_t(x), \quad \phi_t : X \to X \text{ diffeo } (C^\infty \text{ in } t)$$

then $x = \phi_t^{-1}(y) = \phi_{-t}(y) =: \psi_t(y)$

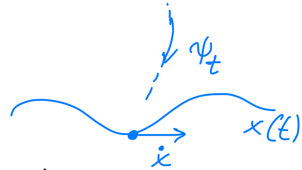
GOAL: Write the Lagrangian in these new coordinates.

Action

$$S = \int \mathcal{L}(y(t), \dot{y}(t), t)\, dt$$



$$\stackrel{!}{=} \int \mathcal{L}(x(t), \dot{x}(t), t)\, dt$$



$$= \int \mathcal{L}\big(\psi_t(y), (d\psi_t)(\dot{y}(t)) + (\dot{\psi}_t)(y(t)), t\big)\, dt$$

EXAMPLE: $X = \mathbb{R}^2 \simeq \mathbb{C}$, $\psi_t(z) = e^{it} z =: w(t)$ ← coordinates depend on time

Suppose $\mathcal{L}(w, \dot{w}, t) = \frac{1}{2} |\dot{w}|^2$

Q: What is the Lagrangian in the moving frame?

Velocity of path in new coordinates $= \underbrace{e^{it}\dot{z}}_{(d\psi_t)(\dot{y}(t))} + \underbrace{i e^{it} z}_{(\dot{\psi}_t)(y(t))}$

(ψ_t is linear here)

Then

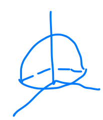
$$\tilde{\mathcal{L}}(z, \dot{z}, t) = \frac{1}{2} \left| e^{it}\dot{z} + i e^{it} z \right|^2$$

$$= \frac{1}{2} |\dot{z} + iz|^2$$

$$= \frac{1}{2} |\dot{z}|^2 + \overbrace{\boxed{\mathrm{Re}(-i\dot{z}\bar{z})}}^{\deg 1} + \overbrace{\boxed{\frac{1}{2}|z|^2}}^{\deg 0}$$

$\mathrm{Re}(-i\dot{z}\bar{z}) = f_A$ Coriolis Force

$\frac{1}{2}|z|^2$: Potential $V = -\frac{1}{2}|z|^2$ Centrifugal force



"Fictitious Forces" They only appear here because the new coordinates are not an inertial frame ....

"Fictitious" name arises b/c Newton knew mech. only works in inertial ref. frames.

$z = x + iy$

$$\mathrm{Re}(-i\dot{z}\bar{z}) = \mathrm{Re}(-i(\dot{x}+i\dot{y})(x-iy)) \rightsquigarrow f_A = -y\dot{x} + x\dot{y} = \begin{vmatrix} x & y \\ \dot{x} & \dot{y} \end{vmatrix}$$

$$\rightsquigarrow \boxed{A = x\,dy - y\,dx}$$

LAGRANGIAN METHOD INCORPORATING TIME: $M = X \times \mathbb{R}$.

spacetime $\nearrow$ $\quad \{x^i\} \quad t$

Phase space: $TM = TX \times T\mathbb{R} = TX \times \mathbb{R}_t \times \mathbb{R}_{\dot{t}}$

If we have a Lagrangian $\mathcal{L}(x, \dot{x})$ as before, we can extend it to this new phase space by

time-indep.

$$\tilde{\mathcal{L}}(x, \dot{x}, \dot{t}) = \mathcal{L}(x, \dot{x}) + \underbrace{\frac{1}{2}\dot{t}^2}_{\text{kinetic in time}}$$

Extended Lagrangian on extended phase sp.

Euler-Lagrange Equations for a path in M $(x^i(\tau), t(\tau))$ are:

$$\begin{cases} \dfrac{d}{d\tau}\left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial \dot{x}^i}\right) = \dfrac{\partial \tilde{\mathcal{L}}}{\partial x^i} \\ \dfrac{d}{d\tau}\left(\dfrac{\partial \tilde{\mathcal{L}}}{\partial \dot{t}}\right) = \dfrac{\partial \tilde{\mathcal{L}}}{\partial t} \rightsquigarrow \dfrac{d}{d\tau}(\dot{t}) = 0 \Rightarrow \dot{t} \text{ is conserved!} \end{cases}$$

i.e., $\tau = t + \text{const} \Rightarrow$ we can parametrize curves in the new space by t!

Makes it easier to move around coords transformations $(x, t) \xmapsto{\Phi} (\phi_t(x), t)$

Simplifies relativity

$$\boxed{\tilde{\mathcal{L}} = \mathcal{L}(\Phi(x,t), d\Phi(x,t), t)}$$

RELATIVISTIC VERSION: (1d space, 1d time)

$M = \mathbb{R}_x \times \mathbb{R}_t$

$TM = T\mathbb{R}_{(x, \dot{x})} \times T\mathbb{R}_{(t, \dot{t})}$

Minkowski metric

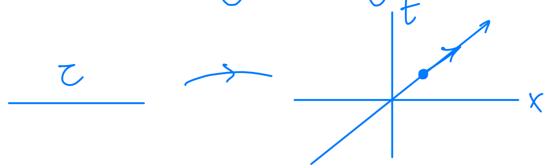
$\eta = dx \otimes dx - dt \otimes dt$

$\{$

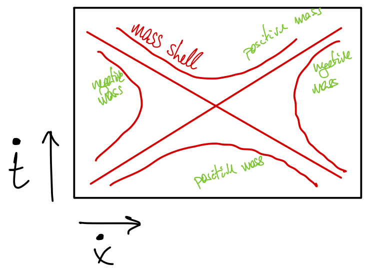
Lagrange's Equations for a path $(x(\tau), t(\tau))$

$$\boxed{T_\eta = \frac{1}{2}\dot{x}^2 - \frac{1}{2}\dot{t}^2}$$

$$\boxed{\frac{d}{d\tau}(\dot{x}) = 0 \ , \quad \frac{d}{d\tau}(\dot{t}) = 0} \Rightarrow \text{Paths are again straight lines}$$



In TX, the constant energy surfaces are:



$$\frac{dx}{dt} = \frac{dx/d\tau}{dt/d\tau} \leq 1 \Rightarrow \text{limit on speed}$$

LECTURE 14

29/02/2024

HAMILTONIAN FORMALISM

Lagrangian $\longrightarrow$ Hamiltonian

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) = \frac{\partial \mathcal{L}}{\partial q}$$

$$\begin{cases} \dot{p} = -\dfrac{\partial H}{\partial q} \\ \dot{q} = \dfrac{\partial H}{\partial p} \end{cases} , \quad p := \frac{\partial \mathcal{L}}{\partial \dot{q}}$$

Legendre transf of $\mathcal{L}$ $\longrightarrow$ $\boxed{H(q,p,t) = p\dot{q} - \mathcal{L}(q,\dot{q},t)}$

Features of Hamiltonian Formalism:

1. Dynamics (i.e., the vector field) is on T^*X instead of TX.

$(x^1, \_, x^n)$
on X

on TX

$(x^1, \_, x^n, \dot{x}^1, \_, \dot{x}^n)$
on TX
for $\dot{x}^1 \frac{\partial}{\partial x^1} + \cdots + \dot{x}^n \frac{\partial}{\partial x^n}$

on T^*X

these are dual to each other

$(x^1, \_, x^n, p_1, \_, p_n)$
on T^*X
for $p_1 dx^1 + \cdots + p_n dx^n$

Main idea: when we take the fiber derivative of $\mathcal{L}$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) \in T^*_q X$$

Namely, the fiber derivative of $\mathcal{L}$ defines a map:

"fiber derivative of $\mathcal{L}$"

$$\begin{aligned} F\mathcal{L} : TX &\longrightarrow T^*X \\ (q, \dot{q}) &\longmapsto \left(q, \frac{\partial \mathcal{L}}{\partial \dot{q}}(q, \dot{q}) \right) \end{aligned}$$

If the above map is an isomorphism, then Lagrangian and Hamiltonian formalisms are equivalent.

EXAMPLE: 1) $\mathcal{L} = -V(q)$ (i.e., no kinetic energy)

$$\frac{\partial \mathcal{L}}{\partial \dot q} = 0 \Rightarrow \quad FL : TX \longrightarrow T^*X$$
$$(q,\dot q) \mapsto (q,0) \quad \leftarrow \underline{\underline{\text{Not}}} \text{ isom.!}$$

2) $\mathcal{L} = T_g = \frac{1}{2} g_{ij}\dot x^i \dot x^j$, g = Riem. metric

$$\frac{\partial \mathcal{L}}{\partial \dot x^i} = g_{ij}(x)\,\dot x^j . \quad \text{Thus:}$$

$$FL : TX \longrightarrow T^*X$$
$$(x^1,\dots,x^n,\dot x^1,\dots,\dot x^n) \longmapsto (x^1,\dots,x^n, g_{1j}\dot x^j, g_{2j}\dot x^j,\dots, g_{nj}\dot x^j)$$

Since g is non-degenerate, FL is an isom. $\forall x$; i.e., $TX \xrightarrow[g]{\simeq} T^*X$

$\Rightarrow$ only need g to be non-deg. metric (weaker than Riem.)

Lag $\simeq$ Ham

ADVANTAGES OF HAMILTONIAN FORMALISM: (Basic case T^*X)

Inputs: $\begin{cases} X = \text{configuration space} \\ H : T^*X \longrightarrow \mathbb{R} \end{cases}$ (in general, $H : T^*X \oplus \mathbb{R}_t \to X$)

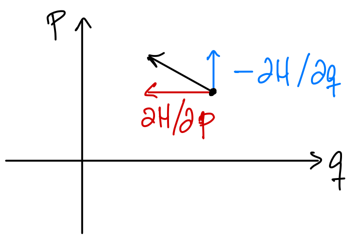
Equations of Motion: $\dot p = -\dfrac{\partial H}{\partial q}$, $\dot q = \dfrac{\partial H}{\partial p}$

$$\begin{tikzcd} TX \arrow[r, "FL"] \arrow[dr, "\mathcal{L}"'] & T^*X \arrow[d, "H"] \\ & X \end{tikzcd}$$

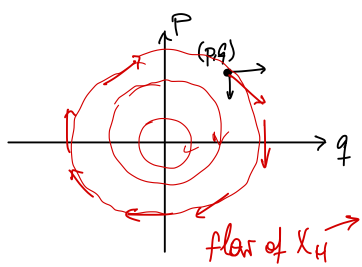
$\textcircled{E}$

Bad? Does not commute

HAMILTONIAN PHASE SPACE



e.g.: $X = \mathbb{R}_q$, $H(q,p) = \frac{1}{2}(p^2+q^2)$



$$X_H = p\frac{\partial}{\partial q} - q\frac{\partial}{\partial p}$$

Hamilt. Vec. Field

$$\phi_t^{X_H}(q,p) = e^{tA}\begin{pmatrix} q \\ p \end{pmatrix},$$

for time t $\quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

obs: $H(q,p) = \frac{1}{2}(p^2+q^2)$

$= \underbrace{\text{"KE"}}_{} + PE$

$T_{g^{-1}} = \frac{1}{2} g^{ij} p_i p_j$ $\qquad$ $T_g = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j$

REMARK: Along a solution of the equations of motion, $\frac{d}{dt} H = 0$ as long as H is independent of time:

$$\frac{d}{dt} H(q(t), p(t)) = \partial_q H \dot{q} + \partial_p H \dot{p} + \cancel{(\partial_t H)}^{=0}$$

$$= \partial_q H \partial_p H + \partial_p H(-\partial_q H)$$

$$= 0.$$

Upshot: as long as H is time-indep, we automatically have a conserved quantity (i.e., H).

REMARK: Hamiltonian phase space is endowed with a natural volume form and it is preserved by the flow.

$$\boxed{\begin{aligned} dp\left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial p}\right) &= b \\ dq\left(a\frac{\partial}{\partial q} + b\frac{\partial}{\partial p}\right) &= a \end{aligned}}$$

In dim $= 2$, $dp \wedge dq$ $\left(= \frac{dp \otimes dq - dq \otimes dp}{2}\right)$ convention?

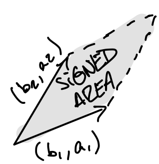
$$(dp \otimes dq)\left(a_1\frac{\partial}{\partial q} + b_1\frac{\partial}{\partial p},\ a_2\frac{\partial}{\partial q} + b_2\frac{\partial}{\partial p}\right) = b_1 a_2$$

No coeff.

$$(dp \wedge dq)\left(a_1\frac{\partial}{\partial q} + b_1\frac{\partial}{\partial p},\ a_2\frac{\partial}{\partial q} + b_2\frac{\partial}{\partial p}\right) = b_1 a_2 - b_2 a_1$$

skew-symmetric 2-tensor = "2-form"

$$= \begin{vmatrix} b_1 & b_2 \\ a_1 & a_2 \end{vmatrix} =$$



Obviously, not necessarily agrees w/ Euclidean vol.
(e.g. multiply $dp \wedge dq$ by constant)

Upshot: Top forms measure (signed) volumes spanned by vectors.

$$\mathrm{vol} := dp_1 \wedge dq^1 \wedge dp_2 \wedge dq^2 \wedge \dots \wedge dp_n \wedge dq^n$$

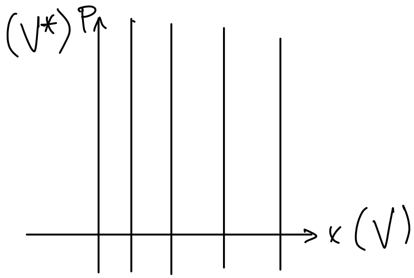
Canonical $2n$ volume form, everywhere nonzero on T^*X

LECTURE 16

HAMILTONIAN STUFF (ctd)

07/03/2024

Note: $V \times V^* \simeq T^*V$. Canonical 2-form $\omega = dp_i \wedge dx^i$ "Area form"



Such canonical form exists on T^*X (and it is called "SIMPLECTIC FORM")

In the Hamiltonian formalism, we need:

- (M, ω) simplectic manifold (Hamiltonian phase space)
- $H \in C^\infty(M, \mathbb{R})$ Hamiltonian function

With this, H generates a vec. field which gives the dynamics

Ex: $M = T^*V = V \times V^*$, $\omega = dp_i \wedge dx^i$

H = any function $H(p_1, -, p_n, x^1, -, x^n)$

MECHANISM: Let $M = \mathbb{R}^{2n}_{(x,p)}$ with $\omega = dp_i \wedge dx^i$ and $V \in T_{(x,p)}M$.

$$\omega(V, \bullet) \in T^*_{(x,p)}M$$

Then, given a Hamiltonian function $H(x,p) \in C^\infty(M, \mathbb{R})$, we have

b/c ω is non-degenerate

$$\begin{tikzcd} TM \arrow[rr, "\omega^\#", "\simeq"'] \arrow[dr] & & T^*M \arrow[dl] \\ & M & \end{tikzcd}$$

This isomorphism is the main tool for the Hamiltonian formalism.

Step 1:
$$\boxed{dH = \frac{\partial H}{\partial x^i} dx^i + \frac{\partial H}{\partial p_i} dp_i}$$

Rmk: $dH \in \Omega^1(M)$ and defines a covector at each pt. in M.

Step 2: Use ω to convert dH into a vector field X_H:

$$\boxed{X_H = -(\omega^{\#})^{-1}(dH)}$$ Hamiltonian vector field of H

convention

[cf. with $\nabla H = g^{-1}(dH)$ $:= \text{grad } H$

$$\begin{tikzcd} TM \arrow[rr, "g", "\simeq"'] \arrow[dr] & & T^*M \arrow[dl] \\ & M & \end{tikzcd}$$

$V \mapsto g(V, \cdot)$]

Ex: $T^*\mathbb{R} \simeq \mathbb{R}_x \times \mathbb{R}_p$
$\omega = dp \wedge dx$

Obs: $\omega(\partial_p, \partial_x) = 1$, $\omega(\partial_x, \partial_x) = 0 = \omega(\partial_p, \partial_p)$
$\omega(\partial_x, \partial_p) = -1$

So, $\omega(\partial_p, \cdot) = dx$ $\Rightarrow$ $\omega^{\#}(\partial_p) = dx$
$\omega(\partial_x, \cdot) = -dp$ $\quad$ $\omega^{\#}(\partial_x) = -dp$

Let $H = H(x,p)$. Then

$$dH = \frac{\partial H}{\partial x} dx + \frac{\partial H}{\partial p} dp \in \Omega^1(\mathbb{R}^2)$$

So,

$$X_H = -(\omega^{\#})^{-1}(dH) = -\left(\frac{\partial H}{\partial x}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\left(-\frac{\partial}{\partial x}\right)\right)$$

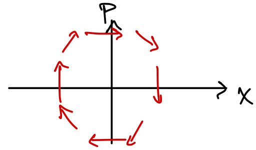
$$\Rightarrow \boxed{X_H = \frac{\partial H}{\partial p}\frac{\partial}{\partial x} - \frac{\partial H}{\partial x}\frac{\partial}{\partial p}}$$

cf. Hamilton's Equations

$$\begin{cases} \dot{x} = \dfrac{\partial H}{\partial p}(x(t), p(t)) \\ \dot{p} = -\dfrac{\partial H}{\partial x}(x(t), p(t)) \end{cases}$$

Say, $H(x,p) = \frac{1}{2}(x^2+p^2)$ (Simple Harmonic Oscillator)

Then, $X_H = p\frac{\partial}{\partial x} - x\frac{\partial}{\partial p}$.



IMPORTANT CONSEQUENCES:

1) ω is automatically preserved by X_H!

$$\dot{\omega} = \frac{d}{dt}(dp_i \wedge dx^i) = d\dot{p}_i \wedge dx^i + dp_i \wedge d\dot{x}^i$$

$$= d\left(-\frac{\partial H}{\partial x^i}\right) \wedge dx^i + dp_i \wedge d\left(\frac{\partial H}{\partial p^i}\right)$$

$$= \cancel{-\frac{\partial^2 H}{\partial x^i \partial x^j} dx^j \wedge dx^i} - \cancel{\frac{\partial^2 H}{\partial x^i \partial p_j} dp_j \wedge dx^i}$$

$$+ \cancel{\frac{\partial^2 H}{\partial p_i \partial x_j} dp_i \wedge dx^j} + \cancel{\frac{\partial^2 H}{\partial p_i \partial p_j} dp_i \wedge dp_j}$$

$\frac{\partial^2 H}{\partial x^i \partial x^j}$ is symmetric but $dx^j \wedge dx^i$ is skew-symmetric $0=$

$0=$

$$= 0. \Rightarrow \omega \text{ is preserved.}$$

This means that ω^2 is preserved

ω^2 is preserved

$\vdots$

$vol_\omega = \frac{1}{n!}\omega^n$ is preserved! $\Rightarrow$ Volume form is preserved by the Hamiltonian flow!

Cor: (Poincaré Recurrence Theorem) If the Hamiltonian flow preserves a bounded region in phase space (due, for example, to energy conservation), then any initial state will return to an arbitrarily small neighborhood from where it started.

LECTURE 17

HAMILTONIANS (ctd.)

12/08/2024

X = configuration space

$T^*X =: M$ cotangent bundle $\leftarrow$ Endowed w/ a 2-form
$$\omega = dp_i \wedge dx^i$$

MAIN CONSTRUCTION: Given any function $H = H(p,x) \in C^\infty(M)$, we can find a vector field $\boxed{X_H := -\omega^{-1}(dH)}$ (Hamiltonian Vector Field)

Under the flow of X_H, ω is preserved $\iff \dot{\omega} = 0$
$$\iff \mathcal{L}_{X_H}\omega = 0$$
Lie Derivative

This gives that $\boxed{\mathcal{L}_{X_H}\left(\frac{\omega^n}{n!}\right) = 0}$ i.e., the volume form of phase space is preserved.

Cor: (Poincaré Recurrence) Eventual return to any neighborhood of initial conditions when flow is preserved in a bounded region.

Hamiltonian form of Nöther's Theorem

Claim: Under the flow of X_H, $\dot{H} = 0$ (i.e., H is conserved along the Hamiltonian vector field).

Hamilton's eqs

Pf: $\frac{dH}{dt}(p,q) = \partial_p H\, \dot{p} + \partial_q H\, \dot{q} = (\partial_p H)(-\partial_q H) + (\partial_q H)(\partial_p H)$
$$= 0$$
$\square$

$\Rightarrow$ X_H is always tangent to the level sets of H
H is automatically preserved by the flow it generates

MOST IMPORTANT CASE: H = Total energy of the system

e.g., $\frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V(x)$

Then X_H = dynamical system; i.e., flow in time.

$\Rightarrow$ H is conserved.

DEVELOP THE ALGEBRAIC STRUCTURE: Let $g \in C^\infty(M)$, then g is affected by the flow of X_H as follows:

$$X_H(g) = \left(dg = \frac{\partial g}{\partial p} dp + \frac{\partial g}{\partial q} dq \right)(X_H)$$

$$= -\omega^{-1}(dH, dg)$$

$$= -\{H, g\} \quad \leftarrow \text{Poisson bracket}$$

Obs: $\omega = dp_i \wedge dq^i \rightsquigarrow \omega^{-1} = -\frac{\partial}{\partial q^i} \wedge \frac{\partial}{\partial p_i}$

$$\{f, g\} \stackrel{\text{def}}{=} \omega^{-1}(df, dg) = \frac{\partial f}{\partial q^i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q^i} \frac{\partial f}{\partial p_i}$$

Properties of $\{\cdot,\cdot\}$:

(i) $\{f, g\} = -\{g, f\}$ skew-symmetric

(ii) Leibniz & (iii) Jacobi... check a book idk...

$$\{f, gh\} \stackrel{\text{def}}{=} \omega^{-1}(df, d(gh)) = \omega^{-1}(df, h\,dg + g\,dh) = h\{f,g\} + g\{f,h\}.$$

$$-\{h, \{f,g\}\} = X_h(\{f,g\}) = X_h(\omega^{-1}(df, dg))$$

$$= \underbrace{(\mathcal{L}_{X_h} \omega^{-1})}_{0}(df, dg) + \omega^{-1}(\underbrace{dX_h(f)}_{-\{h,f\}}, dg) + \omega^{-1}(df, \underbrace{d(X_h(g))}_{-\{h,g\}})$$

$= -\{\{h,f\},g\} - \{f,\{h,g\}\}$

Upshot: (i), (ii), (iii) $\Longrightarrow$ $C^\infty(M)$ is a Poisson Algebra.

APPLICATION OF $\{\cdot,\cdot\}$: on T^*X, take coords q^i, p_i. Then, the motion of a particle is determined by the evolution of $\dot{q}^i$ and $\dot{p}_i$. Thus, the evolution in time is:

HAMILTON'S EQUATIONS OF MOTION:
$$\dot{p}_i = \{p_i, H\}$$
$$\dot{q}^i = \{q^i, H\}$$

!

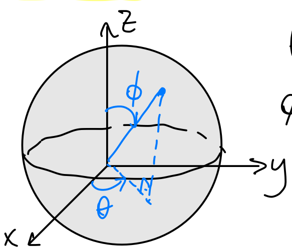
POISSON'S THEOREM: if f,g are conserved by the Hamiltonian flow X_H, then $\{f,g\}$ is also conserved.

$\Longrightarrow$ Can produce more conserved quantities!

Pf: Assumption $\Leftrightarrow$ $\{f,H\} = \{g,H\} = 0$.

Then $\{\{f,g\},H\} = 0$ by Jacobi. $\square$

EXAMPLE: $M = S^2$ with its usual area form



$\theta \in [0, 2\pi]$

$\phi \in [0, \pi]$

$$\omega = (\sin\phi \, d\theta) \wedge d\phi$$

e.g.: interior product gives

$$\partial_\theta \longmapsto \sin\phi \, d\phi$$
$$\partial_\phi \longmapsto -\sin\phi \, d\theta$$

Note: $\int_{S^2} \omega = \int_0^{\pi}\int_0^{2\pi} \sin\phi \, d\theta \, d\phi = 4\pi$ ✓

Hight function: $z = \cos\phi \to dz = -\sin\phi \, d\phi$

$$-\omega^{-1}(dz) = \frac{\partial}{\partial\theta} = \text{HAMILTONIAN VECTOR FIELD}$$

$$= x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \quad \text{in Cartesian coordinates}$$

← tangent to S^2

Upshot: The Hamiltonian vec. fields of the hight function are:

$$\begin{cases} X_z = x\,\partial_y - y\,\partial_x \\ X_y = y\,\partial_z - z\,\partial_y \\ X_x = z\,\partial_x - x\,\partial_z \end{cases}$$

⤳ Generates the rotations...

$$\Rightarrow \{x,y\} = z, \quad \{z,x\} = y, \quad \{y,z\} = x.$$

Upshot: Linear functions on $\mathbb{R}^3$ restricted to S^2 are closed under the Poisson bracket

$\mathfrak{so}(3) = \operatorname{span}_{\mathbb{R}}\langle x, y, z\rangle$ closed under $\{\cdot,\cdot\}$ gives a Lie algebra

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LECTURE 18

HAMILTONIAN STUFF (ctd)

14/03/2024

Hamiltonian formalism: (M,ω) symplectic mnfd ← Analog of $(T^*X, dp_i \wedge dx^i)$

ω allows us to convert $H \in C^\infty(M)$ into $X_H := -\omega^{-1}(dH)$

i.e., (i) need $\omega \in \Omega^2(M)$ to be nondegenerate dim M needs to be even $\left\{ \begin{array}{l} \text{i.e., need } \omega^{\#} : TM \xrightarrow{\simeq} T^*M \\ \qquad\qquad\qquad v \longmapsto \omega(v, \bullet) \end{array} \right.$

(ii) Also need that X_H preserves ω:

$$i_{X_H}\omega = \omega(-(\omega^{\#})^{-1}(dH), \bullet)$$
$$= \omega^{\#}(-(\omega^{\#})^{-1}(dH)) = -dH$$

$$\mathcal{L}_{X_H}\omega \underset{\text{Cartan's formula}}{=} \underbrace{[d, i_{X_H}]}_{\text{"GRADED COMMUTATOR"}}\omega = d\,\boxed{i_{X_H}\omega} + i_{X_H}d\omega$$

$$= \cancel{d(-dH)}^{=0} + i_{X_H}d\omega$$

EXTRA ASSUMPTION: $d\omega = 0$

$$= 0 \checkmark$$

where

$$\Omega^2(M) \ni \omega \xmapsto{\;i_X\;} \omega(X, \bullet) \in \Omega^1(M)$$

and $d : \Omega^2(M) \to \Omega^3(M)$

$[a,b] = ab - (-1)^{|a||b|}ba$ (Koszul sign)

degree of $d \rightsquigarrow +1$
degree of $i_X \rightsquigarrow -1$ $\Rightarrow |d||i_X| = -1$

Upshot: if ω is closed, then ω is preserved by X_H

N.B.: X_H automatically preserves H because

$$X_H(H) = -\omega^{-1}(dH, \bullet)(dH) = 0.$$

$X(f) = i_X df$; b/c ω is skew

All of the above defines the space of states (M, ω).
To obtain dynamics, we need a function $H \in C^\infty(M, \mathbb{R})$.
This determines how the system evolves in time by

$$X_H = -\omega^{-1} dH.$$

Any physical observable (e.g., position, momentum, etc) is a function $f \in C^\infty(M)$.

Every observable f has X_f but H is the one used for the time evolution.

Q: Is X_f a symmetry of the system? system $= (M, \omega, H)$

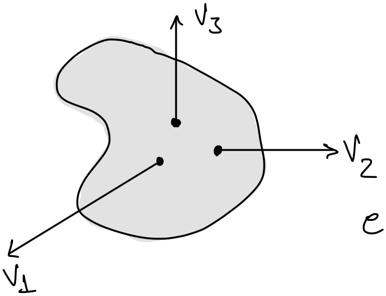
A: X_f is a symmetry when $\boxed{\{f, H\} = 0.}$

LECTURE 19

RIGID BODIES

19/03/2024

In order to study the motion of a rigid body,

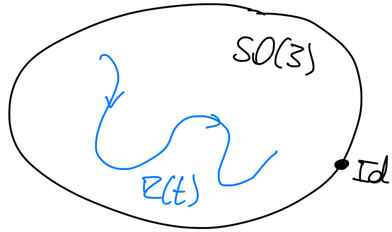


1. Install an orthonormal frame $v := (v_1, v_2, v_3)$. Usually at the center of mass (Moving frame)

2. Compare v with a background fixed frame $e := (e_1, e_2, e_3)$ for $\mathbb{R}^3$:

$$\boxed{v = R e, \quad R \text{ rotation matrix}}$$

Upshot: If CM is fixed, then the motion of the rigid body can be described as a path $v(t)$ in the space of orthonormal frames or, equivalently, as a path $R(t) \in SO(3)$



i.e., the configuration space X is both a group and a manifold.

$X = SO(3)$ (i.e., a Lie group)

GENERALIZATIONS: 1. If we don't fix the CM, then the configuration space is

$v(t) = (\vec{v}_1(t), \vec{v}_2(t), \vec{v}_3(t))$ ← Rotations of the axis

position of the center of mass

So, we need $\big(\underset{\underset{SO(3)}{\cap}}{R(t)}, \underset{\underset{\mathbb{R}^3}{\cap}}{\vec{r}(t)}\big) \in SO(3) \ltimes \mathbb{R}^3$

$(R_2, r_2) \cdot (R_1, r_1)$

$\cancel{\neq (R_1 R_2, r_1 r_2)}$

$\| $

$(R_2 R_1, r_2 + R_2 r_1)$

b/c of this mod., semi-direct product

2. Often, a system's configuration can be described by a transformation applied to some "standard" configuration. In this case, the configuration space is the group of all such transformations.

e.g., a pool P as of volumes. Its conf. space can be described by a diffeo. $\varphi : P \to P$ (incompressible $\to$ φ is volume preserving).

So, conf. space $= \mathrm{Diff}(P)$.

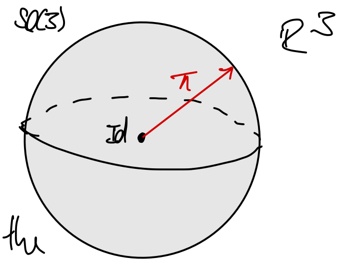
Rmk: $SO(3) = \{ R \in M_{3\times 3} : RR^T = \mathrm{Id} \text{ and } \det R = +1 \}$

6 constraints b/c RR^T is symmetric

R preserves the standard vol. form $dx \wedge dy \wedge dz$.

$\dim SO(3) = 3$

Picture of $SO(3)$: Every vector in the ball of radius π represents rotation by angle $\|u\|$ about the $\frac{u}{\|u\|}$ axis using right-hand-rule.



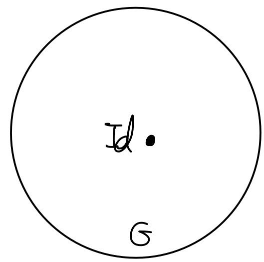
Also, identify antipodal pts on the bdry of the ball.

$$\Rightarrow \boxed{SO(3) = B^3 /_{\partial B^3 \ni x \sim -x}} \quad \mathbb{R}^3$$

1. **Lagrangian phase space**: Consider $X := G = SO(3)$. The configuration space is $TG = T(SO(3))$. ← 6-dimensional

But, there are 2 transformations of G taking a fixed pt. $g \in G$ to the identity Id; namely: right and left translations by g^{-1}:

$$L_{g^{-1}} : G \to G, \quad h \mapsto g^{-1}h \qquad R_{g^{-1}} : G \to G, \quad h \mapsto hg^{-1}$$



$$L_g R_{g^{-1}} : G \longrightarrow G, \quad h \longmapsto ghg^{-1}$$ } Conjugation by $g \in G$

Def: (**Lie group**) G is a Lie group if G is a smooth mnfd and G is a group (i.e., $m: G \times G \to G$, $inv: G \to G$, etc...) and the multiplication and inversion maps are smooth.

Obs: 1) $L_g, R_g, L_g R_{g^{-1}}$ are all diffeos (by the assumption that

G is a Lie group)

2) $[L_g, R_h] = 0 \quad \forall g,h \in G$

3) $L_g R_{g^{-1}} \in \mathrm{Aut}(G)$.
$$\begin{cases} L_g R_{g^{-1}}(hk) = (L_g R_{g^{-1}} h)(L_g R_{g^{-1}} k) \\ L_g R_{g^{-1}}(e) = e \\ [L_g R_{g^{-1}}, \mathrm{Inv}] = 0 \end{cases}$$

Def: $T_e G =: \mathfrak{g}$ is the Lie algebra of G (i.e., it is a linear approx. to G)

Prop:
$$\boxed{TG \simeq G \times \mathfrak{g}}$$

i.e., the tangent bundle of any Lie group is trivial !

$\theta^L : TG \to G \times \mathfrak{g}$
$(g, \dot g) \mapsto (g, (dL_{g^{-1}})(\dot g))$

or

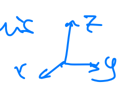
$\theta^R : TG \to G \times \mathfrak{g}$
$(g, \dot g) \mapsto (g, (dR_{g^{-1}})(\dot g))$

Ex: $\mathfrak{g} = \mathrm{Lie}(SO(3)) = \mathfrak{so}(3) = T_e SO(3)$

i.e., if $R(t)R(t)' = \mathrm{Id}$, then, differentiating w.r.t. t,
$$\dot R(t) R(t)^T + R(t) \dot R(t)^T = 0$$
at $t=0$, $R(t) = \mathrm{Id}$, so:
$$\boxed{\dot R(0) + \dot R(0)^T = 0}$$
i.e., $\dot R$ is a skew-symmetric 3×3 matrix

Thus:
$$\boxed{\mathfrak{so}(3) = \{ X \in \mathbb{R}^{3\times 3} : X + X^T = 0 \}}$$

Take a basis for this



$$E_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad E_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\Rightarrow \mathfrak{so}(3) = \mathbb{R}E_x \oplus \mathbb{R}E_y \oplus \mathbb{R}E_z$; i.e., 3d vec. space

$$T\,SO(3) \simeq SO(3) \times \mathfrak{so}(3)$$

$$(R, \dot{R}) \xmapsto{\theta^L} (R, \underline{R^{-1}\dot{R}})$$

$$\begin{cases} RR^T = \mathrm{Id}, \ \det R = 1 \\ \dot{R}R^T + R\dot{R}^T = 0 \end{cases}$$

$L_R : S \mapsto RS$ (linear)

"Angular velocity"

$$\vec{\Omega} = \Omega_x E_x + \Omega_y E_y + \Omega_z E_z$$

LECTURE 20

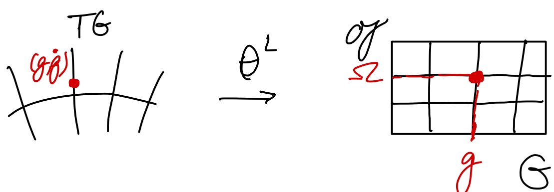
21/03/2024

Rigid Bodies (Brought by Sophus Lie)

Recall: $TG \simeq G \times \mathfrak{g}$

$$\uparrow$$

$$dL_{g^{-1}} =: \theta^L$$



Any pt. in the phase space $(g, \dot{g})$ can be viewed as

$$\left(g, (dL_{g^{-1}})(\dot{g})\right) = \left(g, g^{-1}\dot{g} =: \Omega\right),$$

"Angular Velocity"

$$\Omega = \Omega_x E_x + \Omega_y E_y + \Omega_z E_z$$

(in 3d, Ω is a 3-vector; in 4d, it is a 6-vector)

Rmk: We have 2 conserved quantities

(i) Energy $= \frac{1}{2}\left(I_1 \Omega_1^2 + I_2 \Omega_2^2 + I_3 \Omega_3^2\right)$

(ii) Total angular momentum $= L_x^2 + L_y^2 + L_z^2 = \|\Omega\|^2_{\mathbb{R}^3}$

LECTURE 21

26/03/2024

RIGID BODIES (Brought by Sophus Lie)

We have that $\mathfrak{so}(3) := T_e SO(3) = \{X \in \mathbb{R}^{3\times 3} : X + X^T = 0\}$

Obs: $\dim \mathfrak{so}(n) = \binom{n}{2}$

Isomorphism:

cross-product natural isomorphism "Lie bracket"

$$\left(\mathbb{R}^3, a\times b, (a,b)_{Euc}\right) \xrightarrow{\simeq} \left(\mathfrak{so}(3), [E_a, E_b], -\frac{1}{2}\mathrm{tr}(E_a, E_b)\right)$$

$SO(3)$ acting on the left: $a \mapsto R(a)$

$SO(3)$ acting on the right: $X \mapsto RXR^{-1}$ Adjoint action of $SO(3)$ on $\mathfrak{so}(3)$.

$$e_1 \longmapsto E_1 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$e_2 \longmapsto E_2 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$e_3 \longmapsto E_3 := \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$a \longmapsto E_a$$

Recall: $\mathbb{R}^3 \qquad\qquad \mathfrak{so}(3)$

$a^* := \langle a, \bullet \rangle$

$$a \times b \longmapsto -a \otimes b^* + b \otimes a^*$$

e.g.: $\mathbb{R}^3 \ni u \longmapsto -a\langle b, u\rangle + b\langle a, u\rangle$

e.g.: $e_3 = e_1 \times e_2 \longmapsto -e_1 \otimes e_2^* + e_2 \otimes e_1^*$

Cross Product: $u \times v = g_{Euc}^{-1}\Big(\underbrace{i_v\, i_u\, \underbrace{dx \wedge dy \wedge dz}_{vol_{\mathbb{R}^3}}}_{\text{covector}}\Big)$

$$g_{Euc}: \mathbb{R}^3 \xrightarrow{\simeq} (\mathbb{R}^3)^*$$
$$a \longmapsto \langle a, \bullet \rangle$$

Obs: Because of this, we only have cross products in 3 dimensions.

obs: Cross product is invariant under rotations $R \in SO(3)$
i.e., $(Ru) \times (Rv) = R(u \times v)$.

Rmk: $SO(3)$ acts on $\mathfrak{so}(3)$ via:

$$u \times v \overset{E}{\longmapsto} v \otimes u^* - u \otimes v^*$$

$$R(u \times v) = (Ru) \times (Rv) \longmapsto Rv \otimes (Ru)^* - Ru \otimes (Rv)^*$$
$$= Rv\langle Ru, \bullet\rangle - Ru\langle Rv, \bullet\rangle$$
$$= Rv\langle u, R^{-1}\bullet\rangle - Ru\langle v, R^{-1}\bullet\rangle$$
$$= R\circ\left(v \otimes u^* - u \otimes v^*\right)R^{-1}.$$

Rigid Body: In 3-dim. space, choose an inertial frame $E := (e_1, e_2, e_3)$ providing coordinates $x = (x_1, x_2, x_3)^T$ so that we have the usual kinetic energy

$$T = \frac{1}{2}\left(\dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2\right).$$

"Body frame"

Install an orthonormal frame $F = (f_1, f_2, f_3)$ on the body, so that each constituent particle has a fixed address

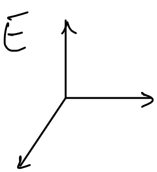
$$a = (a_1, a_2, a_3)^T$$

in F. Then, to describe the motion of the body (assuming CM is fixed at the origin of E) simply express F in terms of E:

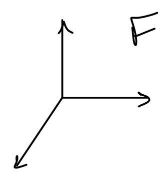
CONVENTION

$$\boxed{F = E\,R(t)} \quad \text{i.e., } (f_1, f_2, f_3) = (e_1\ e_2\ e_3)\,(R),$$

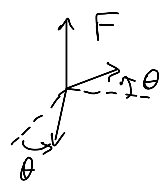
where each column of R consists of E-coordinates of a vector in F. Then, the configuration space coordinate is: $R \in SO(3)$.



$$\begin{pmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \end{pmatrix} \longleftrightarrow$$



$$\begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \longleftrightarrow$$



i.e.: a particle in the body w/ address a (in frame F) then describes a path in E given by $\boxed{x(t) := R(t)\,a}$

It's helpful to view these frames as isomorphisms

$$\left(\mathbb{R}^3, \langle\cdot,\cdot\rangle\right) \underset{E}{\overset{F(t)}{\rightrightarrows}} \left(V, \langle\cdot,\cdot\rangle_V\right)$$

3-dim Euclidean space w/out coords.

CONVENTION

$$\boxed{R(t) := E^{-1} F(t) \in SO(3)}$$

some places use $F^{-1}E$ but this changes all the further left/right conventions.

ANGULAR VELOCITY: Using inertial frame:

$$\Omega_E = E^{-1} \dot{F} F^{-1} E = \dot{R} R^{-1} \in so(3) \simeq \mathbb{R}^3$$

Using the body frame:

$$\Omega_F = F^{-1} \dot{F} F^{-1} F = F^{-1} E E^{-1} \dot{F}$$
$$= R^{-1} \dot{R} \in so(3) \simeq \mathbb{R}^3$$

When the tangent vector $\dot{R} \in T_R SO(3)$ is transported

- by right translation: $\underset{\text{inertial}}{\Omega_E} \in so(3)$
- by left translation: $\underset{\text{Body}}{\Omega_F} \in so(3)$

Thus: $\Omega_E = R \, \Omega_F R^{-1} \xrightarrow[\text{in } \mathbb{R}^3]{\text{as vectors}} \vec{\Omega}_E = R \vec{\Omega}_F$

Upshot: $so(V) \ni$ $\boxed{\Omega = \dot{F} F^{-1}}$

KINETIC ENERGY FUNCTION. $R \in SO(3) \to$ Lag. phase space $= TSO(3)$

So, $(R, \dot{R}) \in T_R SO(3)$, then

$$\mathcal{L} = \mathcal{L}(R, \dot{R}) = T(R, \dot{R})$$

$\mathcal{L}$ should be $\to$ independent of E

$\to$ dependent on F

Prop: The Lagrangian of a free rigid body in $\mathbb{R}^3$ (w/ its CM at origin) is the kinetic energy of a left-invariant metric on $SO(3)$.

This is b/c: if we change inertial frame $E' = EP$, $P \in SO(3)$,

then $F = E'R'$ new coord.

$= EPR' = ER$ old coord $R = PR'$

left multiplication

<u>Note</u>: Change body frame $F' = FQ$, $Q \in SO(3)$

$\Rightarrow F' = ER'$, $FQ = ER'$

$F = ER'Q^{-1} = ER$ old $\rightsquigarrow R = R'Q^{-1}$

Right multiplication

Upshot: Need a left-invariant metric on $SO(3)$.

<u>NOT</u> a bi-invariant one...

i.e., choose <u>ANY</u> pos. def. inner product $\boxed{\kappa(\cdot,\cdot)}$ on $T_e SO(3) = \mathfrak{so}(3) \simeq \mathbb{R}^3$ and left-translate to all other pts

depends on shape of body!

$$\boxed{TSO(3) \ni T(R, \dot{R}) = \frac{1}{2}\kappa\left(R^{-1}\dot{R}, R^{-1}\dot{R}\right)}$$

$\Longrightarrow$ on $\mathfrak{so}(3) \simeq \mathbb{R}^3$, we have 2 inner products:

$$\kappa(\cdot,\cdot) \quad \text{and} \quad \langle \cdot , \cdot \rangle$$

$$\begin{tikzcd} \mathbb{R}^3 \arrow[r, bend left, "\kappa"] \arrow[r, bend right, "\langle\cdot,\cdot\rangle =: \kappa_0"'] & (\mathbb{R}^3)^* \end{tikzcd}$$

$$\kappa(\cdot,\cdot) = \langle I(\cdot), \cdot \rangle$$

where $\boxed{I := \kappa_0^{-1}\kappa}$ is such that $\langle I(\cdot), \cdot \rangle = \langle \cdot, I(\cdot) \rangle$

(moment of inertia tensor of the body)

i.e., $\underbrace{I \text{ is symmetric!}}$

$\Downarrow$ Spectral Theorem

$\exists$ body frame (called "principal frame") in which

$$I = \begin{pmatrix} I_1 & & 0 \\ & I_2 & \\ 0 & & I_3 \end{pmatrix},$$

$I_1, I_2, I_3 \in \mathbb{R}$ ("principal moments" of inertia)

Upshot: Any body has 3 preferred principal axes in its frame.

Thus, in the principle frame:

$$T(R,\dot{R}) = \frac{1}{2}\kappa\left(R^{-1}\dot{R}, R^{-1}\dot{R}\right)$$

$\Omega_F = R^{-1}\dot{R}$

$$\overset{\text{Principal frame}}{=} \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right).$$

LECTURE 22

28/03/2023

EQUATIONS OF MOTION FOR RIGID BODIES

For a rigid body, $F = E\,R(t)$, $\quad \Omega_F = R^{-1}\dot{R} \in so(3) \simeq \mathbb{R}^3$

$$\vec{\Omega}_E = R(\vec{\Omega}_F)$$

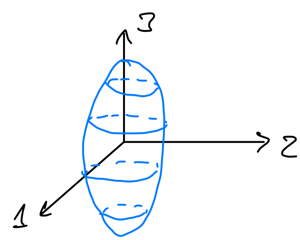
Thus, the (rotational) kinetic energy is given by

$$T(R,\dot{R}) = \frac{1}{2}\kappa\left(R^{-1}\dot{R}, R^{-1}\dot{R}\right) \overset{\text{def}}{=} \frac{1}{2}\left(\underset{\substack{\uparrow \\ \text{Moment of inertia tensor} \\ \text{(symmetric)}}}{I}\vec{\Omega}_F, \vec{\Omega}_F\right)$$

Diagonalize I

$$= \frac{1}{2}\left(I_1\Omega_1^2 + I_2\Omega_2^2 + I_3\Omega_3^2\right)$$

in the principal frame $F_{\text{principal}}$:



Compute I from the mass distribution: Suppose a body $B \subset \mathbb{R}^3$ has mass density $\rho(a)\, da_1 \wedge da_2 \wedge da_3$. Then, the total mass is given by

$$M := \iiint \rho(a) \underbrace{da_1 \wedge da_2 \wedge da_3}_{=: d^3a} .$$

So, a volume element at $a \in B$ has kinetic energy

$$\frac{1}{2}\rho(a)\langle v,v\rangle = \frac{1}{2}\rho(a)\langle \dot{R}a, \dot{R}a\rangle = \frac{1}{2}\rho(a)\langle R^{-1}\dot{R}a, R^{-1}\dot{R}a\rangle$$

$\mathfrak{so}(3) \simeq \mathbb{R}^3$

$$= \frac{1}{2}\rho(a)\langle \Omega_F(a), \Omega_F(a)\rangle$$

$$= \frac{1}{2}\rho(a)\langle \underbrace{\vec{\Omega}_F \times a}, \vec{\Omega}_F \times a\rangle$$

$$\vec{\Omega}_F \times a = -a \times \vec{\Omega}_F = -E_a(\vec{\Omega}_F)$$

$$E_a = a_1\begin{pmatrix} & & \\ & & -1 \\ & 1 & \end{pmatrix} + a_2\begin{pmatrix} & & 1 \\ & & \\ -1 & & \end{pmatrix} + a_3\begin{pmatrix} & -1 & \\ 1 & & \\ & & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix}$$

$$\Rightarrow T = \frac{1}{2}\int \rho(a)\underbrace{\langle E_a(\vec{\Omega}_F), E_a(\vec{\Omega}_F)\rangle}_{(E_a(\vec{\Omega}_F))^T E_a(\vec{\Omega}_F) = \Omega_F^T(E_a^T E_a)\vec{\Omega}_F}\, d^3a$$

Thus,

$$\boxed{I = \int \rho(a)\, E_a^T E_a\, d^3a}$$

i.e.,

$$\boxed{I = \int \rho(a)\begin{pmatrix} a_2^2 + a_3^2 & -a_1a_2 & -a_1a_3 \\ -a_1a_2 & a_1^2 + a_3^2 & -a_2a_3 \\ -a_1a_3 & -a_2a_3 & a_1^2 + a_2^2 \end{pmatrix} d^3a}$$

Equations of Motion: for $T = \frac{1}{2}\langle I\Omega_F, \Omega_F\rangle$, the geodesic flow is given on $(\mathfrak{so}(3), \kappa)$ and defined by the following equations:

left-invariant Riem. metric

$$\boxed{I\dot{\vec{\Omega}}_F = (I\vec{\Omega}_F)\times\vec{\Omega}_F} \quad (A)$$

In momentum space, $\vec{J}_F := I\vec{\Omega}_F$, thus

$\vec{\Omega}_E = R\vec{\Omega}_F$
$\vec{J}_E = R\vec{J}_F$

$$\boxed{\dot{J}_F = [I^{-1}J_F, J_F]}$$

$[\cdot,\cdot]$ Lie bracket

Expanding (A), we obtain

$$\begin{cases} I_1\dot{\Omega}_1 = (I_2 - I_3)\,\Omega_2\Omega_3 \\ I_2\dot{\Omega}_2 = (I_3 - I_1)\,\Omega_3\Omega_1 \\ I_3\dot{\Omega}_3 = (I_1 - I_2)\,\Omega_1\Omega_2 \end{cases}$$

(Euler Equations)

LECTURE 23

INTEGRABLE SYSTEMS

02/04/2024

Def: Let (V^{2n}, ω) be a symplectic vector space. A subspace $U \subset V$ is

- **isotropic**: when $\omega|_U \equiv 0$ (i.e., $\omega(u_1, u_2) = 0 \ \forall u_1, u_2 \in U$)

e.g.: $\mathrm{span}(u)$, $\omega(u,u) = 0$

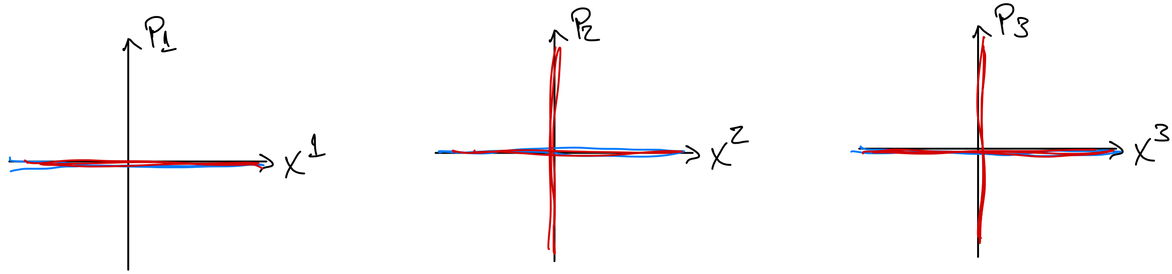
$\mathrm{span}(\partial_{x^1}, \partial_{x^2})$, $\mathrm{span}(\partial_{x^1}, \partial_{x^2}, \partial_{x^3})$.

• **Lagrangian**: when it is isotropic of maximal dimension (i.e., $\dim U = n$).

$\iff U^{\perp\omega} = U$, where

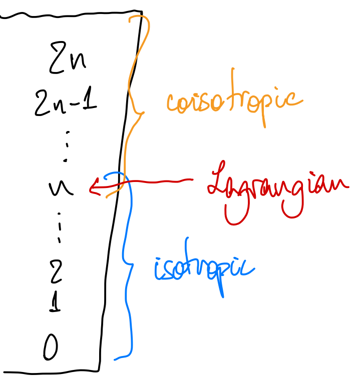
$$U^{\perp\omega} := \{ v \in V : \omega(v,u) = 0 \ \forall u \in U \} \quad \left(\begin{smallmatrix}\text{symplectic}\\ \text{orth. compl.}\end{smallmatrix}\right)$$

• **coisotropic**: when $U^{\perp\omega}$ is isotropic



$$\omega = dp_1 \wedge dx^1 + dp_2 \wedge dx^2 + dp_3 \wedge dx^3$$

$U^{\perp\omega}$ of $U = \text{span}\, \partial_{x^1}$



Def: $Y \subset (X^{2n}, \omega)$ submnfd is called

• *isotropic*: when $T_pY \subset T_pX$ $\forall p$ is isotropic

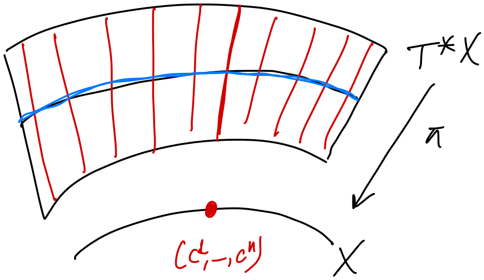
• *Lagrangian*: "

• *coisotropic*: "

Ex: T^*X, $\omega = dp_i \wedge dx^i$. The cotangent fibers $x^1 = c^1, \dots, x^n = c^n$

$$\begin{array}{c} T^*X \\ \downarrow \pi \\ X \end{array} \quad (x^1, \dots, x^n)$$

• $T(T^*_c X) = \text{span}\left(\frac{\partial}{\partial p_1}, \dots, \frac{\partial}{\partial p_n}\right)$



form a Lagrangian foliation

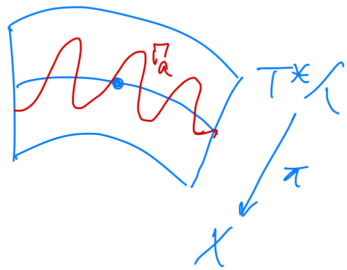
- Zero section
  $Z := \{p_1 = \dots = p_n = 0\}$
  $TZ = \mathrm{span}\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n}\right)$

$$\Omega^1(X) \ni a = a_1 dx^1 + \dots + a_n dx^n$$

The graph of a $\Gamma_a := \{p_1 = a_1(x), \dots, p_n = a_n(x)\}$

is Lagrangian $\Longleftrightarrow$ $\boxed{da = 0}$

(basically a PDE)



When solving for integral curves of Ham. system

$$((M,\omega),\ H \in C^\infty(M,\mathbb{R})) \rightsquigarrow X_H := -(\omega^\#)^{-1}(dH)$$

Helpful to find other conserved quantities: $J, K, \dots$

i.e., $0 = \{J, H\} = \{K, H\} = \dots$

Note: $\{f, g\} = \omega(X_f, X_g)$. So,

1) $\underbrace{\{f, g\} = 0}_{\text{Poisson commuting}} \Longleftrightarrow \underline{\omega(X_f, X_g) = 0}$ some form of isotropy!

2) $[X_f, X_g] = X_{\{f,g\}} \rightsquigarrow \{f, g\} = 0 \Longrightarrow [X_f, X_g] = 0.$

Def: A system $((M,\omega), H)$ is **completely integrable** when there exist functions $f_1 := H, f_2, \dots, f_{\cancel{max}\,n}$ s.t.

1) $\{f_i, f_j\} = 0 \ \forall i,j$ (f's are involutions)

2) $(df_1, \dots, df_{\cancel{max}\,n})$ is linearly indep; i.e., $df_1 \wedge \dots \wedge df_{\cancel{max}\,n}$ is nowhere zero.

Consequence: 1) $\Rightarrow \omega(X_{f_i}, X_{f_j}) = 0 \ \forall i,j$ (isotropic)

2) $\Rightarrow (X_{f_1}, \dots, X_{f_{max}})$ lin. indep.

Thus, $max = \frac{1}{2}\dim M = \frac{1}{2} 2n = n$

$\boxed{\text{Ex}}$: $U \subset X$, $(x^1, \dots, x^n)$

$$\underset{x^1,\dots,x^n}{T^*U = U} \times \underset{p_1,\dots,p_n}{\mathbb{R}^n}$$

$$\left.\begin{aligned} H &= x^1 \\ f_2 &= x^2 \\ &\vdots \\ f_n &= x^n \end{aligned}\right\} \text{completely integrable system}$$

Consequence: Ham. vec. fields of $f_1, \dots, f_n$

- span a Lagrangian subspace at each pt.
- commute (i.e., $[X_{f_i}, X_{f_j}] = 0$)

$\Rightarrow$ Lagrangian foliation by flows!

$$p \longmapsto \phi_{f_n}^{t_n} \cdots \phi_{f_1}^{t_1}\, p$$

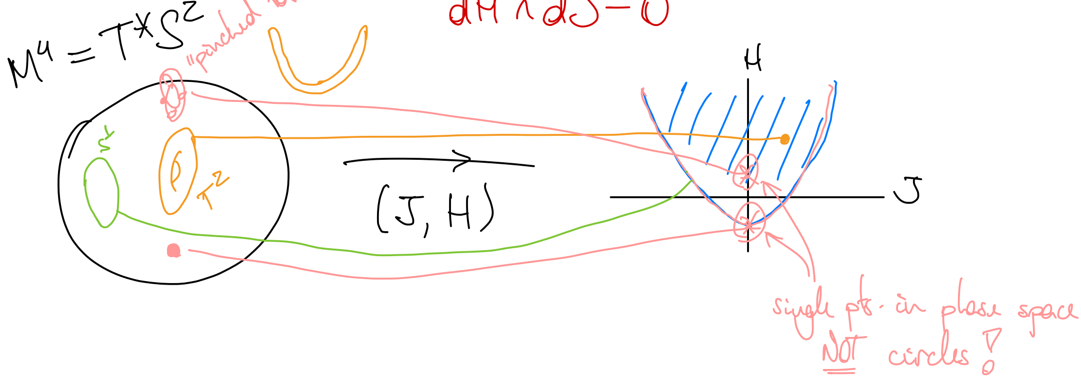
$$\mathbb{R}^n \longrightarrow (M, \omega)$$

Ex: Spherical pendulum $T^*S^2 = (M^4, \omega)$

$$H = KE + PE \qquad \text{s.t.} \qquad \{J, H\} = 0$$
$$J = M_z$$

Completely integrable system away from the pts. where

$$dH \wedge dJ = 0$$



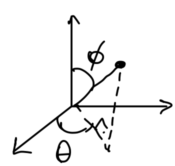
LECTURE 24

QUANTIZATION

04/04/2024

Ex: $(S^2, \omega = \sin\phi \, d\theta \wedge d\phi)$ Symplectic 2-mnfd

S^1-action on S^2 by rotation by z-axis:



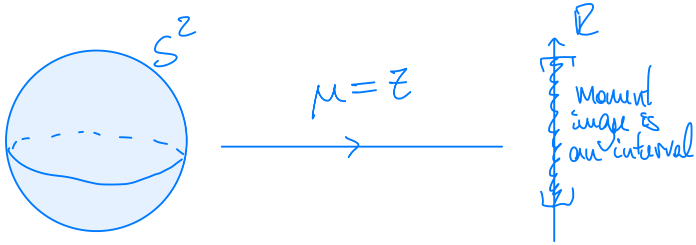
$$\rho: S^1 \times S^2 \longrightarrow S^2$$
$$(e^{it}, (\theta, \phi)) \longmapsto (\theta + t, \phi)$$

Vector field generating the flow $X = \frac{\partial}{\partial \theta}$.

Thus, the Ham. vector field is:

$$\omega\left(\frac{\partial}{\partial\theta}\right) = \sin\phi\, d\phi = -d(\underbrace{\cos\phi}_{\text{Ham. fct.}})$$

Upshot: S^1-action is Hamiltonian with moment map $\boxed{\cos\phi}$.



Height fct. z = conserved quantity

ACTION-ANGLE COORDINATES: instead of (θ,ϕ), use (θ, z). Then

$$\omega = dz \wedge d\theta.$$

$\{f^i, f^j\} = 0$

Thm: (Liouville) Let $f^1, \ldots, f^n$ be fcts. in INVOLUTION on (M^{2n}, ω) and s.t.

$$df^1 \wedge \cdots \wedge df^n$$

is nonvanishing. Then, $\forall p \in M$, $\exists U$ and fcts $a_1, \ldots, a_n$ s.t.

"angle variables"

$$\omega = df^i \wedge da_i$$

local theorem

Completely Integrable System

$\hookrightarrow$ action variables are fixed (i.e., conserved) motion is just linear motion in $a_1, \ldots, a_n$.

Ex: $\Big(\underset{(\theta_1,\phi_1)}{S^2} \times \underset{(\theta_2,\phi_2)}{S^2},\ C\sin\phi_1\, d\theta_1 \wedge d\phi_1 + D \sin\phi_2\, d\theta_2 \wedge d\phi_2\Big)$

4-symplectic mnfd. Two symmetries:

$$e^{it_1}(\theta_1,\phi_1,\theta_2,\phi_2) = (\theta_1+t_1,\phi_1,\theta_2,\phi_2)$$

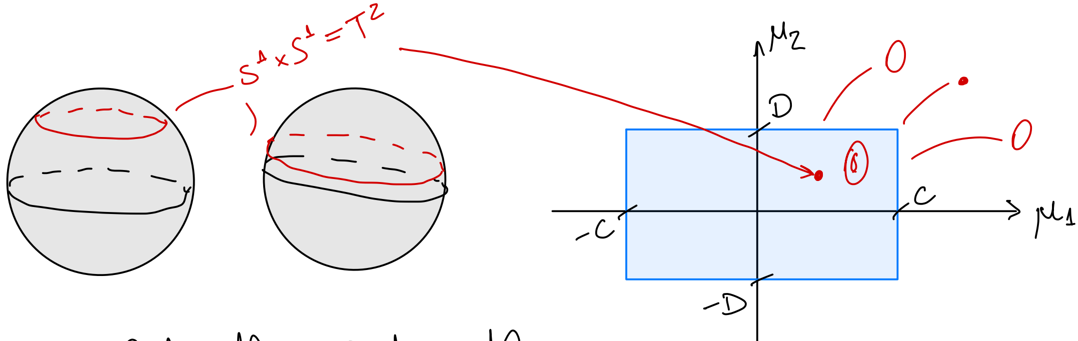
$$e^{it_2}(\theta_1,\phi_1,\theta_2,\phi_2) = (\theta_1,\phi_1,\theta_2+t_2,\phi_2)$$

$\Rightarrow$ 2 commuting S^1 symmetries

i.e., $(S^1\times S^1) \circlearrowright (S^2\times S^2, \omega)$

Moment Map: $S^2\times S^2 \xrightarrow{(\mu_1,\mu_2)} \mathbb{R}^2$

$$(\theta_1,\phi_1,\theta_2,\phi_2) \longmapsto (C\underset{\cos\phi_1}{z_1}, D\underset{\cos\phi_2}{z_2})$$



$$\omega = C\,dz_1\wedge d\theta_1 + D\,dz_2\wedge d\theta_2$$

Note: $\omega(\partial_{\theta_1},\partial_{\theta_2}) = 0 \longrightarrow$ Lagrangian fibration

---

PREQUANTIZATION PROBLEM. (M,ω) symplectic. Can we find a prequantum line bundle for ω?

L^{2n+2} bundle of 1-dim vec. space$/\mathbb{C}$

$\pi\downarrow$ i.e., $\forall p\in M\ \exists U$ nbhd s.t.

M^{2n} $\quad\pi^{-1}(U)\simeq U\times\mathbb{C}$

isom. $\searrow$ $\swarrow$ isom.

$$U$$

Equip L w/

1) Hermitian inn. product h ← "unitary connection"
2) Connection preserving h $\nabla : \Gamma(L) \to \Gamma(T^*M \otimes L)$

<u>Curvature of ∇</u>: $F(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

<u>Obs.</u> $\frac{F}{2\pi i} \in \Omega^2(M, \mathbb{R})$.

If $\boxed{\frac{F}{2\pi i} = \omega}$, then "$\omega$ is prequantized by (L, h, ∇)"

i.e., $\in \mathbb{Z}$

<u>Thm</u>: (Kostant) ω can be prequantized iff it has integral 2d area on any compact 2-cycle

$$\iff [\omega] \in H^2(M, \mathbb{Z})$$