LECTURE 1

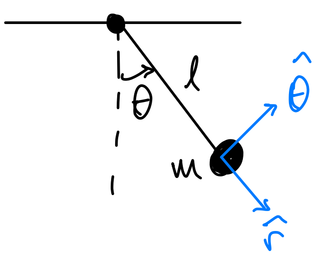
08/09/2023

INTRODUCTION

Prob = 0 to find an ODE w/ solution expressible in elementary terms.

- How to plot phase diagrams? Dimensions?
- In this course, we'll do ODEs w/ INDEPENDENT VARIABLE: TIME t.

* EXAMPLE 1: PENDULUM



ODE: $\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2 \underbrace{\sin\theta}_{\text{nonlinear}} = 0$

$\left(m\vec{a} - \sum \vec{F}\right)\cdot\hat{\theta} = 0$

$\vec{F}_{\text{friction}} = -b\vec{v}$, $\gamma = \dfrac{b}{2m\ell}$

This is a 2-dimensional problem!

Define $x_1 = \theta$, $x_2 = \dot{\theta}$. Then we get the following system:

$$! \quad \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -2\gamma x_2 - \omega_0^2 \sin x_1 \end{cases}$$

* **OBS**: In general, we can always write equations of the form

$$\frac{d^n x}{dt^n} + \dots = 0$$

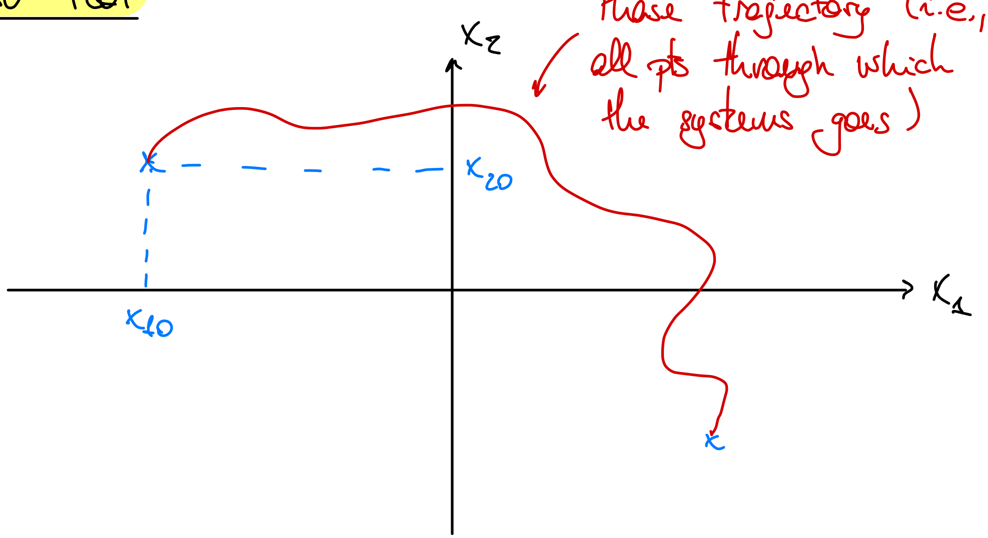
as a system

This is how we solve problems in the computer

$$\begin{cases} \dot{x}_1 = f_1(x_1, \_\_, x_n) \\ \dot{x}_2 = f_2(x_1, \_\_, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \_\_, x_n) \end{cases}$$

n-dimensional problem $\Rightarrow$ n axis in phase plot.

* **PHASE PLOT**:



Write

$$\vec{x} := \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \Longrightarrow \dot{\vec{x}} = \vec{f}(\vec{x}) \quad \text{with} \quad \vec{f} := \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}$$

state vector — "velocity"

* **EXAMPLE 2**: SHO (e.g., when pendulum becomes linear and undamped: $\theta \ll 1 \Rightarrow \sin\theta \approx \theta$ and $\gamma = 0$). Then, we have

ODE: $\ddot{\theta} + \omega_0^2 \theta = 0$

$$\Rightarrow \begin{cases} \theta(t) = \theta_0 \cos(\omega_0 t - \phi) \\ \dot{\theta}(t) = -\omega_0 \theta_0 \sin(\omega_0 t - \phi) \end{cases}$$

So,

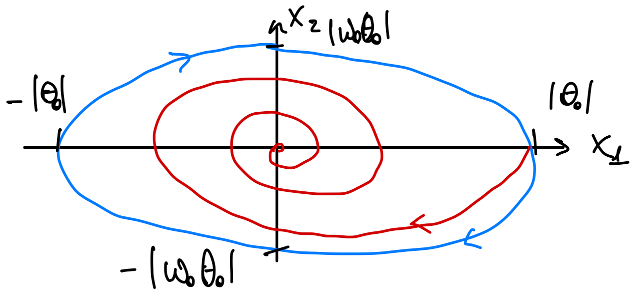
$$\theta^2 + \frac{\dot{\theta}^2}{\omega_0^2} = x_1^2 + \frac{x_2^2}{\omega_0^2} = \theta_0^2 \, .$$

Eq. of ellipse

Phase plot:

Undamped

Damped



Obs: Closed $\iff$ Periodic

Now, **WITH DRIVING** (and undamped),

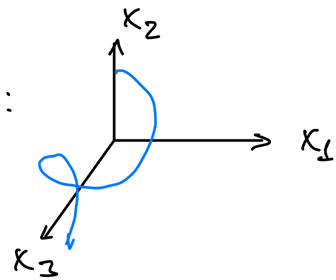
$$\ddot{\theta} + \omega_0^2 \sin\theta = \beta \cos(\omega t)$$

State vector is $\vec{x} := \vec{f}(\vec{x}, t)$

time is a variable now

$$\Rightarrow \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\omega_0^2 \sin x_1 + \beta \cos x_3 \\ \dot{x}_3 = \omega \end{cases}$$

Thus, phase plot is 3-dim.:



LECTURE 2

1-DIM. SYSTEMS

13/09/2023

RECALL:

$$\begin{cases} \dot{x}_1 = f_1(x_1, \ldots, x_n) \\ \vdots \\ \dot{x}_n = f_n(x_1, \ldots, x_n) \end{cases}$$

* **1 - DIMENSIONAL SYSTEMS**: Suppose we have $\dot{x} = f(x)$: "autonomous system" $\Leftrightarrow$ no t dependence in f.

$$\ddot{\theta} + 2\gamma\dot{\theta} + \omega_0^2 \sin\theta = 0$$

where γ is very very big (hyper damping).

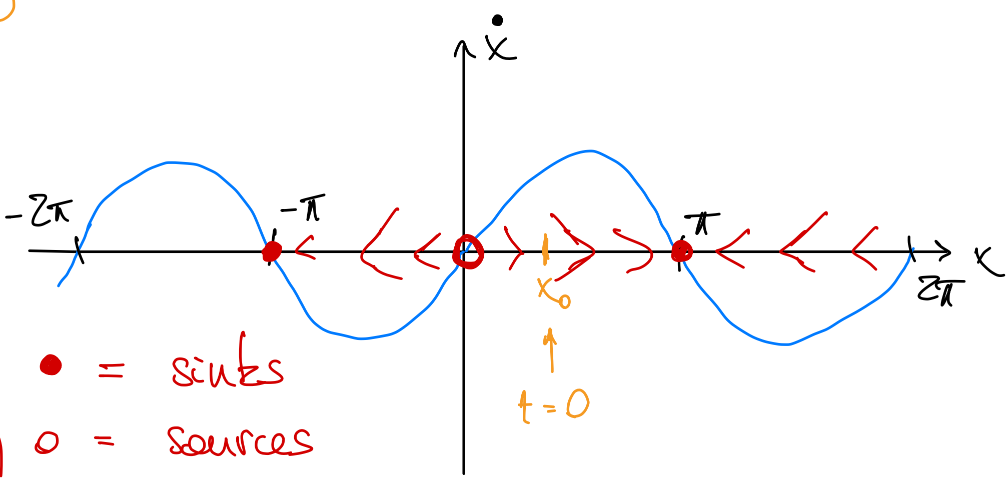
Thus, the ODE takes the form of

$$\ddot{\theta} \simeq -\frac{\omega_0^2}{2\gamma}\sin\theta.$$

* **EXAMPLE**: $\boxed{\dot{x} = \sin x} =: f(x)$

$$t = \log\left|\frac{\frac{1}{\sin x_0} + \frac{1}{\tan x_0}}{\frac{1}{\sin x} + \frac{1}{\tan x}}\right|$$

Say $x(0) =: x_0$



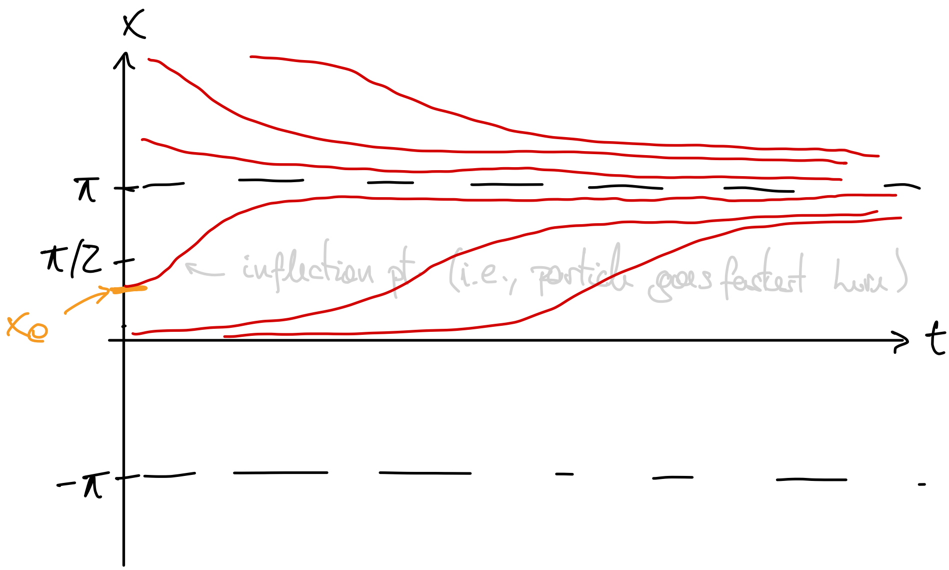
$\left\{\begin{array}{l} \bullet = \text{sinks} \\ \circ = \text{sources} \end{array}\right.$

Def: x^* such that $f(x^*) = 0$ are called fixed points.

→ Can be stable fixed pts. (sinks)

→ Can be unstable fixed pts. (sources)

PICTORIALLY:

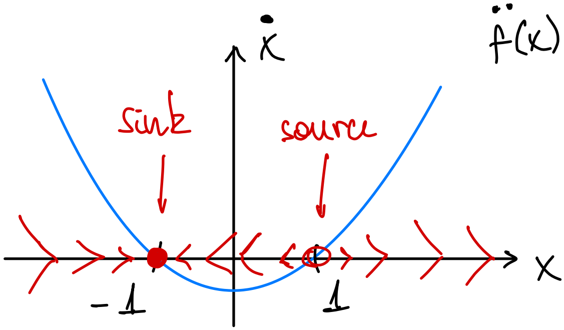


* NOTATION: $x(t) \rightsquigarrow \varphi$ trajectory

point $\rightsquigarrow \varphi$ point.

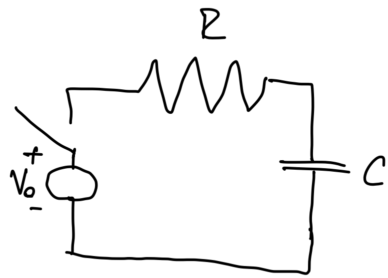
many trajectories $x(t) \rightsquigarrow \varphi$ portrait

EXAMPLE: $\dot{x} = \underbrace{x^2 - 1}_{= f(x)}$. First: always plot $f(x)$



fixed points: $\{\pm 1\}$.

EXAMPLE: RC - circuit



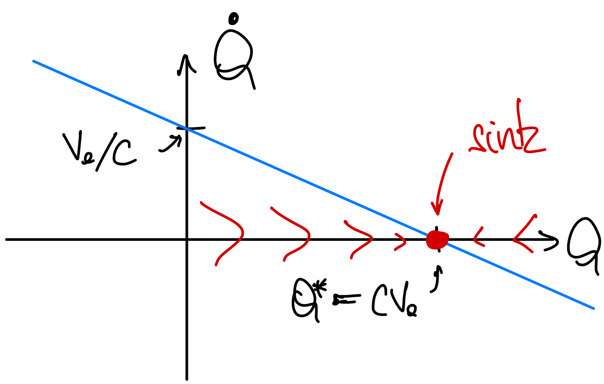
- Voltage across resistor is:

$$RI = R\dot{Q}$$

- Capacitor: Q/C.

From Kirchhoff's law, we obtain

$$R\dot{Q} + \frac{Q}{C} - V_0 = 0 \iff \boxed{\dot{Q} = \frac{V_0}{R} - \frac{Q}{RC}}$$

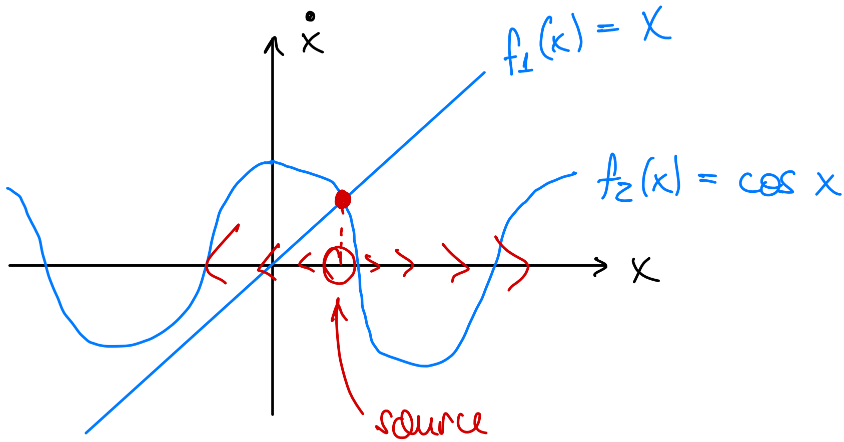


fixed pt.:

$$Q^* = CV_0$$

EXAMPLE: $\dot{x} = x - \cos x =: f(x)$

Fixed pts.: $x - \cos x = 0$



$*$ **LINEAR STABILITY ANALYSIS:** The goal is to study behaviour near equilibria. For that, let $\varepsilon := x - x^*$, where x^* is a fixed pt. of the ODE. Thus,

$$\dot{x} = \dot{\varepsilon} - f(x^* + \varepsilon)$$

Assume it's small & smooth enough to Taylor expand

$$= f(x^*) + \varepsilon f'(x^*) + O(\varepsilon^2)$$

So, at the end, we have the following equation:

$$\dot{\varepsilon} = \sigma \varepsilon, \quad \sigma := f'(x^*)$$

$$\Longrightarrow \quad \varepsilon(t) = \varepsilon_0 e^{\sigma t}$$

Upshot:

- if $\sigma = f(x^*) > 0 \Rightarrow$ growth (source)
- if $\sigma = f(x^*) < 0 \Rightarrow$ decay (sink)

EXAMPLE: $\boxed{\dot{x} = \sin x}$.

fixed pts.: $x_n^* = n\pi, \ n \in \mathbb{Z}$.

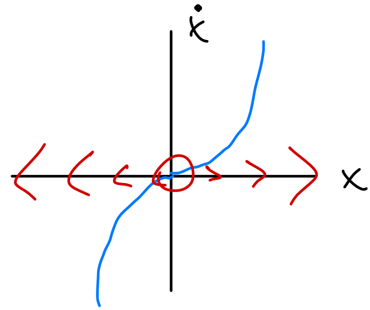
So,

$$f'(x_n^*) = \cos(n\pi) = (-1)^n .$$

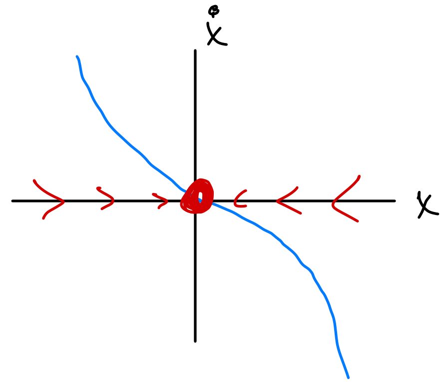
This means that

$$\begin{cases} n \text{ even} \rightsquigarrow \sigma = 1 & \text{(source)} \\ n \text{ odd} \rightsquigarrow \sigma = -1 & \text{(sink)} \end{cases}$$

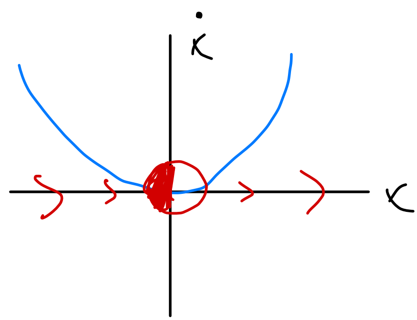
EXAMPLE: $\dot{x} = x^3$



EXAMPLE: $\dot{x} = -x^3$



EXAMPLE: $\dot{x} = x^2$

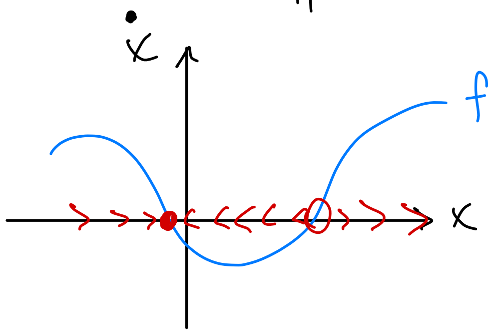


Lecture 3

15/09/2023

1-Dim. Flows

Recall: Suppose $\dot{x} = f(x)$, where



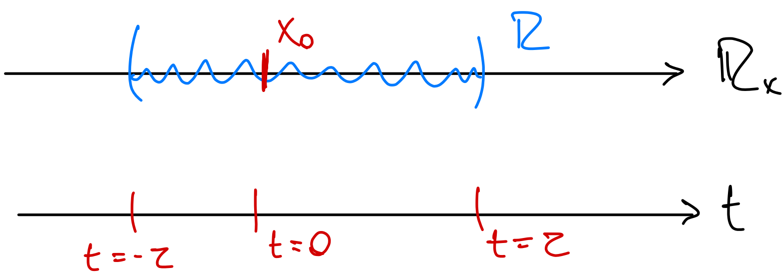
Say we have an initial condition

$x_0 := x(0) \in R \subset \mathbb{R}$

open

$$(\text{IVP}) \qquad \dot{x} = f(x), \quad x(0) = x_0$$

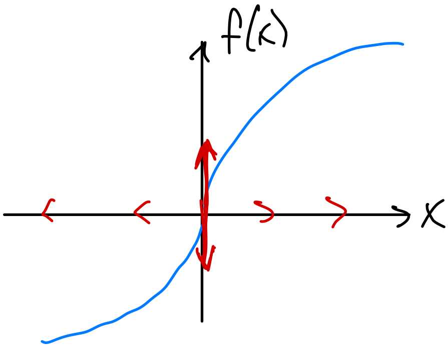
Thm: If f and f' are continuous, then there exists a unique solution to (IVP) on some time interval $(-\tau, \tau)$

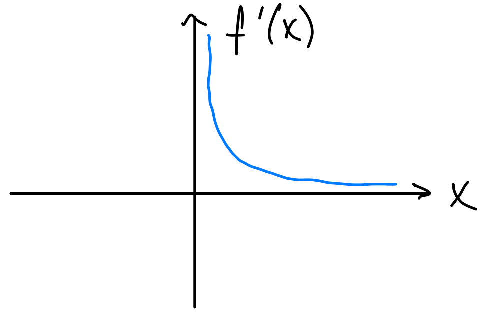


Ex: $\dot{x} = x^{1/3}$.

$f(x) = x^{1/3}$

$f'(x) = \frac{1}{3} x^{-2/3}$





Say $x_0 = x(0) = 0$. (IVP)

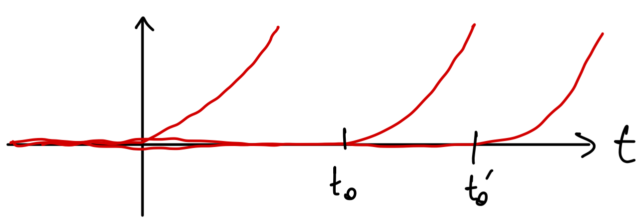
Then $x(t) = 0$ is a solution.

But also:

$$\frac{dx}{dt} = x^{1/3} \rightsquigarrow \int_0^x x^{-1/3}\,dx = \int_0^t dt$$

$$\Rightarrow \quad x(t) = \left(\frac{2}{3}t\right)^{3/2}$$

We have ∞-many solutions, actually:



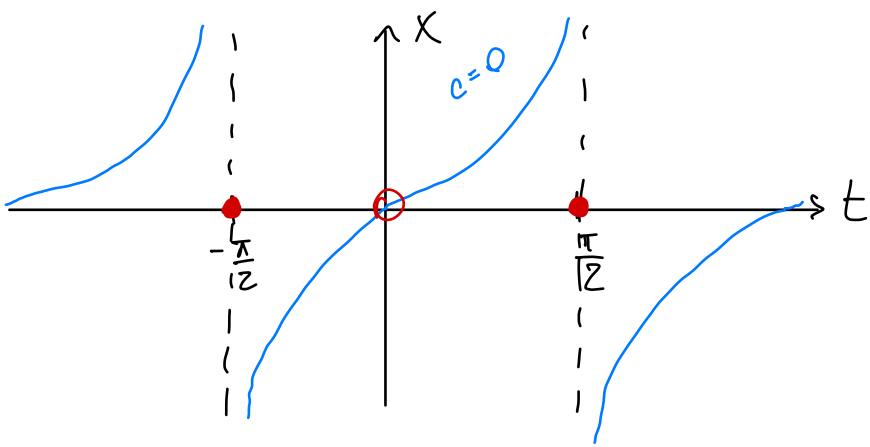
Ex: (IVP) $\begin{cases} \dot{x} = 1 + x^2 & \text{on } R = \mathbb{R} \\ x(0) = x_0 \end{cases}$

Now,

$$\frac{dx}{dt} = 1 + x^2 \rightsquigarrow \int \frac{dx}{1+x^2} = \int dt$$

$$\Rightarrow \quad \arctan x = t + C$$

$$x = \tan(t + C), \quad C \in \mathbb{R}$$

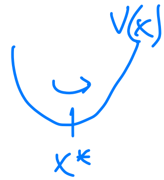


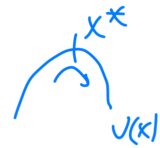
REMARK: 1^{st} order systems don't oscillate between fixed pts..

NOTE: Potentials in 1d systems

POTENTIAL

	$f(x)$	$V(x)$
STABLE FIXED PT.	$f(x^*) = 0$ LSA: $f'(x^*) < 0$	$\left.\frac{dV}{dx}\right\|_{x^*} = 0$ $\left.\frac{d^2V}{dx^2}\right\|_{x^*} > 0$
UNSTABLE FIXED PT.	$f(x^*) = 0$ LSA: $f'(x^*) > 0$	$\left.\frac{dV}{dx}\right\|_{x^*} = 0$ $\left.\frac{d^2V}{dx^2}\right\|_{x^*} < 0$



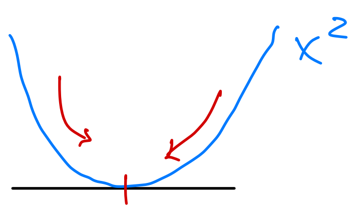


So, we define $\boxed{\dot{x} = f(x) = -\frac{dV}{dx}}$

Ex: Hooke's Law

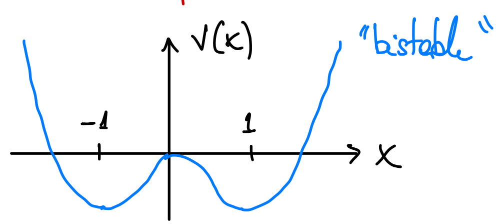
$F = -kx$

$F_{friction} = -bv$



$f(x) = x - x^3$

$V(x) = -\frac{x^2}{2} - \frac{x^4}{4}$



* **SOLVING THIS ON THE COMPUTER**:

Euler Method

$$\dot{x} = \frac{dx}{dt} = \lim_{h \to 0} \frac{x(t+h) - x(t)}{h}$$

$$\Rightarrow x(t+dt) = x(t) + h f(x)$$

Backward Euler Method: $\lim_{h \to 0} \frac{x(t) - x(t-h)}{h}$.

Way better: Runge - Kutta method.

scipy.integrate.solve_ivp

LECTURE 4

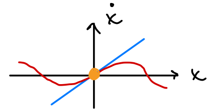
20/09/2023

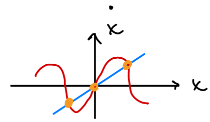
BIFURCATIONS

Poincaré c1895: "Slight change in the parameters leading to an abrupt change in the qualitative behavior of the solution."

$\boxed{\text{Ex}}$: $\dot{x} = x - r\sin x$

$$\begin{cases} r < 1 \rightsquigarrow 1 \text{ fixed point} \\ r > 1 \rightsquigarrow 3 \text{ fixed points} \end{cases}$$



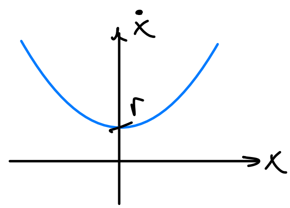


$\Rightarrow$ $r = 1 =: r_c \rightsquigarrow$ BIFURCATION PT.

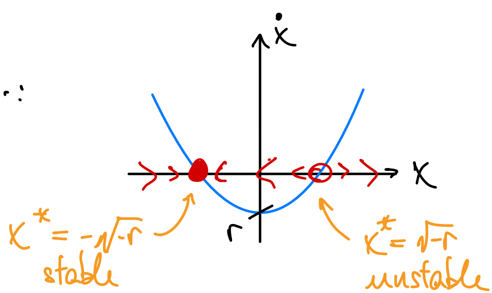
critical

$\boxed{\text{Ex}}$: $\dot{x} = r + x^2$ (SADDLE - NODE BIFURCATION)

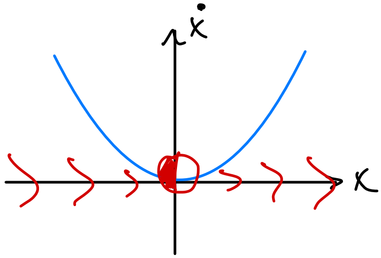
- $\underline{r > 0}$ $\rightsquigarrow$ zero fixed pts



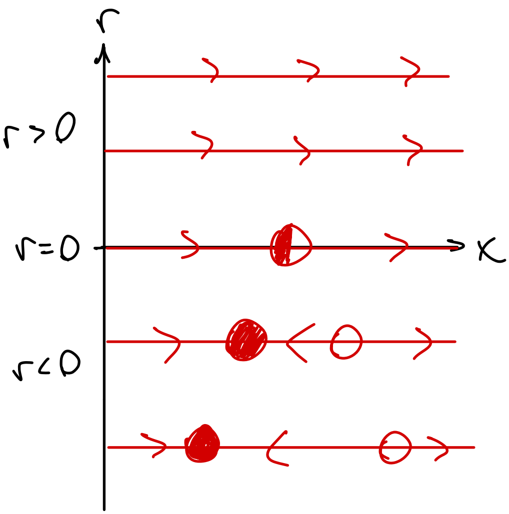
- $\underline{r < 0}$ $\rightsquigarrow$ 2 fixed pts.: $\{\pm\sqrt{-r}\}$



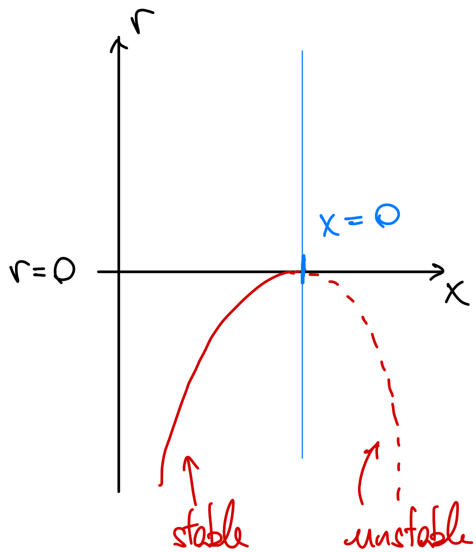
• $\underline{r = 0}$ (critical value) $\rightsquigarrow$ 1 fixed pt. $\{x^* = 0\}$



So, we have the following **BIFURCATION DIAGRAM**:

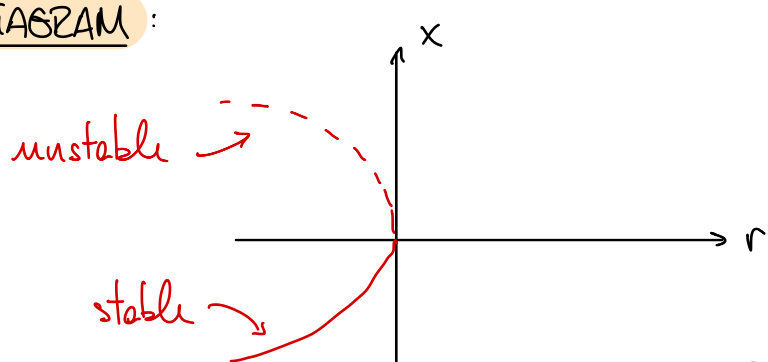


OR $\rightsquigarrow$

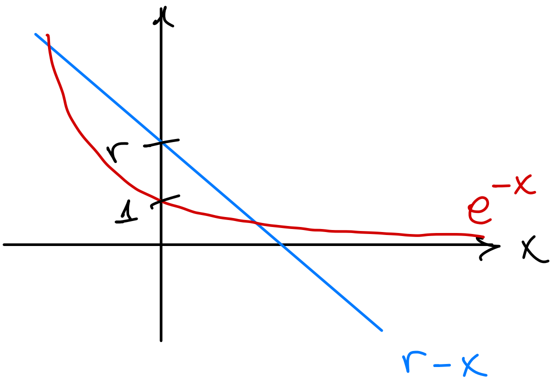


BIFURCATION DIAGRAM:

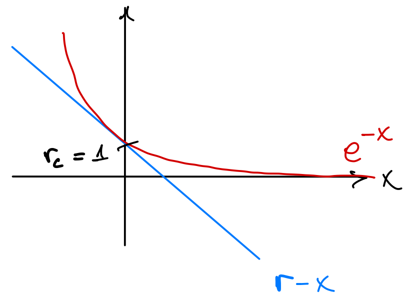
SADDLE - NODE BIFURCATION (i.e., no fixed pt. and then fixed pts appear)



Ex: $\dot{x} = r - x - e^{-x}$



Quite obvious that $r = 1$ is a critical pt..



So, for fixed pts.:

$$r - x_* = e^{-x_*} \implies \boxed{x_* = 0}$$

Now, to find the critical pt. r_c, plug x_* back into the RHS of the ODE:

$$r_c = x_* + e^{-x_*} = 0 + 1 \implies \boxed{r_c = 1}$$

! IMPORTANT !

* GENERAL FORM OF SADDLE-NODE BIFURCATIONS:

Setting $RHS(ODE) =: f(x)$, when $x \approx x_*$ and $r \approx r_c$, we find that

$$f(x,r) = \underbrace{f(x_*, r_c)}_{=0} + (x - x_*) \underbrace{\frac{\partial f}{\partial x}\Big|_{x=x_*, r=r_c}}_{=0}$$

$$+ \frac{1}{2}(x - x_*)^2 \frac{\partial^2 f}{\partial x^2}\Big|_{x=x_*, r=r_c}$$

$$+ (r - r_c) \frac{\partial f}{\partial r}\Big|_{\substack{r=r_c \\ x=x_*}} + O(x^2, r^2)$$

Now, define

$$a := \frac{1}{2} \frac{\partial^2 f}{\partial x^2}\Big|_{\substack{x=x_* \\ r=r_c}}, \quad b := \frac{\partial f}{\partial r}\Big|_{\substack{r=r_c \\ x=x_*}}.$$

Then

$$\boxed{f(x, r_c) = a(x - x_*)^2 + b(r - r_c)}$$

■

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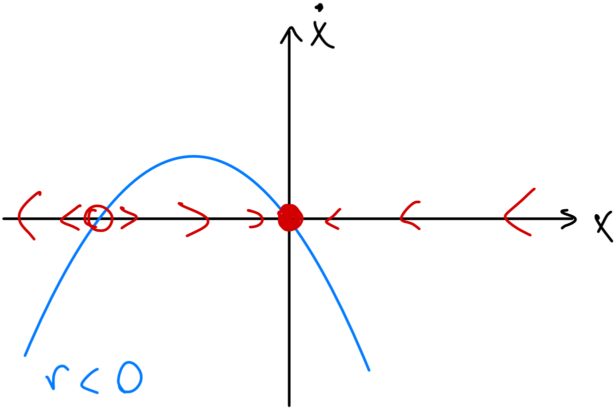
* TRANSCRITICAL BIFURCATION:

Ex: $\dot{x} = rx - x^2 = x(r - x)$.

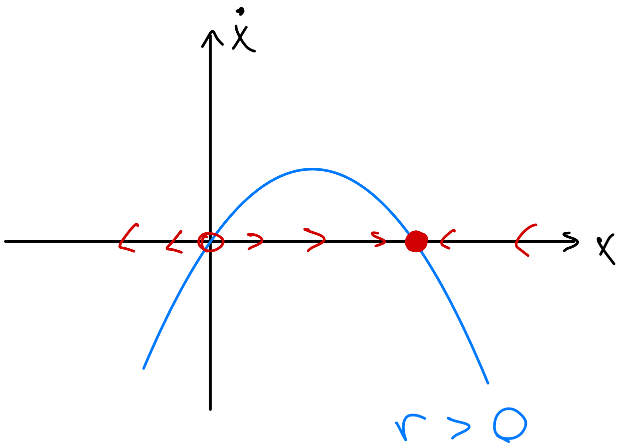
Then, the fixed pts. are

$$x_* = 0 \quad \text{and} \quad x_* = r.$$



$x_* = 0 \rightsquigarrow$ stable

$x_* = r \rightsquigarrow$ unstable

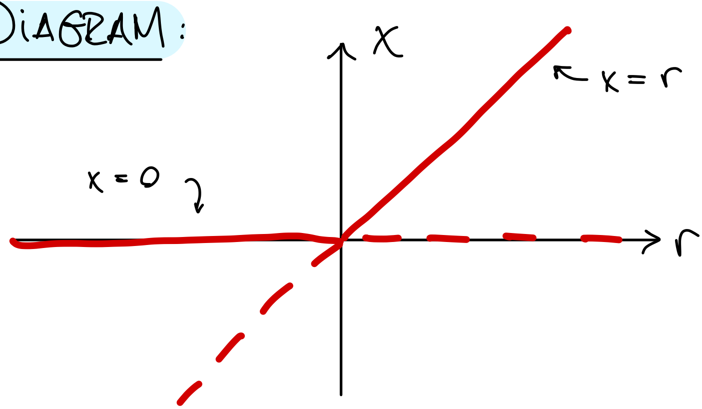


$x_* = 0 \rightsquigarrow$ unstable

$x_* = r \rightsquigarrow$ stable

- BIFURCATION DIAGRAM:

TRANSCRITICAL BIFURCATION



Ex: $\dot{x} = x(1-x^2) - a(1-e^{-bx})$

$$x_*(1-x_*) = 0 \rightsquigarrow e^{-bx} = 1 \checkmark$$

$\Rightarrow$ $x_* = 0$ is always a fixed pt.

So, expand the RHS of ODE:

$$f(x) = x - \cancel{x^3}^{\,\to O(x^3)} - a\left(1 - 1 + bx - \frac{b^2x^2}{2}\right) + O(x^3)$$

$$= (1-ab)x + \frac{ab^2x^2}{2} + O(x^2)$$

Obs: Don't go over order of x^2 since the ODE only goes up to x^2.

LECTURE 5

22/09/2023

BIFURCATIONS (ctd.)

Recall: We saw SADDLE-NODE BIFURCATION (SDB) (fixed pts. either appear suddenly, or annihilate each other). $\rightsquigarrow$ $\dot{x} = r + x^2$ NORMAL FORM OF SNB

We also saw TRANSCRITICAL BIFURCATION (TCB) (2 fixed pts. - one of them always at zero - that swap stability during the bifurcation) ⇝ $\dot{x} = rx - x^2$

NORMAL FORM of TCB

Ex: TRANSCRITICAL BIFURCATION:

$$\dot{x} = r \log x + x - 1$$

$\Rightarrow$ fixed pts: $x_* = 1$ (always a fixed pt.)

$x_* \neq 0$ so, change variables so one of the fixed pts. is at 0

Now, set $u := x - 1 \Rightarrow \dot{u} = r \log(1+u) + u$

Thus, near the bifurcation, we have

$$\log(1+u) \approx u - \frac{u^2}{2}.$$

$$\Rightarrow \dot{u} \approx r\left(u - \frac{u^2}{2}\right) + u$$

$$= (r+1)\,u - \frac{r}{2}u^2$$

Now, set $u := Ax \Rightarrow \cancel{A}\dot{x} = (1+r)\cancel{A}x - \frac{r}{2}A^{\cancel{2}}x^2$

Choose $A = \frac{2}{r}$, $R = 1 + r$ $\Rightarrow$ $\boxed{\dot{x} = Rx - x^2}$

Normal form!

! **IMPORTANT**: Always want to write the ODE in the "normal form" close to the bifurcation.

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* **PITCHFORK BIFURCATION**: (PFB) There are 2 types of PFB

1) **SUPERCRITICAL**: (Super PFB)

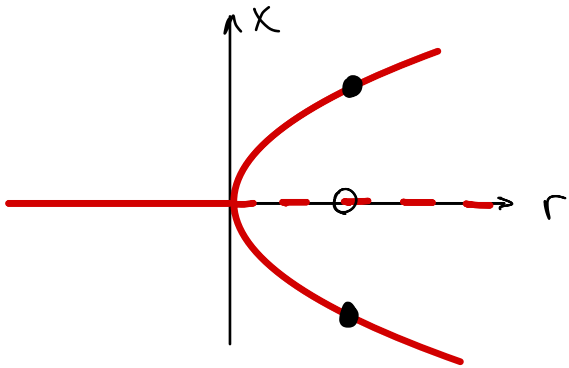
Normal form $\rightsquigarrow$ $\dot{x} = rx - x^3$!

Obs: has the following reflection symmetry $x \mapsto -x$

- $r < 0$: $x_* = 0$ (only stable fixed pt)
- $r > 0$: $x_* = 0$ (unstable fixed pt.)

  $x_* = \pm\sqrt{r}$ (stable fixed pts.)

Bifurcation diagram for SuperPFB:



At the critical pt: $r_c = 0$, then

$$\frac{dx}{x^3} = -dt \rightsquigarrow x = \pm\left(\frac{1}{x_0^2} + 2t\right)^{-1/2} \underset{t \nearrow +\infty}{\sim} \pm\frac{1}{\sqrt{2t}}$$

Ex: (SuperPFB) $\dot{x} = -x + \beta \tanh(x)$

For $\beta > 1$: x_* s.t.

$$x_* = \beta \tanh(x_*)$$

$$\Longrightarrow \boxed{\beta = \frac{x_*}{\tanh(x_*)}}$$

2) SUBCRITICAL: (Sub PFB)

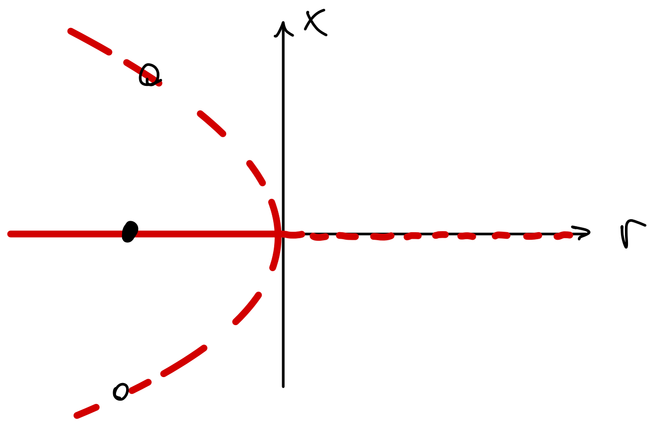
Normal form ⟶ $\boxed{\dot{x} = rx + x^3}$

- $\underline{r < 0}$: $x_* = 0$ (stable fixed pt.)

  $x_* = \pm\sqrt{r}$ (unstable fixed pts.)

- $\underline{r > 0}$: $x_* = 0$ (unstable fixed pt.)

Bifurcation diagram for Sub PFB:



At the critical pt. $r_c = 0$, we have that

$$\frac{dx}{x^3} = dt \rightsquigarrow x = \pm\sqrt{\frac{2x_0^2}{1-2x_0(t-t_0)}} \xrightarrow{t \to t_0 + \frac{1}{2x^2}} +\infty$$

"BLOW UP"

Obs: If $|x_{initial}| > \sqrt{-r}$, then $|x|$ grows and, eventually, $rx \ll x^3 \Rightarrow \dot{x} \approx x^3$.

$\boxed{\text{Ex}}$: $\dot{x} = rx + x^3 - x^5 = x(r + x^2 - x^4)$

$\Rightarrow x_* = 0$ always a fixed pt.

Now, solve the "quadratic":

$$x_*^2 = \frac{1 \pm \sqrt{1+4r}}{2} \Rightarrow x_* = \pm\sqrt{\frac{1 \pm \sqrt{1+4r}}{2}}$$

4 cases to consider

- $r = -1/4$

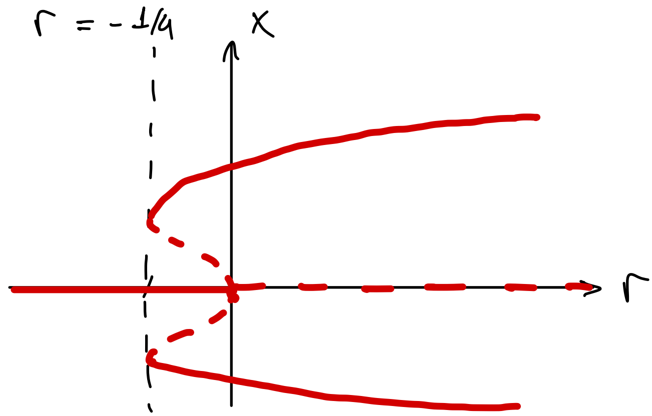
$r < -\frac{1}{4} \Rightarrow \sqrt{1+4r} \notin \mathbb{R} \Rightarrow$ no additional fixed pts.

- $r = 0$

- $r > 0 \Rightarrow x_* = \pm\sqrt{\dfrac{1+\sqrt{1+4r}}{2}}$

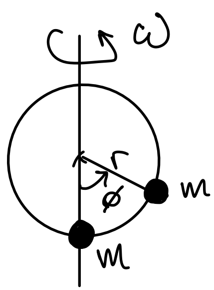
- $-\frac{1}{4} < r < 0$

Bifurcation diagram:



2 saddle-node and
1 subcritical pitchfork bifurcation

Ex:



(ODE) $\underbrace{mr\ddot{\phi}}_{\text{Azimuthal acceleration}} = \underbrace{-b\dot{\phi}}_{\text{Damping}} \underbrace{- mg\sin\phi}_{\text{Weight}}$

$\underbrace{+ mr\omega^2 \sin\phi\cos\phi}_{\text{Centrifugal}}$

Non-dimensionalize this bad boy: say $t = T\tau$.

Then, $\dfrac{mr}{T^2}\dfrac{d^2\phi}{d\tau^2} + \dfrac{b}{T}\dfrac{d\phi}{d\tau} = \ldots$ $\Big\}$ Pitchfork bifurcation.

Lecture 6 — Catastrophes

27/09/2023

Recall: Studying initial value problems of the form

$$\dot{x} = f(x, r), \quad x(0) = x_0 .$$

Parameter $r \in \mathbb{R}$ (can produce bifurcations)

Bifurcations:

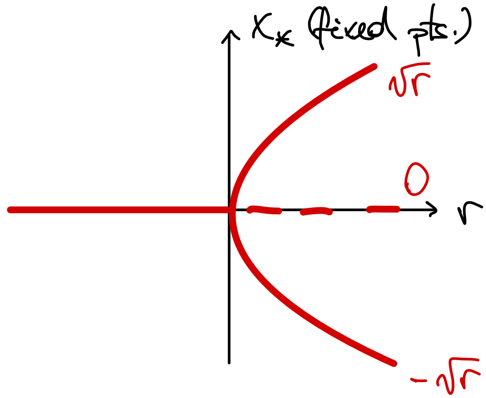
- Saddle-node bifurcation: $\dot{x} = r \pm x^2$
- Transcritical bifurcation: $\dot{x} = rx - x^2$
- Pitchfork bifurcation: $\dot{x} = rx \pm x^3$

---

* **IMPERFECT BIFURCATIONS:** Add an imperfection to the normal forms. Take, for example, the pitchfork bifurcation:

$$\dot{x} = h + rx - x^3$$

$\uparrow$ imperfection



For each value of $r \in \mathbb{R}$, changing the value of h will affect the overall behavior of the solution (e.g., the number / location / stability of fixed pts. change w/ h).

More specifically,

$$f(x_*, r, h) = 0$$

$$\Rightarrow h + rx - x^3 = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = r - 3x^2 = 0$$

$$\Rightarrow \quad x_{max} = \pm\sqrt{\frac{r}{3}} \, .$$

Thus, the "critical value" of h is

$$h_c = -r\sqrt{\frac{r}{3}} + \left(\frac{r}{3}\right)^{3/2}$$

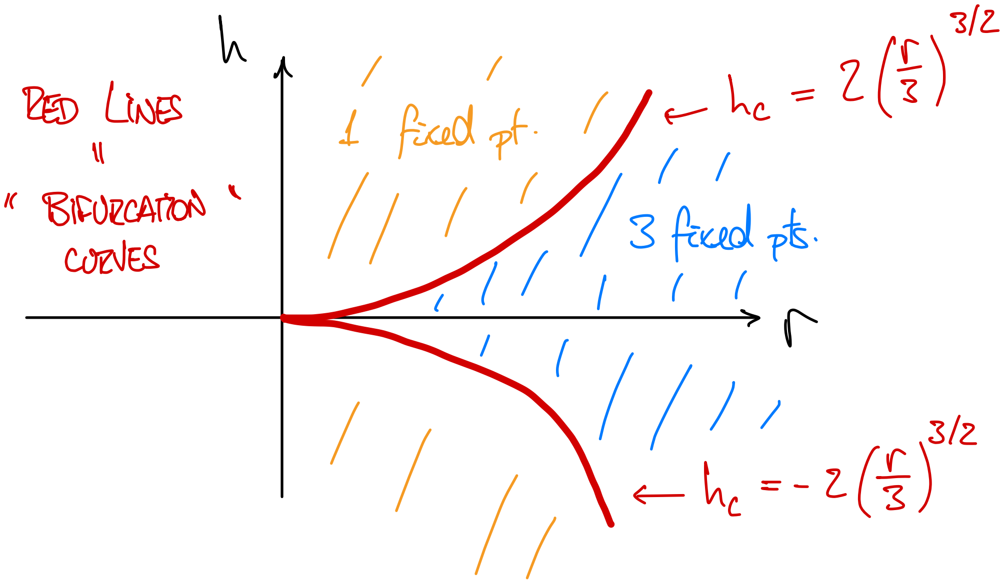
$$= \sqrt{\frac{r}{3}}\left(\frac{r}{3} - r\right)$$

$$= 2\left(\frac{r}{3}\right)^{3/2} .$$

Given $h \in \mathbb{R}$,

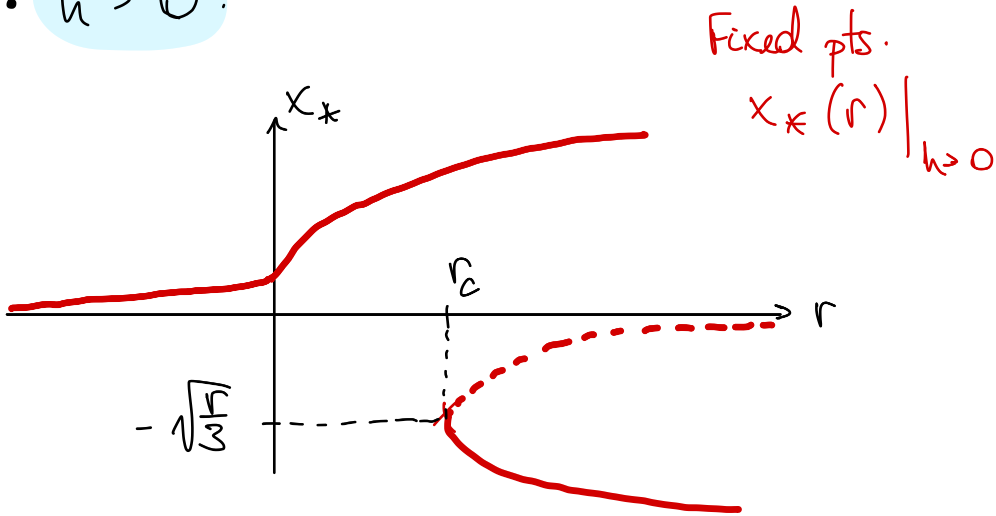
$$r_c = 3\left(\frac{h}{2}\right)^{2/3} .$$

Therefore, we can create a "STABILITY DIAGRAM"



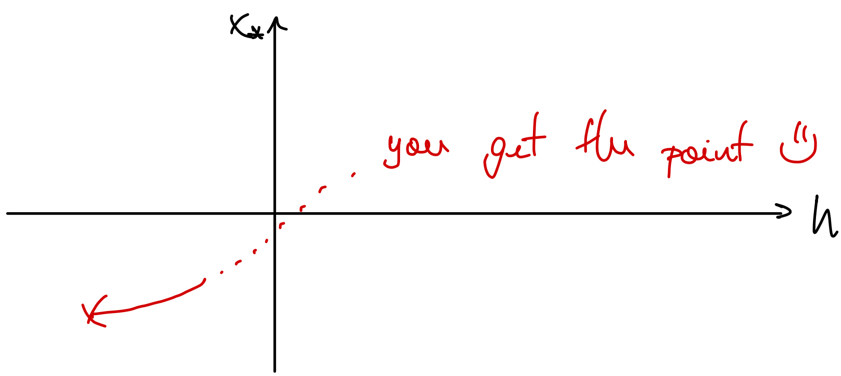
Thus, the bifurcation diagram for a given $h \in \mathbb{R}$ is:

- $h > 0$:



- $h < 0$: Just "flips" the diagram above

Can also plot a bifurcation diagram fixing r and varying h: fix r

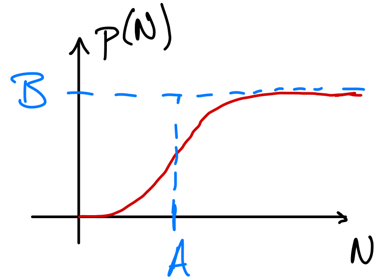


* **INSECT OUTBREAK**: Governed by

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) \quad \left(\text{LOGISTIC EQUATION}\right)$$

WITHOUT PREDATION

Predation: $p(N) = \dfrac{BN^2}{A^2+N^2}$



$N \nearrow \infty \Rightarrow p(N) \approx B$

$$\dot{N} = -B$$

Now, combine the logistic Equation w/ the predation (who let the dogs out?):

$$\dot{N} = RN\left(1 - \frac{N}{K}\right) - \frac{BN^2}{A^2+N^2}$$

WITH PREDATION

↳ 4-dimensional parameter space. But:

$[R] = s^{-1}$; $[K] = \text{bugs}$; $[A] = \text{bugs}$; $[B] = \text{bug}\cdot s^{-1}$

$\Rightarrow$ Effective 2-dimensional problem

(BUCKINGHAM Π THEOREM)

Lecture 7

Flows on the Circle

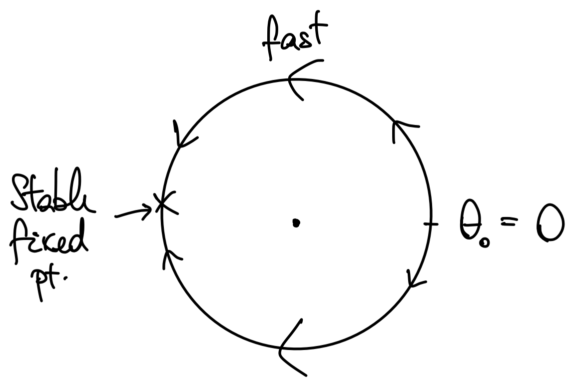
29/09/2023

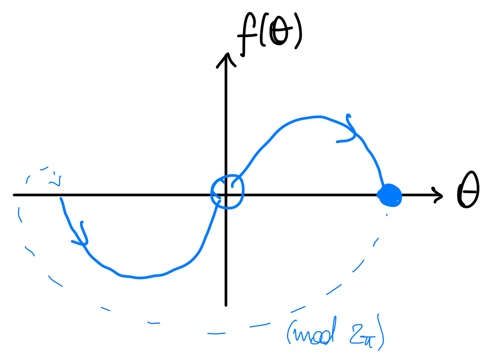
* **Flows on the circle**: for periodic functions of the form

$$\dot{\theta} = f(\theta), \quad \text{where } f(\theta) = f(\theta + 2\pi n) \;\; \forall n \in \mathbb{Z}.$$

Convention: $-\pi \leq \theta < \pi$.

Ex: $\dot{\theta} = \sin\theta$

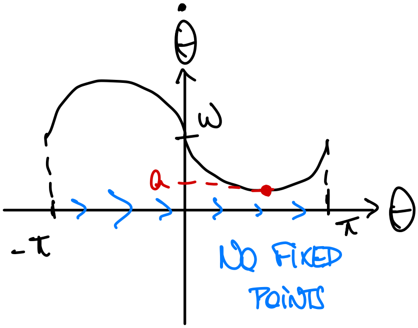


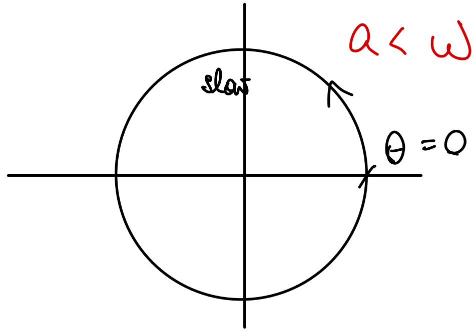


Ex: $\dot{\theta} = \omega \Rightarrow \theta(t) = \omega t + \theta_0$ (uniform oscillator)

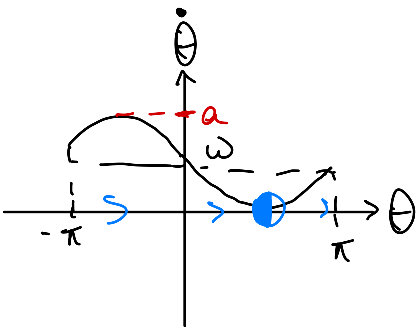
Ex: $\dot{\theta} = \omega - a\sin\theta$, $a > 0$, $\omega > 0$.

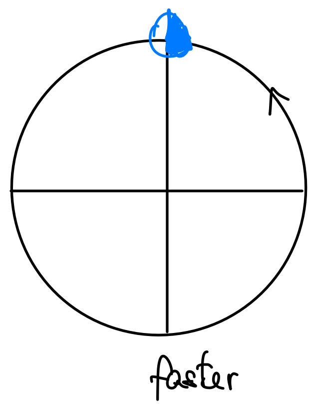
(nonuniform oscillation)



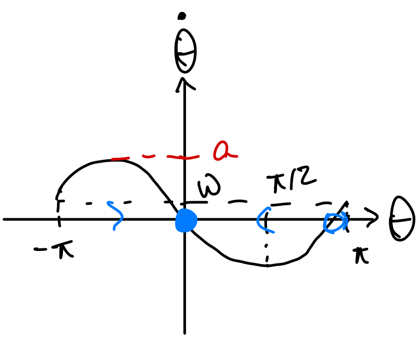


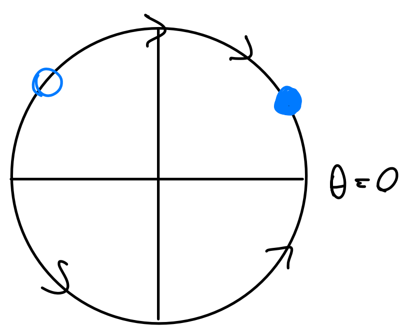
increase a





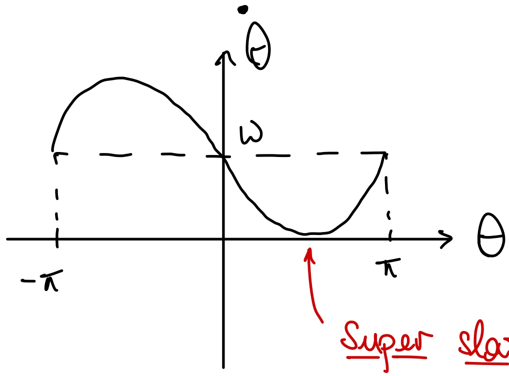
increase a





Saddle-Node Bifurcation

* GHOSTS & BOTTLENECKS: Consider, for example:



Super slow here (near $\pi/2$)

$a < \omega$

$a \approx \omega$

• **Compute the period T:**

$$\frac{d\theta}{dt} = f(\theta) \quad \Longrightarrow \quad \frac{d\theta}{f(\theta)} = dt$$

Then,

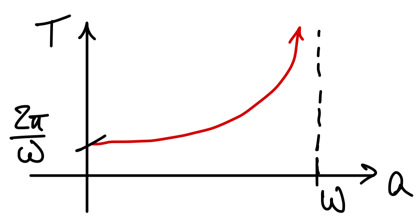
$$T = \int_{-\pi}^{\pi} \frac{d\theta}{\omega - a\sin\theta}$$

$u = \tan\left(\frac{\theta}{2}\right)$

$d\theta = \frac{2du}{1+u^2}$, $\sin\theta = \frac{2u}{1+u^2}$

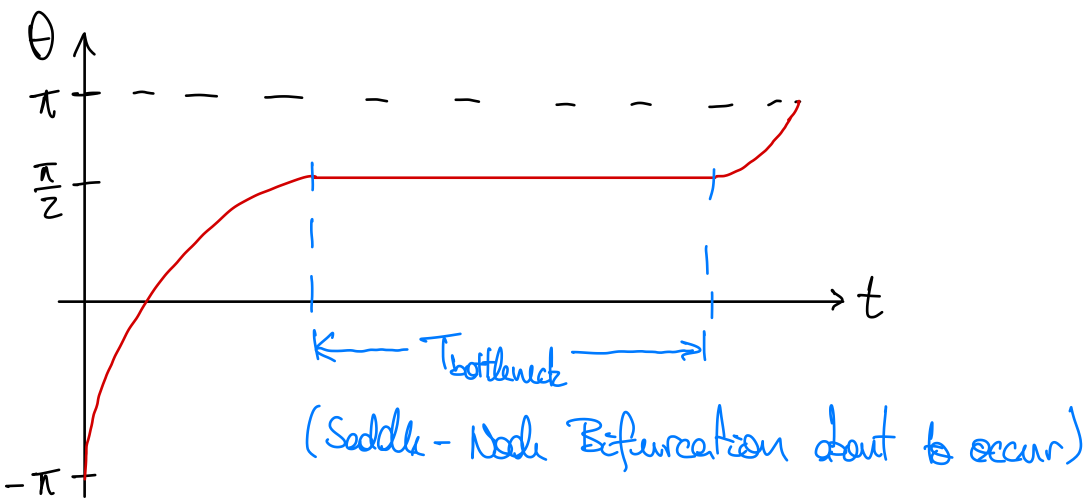
"GHOST OF SADDLE-NODE BIFURCATION"

$$\Rightarrow \boxed{T = \frac{2\pi}{\sqrt{\omega^2 - a^2}}}$$



Then, we observe that the particle spends a lot of time near what would be a stable fixed pt. (if we slightly changed the parameter).

Thus:



$$a \approx \omega \Rightarrow T = \frac{2\pi}{\sqrt{\omega^2-a^2}} = \frac{2\pi}{\sqrt{(\omega-a)(\omega+a)}}$$

$(\omega-a)(\omega+a) \approx 2a(\omega-a)$

$$\Rightarrow \boxed{T \approx \frac{2\pi}{\sqrt{2a(\omega-a)}}}.$$

Thus

$$f(\theta) = \omega - a \sin\left(\frac{\pi}{2} + \phi\right)$$

$$= \omega - a\cos\phi$$

$$\approx \omega - a\left(1 - \frac{\phi^2}{2}\right)$$

$$= \omega - (\omega - r)\left(1 - \frac{\phi^2}{2}\right)$$

$$= r + \frac{\phi^2}{2}.$$

Set $r := \omega - a \Rightarrow a = \omega - r.$

$$X := \frac{a\phi}{2}, \quad R := \frac{ar}{2}$$

$$\Rightarrow \boxed{\dot{X} = R + X^2}$$ Normal form for saddle-node bifurcation

We then obtain that:

$$\frac{dX}{dt} = R + X^2 \Rightarrow T = \int_{\mathbb{R}} \frac{dX}{R + X^2} = \frac{\pi}{\sqrt{R}}.$$

But recall that

$$R = \frac{ar}{2} = \frac{a(\omega - a)}{2} \Rightarrow \boxed{T_{\text{saddle-node}} = \frac{\sqrt{2}\,\pi}{\sqrt{a(\omega-a)}}}.$$

same as before

Ex: ("Horny Fireflies")

$$\dot{\theta} = \omega, \quad \theta_0 = 0 : \text{ flash}$$

Stimulus: $\dot{H} = \Omega$.

When

$$0 < H - \theta < \pi, \quad \omega < \dot{\theta}$$

$$0 > H - \theta > -\pi, \quad \omega > \dot{\theta}$$

So, since they want to mate:

$$-A < \dot{\theta} - \omega < A.$$

Thus: $\boxed{\dot{\theta} = \omega + A\sin(H - \theta)}$

Introduce $\phi := H - \Theta$, $\quad \dot\phi = \dot H - \dot\Theta = \Omega - \dot\Theta$

$$\Rightarrow \dot\phi = \Omega - \omega - A\sin\phi .$$

unitless

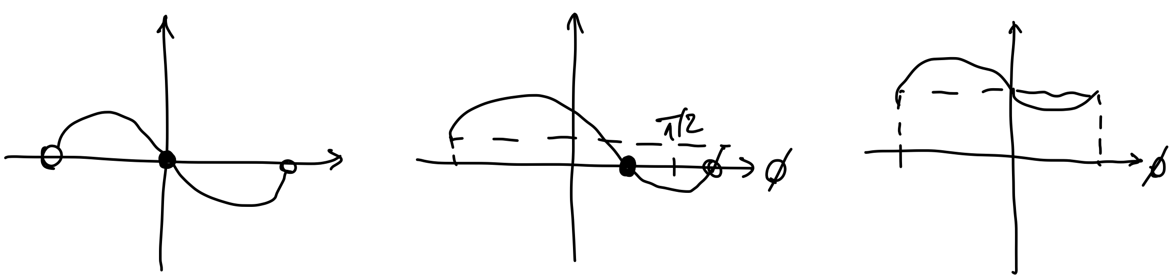
"Non-dimensionalize" time: $\quad t := T\tau$

$$\Rightarrow \frac{d}{dt} = \frac{1}{T}\frac{d}{d\tau}$$

$$\Rightarrow \dot\phi = T(\Omega - \omega) - AT\sin\phi$$

Let $T =: 1/A$ and $\mu := T(\Omega - \omega)$.

$$\dot\phi = \mu - \sin\phi$$



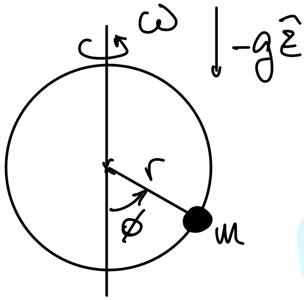
$\mu = 0$ $\qquad\qquad$ $0 < \mu < 1$ $\qquad\qquad$ $\mu > 1$

Lecture 8

04/10/2023

Non-Dimensionalization

One last kiss, my dear 1-dim. systems: Consider the following system w/ 5 parameters:



Damping $= b$

(1) Step 1: Nondimensionalize

2 ways → If you know the ODE

→ If you don't know the ODE ⇒ Buckingham's Π - "theorem"

Do this one → (suppose we forgot papa Newton)

ODE for ϕ:

$$\underbrace{m r \ddot{\phi}}_{\text{inertial}} = \underbrace{-b\dot{\phi}}_{\text{damping}} - \underbrace{mg\sin\phi}_{\text{gravity}} + \underbrace{m r \omega^2 \sin\phi\cos\phi}_{\text{centrifugal acceleration}} .$$

Task: Nondimensionalize the above ODE.

RULE: We want all the variables to be $O(1)$ after nondimensionalization (i.e., $\phi, t, \dot{\phi}, \ddot{\phi}$).

Define "REPRESENTATIVE REFERENCE QUANTITIES".

- Representative time scale T: $\boxed{\tau := \frac{t}{T}}$.

Substitute in

$$\frac{d\phi}{dt} = \frac{1}{T} \underbrace{\frac{d\phi}{d\tau}}_{O(1)} , \qquad \frac{d^2\phi}{d\phi^2} = \frac{1}{T^2} \underbrace{\frac{d^2\phi}{d\tau^2}}_{O(1)}$$

Denote $\dot{\phi} = \frac{d\phi}{d\tau}$. We now have the following equation

$$\frac{mr}{T^2} \ddot{\phi} = - \frac{b}{T} \dot{\phi} - mg \sin\phi + mr^2 \omega \cos\phi \sin\phi$$

/ "Divide by what annoys you" – Nico Grsgrs

In this case, that is b/T.

$$\frac{mr}{bT}\ddot{\phi} = -\dot{\phi} - \frac{mgT}{b}\sin\phi + \frac{mr\omega^2 T}{b}\cos\phi\sin\phi$$

Define $T := \dfrac{b}{mg}$. Then

$$\frac{m^2 g r}{b^2}\ddot{\phi} = -\dot{\phi} - \sin\phi + \frac{r\omega^2}{g}\cos\phi\sin\phi$$

Define $\varepsilon := \dfrac{m^2 g r}{b^2}$ and $\gamma = \dfrac{r\omega^2}{g}$ so that

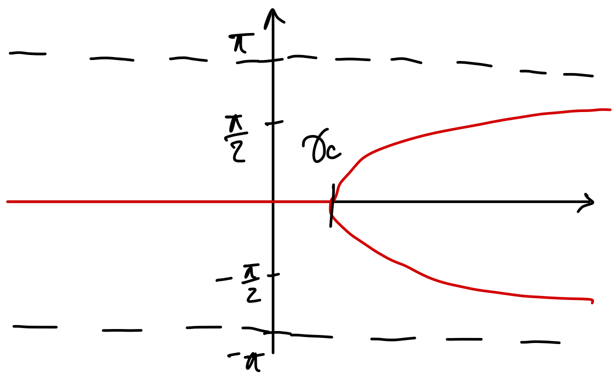
Non-dimensional

$$\boxed{\varepsilon\ddot{\phi} + \dot{\phi} = (\gamma\cos\phi - 1)\sin\phi}$$

Non-dimensionalized equation

Now... fixed pts.: $\phi_* = 0$ (always)

$$\gamma \cos\phi_* = 1 \quad \text{(sometimes)}$$



$\gamma > 1$

etc etc ...

Warning: When we wrote the non-dimensional equation, we lost a "transient" in the beginning

$$\dot\phi = f(\phi,\gamma), \quad \phi_0 = A, \quad \dot\phi_0 = B \neq 0.$$

Define $\psi := \dot\phi$, so we can write the 2^{nd} order ODE as 2 first order ODEs:

$$\varepsilon\ddot\phi + \dot\phi = \sin\phi(\gamma\cos\phi - 1) =: f(\phi)$$

$$\begin{cases} \dot\psi = -\dfrac{\dot\phi + f(\phi)}{\varepsilon} = \dfrac{f(\phi) - \psi}{\varepsilon} \\ \dot\phi = \psi \end{cases}$$

So, as $\varepsilon \to 0$, $\dot\psi \nearrow \infty$.

Buckinham's Π theorem: If $\exists N$ parameters

$$f(q_1, \ldots, q_N) = 0,$$

and R units, then we can construct $N-R$ non-dimensional numbers to complitely describe the system.

$$\begin{array}{c|ccccc} & m & g & r & \omega & b \\ \hline M & & & & & \\ L & & & & & \\ T & & & & & \end{array}$$

$$\begin{array}{c} M \\ L \\ T \end{array} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & -2 & 0 & -1 & -1 \end{pmatrix} =: A$$

$$\operatorname{rank} A = R$$

New chapter

LECTURE 9

2D SYSTEMS

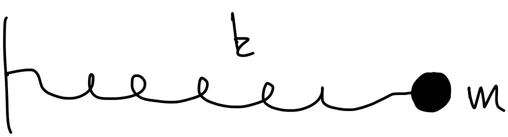
06/10/2023

General form:
$$\begin{cases} \dot{x} = ax + by \\ \dot{y} = cx + dy \end{cases}.$$

Write this as an eigenvalue problem:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

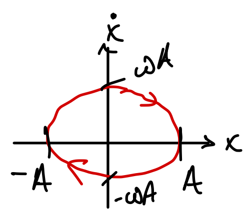
$\boxed{\text{Ex}}$: Mass on a spring



$$m\ddot{x} = -kx$$
$$\Rightarrow \ddot{x} + \omega^2 x = 0, \quad \omega^2 = k/m.$$

So, as 1^{st} order system
$$\begin{cases} \dot{x} = v \\ \dot{v} = -\omega^2 k \end{cases}$$

$$x(t) = A\sin(\omega t) + B\cos(\omega t)$$
$$\dot{x}(t) = v(t) = \omega A\cos(\omega t) - \omega B\sin(\omega t)$$



Now, if

$$\boxed{A = \begin{pmatrix} a & 0 \\ 0 & -1 \end{pmatrix}} \rightsquigarrow \dot{\vec{x}} = A\vec{x}$$

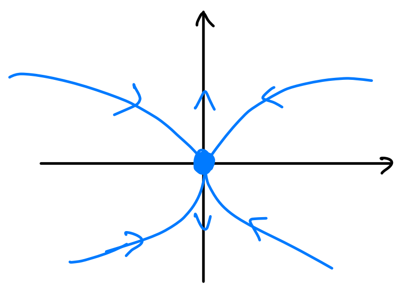
$$\Rightarrow \begin{cases} \dot{x}_1 = a x \\ \dot{x}_2 = -x_2 \end{cases} \rightsquigarrow \begin{cases} x_1(t) = x_0 e^{at} \\ x_2(t) = \tilde{x}_0 e^{-t} \end{cases}$$

- <u>Case 1</u>: $a < -1 \Rightarrow x_1$ decays faster than x_2.

<u>Slope</u>: $\dfrac{dx_2}{dx_1} = \dfrac{\dot{x}_2}{\dot{x}_1} = \dfrac{\tilde{x}_0 e^{-t}}{a x_0 e^{at}} \propto e^{-(1+a)t}$,

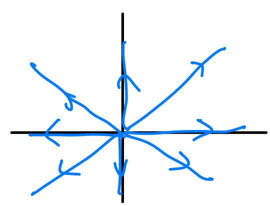
but $(1+a) < 0$, so,

$t \to -\infty$: "Horizontal" stable node



- <u>Case 2</u>: $a = -1$, then

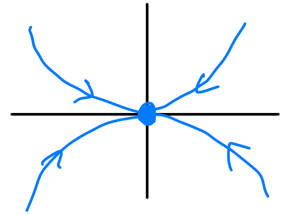
$$\begin{aligned} x_1 &= x_0 e^{-t} \\ x_2 &= \tilde{x}_0 e^{-t} \end{aligned} \Rightarrow \text{" Star "}$$



- **Case 3:** $-1 < a < 0$

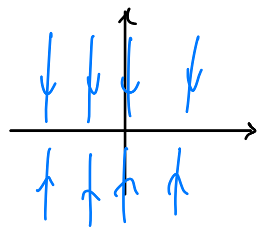
$x_1 = x_0 e^{at}$
$x_2 = \tilde{x}_0 e^{-t}$

Stable Node



- **Case 4:** $a = 0$

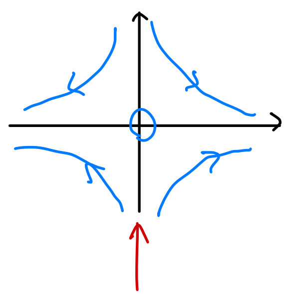
$x_1 = x_0$
$x_2 = \tilde{x}_0 e^{-t}$ $\rightsquigarrow$ "Line of stable fixed points"



- **Case 5:** $a > 0$ $\rightsquigarrow$ x grows

$\Rightarrow$ "Saddle"

The potential here looks like a Pringles →



y-axis = "Stable manifold"

x-axis = "Unstable manifold"

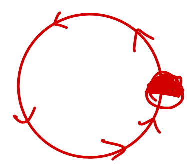
I'm sorry, little one

- **ATTRACTING FIXED POINT**: all trajectories tend to this fixed pt. $\vec{x}_*$ as $t \nearrow \infty$.

In addition: globally attracting if all trajectories end up at $\vec{x}_*$. Locally attracting if only attracts locally.

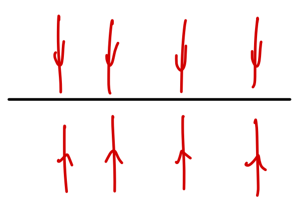
- **LYAPUNOV STABLE**: start in the vicinity of $\vec{x}_*$, means that you stay in the vicinity.

! WARNING: attracting but not Lyapunov exist; e.g

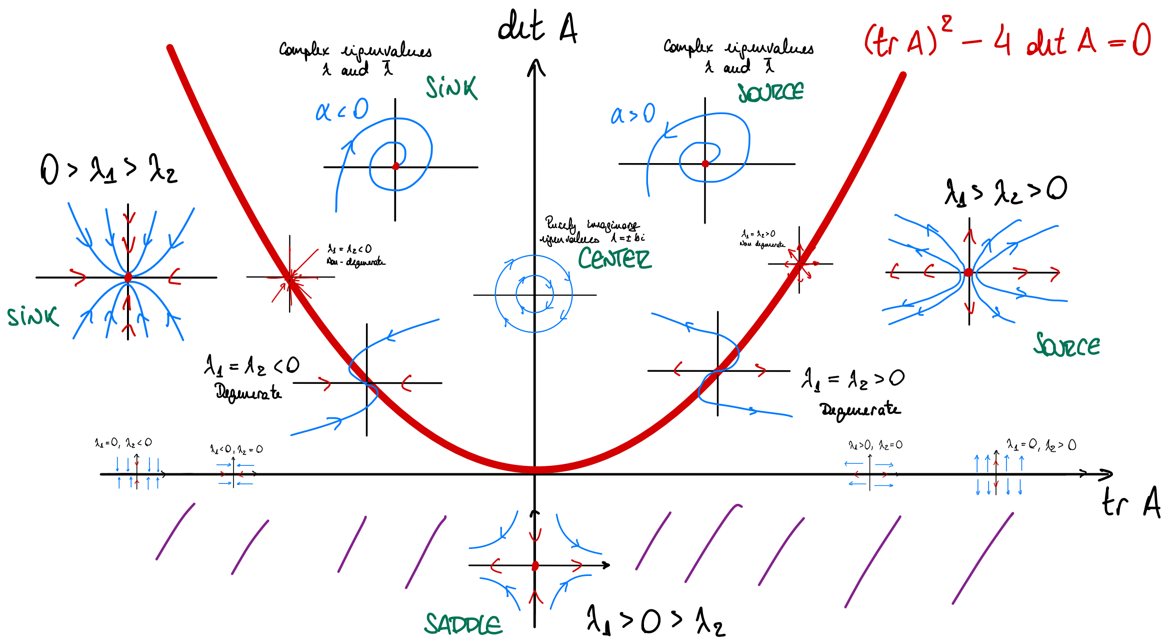


- **STABLE FIXED POINT**: Lyapunov + attracting.

- **NEUTRALLY ATTRACTING**: Lyapunov not attracting $\rightsquigarrow$



SUMMARY: Dream about this again **! IMPORTANT**



LECTURE 10

2D FLOWS (ctd)

11/10/2023

Recall: for

$$\dot{x} = Ax, \quad x(0) = (x_0, y_0),$$

the unique solution is given by

$$x(t) = e^{tA} x(0).$$

$$\iff \quad x(t) = e^{tA} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

$$e^{tA} := \sum_{n=0}^{\infty} \frac{1}{n!} t^n A^n .$$

If A is diagonal, e.g.,

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix},$$

then

$$e^{tA} = \begin{pmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{pmatrix}$$!! ☺

If A is not diagonal, get screwed... but always remember that $e^{tA} \in SO(2)$ (in this case).

Oh wait! Jordan decomposition!

If $A = CJC^{-1}$, $e^{tA} = C\, e^{tJ} C^{-1}$

changes of basis

(SO(2) is a group...)

Jordan Canonical Form !! ☺

Thm: (C. Jordan, ~1909) Can write $A \in \mathbb{R}^{n\times n}$ as $A = CJC^{-1}$, where C are changes of basis and J is of the form

$$J = \begin{pmatrix} B_1 & & 0 \\ & \ddots & \\ 0 & & B_k \end{pmatrix}$$

where B_j's are of blocks of the form

$$B_j = \begin{pmatrix} \lambda_j & 1 & & 0 \\ & \lambda_j & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_j \end{pmatrix}_{l_j \times l_j} = \mathrm{Id}_{l_j\times l_j}\lambda_j + \underset{\text{nilpotent}}{N} .$$

Ex: Suppose λ is an eigenvalue of A with algebraic multiplicity m. Then

$$\dim\left(\ker (A-\lambda \mathrm{Id})^{m-1}\right) = \#\text{ of Jordan blocks of length} \leq m-1 .$$

Def: The origin is called a <u>HYPERBOLIC EQUILIBRIUM</u> for the ODE $\dot{x} = Ax$ if all eigenvalues of A have nonzero real part.

Call this property "hyperbolicity".

Claim: Hyperbolicity is generic in $\mathbb{R}^{n\times n}$ (i.e., the set of hyperbolic $n\times n$ matrices is open and dense in $\mathbb{R}^{n\times n}$). Karlsen (2003)

LECTURE 11

2D NONLINEAR SYSTEMS

13/10/2023

Consider the system

$$\begin{cases} \dot{x} = x + e^{-y} \\ \dot{y} = -y \end{cases} \quad \text{Nonlinear!} \qquad \rightsquigarrow \qquad \underbrace{y(t) = y_0 e^{-t}}_{\text{decays}}$$

So, as $t \nearrow \infty$, $y \to 0$, thus $\dot{x} \approx x + 1$ for large enough t. This gives $x = Ae^t - 1$, which grows with time.

Fixed points: $y_* = 0$

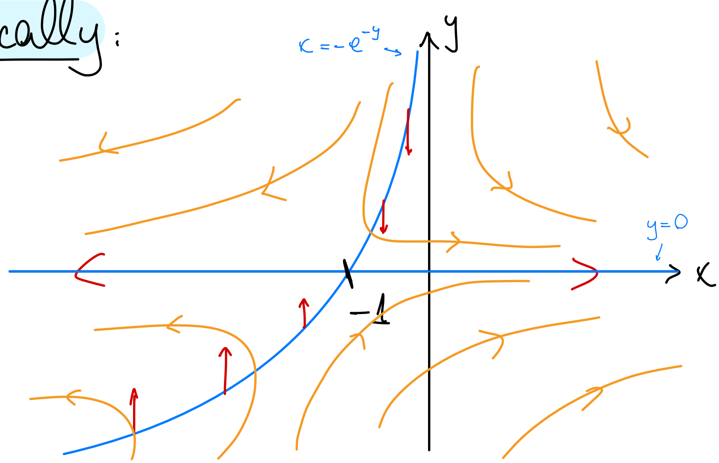
$$x_* = -e^{-y_*} = -1$$

Nullclines: lines where $\dot{x} = 0$ (flows vertically) or $\dot{y} = 0$ (flows horizont.)

$$\dot{y} = 0 \Rightarrow \boxed{y = 0}$$

$$\dot{x} = 0 \Rightarrow \boxed{x = -e^{-y}}$$

Graphically:



PICARD'S THEOREM (1955): Consider the system

(IVP) $\quad \dot{y} = f(x,y), \quad y(t_0) = y_0,$

where f is continuous in x and L-Lipschitz in y on $D \subset \mathbb{R}^n$. Then, $\exists a > 0$ such that (IVP) has a unique solution on $[x-a, x+a]$.

$\Rightarrow$ Solutions can't intersect! (violates uniqueness)

* **STABILITY**: Consider the C^1 system

$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases}$$

Assume (x_*, y_*) is such that $f(x_*, y_*) = g(x_*, y_*) = 0$.

Define $u := x - x_*$ and $v := y - y_*$. Then,

$$\dot{u} = \underbrace{\cancel{f(x_*, y_*)}}_{=0} + \left.\frac{\partial f}{\partial x}\right|_{(x_*,y_*)} u + \frac{1}{2!} \left.\frac{\partial^2 f}{\partial x^2}\right|_{(x_*,y_*)} u^2$$

$$+ \left.\frac{\partial f}{\partial y}\right|_{(x_*,y_*)} v + O(v^2, uv)$$

Neglect

$$\approx \left.\frac{\partial f}{\partial x}\right|_{(x_*, y_*)} u + \left.\frac{\partial f}{\partial y}\right|_{(x_*, y_*)} v$$

Similarly,

$$\dot{v} = \left.\frac{\partial g}{\partial x}\right|_{(x_*, y_*)} u + \left.\frac{\partial g}{\partial y}\right|_{(x_*, y_*)} v$$

The linearization of the system around the fixed point is:

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \overbrace{\begin{pmatrix} \left.\frac{\partial f}{\partial x}\right|_{(x_*, y_*)} & \left.\frac{\partial f}{\partial y}\right|_{(x_*, y_*)} \\ \left.\frac{\partial g}{\partial x}\right|_{(x_*, y_*)} & \left.\frac{\partial g}{\partial y}\right|_{(x_*, y_*)} \end{pmatrix}}^{=: A} \begin{pmatrix} u \\ v \end{pmatrix}$$

Ex: $\begin{cases} \dot{x} = x + e^{-y} = f(x,y) \\ \dot{y} = -y = g(x,y) \end{cases} \rightsquigarrow \begin{array}{l} x_* = -1 \\ y_* = 0 \end{array}$

$$A = \begin{pmatrix} 1 & -1 \\ 0 & -1 \end{pmatrix} \rightsquigarrow \begin{array}{l} \operatorname{tr} A = 0 \\ \det A = -1 \end{array} \Rightarrow \boxed{\text{SADDLE at } (-1, 0)}$$

slanted according to eigenvecs.

$\square$

Ex:

$$\begin{cases} \dot{x} = -y + ax(x^2+y^2) =: f(x,y) \\ \dot{y} = x + ay(x^2+y^2) =: g(x,y) \end{cases}$$

Fixed point: $(x_*, y_*) = (0,0)$

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$\Downarrow$

Center ?

Confirm: $\operatorname{tr} A = 0$, $\det A = 1$ $\Rightarrow$ Center at $(0,0)$. $\leftarrow$ Linearization

But GLOBALLY it doesn't look like a center, clearly.

Upshot: LSA predicts

- saddle
- stable or unstable nodes
- stable or unstable spirals

Anything else...
need to study case-by-case.

LECTURE 12

18/10/2023

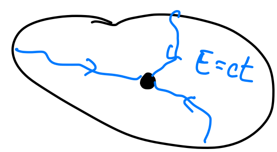
CONSERVATIVE SYSTEMS

CONSERVATIVE SYSTEMS: some function $E(x,y)$ such that $\dot{E} = 0$ along trajectories.

PROPERTIES:

(1) No attracting fixed points.

Can't happen →

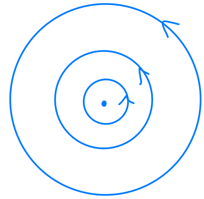


(2) Fixed points can only be saddles or centers!

(3) For all trajectories, $E = \text{const.}$ along them

(4) If $\vec{x}_*$ is an isolated center, then all trajec-

tories in the vicinity are closed.



* **REVERSIBLE SYSTEMS**: Systems that are invariant under $t \mapsto -t$.

Consider $m\ddot{x} = f(x)$, then, by the chain rule

$$\dot{x} \longmapsto -\dot{x}$$
$$\ddot{x} \longmapsto \ddot{x}$$

* **More general definition**: a reversible system is a system that is invariant under $t \mapsto -t$ and $y \mapsto -y$

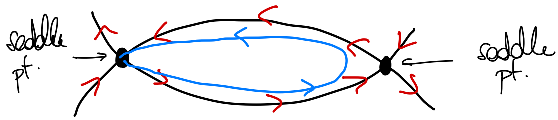
$$\begin{cases} \dot{x} = f(x,y) \\ \dot{y} = g(x,y) \end{cases} \quad \overset{t \mapsto -t}{\underset{y \mapsto -y}{\rightsquigarrow}} \quad \begin{cases} -\dot{x} = f(x,-y) \overset{!}{=} -f(x,y) \\ +\dot{y} = g(x,-y) \overset{!}{=} g(x,y) \end{cases}$$

$$\Longleftrightarrow \begin{bmatrix} f \text{ needs to be ODD on } y \\ g \text{ needs to be EVEN on } y \end{bmatrix}$$

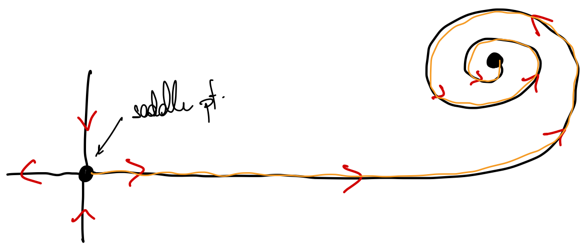
- In reversible systems, fixed points are usually saddles and centers

- Tend to find HOMOCLINIC ORBITS

returns to the same fixed pt



or HETEROCLINIC ORBITS



EXAMPLE:
$$\begin{cases} \dot{x} = y - y^3 =: f(x,y) \text{ odd on } y \\ \dot{y} = -x - y^2 =: g(x,y) \text{ even on } y \end{cases}$$

Fixed points: $f(x,y) = 0 \Rightarrow y_* = 0$ or $y_* = \pm 1$

- For $y_* = 0$, $x_* = 0$. So,

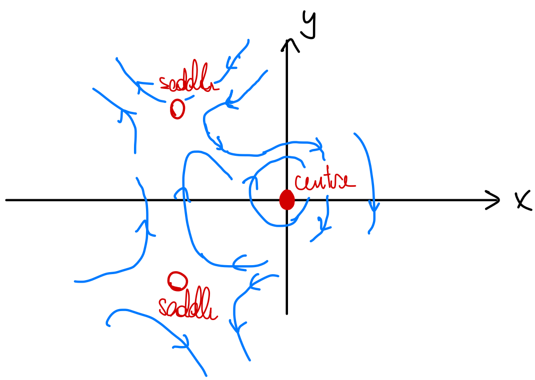
$$A = \begin{pmatrix} 0 & 1-3y^2 \\ -1 & -2y \end{pmatrix} \underset{\substack{x_*=0 \\ y_*=0}}{=} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\left.\begin{aligned} \det A &= 1 \\ \operatorname{tr} A &= 0 \end{aligned}\right\} \Rightarrow \underline{\text{center}}$$

- For $y_* = \pm 1$, $x_* = -1$. So,

$$A = \begin{pmatrix} 0 & -2 \\ -1 & \mp 2 \end{pmatrix} \rightsquigarrow \begin{aligned} \det A &= -2 \\ \operatorname{tr} A &= \mp 2 \end{aligned} \Rightarrow \underline{\text{saddles}}$$

Upshot:



$\blacksquare$

Def: (Reversible system) Any "nontrivial" mapping $\vec{x} \longmapsto R(\vec{x})$ s.t. $R^2 = id \Rightarrow$ system is reversible. ???

↖ Very bad definition?

EXAMPLE:

$$\begin{cases} \dot{x} = -2\cos x - \cos y \\ \dot{y} = -\cos x - 2\cos y \end{cases}$$

Fixed points: $\begin{cases} 2\cos x_* + \cos y_* = 0 \\ \cos x_* + 2\cos y_* = 0 \end{cases} \Rightarrow \begin{cases} \cos y_* = 0 \\ \cos x_* = 0 \end{cases}$

$\Rightarrow$ $x_* = \pm \frac{\pi}{2}$ and $y_* = \pm \frac{\pi}{2}$.

Need to evaluate the 4 cases separately:

$$A = \begin{pmatrix} 2\sin x & \sin y \\ \sin x & 2\sin x \end{pmatrix}$$

- For the case: $x_* = y_* = -\frac{\pi}{2}$, we have

$$A = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} \rightsquigarrow \begin{aligned} \det A &= 5 \\ \operatorname{tr} A &= -4 \end{aligned}$$

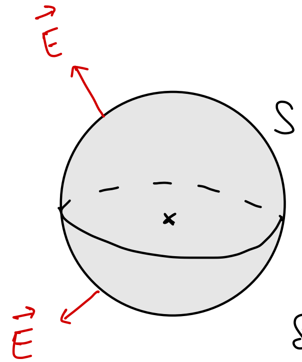
$\Downarrow$

$\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right)$ is a stable node

LECTURE 13

INDEX THEORY

20/10/2023



Consider a closed surface $S \subset \mathbb{R}^3$.

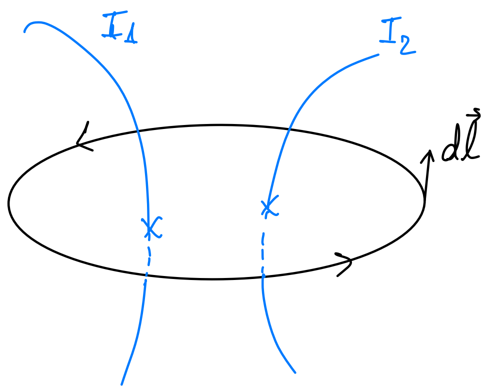
Electric field:

$$\phi = \iint \vec{E} \cdot d\vec{a}.$$

Say q is the enclosed charge, then

$$\phi = q/\varepsilon_0.$$

Now, if there is a loop ℓ.

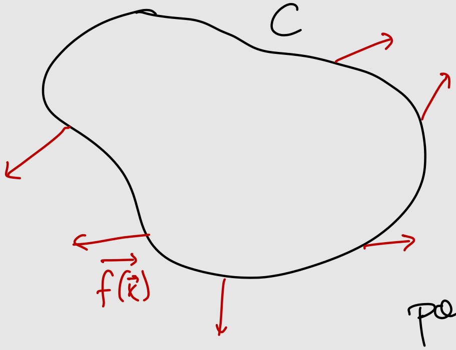


Magnetic field $\vec{B}$

$$\int \vec{B} \cdot d\vec{\ell} = \mu_0 \sum_k I_k.$$

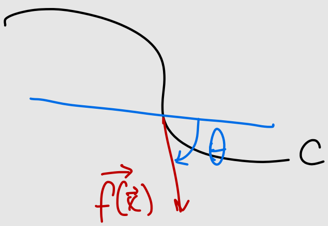
Def: (INDEX OF A CURVE) Consider $\dot{\vec{x}} = f(\vec{x})$.

Let C be a closed curve that does not go through any fixed point of the system (i.e., $\vec f(\vec x) \neq 0$ along C). Let



$$\theta = \arctan\left(\frac{\dot y}{\dot x}\right) = \arctan\left(\frac{f_y}{f_x}\right)$$

Note that θ varies continuously since f smooth and no fixed points lie on C.



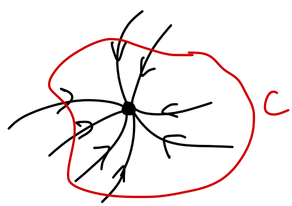
Moving counterclockwise, check how much θ has changed:

$$\Delta\theta = 2\pi \underbrace{I_C}$$

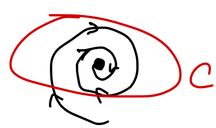
$\mathbb{Z} \ni I_C$ is the index of the curve.

Ex: Suppose C encircles

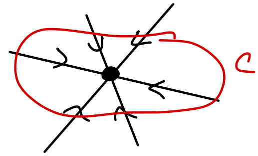
- Stable node : $I_C = 1$



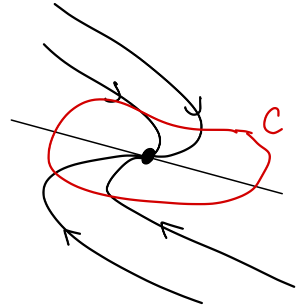
- Stable spiral : $I_C = 1$



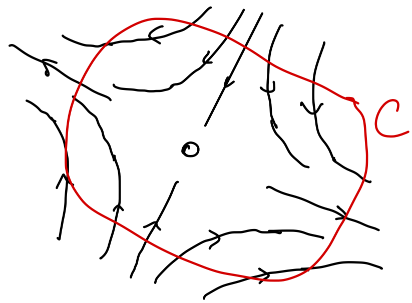
- **Attractive pt.:** $I_C = 1$



- **Degenerate star:** $I_C = 1$



- **Saddle:** $I_C = -1$

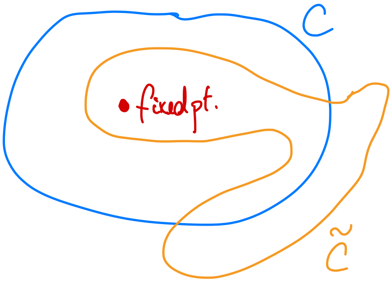


Upshot: If the curve encircles only 1 fixed pt.

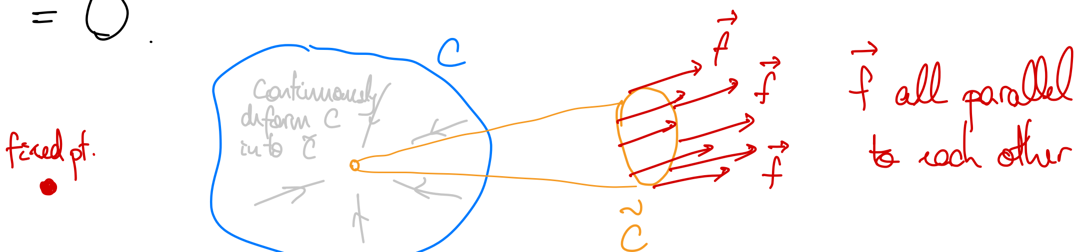
$$I_C = \begin{cases} -1, & \text{if fixed pt. is a saddle} \\ \ldots & \end{cases}$$

Properties of Index

1) If C and $\tilde{C}$ can be continuously deformed to each other $C \leftrightarrow \tilde{C}$ (i.e., homotopic) without passing through fixed points, then $I_C = I_{\tilde{C}}$.



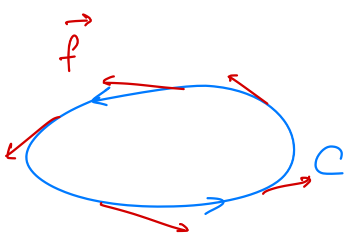
2) If C does not encircle any fixed point, then $I_C = 0$.



3) Transformation $t \mapsto -t$ leaves I_C unchanged.

($\dot{x} \mapsto -\dot{x}$ and so on

4) If C is a trajectory, then $I_C = 1$



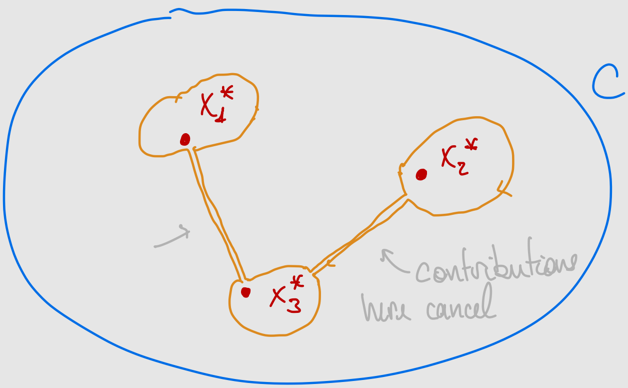
$\Rightarrow I_C = 1$.

* INDEX OF A FIXED POINT:

If C encloses an <u>isolated</u> fixed point, then there are

- Saddles $\Rightarrow I_C = +1$
- All others $\Rightarrow I_C = -1$

Thm: If C encircles more than one fixed point



$$I_C = \sum_k I_k \, .$$

Thm: Any closed orbit in phase plane must encircle at least one fixed point and so $I_C = +1$.

Lecture 14

25/10/2023

Limit Cycles

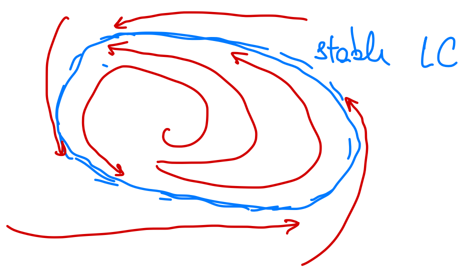
Isolated closed orbits

Suppose $x(t)$ is a solution to $\dot{x} = f(x)$ and $\mathrm{Dom}(f) = D$. Then

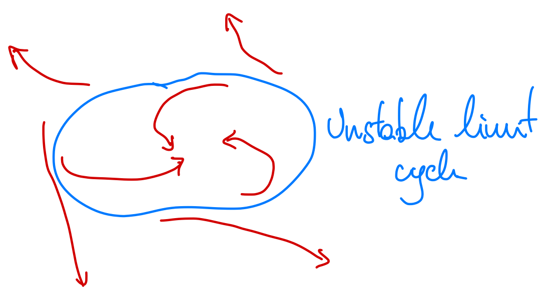
$$\omega(x) = \{ y \in D : \exists\, t_j \nearrow \infty \text{ s.t. } \lim_{j\to\infty} x(t_j) = y \}$$

$$\alpha(x) = \{ y \in D : \exists\, t_j \searrow \infty \text{ s.t. } \lim_{j\to\infty} x(t_j) = y \}$$

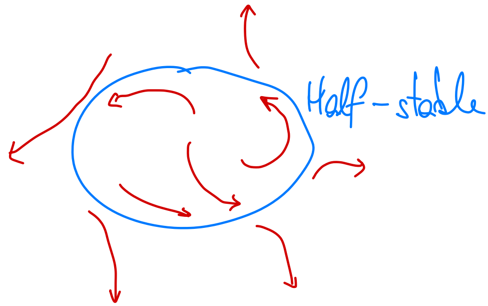
Stable Limit Cycles:



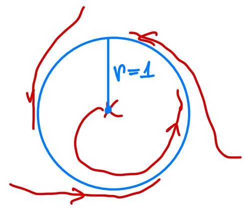
Unstable Limit Cycle:



Half-stable limit cycle:



Ex: $\begin{cases} \dot{r} = r(1-r^2) \\ \dot{\theta} = 1 \end{cases}$ $\rightsquigarrow$



Ex: van der Pol $\quad \ddot{x} + \underbrace{\mu \dot{x}(x^2-1)}_{\text{Non-linear damping}} + x = 0$

$|x| > 1 \rightsquigarrow$ damping

$|x| < 1 \rightsquigarrow$ amplifying

$\Rightarrow$ Half stable limit cycle

* **RULING OUT CLOSED ORBITS**

(1) Index theory: closed orbit $\Rightarrow I_c = +1$

(2) If gradient system $\dot{\vec{x}} = f(\vec{x})$ w/ $f = -\nabla V$, V $\mathbb{R}$-valued.

(3) If Lyapunov fct. $V(\vec{x})$ $\mathbb{R}$-valued and $\exists \vec{x}_*$ s.t.

$$V > 0 \text{ if } \vec{x} \neq \vec{x}_*$$
$$V(\vec{x}_*) = 0$$
$$\dot{V}(\vec{x}_*) = 0$$

(4) If $\dot{\vec{x}} = \vec{f}(\vec{x})$ defined on a simply connected subset and $\exists g \in C^1(D)$ s.t. $g(\vec{x})\dot{\vec{x}}$ has the same sign throughout D, then no limit cycle.

Thm: (Poincaré - Bendixon) Consider $\dot{x} = f(x)$ with $f \in C^1(D)$. Let $x \in D$. If $\omega(x)$ contains no fixed points and is contained in a compact set, then $\omega(x)$ is a periodic orbit s.t.

i.e., there exists a periodic solution $y(t)$

$$\omega(x) = \{ y(t) : t \in [0, T] \}.$$

$y(t+T) = y(t)$, $T > 0$

Period

Loosely speaking: if $x \in D \subset K^{\text{compact}}$ $\dot{x} = f(x)$ is $C^1(D)$, K does not contain fixed pts, $\exists$ a trajectory C confined in K, then C is either a closed trajectory or it tends towards one as $t \nearrow \infty$.

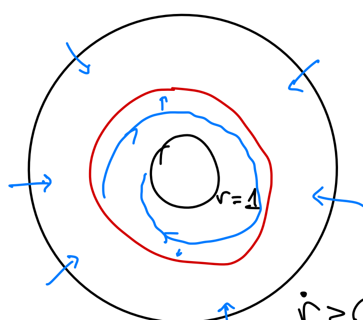
Ex: Logistic Eq.: $\begin{cases} \dot{r} = r(1-r^2) + \mu \cos\theta \\ \dot{\theta} = 1 \end{cases}$

(polar coördinates)

Clearly, domain is contained in a compact, it is C^1, does not contain fixed pts.. Lastly, need to check we have a confined trajectory

WTF: r_{min} s.t.

$$\dot{r}\big|_{r=r_{min}} = r_{min}(1-r_{min}^2) + \mu r_{min}\cos\theta$$



$$\Rightarrow 1 - r_{min}^2 - \mu > 0 \rightsquigarrow \boxed{r_{min} = \sqrt{1-\mu}} \quad (\mu < 1)$$

$$\boxed{r_{max} = \sqrt{1+\mu}}$$

Thus, $\exists$ a solution w/ radius confined between $\sqrt{1-\mu} < r < \sqrt{1+\mu}$ $\Rightarrow$ Poincaré-Bendixon applies.

——— // ———

RELAXATION OSCILLATIONS

Consider the van der Pol system

Weakly non-linear

$$\ddot{x} + \mu \dot{x}(x^2 - 1) + x = 0 \quad \text{w/} \quad \mu \gg 1$$

Change of Variables: note that

$$\dot{x}(x^2-1) = \frac{d}{dt}\Big(\underbrace{\frac{x^3}{3} - x}_{=:\,F(x)}\Big)$$

$$\Rightarrow \quad \ddot{x} + \mu\dot{x}(x^2-1) = \frac{d}{dt}\Big(\underbrace{\dot{x} + \mu F(x)}_{=:\,w(x)}\Big) = -x$$

$$\Rightarrow \quad (\text{vdP}) \iff \begin{cases} \dot{w} = -x \\ \dot{x} = w - \mu F(x) \end{cases}$$

Change variables: $y := \frac{w}{\mu}$. Then

$$(vdP) \leftrightsquigarrow \begin{cases} \dot{x} = \mu(y - F(x)) \\ \dot{y} = -\frac{x}{\mu} \end{cases}$$

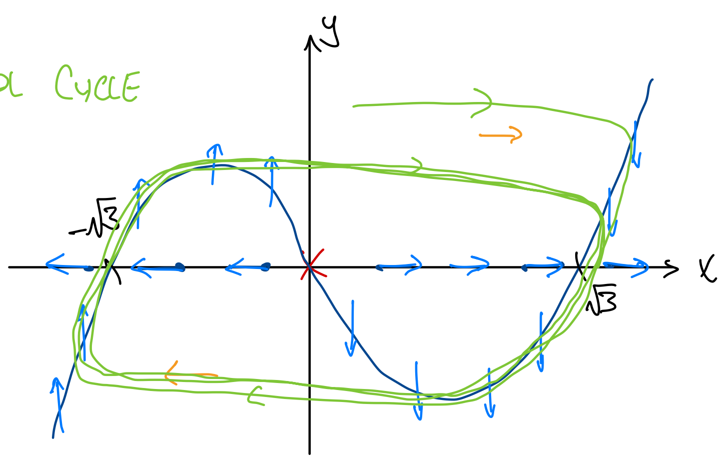
Fixed points: $x_* = y_* = 0$

Nullclines:

$\dot{y} = 0 \rightsquigarrow \boxed{x = 0}$ here solution only grows horizontally

$\dot{x} = 0 \rightsquigarrow \boxed{y = F(x)}$ here solution only grows vertically

VAN DER POL CYCLE



LECTURE 15

27/10/2023

STRONGLY NONLINEAR VAN DER POL

WEAKLY NONLINEAR OSCILLATOR:

$$\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0, \quad 0 < \varepsilon \ll 1$$

weakly

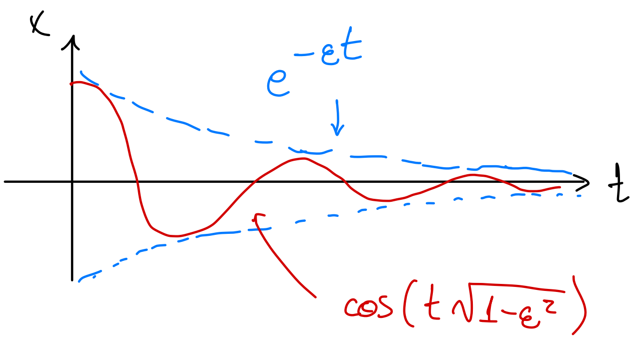
$h \in C^\infty$

E.g.: van der Pol $\rightsquigarrow h(x,\dot{x}) = \dot{x}\,(x^2 - 1)$

Duffing $\rightsquigarrow h(x,\dot{x}) = x^3$

Damped Harmonic Oscillator: $\ddot{x} + 2\varepsilon\dot{x} + x = 0$

$h(x,\dot{x}) = 2\dot{x}$



Initial conditions: $\left.\begin{array}{l} x_* = 0 \\ \dot{x}_* = 1 \end{array}\right\} \Rightarrow x(t) = \frac{1}{\omega} e^{-\varepsilon t} \sin(\omega t),$

$$\omega = \sqrt{1-\varepsilon^2}.$$

$\varepsilon \ll 1$: $x(t) = x_0(t) + \varepsilon x_1(t) + O(\varepsilon^2)$

Regular Perturbation Theory :

1) Do the expansion

2) Plug in original value

3) Solve order-by-order.

Doing this, we obtain:

1) $\ddot{x} = \ddot{x}_0(t) + \varepsilon \ddot{x}_1(t) + O(\varepsilon^2)$

$$2\varepsilon \dot{x} = 2\varepsilon(\dot{x}_0(t)) + O(\varepsilon^2)$$

$$x = \dots$$

2) $\ddot{x}_0(t) + x_0(t) + \varepsilon(\ddot{x}_1 + 2x_0 + x_1) + O(\varepsilon^2) = 0$

3) Order 1:

$$\ddot{x}_0 + x_0 = 0 \Rightarrow x_0 = A\cos t + B\sin t$$

$$x_* = 0 \Rightarrow A = 1$$

$$\dot{x}_* = 1 = \dot{x}_0(0) + \varepsilon \dot{x}_1(0)$$

$$\Rightarrow \dot{x}_0(0) = 1$$

$$\dot{x}_n(0) = 0 \quad \forall n \geq 1$$

$$\Rightarrow x_0 = B\sin t, \quad \dot{x}_0 = -B\cos t$$

$$\dot{x}_0 = \dot{x}_1 = 1 = B \Rightarrow \boxed{x_0 = \sin t}$$

Order ε:

$$\ddot{x}_1 + x_1 + 2x_0 = 0 \Rightarrow \ddot{x}_1 + x_1 = -2(A\cos t + B\sin t)$$

$$\rightsquigarrow \ddot{x}_1 + x_1 = -2\sin t$$

$$x_0 = -t\sin t \quad (\text{secular growth})$$

Thus: $$\boxed{x(t) = \sin t - \varepsilon t \sin t + O(\varepsilon^2)}$$

Only valid for small ε.

Order of discrepancy is $O(1/\varepsilon)$.

! Obs: diverges very fast and only useful for small $\varepsilon \ll 1$.

Fix this w/ a better method

* **Multiple Scale Analysis**: Define 2 fcts that depend on time:

$$\tau(t) := t \quad \text{(fast time scale)}$$

$$T(t) := \varepsilon t \quad \text{(slow time scale)}$$

$0 < \varepsilon \ll 1$

Use τ and T as independent time variables:

$$x(t) \longmapsto x(\tau(t), T(t))$$

$$\frac{dx}{dt} \xmapsto{\text{Chain Rule}} \frac{\partial \tau}{\partial t}\frac{\partial x}{\partial \tau} + \frac{\partial T}{\partial t}\frac{\partial x}{\partial T} = \frac{\partial x}{\partial \tau} + \varepsilon \frac{\partial x}{\partial T}$$

$$\frac{d^2x}{dt^2} \xmapsto{\text{Chain Rule}} \frac{\partial^2 x}{\partial \tau^2} + 2\varepsilon \frac{\partial^2 x}{\partial \tau \partial T} + \cancel{O(\varepsilon^2)} \to \text{Neglect}$$

Say $\frac{\partial x}{\partial \tau} = x_{\cdot,\tau}$ and $\frac{\partial x}{\partial T} = x_{\cdot,T}$. Then

$x = x_0 + \varepsilon x_1 + O(\varepsilon^2)$. So,

$$\ddot{x} + 2\varepsilon \dot{x} + x = 0 \rightsquigarrow x_{0,\tau\tau} + \varepsilon x_{1,\tau\tau} + 2\varepsilon x_{0,\tau T}$$
$$+ 2\varepsilon x_{0,\tau}$$
$$+ x_0 + \varepsilon x_1 + O(\varepsilon^2) = 0$$

$O(1)$: $x_{0,\tau\tau} + x_0 = 0 \Rightarrow \boxed{x_0 = A(T)\cos(\tau) + B(T)\sin\tau}$

$O(\varepsilon)$: $x_{1,\tau\tau} + x_1 = -2\varepsilon(x_{0,\tau T} + x_{0,\tau})$

$$= \dots$$

$$= -2\left[-(A'+A)\sin\tau + (B'+B)\cos\tau\right]$$

Thus: there'll be secular growth unless

$$A' + A = 0 \Rightarrow A(T) = A_* e^{-T}$$

$$B' + B = 0 \Rightarrow B(T) = B_* e^{-T}$$

So,

$$x = (A_* \cos\tau + B_* \sin\tau)\, e^{-T} + O(\varepsilon^2)$$

Initial conditions $A_* = 0$, $B_* = 1$. Then

$$\boxed{x(t) = e^{-T}\sin\tau + O(\varepsilon^2)}$$

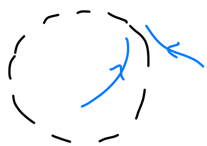
Different from $\frac{1}{\omega} e^{-\varepsilon T}\sin(\omega t)$ but MUCH better approximation.

LECTURE 16

LIMIT CYCLES & vdP

01/11/2023

LIMIT CYCLES: $\ddot{x} + x + \varepsilon h(x, \dot{x}) = 0$, with $|\varepsilon| \ll 1$.



REGULAR PERTURBATION THEORY:

$$x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + O(\varepsilon^3)$$

MULTIPLE SCALE ANALYSIS: Fast time $\tau(t) := t$

slow time $T(t) := \varepsilon t$

Indep. variables: $\frac{d}{dt} = \partial_\tau + \varepsilon \partial_T$

$$\Rightarrow \frac{dx}{dt} = (\partial_\tau + \varepsilon \partial_T)(x_0 + \varepsilon x_1) + O(\varepsilon^2)$$

$$= x_{0,\tau} + \varepsilon (x_{0,T} + x_{1,\tau}) + O(\varepsilon^2)$$

$$\Rightarrow \frac{d^2x}{dt^2} = (\partial_\tau + \varepsilon \partial_T)\left[x_{0,\tau} + \varepsilon (x_{0,T} + x_{1,\tau}) + O(\varepsilon^2)\right]$$

$$= x_{0,\tau\tau} + \varepsilon\left(2x_{0,\tau T} + x_{1,\tau\tau}\right) + O(\varepsilon^2)$$

Solving on each order:

$O(1)$: $\quad x_{0,\tau\tau} + x_0 = 0 \Rightarrow x_0 = A(T)\cos(\tau + \phi(T))$

$O(\varepsilon)$: $\quad 2x_{0,\tau T} + x_{1,\tau\tau} + x_1 + h = 0$

$$x_{1,\tau\tau} + x_1 = -2x_{0,\tau T} - h \quad (*)$$

where

$$x_{0,\tau T} = \partial_T\left(-A(T)\sin(\tau + \phi(T))\right)$$

$$= -A'(T)\sin(\tau + \phi(T))$$

$$-A(T)\phi'(T)\cos(\tau + \phi(T))$$

Plug this into $(*)$ to get:

$$x_{1,\tau\tau} + x_1 \overset{(**)}{=} -2A'(T)\sin(\tau + \phi(T))$$

$$-2A(T)\phi'(T)\cos(\tau + \phi(T))$$

$$-h(x,\dot{x})$$

<u>Ex</u>: van der Pol $\rightsquigarrow h(x,\dot{x}) = \dot{x}(x^2 - 1)$

Then

$O(1)$: $h = x_{0,z}(x_0^2 - 1) + O(\varepsilon)$

Plug in $\begin{cases} x_0 = A(T)\cos(z+\phi(T)) \\ x_{0,z} = -A(T)\sin(z+\phi(T)) \end{cases}$ into

We get:

$$h = -A^3 \sin(z+\phi)\cos^2(z+\phi) + A\sin(z+\phi) + O(\varepsilon)$$

$$\sin(\cdot)\cos^2(\cdot) = \frac{1}{4}\left[\sin(\cdot) + \sin(3\,\cdot)\right]$$

$$= \left(-\frac{A^3}{4} + A\right)\sin(z+\phi) + (\text{smth.})\cdot\cancel{\sin(3z+3\phi)} + \cancel{O(\varepsilon)}$$

Neglect b/c has ≠ freq. from RHS of (***)

Neglect

Plug in (**):

$$x_{1,zz} + x_1 = \left(-2A' - \frac{A^3}{4} + A\right)\sin(z+\phi)$$

$$- 2A\phi'\cos(z+\phi) + \left(\begin{matrix}\text{non-resonant}\\ \text{terms}\end{matrix}\right) + O(\varepsilon)$$

Matching coeffs.:

$$\Longrightarrow \boxed{A' \overset{(1)}{=} \frac{-A^3}{8} + \frac{A}{2}} \quad ; \quad \boxed{A\phi' = 0} \Rightarrow \phi' = 0$$

Stable fixed pt. at $A=2$

Unstable fixed pt. at $A=0$

Moreover, we can separate variab. & integrate (1):

(note: $A(0)=1$) $\rightsquigarrow$ $\boxed{A(T) = \dfrac{2}{\sqrt{1+3e^{-T}}}}$

$$\Rightarrow \boxed{x = \frac{2}{\sqrt{1+3e^{-\varepsilon T}}} \cos(t+\phi_0) + O(\varepsilon)}$$

<u>MORE GENERALLY</u>: $h = \dfrac{\alpha_0}{2} + \displaystyle\sum_{n=1}^{\infty} [\alpha_n \cos(n\theta) + \beta_n \sin(n\theta)]$

$\theta = \tau + \phi(T)$

$x_0 = A(T) \cos(\underbrace{\tau + \phi}_{\theta})$. Then, by analysis...

$$\alpha_n = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \cos(n\theta)\, d\theta$$

$$\beta_n = \frac{1}{\pi} \int_0^{2\pi} h(\theta) \sin(n\theta)\, d\theta$$

$$\Rightarrow \quad h = \alpha_1 \cos(\tau+\phi) + \beta_1 \sin(\tau+\phi) + \begin{pmatrix} \text{non-resonant} \\ \text{junk} \end{pmatrix}$$

Then,

$$X_{1,\tau\tau} + X_1 = \left(-2A' - \beta_1\right)\sin(\tau+\phi)$$
$$+ \left(-2A\phi' - \alpha_1\right)\cos(\tau+\phi)$$
$$+ \begin{pmatrix} \text{non-resonant} \\ \text{junk} \end{pmatrix}$$

Thus:

$$\begin{cases} A' = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} h(\theta)\sin\theta \, d\theta \\ \\ A\phi' = \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} h(\theta)\cos\theta \, d\theta \end{cases}$$

"AVERAGED EQUATIONS"

!

Ex: DUFFING OSCILLATOR

$$\ddot{x} + x + \varepsilon x^3 = 0 \,, \quad |\varepsilon| \ll 1$$

$$\hookrightarrow h = x^3 = A(T)^3 \cos^3(\tau+\phi) + O(\varepsilon)$$

Using the averaged equations

$$A' = A(T)^3 \cdot \frac{1}{2\pi}\int_0^{2\pi} \sin\theta \cos^3\theta \, d\theta = 0,$$

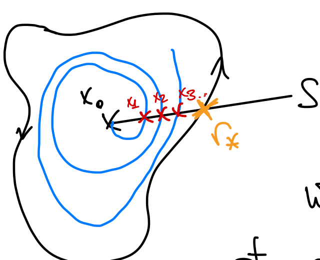
$$A\phi' = A(T)^3 \frac{1}{2\pi}\int_0^{2\pi} \cos^4\theta \, d\theta = \frac{3A(T)^3}{8}$$

$$\phi'(T) = \frac{3}{8}A(T)^2 \rightsquigarrow \omega := \frac{d\theta}{dt} = 1 + \varepsilon\phi'.$$

LECTURE 18 — CHAOS

15/11/2023

STABILITY OF LIMIT CYCLES: Record crossings of trajectories w/ S.



$r_k \rightsquigarrow r_{k+1}$ one cycle later

POINCARÉ MAP: $P(r_k) = r_{k+1}$. Then, whenever we reach a cycle, we have a fixed pt. of the Poincaré map:

$$\boxed{P(r_*) = r_*}$$

Easy to compute numerically (hard analytically)

LINEAR STABILITY ANALYSIS: pick $\vec{r}_0$ in vicinity of $\vec{r}_*$ (i.e., let $\varepsilon_0 > 0$ and set $\vec{r}_0 := \vec{r}_* + \varepsilon_0$). Iterate:

Taylor expand; Vector field derivative

$$\vec{r}_1 := \vec{r}_* + \varepsilon_1 = P(\vec{r}_* + \vec{\varepsilon}_0) = P(\vec{r}_*) + \underbrace{DP\Big|_{r=r*}}_{(n-1)\times(n-1) \text{ matrix}} \varepsilon_0$$

Thus $\vec{r}_* + \varepsilon_1 = \vec{r}_* + DP\big|_{\vec{r}=\vec{r}_*} \varepsilon_0$

$$\Rightarrow \varepsilon_1 = DP\big|_{\vec{r}=\vec{r}_*} \varepsilon_0 \, ,$$

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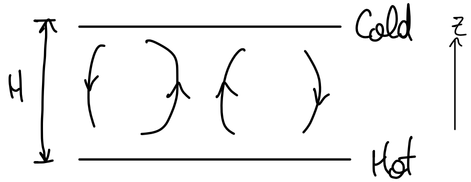
* **CHAOS**: **FORCED - DISSIPATIVE CHAOS** (Lorenz 1963)

Not the mathematician! (a meteorologist......)

Convective system: Temperature:



$$T = \frac{\Delta T}{H} z + \Theta$$

$$\partial_t T = \kappa \nabla^2 T$$

Then, we get

$$\boxed{\partial_t \theta = \partial_x \theta \, v_x + \partial_z \theta \, v_z + \kappa \nabla^2 \theta + \frac{\Delta H}{H} v_z}$$

Also have the vorticity $\xi = \partial_z v_x - \partial_x v_z$

$$\boxed{\partial_t \xi = \partial_x \xi \, v_x + \partial_z \xi \, v_z + \nu \nabla^2 \xi + g \alpha \, \partial_x \theta}$$

Density $\Delta\rho = -\alpha\,\Delta\theta$. Define

$$-v_z = \partial_x \psi \quad \text{and} \quad v_x := \partial_x \psi .$$

Then $\boxed{\nabla^2 \psi = \xi}$ ...... get

LORENZ EQUATIONS

$$\dot{X} = \sigma(Y - X)$$

$$\dot{Y} = rX - Y - XZ$$

$$\dot{Z} = XY - bY$$

$\sigma = \frac{\nu}{\kappa}$ Prandtl number

$r = \frac{Ra}{Ra_c}$ Rayleigh number $\left(\begin{array}{l} r > 1 \Rightarrow \text{chaos} \\ r < 1 \Rightarrow \text{stable} \end{array} \right)$

$b = \frac{4}{1 + a^2}$ ratio of length/height of plates.

FEATURES OF LORENZ EQUATIONS:

- Clearly, nonlinear
- Invariant under $(X, Y, Z) \mapsto (-X, -Y, Z)$

LECTURE 19

LORENZ ATTRACTOR

17/11/2023

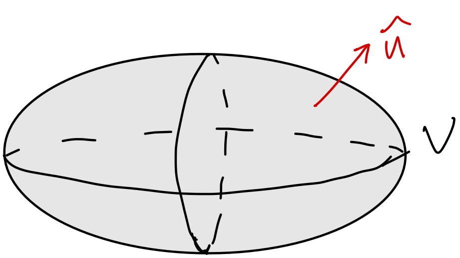
Lorenz Equations ('63):

$$\begin{cases} \dot{x} = \sigma(y-x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

Rayleigh Number: $Ra = \dfrac{g\alpha(\Delta T)H^3}{\nu\kappa}$.

$$r = \frac{Ra}{Ra_c}, \quad \sigma = \frac{\nu}{\kappa}, \quad b = \frac{4}{1+a^2}$$

- System of 1^{st} order, nonlinear, coupled ODEs.
- Symmetry under $(x,y,z,t) \longmapsto (-x,-y,z,t)$.
- Phase-space contraction: volume V changes as



$$\dot{V} = \iint \underbrace{\dot{\vec{x}}}_{(\dot{x},\dot{y},\dot{z})} \cdot \hat{n}\, dS$$

$$\overset{\text{Green}}{=} \iiint \operatorname{grad} \dot{\vec{x}}\, dV$$

$\text{grad}\,\dot{\vec{x}} = \partial_x \dot{x} + \partial_y \dot{y} + \partial_z \dot{z}$

$$= \iiint \underbrace{(-\sigma - 1 - b)}_{<0}\, dV$$

$$= -(\sigma - 1 - b)\, V$$

Thus,

non-dimensional time...

$$\boxed{V(\tau) = V_0\, e^{-(1+\sigma+b)\tau}}$$

i.e., volume shrinks

This means there are no unstable nodes and no unstable cycles. Moreover, there are no quasi-periodic orbits (i.e., nothing of the form $x = A_1 \cos t + A_2 \cos \sqrt{2}t$) ← cannot happen

We have shrinkage ! (i.e., sinks) We also have stable cycles, and also saddles !

Fixed Points:

$$\begin{cases} x_* = y_* \\ r x_* - y_* - x_* z_* = 0 \\ x_* y_* = b z_* \end{cases} \leadsto \begin{cases} x_* = y_* \\ (r-1) x_* = x_* z_* \\ b z_* = x_*^2 \end{cases}$$

Thus, the 3 fixed points are:

$$\boxed{x_* = y_* = z_* = 0}$$

$$\boxed{\begin{aligned} &z_* = r-1 \\ &x_* = y_* = \pm\sqrt{b(r-1)} \quad \text{only if } r>1. \end{aligned}}$$

<u>Stability</u>: Linearize $\longrightarrow$ $A = \begin{pmatrix} -\sigma & \sigma & 0 \\ r-z & -1 & -x \\ y & x & -b \end{pmatrix}$

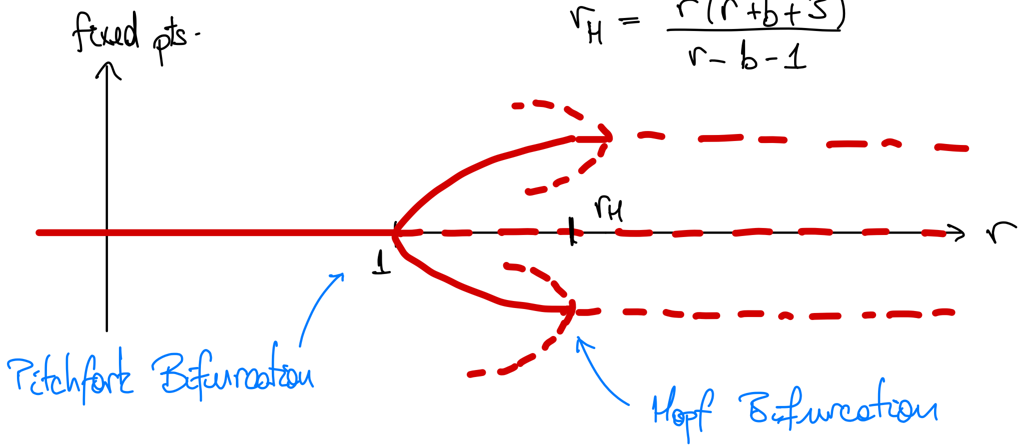
(i) <u>$x_* = y_* = z_* = 0$</u>: $\begin{pmatrix} -\sigma & \sigma & 0 \\ r & -1 & 0 \\ 0 & 0 & -b \end{pmatrix}$

stable on z

(ii) <u>Other fixed pts.</u>:

$$\begin{pmatrix} -\sigma & \sigma & 0 \\ 1 & -1 & \mp\sqrt{b(r-1)} \\ \pm\sqrt{b(r-1)} & \pm\sqrt{b(r-1)} & -b \end{pmatrix}$$

$$r_H = \frac{r(r+b+3)}{r-b-1}$$



LECTURE 20

CHAOS THEORY

22/11/2023

* Looking at the behavior of $x(t), y(t), z(t)$ as time goes by, there is no periodicity apparent. (Proven rigorously in '99).

* Fractal dimension (Hausdorff dimension > 2)

* <u>Sensitivity to initial conditions</u> (not varies cont. w/ initial conditions) : say $\vec{x}_0 = (0,1,0)$ initial condition and consider another ε-close initial condition $\vec{\tilde{x}}_0 = \vec{x}_0 + \varepsilon \hat{x}$, $\varepsilon = 10^{-10}$.

Then $\underbrace{\delta := \varepsilon e^{\lambda t}}$, λ = Lyapunov exponent.

i.e., the separation between solutions w/ very close init. conditions gets exponentially large as time passes.

* **Time horizon**: Define an absolute tolerance a.
When $\delta < a$, we are good; when $\delta \geq a$, we are sad.

$$t_{horizon} \approx O\left(\frac{1}{\lambda} \log\left(\frac{a}{\|\varepsilon\|}\right)\right).$$

CHAOS:

1) Long-term aperiodicity → no stable fixed pts / limit cycles
2) Deterministic → no randomness
3) Sensitivity to initial conditions → $\lambda > 0$ (Lyapunov)

ATTRACTOR: A closed set A is an attractor if A is (forward) invariant (start in A, stay in A); A attracts an open set U of initial conditions, $U \supset A$ (smallest U = basin of attraction). If $\vec{x}_0 \in U$, "distance" between $\vec{x}_0$ and $A \xrightarrow[t\to\infty]{} 0$. Finally, A needs to be minimal in the sense that it is the "smallest" set possible.

E.g.: if ϕ_t is the flow of Lorenz system, and U is a forward invariant open set (torus of genus 2 containing 0 but not the fixed pts). Then,

$$A = \bigcap_{t\geq 0} \phi_t(U).$$

An attractor is **STRANGE** if we have that sensibility to initial conditions.

LECTURE 21

24/11/2023

1D MAPS

LOGISTIC MAP: $x_{n+1} = \underbrace{r x_n (1-x_n)}_{=: f(x_n)}$, $r \in [0,4]$.

Fixed pts: $x_* = f(x_*)$

Define $x_n = x_* + \varepsilon_n$

$$\rightsquigarrow x_{n+1} = x_* + \varepsilon_{n+1} = f(x_* + \varepsilon_n)$$

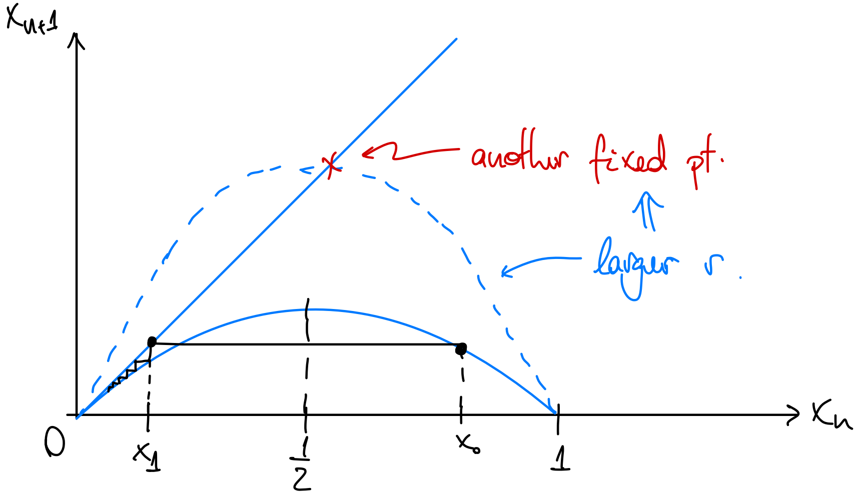
$$\approx \underbrace{f(x_*)}_{= x_*} + \varepsilon_n f'(x_*)$$

$$\Rightarrow \varepsilon_{n+1} = \varepsilon_n f'(x_*) \Rightarrow \boxed{\varepsilon_{n+1} = \lambda \varepsilon_n, \quad \lambda = f'(x_*)}$$

So, $\begin{cases} |\lambda| > 1 \Rightarrow \varepsilon_n \text{ grows} \Rightarrow \text{unstable} \\ |\lambda| < 1 \Rightarrow \varepsilon_n \text{ decays} \Rightarrow \text{stable} \\ |\lambda| = 1 \Rightarrow \text{inconclusive} \end{cases}$

$\lambda = 0 \Rightarrow$ superstable ($\sim$ exp. decay).

COBWEBBING:



Fixed points: $f(x) = rx(1-x)$

$$f(x_*) \stackrel{!}{=} x_* = r x_* (1 - x_*) \rightsquigarrow x_* = 0$$

Or: $1 = r(1-x_*) \longrightarrow x_* = \frac{r-1}{r}$ only for $r > 1$.

$f'(x) = r(1-2x)$; $\underset{\text{first fixed pt}}{f'(0)} = r \longrightarrow$ stable for $r < 1$
unstable for $r > 1$

second fixed pt.

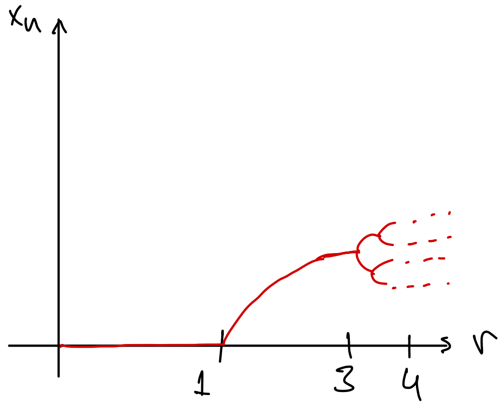
$f'\left(\frac{r-1}{r}\right) = 2 - r \Rightarrow -1 < 2 - r < 1 \Rightarrow 1 < r < 3$.

If $r > 3$, we have a period-2 cycle (2-cycle).

i.e., $\exists\, p, q$ s.t. $\begin{cases} f(f(p)) = p. \\ f(f(q)) = q. \end{cases}$

$$\rightarrow f(f(x_*)) = x_*$$

GRAPHICALLY: orbit diagram



FLIP BIFURCATION: unstable fixed pt. became 2 stable fixed pts. This produced a 2-cycle. Denote these 2 fixed pts. by p and q. Thus,

$$f(p) = q \;, \quad f(q) = p$$

$$f(f(p)) = p \;, \quad f(f(q)) = q$$

STABILITY OF P, Q FOR $f^2 = f \circ f$:

$$x_{n+1} = r x_n (1 - x_n)$$

$$x_{n+2} = r x_{n+1} (1 - x_{n+1}) = \underbrace{r x_n (1 - x_n) [1 - r x_n (1 - x_n)]}_{\text{RHS} = f^2} \stackrel{?}{=} x_n$$

Set $f^2(x_*) \overset{!}{=} x_*$. Then

$$x_* = 0 \qquad \text{or} \qquad x_* = \frac{r-1}{r}$$

So,

2nd order poly.

$$\text{RHS} = P_2(x_n)\, x_n \left(x_n - \frac{r-1}{r}\right) \overset{!}{=} 0$$

Finding these roots..., we find

$$\boxed{p, q = \frac{r+1 \pm \sqrt{(r-3)(r-1)}}{2r}}\,.$$

LYAPUNOV EXP.: $\lambda = \frac{d}{dx}\left(f^2(\dots)\right)$

$$= \dots\dots$$

$$= r\left[1 - 2(p+q) + 4pq\right].$$

$$= \dots$$

$$= 4 + 2r - r^2$$

$$|\lambda| < 1 \Rightarrow |4+2r-r^2| < 1 \rightsquigarrow 3 < r < 1+\sqrt{6}$$

Lecture 22

29/11/2023

Fractals

"Def." (Fractals) Shape that has fine structure at arbitrarily small scales.

Cantor set .... $\dim_H(C) = \frac{\log 2}{\log 3} \simeq 0.63$

Box dimension:

$$N(\varepsilon) \simeq \frac{1}{\varepsilon^d}, \quad \varepsilon \to 0$$

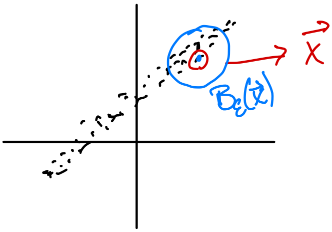
$$d = \lim_{\varepsilon \to 0} \frac{\log(N(\varepsilon))}{\log(1/\varepsilon)}.$$

e.g., smooth line: $d = \lim_{\varepsilon \to 0} \frac{\log(L/\varepsilon)}{\log(1/\varepsilon)} = 1$

e.g., Cantor set: nth iteration $C_n \to 2^n$ intervals of length $(1/3)^n$.

If $\varepsilon_n = \left(\frac{1}{3}\right)^n$, then $N(\varepsilon_n) = 2^n \Rightarrow d = \lim_{\varepsilon_n \to 0} \frac{\log(2^n)}{\log(3^n)} \simeq 0.63$

Compute $\dim_H$ w/ a computer:



Pick one of the pts., say $\vec{x}$. Then consider $B_\varepsilon(\vec{x})$ and count how many $\vec{x}_i$'s are in $B_\varepsilon(\vec{x})$. Then

$$\underbrace{N_{\vec{x}}(\varepsilon)}_{\text{"Pointwise dimension"}} \approx \varepsilon^d.$$

Take average of a bunch of the N's for various $\vec{x}$'s, then we get $C(\varepsilon) \approx \varepsilon^d$ "Correlation dimension"