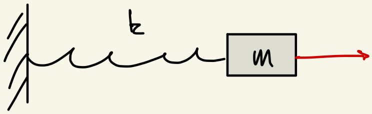
Lecture 1: Introduction

The most important ODE: Newton's Law $F = ma$



$x(t)$ position at time t

ODE because $v(t) = x'(t)$ and $a(t) = x''(t)$.

So, the force can depend on time, position, velocity, etc; i.e., Force $= F(x(t), x'(t), t, \dots)$.

Ex: Hooke's Law For a spring $F(x) = -kx$.
So, the equation is $mx''(t) = -kx(t)$.
↳ For a diff. equation, the unknowns are functions.

Ex: Pendulum Force $\propto$ - const$\cdot \sin\theta$
$$\Rightarrow m\theta'' = k \sin\theta$$

Solve an ODE. "educated guesses"

Consider $x''(t) = -\frac{k}{m} x(t)$. Try something w/ cos or sin.

Guess $x(t) = \cos(\omega t)$, $\omega > 0$ (parameter for later). Plug in:
$x'' = -\omega^2 \cos(\omega t) = -\omega^2 x$. Choose $\omega = \sqrt{\frac{k}{m}}$. It works!
$x(t) = \cos\left(\sqrt{\frac{k}{m}}\, t\right)$ is a solution for the ODE. Other solutions: $\sin(t\sqrt{k/m})$, $\cos^*((t-t_0)\sqrt{k/m})$. More generally,
$$x(t) = A\cos\left(t\sqrt{\tfrac{k}{m}}\right) + B\sin\left(t\sqrt{\tfrac{k}{m}}\right)$$
are solutions for $A, B \in \mathbb{R}$.

* Can translate the origin (starting time)

Q: Are these all the solutions? Yes, but this needs a proof.

What determines A and B? Suitable initial values.

This is a 2-dimensional solution space.

Def: (ODE) An ODE is an equation of the form

Implicit form → $$F(t, x(t), x'(t), \ldots, x^{(k)}(t)) = 0$$ Order of the ODE

Independent variable

Dependent variable (k-th derivative)

- x is a vector valued function on an open interval on $\mathbb{R}$, which is k-times continuously differentiable.

We can (sometimes) solve for $x^{(k)} = G(t, x, x', \ldots, x^{(k-1)})$ "in the explicit form."

Def: (CLASSICAL SOLUTION) A solution to the ODE written in the above definition is a function $\phi: I \subset \mathbb{R} \to \mathbb{R}$ which is C^k and such that $F(t, \phi, \phi', \ldots, \phi^{(k)}) = 0, \ \forall t \in I$.

NON-EXAMPLE: Take $x + y \cdot y' = 0$. The equation $y = \sqrt{-(1+x^2)}$ is not a solution because it's not a real valued function.

NON-EXAMPLE: Take $\left|\frac{dy}{dx}\right| + |y| + 1 = 0$ has no solutions.

- Usually, the independent variable is time, but it could be literally anything.
- In physics, we can write $\dot{x} = \frac{dx}{dt}$ (Just clearly explain it clearly and be consistent).

> **Def:** (GENERAL SOLUTION OF AN ODE) The general solution of an ODE is a formula for all possible solutions.

Ex: For the spring, $mx'' + kx = 0$, the general solution is $\phi(t) = A\cos\left(t\sqrt{k/m}\right) + B\sin\left(t\sqrt{k/m}\right)$.

STANDARD TRICK: go from a higher order ODE to a system of 1^{st} order ODEs.

Ex: $mx'' = -kx - cx'$, where $m, k, c > 0$. Introduce $x_1 := x$ and $x_2 := x'$. So, write

friction constant: c

$$\begin{cases} x_1' = x_2 \\ m x_2' = -kx_1 - cx_2 \end{cases} \longleftrightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -\frac{k}{m}x_1 - \frac{c}{m}x_2 \end{pmatrix}.$$

More generally, if $F(t, x, x', \dots, x^{(k)}) = 0$, write

$$\underbrace{\begin{cases} x_1' = x_2 \\ \vdots \\ x_{k-1}' = x_k \end{cases}}_{k \text{ equations}} \quad \text{and} \quad F(t, x_1, x_2, \dots, x_k) = 0.$$

Lecture 2:

MAT267

Philosophy of the course:

Vector field on $\mathbb{R}^n$; C^1-diffeo. / C^r / C^∞

geometry / physics / biology / etc... $\longrightarrow$ Modeling $\longrightarrow$ ODE $\xrightarrow{\text{solve}}$ Solution

Predictions

Ex: 1. $x' = 0$ on $\mathbb{R}$.

General solution: $x(t) = c$, $c \in \mathbb{R}$ (1-dim solution space)

2. $x' = f(t)$

General solution: By FTC, $x(t) = x_0 + \int_0^t f(s)\, ds$

3. Let $n = 1$. $x' = ax$, $a \in \mathbb{R}$ fixed constant

General solution: guess $x(t) = e^{at}$ is a solution.

But $x(t) = c\, e^{at}$, $c \in \mathbb{R}$ is solution.

Pf: (No other solution) Consider $y(t) = e^{-at} x(t)$.

Assume x is a solution, so that $x' = ax$. Note $y \in C^1$ (b/c $x \in C^1$). Compute

$$\begin{aligned} y'(t) &= -a e^{-at} x(t) + e^{-at} x'(t) \\ &= -a e^{-at} x(t) + e^{-at} \cdot a x(t) \\ &= 0. \end{aligned}$$

$\Rightarrow y(t) = \text{const.} := c \Rightarrow x(t) = c e^{at}$. $\square$

↳ It's easier to guess a solution and check.

What is the "correct" c? Solve an initial value problem.

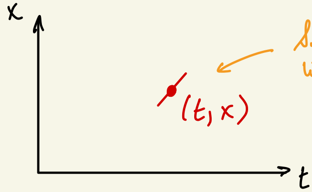
$x' = ax$; $x(0) = u_0$ ← get c

Could be any

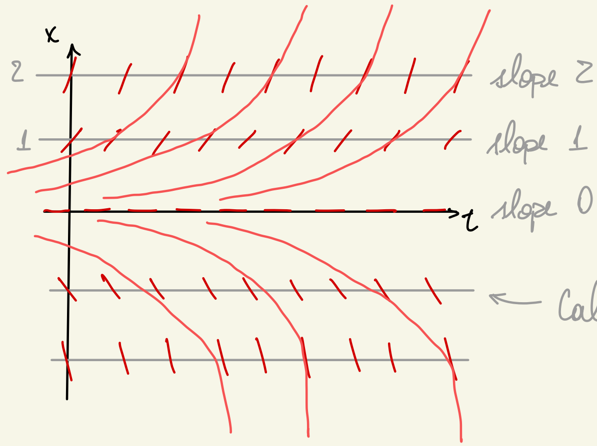
Pictures: $x' = ax \longrightarrow x(t) = ce^{at}$ (say $a=1$)

$\Rightarrow x' = x$

For $x' = f(t,x)$:



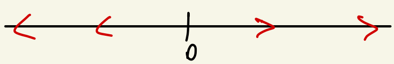
Now, for $x' = x$ (autonomous)



- Solutions to the ODE are curves whose tangent at point (t,x) is $f(t,x)$ $\forall (t,x)$ in the graph of curve

$\Rightarrow$ Solving a equation = finding a function tangent to the slope field

Now,

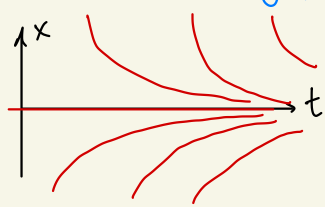


(phase / portrait line)

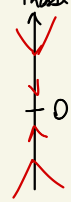
Unstable ("source")

Equilibrium / stationary point

If $a<0$, we have



Phase portrait



stable Equilibrium "sink"

→ The equation $x'=ax$ is **stable** if $a\neq 0$.

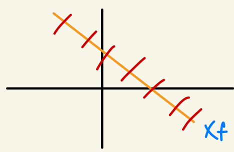
↳ Meaning, if a is replaced by another constant $b\neq a$ with the same sign of a, the qualitative behaviour of the solution does not change.

(Poincaré, 1885)

* But, if $a=0$, then the slightest change in a (in any sign direction) leads to a radical change in the behaviour of solutions.

⇒ We have a **bifurcation** at $a=0$ in the one-parameter family of equations $x'=ax$.

Ex: $y' = x+y$



$x+y=3$ (non-horizontal isocline)

EXAMPLE: LOGISTIC EQUATION: Consider

$$\boxed{x' = ax\left(1-\frac{x}{N}\right), \quad x\in\mathbb{R}},$$

Exponential populational growth

Changing units

where a and N are fixed constants. WLOG, set $N=1$, then

growth rate

carrying capacity

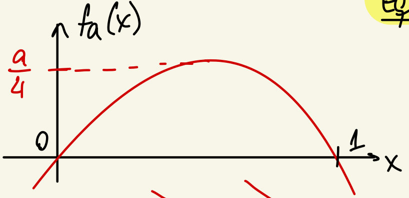
$$\boxed{x' = f_a(x) = ax(1-x)}$$

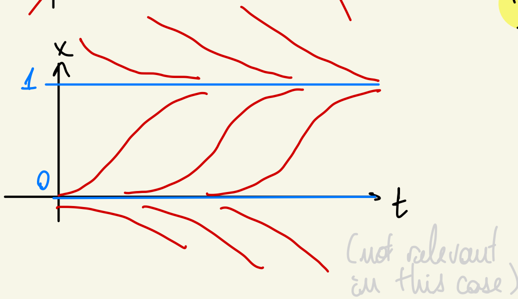
Autonomous

1st order

Nonlinear function of x

Picture:

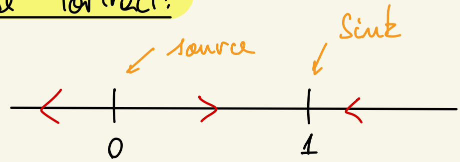




Equilibrium points: $ax(1-x) \overset{!}{=} 0 \to x = 0, 1.$

So, stable eq. points are $x(t) = 0, 1$.

Phase Portrait:



Solve: $x' = ax(1-x)$. Already have 2 solutions: $x(t) \equiv 0$ and $x(t) \equiv 1$ (note that linear combinations of these are NOT solutions b/c f_a is not linear). Now, whenever

$$\frac{dx}{dt} = f(t)\,g(x)$$

with f, g continuous, we can use **SEPARATION OF VARIABLES**.

- **Formally,**

$$\frac{dx}{g(x)} = f(t)\,dt \to \int \frac{dx}{g(x)} = \int f(t)\,dt$$

$$\to G(x) = F(t) + C, \quad G' = \frac{1}{g}, \quad F' = f$$

$\to$ Solve for x:

$$x = G^{-1}(F(t) + C)$$

- **Justification,** we have that

$$\frac{dx}{dt} = f(t)\,g(x) \to \frac{x'(t)}{g(x(t))} = f(t) \Rightarrow \int \frac{x'(t)}{g(x(t))}\,dt = \int f(t)\,dt$$

Change Variables $\to \int \frac{dx}{g(x)}$

$$\Rightarrow G(x(t)) = F(t) + C$$

Take derivatives: $\frac{d}{dt} G(x(t)) = \frac{x'(t)}{g(x(t))} = f(t)$.

So, the solution to $G(x) = F(t) + C$ is a solution to the original ODE. We are solving around (t_0, x_0) s.t. $g(x_0) \neq 0 \Rightarrow g(x) \neq 0$ in some neighborhood of x_0. So, $G'(x) = \frac{1}{g(x)}$. The solution passing through (t_0, x_0) corresponds to $C = G(x_0) - F(t_0)$.

$\square$

REMARK. if f, g are continuous, the equation with separable variables has a UNIQUE solution in a neighborhood of any point (t_0, x_0) such that $g(x_0) \neq 0$.

For the Logistic Equation $x' = ax(1-x)$,

$$\frac{dx}{dt} = ax(1-x) \Rightarrow \int \frac{dx}{x(1-x)} = \int a\, dt$$

$\frac{1}{x(1-x)} = \frac{A}{x} + \frac{B}{1-x}$

$= \frac{A(1-x)+Bx}{x(1-x)} \quad \forall x$

$x = 0 \to A = 1$

$x = 1 \to B = 1$

$$\Rightarrow \int \frac{1}{x}\, dx + \int \frac{1}{1-x}\, dx = a(t+c)$$

$$\Rightarrow \frac{1}{a}\left(\log|x| - \log|1-x|\right) = t + C$$

$$\Rightarrow \frac{1}{a} \log\left|\frac{x}{1-x}\right| = t + C$$

$$\Rightarrow \left|\frac{x}{1-x}\right| = e^{a(t+c)} = |C_2| e^{at}$$

$$\Rightarrow \frac{x}{1-x} = C_2 e^{at} \Rightarrow x = C_2(1-x)e^{at}$$

$$\Rightarrow \boxed{x = \frac{C_2 e^{at}}{1 + C_2 e^{at}}}.$$

We have to assume $x \neq 1, 0$.

- If $x = 0$, $C_2 = 0$
- If $x = 1$, "$C_2 = \infty$"

Write $x(t) = \dfrac{e^{at}}{C_3 + e^{at}}$ → $C_3 = 0 \to x(t) = 1$

→ $x(t) \equiv 0$??? "$C_3 = \infty$"

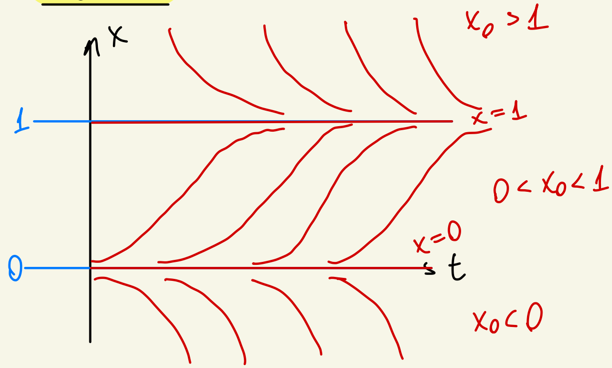
<u>Better parametrization</u>: identify C_2 in terms of the initial values. Note that

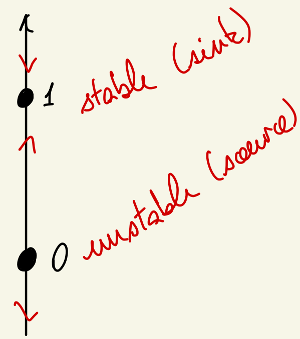
$$x(0) = \frac{C_2}{C_2 + 1} \Rightarrow C_2 = \frac{x(0)}{1 - x(0)}.$$

Thus

$$\boxed{x(t) = \frac{C_2 e^{at}}{1 + C_2 e^{at}} = \frac{x(0)e^{at}}{1 - x(0) + x(0)e^{at}} = \frac{x(0)}{(1-x_0)e^{-at} + x(0)}}.$$

Picture:





* **APPENDIX:** Can solutions cross? If they did, could only do it tangentially.

"<u>Thm</u>": (Existence and Uniqueness simplified) Solving $\underbrace{x' = F(t, x)}_{\text{system of } n \text{ eq.}}$, $x \in \mathbb{R}^n$, with initial value $x_0 = t_0$.

If F is C^1 on $\overset{\text{time}}{I} \times \overset{\text{domain}}{U}$, then, for $t_0 \in I$, $\forall x_0 \in U$ there <u>exists</u> a <u>unique</u> solution $x(t)$ with $x(t_0) = x_0$. Note that $x(t)$ is defined on a time interval J (which depends on t_0, and x_0).

smooth / $I = \mathbb{R}$ / $U = \mathbb{R}$

In our case, $x' = \overbrace{ax(1-x)}^{\text{smooth}}$. So, there exists a solution for some time and it's unique.

- If solutions crossed, at the crossing (or starting) points, we would have more than one solution!!

↳ // ⟷ Existence & Uniqueness

Moreover, since the solutions we found cover all of $\mathbb{R}^2$, this is the general solution

EXAMPLE: $\dfrac{dy}{dx} = \sqrt{|y|}$. Note that $y \equiv 0$ is a solution

Separation of variables: $\dfrac{dy}{\sqrt{|y|}} = dx \overset{y>0}{\Longrightarrow} 2\sqrt{y} = x - C$

$\Rightarrow y = \frac{1}{4}(x-C)^2$, $x \in \underbrace{(C, \infty)}_{y>0}$ is a solution $\forall C \in \mathbb{R}$.

Similarly, (for $y<0$), $y = -\frac{1}{4}(x-C)^2$ on $x \in \overbrace{(-\infty, C)}^{y<0}$ is a solution $\forall C \in \mathbb{R}$. So, combine everything to create

INFINITELY MANY SOLUTIONS: $x \in \mathbb{R}$, $\forall a < b$

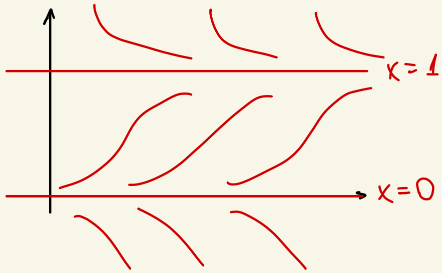
$$y_{ab}(x) = \begin{cases} -\dfrac{(x-a)^2}{4}, & x < a \\ 0, & a \leq x \leq b \\ \dfrac{(x-b)^2}{4}, & x > b \end{cases}$$

$\Rightarrow$ The initial value problem $\begin{cases} y' = \sqrt{|y|} \\ y(0) = 0 \end{cases}$ has infinitely many solutions!

LECTURE 3: LINEAR SYSTEMS OF ODEs

Recall from last time:

$$x' = ax(1-x) \longrightarrow$$ 2 equilibrium points $x=0$ and $x=1$.



Set $f_a(x) = ax(1-x)$

- Near $x=0$, if $x<0$, then $f_a(x) < 0 \Rightarrow$ slopes are negative $\Rightarrow$ "solutions go down"
- If $x>0$, $f_a(x) > 0 \Rightarrow$ slopes are positive $\Rightarrow$ "solutions go up"

$x=0$ is a source

Similarly, we can read off that $x=1$ is a sink.

Take $f_a'(x) = a - 2ax$. So $f_a'(0) = a > 0 \rightarrow$ slopes increase through zero as x passes through zero.

slopes <0 below 0, >0 above 0

Similarly, $f_a'(1) = -a < 0 \Rightarrow x=1$ is a sink.

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* LINEAR SYSTEMS OF ODEs:

$$X' = A(t)X + f(t)$$

$A(t)$: given $n\times n$ matrix — Matrix of coefficients

$f(t)$: Inhomogeneity (given function from interval in $\mathbb{R}^n$)

- If A is a constant matrix (not t dependent), then the equation is called "CONSTANT COEFFICIENT ODE".
- If $f(t) = 0$, then ODE is called "HOMOGENEOUS".
- If $X' = F(X)$, and $\exists \vec{x}_0$ such that $F(\vec{x}_0) = 0$, then $\vec{x}_0$ is an equilibrium point for the system.
  - $\Rightarrow$ If $F(\vec{x}) = A\vec{x}$, A = constant, then zero is always an equilibrium point.
  - $\Rightarrow$ If $\det A = 0$ (but $A \neq 0$), then there exists a straight line of equilibrium points.
- Start with 2x2 matrices.

$\boxed{\text{Ex}}$: Consider

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

If we could diagonalize, $\tilde{x}' = a\tilde{x}$ and $\tilde{y}' = b\tilde{y}$, which we know. $(\tilde{x}, \tilde{y})$ change of variables.

$\rightarrow$ Two approaches:

1) For $n = 1$, $x' = ax$, $a \in \mathbb{R}$ has solution $x(t) = e^{at} x_0$. So, we guess that $x(t) = e^{tA} x_0$, $x_0 \in \mathbb{R}$, solves the given equation. But... what is "e^{tA}"?

Def: (EXPONENTIAL) set, through the power series:

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$

$\hookrightarrow$ Need to converge $\forall t$ and it must be differentiable:

$$\frac{d}{dt} e^{tA} = A e^{tA},$$

and obey some more properties of the exponential.

So, assuming all of this works, $x(t) = x_0 e^{tA}$ is a solution of

$$\begin{cases} x' = Ax \\ x(0) = x_0 \end{cases},$$

and this is the general solution. (same proof as for $n=1$).

"ANSATZ" (attempt in german)

2) <u>Educated guess</u>: exponential growth and decay.

So, we seek solutions of the form $x(t) = e^{\lambda t}\vec{v}$, λ is unknown parameter.

We have to determine λ and $\vec{v}$.

vector

scalar

constant vector in $\mathbb{R}^n$

So, check: $x(t) \in C^1$ ✓

$$\begin{cases} x'(t) = \lambda e^{\lambda t}\vec{v} \\ Ax = A(e^{\lambda t}\vec{v}) = e^{\lambda t} A\vec{v} \end{cases} \Rightarrow$$

$x(t) = e^{\lambda t}\vec{v}$ is a solution to $x' = Ax \Leftrightarrow e^{\lambda t}\lambda\vec{v} = e^{\lambda t}A\vec{v}$ $\forall t$.

$\Leftrightarrow A\vec{v} = \lambda\vec{v}$; i.e., if λ eigenvalue w/ eigenvector $\vec{v}$.

Apply this to the example:

$$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}; \text{ i.e., } \begin{cases} x' = 2x + 3y \\ y' = x \end{cases}.$$

Find the eigenvalues / eigenvectors of A.

$$\det(A - \lambda I) = \det\begin{pmatrix} 2-\lambda & 3 \\ 1 & -\lambda \end{pmatrix} = (2-\lambda)(-\lambda) - 3 \stackrel{!}{=} 0$$

$$\Rightarrow \lambda^2 - 2\lambda - 3 = 0 \Rightarrow \lambda_1 = 3 \text{ and } \lambda_2 = -1.$$

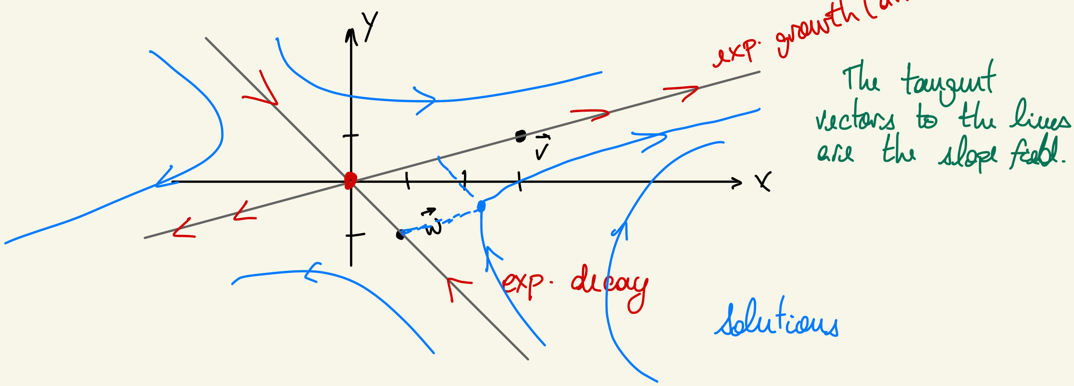
$$(A - \lambda_1 I)\vec{v} = \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{v} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$$

$$(A - \lambda_2 I)\vec{w} = \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \stackrel{!}{=} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \vec{w} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

So, we constructed two solutions to $X' = AX$:

$$x_1(t) = e^{3t}\begin{pmatrix}3\\1\end{pmatrix} \;;\; x_2(t) = e^{-t}\begin{pmatrix}1\\-1\end{pmatrix}.$$

Now, we can draw the phase portrait:



Observe that $x(t) = a_1 x_1(t) + a_2 x_2(t)$ is also a solution. Solving $X' = AX$ is equivalent to finding curves tangent to the vector field $f(x) = Ax$.

* **SUPERPOSITION PRINCIPLE**: Suppose $x_1(t)$ solves

$$x' = A(t)x + f_1(t)$$

and $x_2(t)$ solves

$$x' = A(t)x + f_2(t).$$

Let $a_1, a_2 \in \mathbb{R}$. Then

$$x(t) = a_1 x_1(t) + a_2 x_2(t)$$

solves

$$x' = A(t)x + a_1 f_1(t) + a_2 f_2(t)$$

Pf: Plug in.

Consequences:

1) The solutions of the homogeneous equation
$$x' = A(t)x$$
form a vector space.

2) The general solution to
$$x' = A(t)x + f(t)$$
is given by $x(t) = x_{genhom}(t) + y(t)$, where $x_{genhom}(t)$ is the general solution of $x' = A(t)x$ (homogeneous equation) and $y(t)$ is one particular solution of $x' = A(t)x + f(t)$.

—————— // ——————

Lecture 4: Systems of ODEs (continued)

Consider $x' = Ax$; $x \in \mathbb{R}^n$ and $A \in \mathbb{R}^{n\times n}$ and A is constant (i.e., no t-dependence). Also, assume that $v_1, \dots, v_n$ are linearly independent eigenvectors of A, i.e., $Av_i = \lambda_i v_i$, some $\lambda_1, \dots, \lambda_n \in \mathbb{R}$.

- We saw: $x_i(t) = e^{\lambda_i t} v_i$ are solutions with initial value $x_i(0) = v_i$.
- Superposition Principle: get an n-dim. vector space of solutions:

$$x(t) = a_1 x_1(t) + \cdots + a_n x_n(t); \quad a_i \in \mathbb{R}.$$

Claim: This $x(t)$ is the general solution.

Pf 1: (Nuke to kill a bug) Existence and uniqueness theorem.

Pf 2: By assumption $\{v_1, \dots, v_n\}$ is a basis for $\mathbb{R}^n$ (A diagonalizable). Fix $a_1, \dots, a_n \in \mathbb{R}$ and let

$$y(t) = a_1 e^{\lambda_1 t} v_1 + \cdots + a_n e^{\lambda_n t} v_n.$$

By the superposition principle, y solves the initial value problem (IVP)

$$\begin{cases} x' = Ax \\ x(0) = a_1 v_1 + \cdots + a_n v_n \end{cases}$$ (plug in $t=0$ in y)

will be true in general but w/ a different proof (other assumptions)

Now, we have to prove that it is the <u>only</u> solution of this IVP (solutions are uniquely determined by their initial values).

Let $z(t)$ be another solution of the same IVP. Since $\{v_1, \dots, v_n\}$ is a basis of $\mathbb{R}^n$, we can write

$$z(t) = b_1(t) v_1 + \cdots + b_n(t) v_n,$$ b_i real-valued function

(time-dependent and uniquely determined by z)

We also know that $b_i(0) = a_i$ because $z_0 = x_0 = y(0)$.

Now,

$$z'(t) = b_1'(t) v_1 + \cdots + b_n'(t) v_n$$

$$\begin{aligned} Az(t) &= A\,(b_1(t) v_1 + \cdots + b_n(t) v_n) \\ &= b_1(t) \lambda_1 v_1 + \cdots + b_n(t) \lambda_n v_n \end{aligned}$$

By assumption, $Az(t) = z'(t)$. So,

$$b_i'(t) = \lambda_i b_i(t); \qquad b_i(0) = a_i$$

$$\Rightarrow b_i(t) = e^{\lambda_i t} a_i \Rightarrow z = y.$$

$\square$

→ This wouldn't be true if A depended on t in a "non-differentiable" manner.

REMARK: $b_i(t)$ are differentiable because they are the composition of two differentiable maps: $z(t)$ and a linear projection with constant coefficients (to project the coefficient of v_i). However, note that this projection may not be an orthogonal projection (will not be if $\{v_i\}$ is not an orthonormal basis).

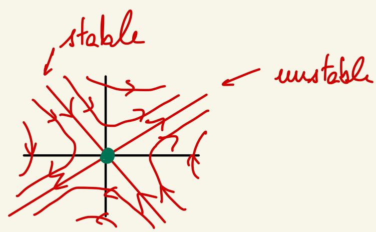
* CONCLUSION: the solution to $x' = Ax$ are uniquely determined by the initial value $x(0) = x_0 \in \mathbb{R}^n$.

—— // ——

EXAMPLE 1: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$

$\lambda_1 = 3 \longrightarrow v_1 = (3, 1)$

$\lambda_2 = -1 \longrightarrow v_2 = (1, -1)$



This is called a "SADDLE POINT"

These are all the solutions.

Def: We say that the system $X' = AX$ has a **SADDLE POINT** at zero if A has a positive and a negative eigenvalue (say $\lambda_2 < 0 < \lambda_1$).

EXAMPLE 2: Assume $\lambda_1, \lambda_2 > 0$. Take $B = A + 2I = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$.

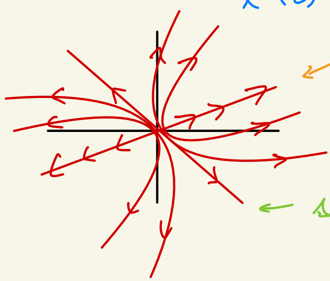
$\lambda_1 = 5 \to v_1 = (3, 1)$; $\lambda_2 = 1 \to v_2 = (1, -1)$

So, all solutions grow exponentially over time

$$X(t) = a_1 e^{5t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} + a_2 e^{t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

faster

slower



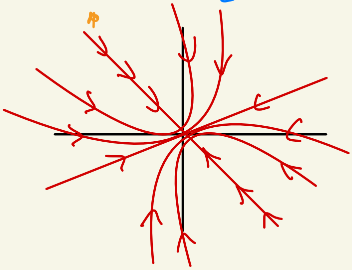
"UNSTABLE NODE"

So, integral curves (solutions) – not on any line defined by eigenvalues – are tangent to the slow direction.

EXAMPLE 3: Let $C = A - 5I = \begin{pmatrix} -3 & 3 \\ 1 & -5 \end{pmatrix}$.

$\lambda_1 = -2 \to v_1 = (3, 1)$ * slower

$\lambda_2 = -6 \to v_2 = (1, -1)$ * faster to zero



"SINK OR STABLE NODE" at origin

All solutions decay exponentially to 0.

* **COMPLEX EIGENVALUES**: Almost always identify $\mathbb{C} \simeq \mathbb{R}^2$. Now,

LEMMA: Let $A \in \mathbb{R}^{n\times n}$. Suppose v is an eigenvector of A associated to eigenvalue λ. If λ is <u>not</u> real (i.e., $\alpha + i\beta$, $\beta \neq 0$), then

1) v is not real. In fact, $\mathrm{Re}(v)$ and $\mathrm{Im}(v)$ are linearly independent.
2) $\bar{\lambda}$ is also and eigenvalue with eigenvalue $\bar{v}$.

Pf: 2) $Av = \lambda v \iff \overline{Av} = \overline{\lambda v}$ (in $\mathbb{C}^n$)

A is real $\iff A\bar{v} = \bar{\lambda}\bar{v}$.

1) Suppose $\lambda = \alpha + i\beta$, $\beta \neq 0$. Suppose $Av = \lambda v$, and $v = u + iw$, $u, w \in \mathbb{R}^n$. We claim that u and w are linearly independent. Assume not. Then $u = s v_0$ and $w = t v_0$, for some $s, t \in \mathbb{R}$ and $v_0 \in \mathbb{R}^n$. So, $v = u + iw = \underbrace{(s+it)}\, v_0$.

Complex multiple of v_0

Hence v_0 is an eigenvector (b/c v is an eigenvector), i.e., $\underbrace{\underset{\in \mathbb{R}^{n\times n}}{A}\,\underset{\in \mathbb{R}^n}{v_0}}_{\text{Real}} = \lambda v_0 = \underbrace{\underset{\notin \mathbb{R}}{(\alpha + i\beta)}\,\underset{\in \mathbb{R}^n}{v_0}}_{\text{Not real}}$

$\hookrightarrow // \hookleftarrow$

$\Rightarrow \lambda \in \mathbb{C}\setminus\mathbb{R} \Rightarrow v$ is complex $\square$

EXAMPLE 4: (NO REAL EIGENVALUES) Let

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

The eigenvalues are:

$$\det(J - \lambda I) = \det\begin{pmatrix} -\lambda & -1 \\ 1 & -\lambda \end{pmatrix} = \lambda^2 + 1 \overset{!}{=} 0$$

$\Rightarrow \lambda_1 = i$ and $\lambda_2 = -i$.

The eigenvectors:

λ_1: $(J - iI) = \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \Rightarrow v_1 = \begin{pmatrix} i \\ 1 \end{pmatrix} \in \ker(J - iI)$

λ_2: $v_2 = \overline{v_1} = \begin{pmatrix} -i \\ 1 \end{pmatrix}$

* Complex solutions are $z(t) = e^{it} v_1$ and $\overline{z(t)} = e^{-it} v_2$.

Claim: z is a complex solution of $x' = Ax$ if and only if $z_{re} = \mathrm{Re}(z)$ and $z_{im} = \mathrm{Im}(z)$ are solutions of $x' = Ax$ $(A \in \mathbb{R}^{n \times n})$.

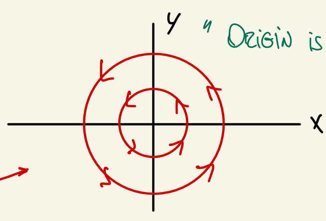
Pf: $\underline{z_{Re}'(t)} + i\,\underline{z_{Im}'(t)} = z'(t) = Az(t) = A(z_{Re}(t) + i z_{Im}(t))$
$= \underline{A z_{Re}(t)} + i\,\underline{A z_{Im}(t)}$ $\square$

Now, $\mathrm{Re}(z) = \frac{1}{2}\left(z(t) + \overline{z(t)}\right) = \frac{1}{2}\left(e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} + e^{-it}\begin{pmatrix} -i \\ 1 \end{pmatrix}\right)$

$$= \frac{1}{2}\begin{pmatrix} i(e^{it} - e^{-it}) \\ e^{it} + e^{-it} \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}.$$

$$\mathrm{Im}(z) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}.$$

"ORIGIN IS A 'CENTER'"



If A has eigenvalues $\pm ai$, $a \neq 0$ we say $x' = Ax$ has a center at 0.

Lecture 5: Other Techniques

$$F(x,y) \longrightarrow F(tx,ty) = t^{\alpha} F(x,y)$$

then F is homogeneous of degree $\alpha \in \mathbb{R}$.

* Consider the ODE

$$P(x,y)\,dx + Q(x,y)\,dy = 0 \qquad (*)$$

such that P and Q have the same degree of homogeneity. Then

$$y = xu \quad (u = x/y), \quad dy = u\,dx + x\,du$$

or $x = yv \quad (v = x/y), \quad dx = v\,dy + y\,dv$

changes $(*)$ into a separable ODE.

Ex: Consider

$$\left[x e^{y/x} - y\sin(y/x)\right] dx + x \sin\left(\frac{y}{x}\right) dy = 0.$$

Let $u = y/x \rightarrow y = xu$. So, we have

$$\left(x e^{u} - xu \cancel{\sin(u)}\right) dx + x\sin u\left(x\,du + \cancel{u\,dx}\right) = 0$$

$$\cancel{x} e^{u}\,dx + x^{\cancel{2}} \sin u\,du = 0 \qquad x \neq 0$$

$$-\frac{dx}{x} = \frac{\sin u}{e^{u}}\,du$$

$$-\log|x| = -\frac{1}{2}\left(\frac{\sin u + \cos u}{e^{u}}\right) + \underline{\underline{C}}$$

$$\Rightarrow \boxed{\log(x^2) - \frac{\sin\left(\frac{y}{x}\right) + \cos\left(\frac{x}{y}\right)}{e^{y/x}} = \underline{\underline{C}}}$$

Not the same! But they're arbitrary

RECALL: system with no real eigenvalues

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad X' = JX.$$

We have $\lambda_1 = i$ and $\lambda_2 = -i = \overline{\lambda_1}$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$v = (i, 1) \qquad\qquad \bar{v} = (-i, 1)$$

So,

$$\underbrace{z(t)}_{\text{complex solution}} \underset{\mathbb{C}}{=} e^{it}\begin{pmatrix} i \\ 1 \end{pmatrix} = (\cos t + i \sin t)\left[\begin{pmatrix} 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]$$

$$= \underbrace{\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}}_{Re(z)} + i\underbrace{\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}_{Im(z)}$$

linearly independent solutions

and $\bar{z}(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} - i\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$.

NOTE: the general solution of the system is

$$\boxed{x(t) = a\begin{pmatrix} -\sin t \\ \cos t \end{pmatrix} + b\begin{pmatrix} \cos t \\ \sin t \end{pmatrix}}.$$

Pf: Suppose $y(t) = \begin{pmatrix} u(t) \\ v(t) \end{pmatrix}$ is a solution. Let

$$f(t) = (u(t) + i v(t))e^{-it}.$$

Differenciate, use that $y(t)$ is a solution

$$f'(t) = 0 \Rightarrow u(t) + i v(t) = \eta e^{it}, \quad \eta \in \mathbb{C}.$$

$\Rightarrow$ y is a linear combination of $Re(z)$ and $Im(z)$. $\square$

EXAMPLE 5: Consider $A = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$ and $X' = AX$.

The eigenvalues are $\lambda_1 = 2+i$ and $\lambda_2 = \bar{\lambda}_1 = 2-i$.

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$v = (i, 1) \quad \text{and} \quad \bar{v} = (-i, 1)$$

So, the general complex solution is

$$z(t) = c_1 e^{\lambda_1 t} v + c_2 e^{\bar{\lambda}_1 t} \bar{v}; \quad c_1, c_2 \in \mathbb{C}$$

The general real solution is

$$x(t) = \mathrm{Re}(z(t)) = \frac{1}{2}\left(c e^{\lambda_1 t} v + \bar{c} e^{\bar{\lambda}_1 t} \bar{v}\right)$$

(True, but not very explicit...)

The general explicit real solution is

$$z = e^{\lambda t} v; \quad \bar{z} = e^{\bar{\lambda} t} \bar{v} \Rightarrow 2 \text{ real solutions:}$$

$$\begin{cases} x_1(t) = \mathrm{Re}(z(t)) = \frac{1}{2}(z + \bar{z}) \\ x_2(t) = \mathrm{Im}(z(t)) = \frac{1}{2i}(z - \bar{z}) \end{cases}$$

Expand to find x_1 and x_2:

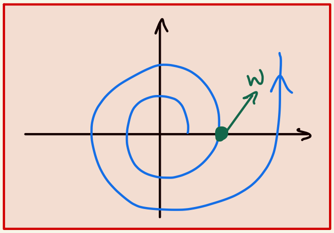
$$z(t) = e^{\lambda t} v = e^{(2+i)t}\left[i\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right]$$

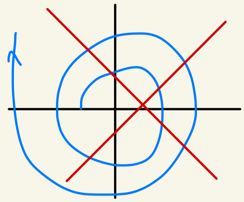
$$= e^{2t}(\cos t + i \sin t)\left[\begin{pmatrix}0\\1\end{pmatrix} + i\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$= \underbrace{e^{2t}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}}_{\mathrm{Re}(z) =: x_1} + i \underbrace{e^{2t}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}}_{\mathrm{Im}(z) =: x_2}$$

Thus, general solution: $x(t) = c_1 x_1(t) + c_2 x_2(t), \quad c_1, c_2 \in \mathbb{R}$.

Phase portrait: $x_1(t) = \underset{\text{radius grows}}{e^{2t}} \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$ ← 2π-periodic





To decide between counterclockwise or clockwise, take an "easy vector" $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and plug it

$$x' = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \underbrace{\begin{pmatrix} 2 \\ 1 \end{pmatrix}}_{=w}$$

$= w$ tangent to the solution.

EXAMPLE 6: (REPEATED EIGENVALUES) $x' = Ax$, $A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$,

whose eigenvalues are $\lambda = 1$ (Double)

ALGEBRAIC MULTIPLICITY

the eigenvectors are $v = (1, 2)$

$2 - 1 = 1$ linearly independent eigenvectors

GEOMETRIC MULTIPLICITY

get a 1-dim space of solutions

$$x(t) = c e^{\lambda t} \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad c \in \mathbb{R}$$

But we expected a 2-dim solution space!

$\Rightarrow$ generalized eigenvectors

Solve $(A - \lambda I) w = v \in \operatorname{im}(A - \lambda I)$, i.e., $(A - \lambda I)^2 w = 0$

$w \in \operatorname{Ker}(A - \lambda I)^2 \setminus \operatorname{Ker}(A - \lambda I) \Rightarrow w = (0, -1)$

So, $v = (1, 2)$ and $w = (0, -1)$

$Av = \lambda v$ and $Aw = \lambda w + v$.

We thus seek a solution of the form

$$x(t) = \alpha(t)\, v + \beta(t)\, w, \quad \beta \neq 0.$$

LECTURE 6: GENERALIZED EIGENVALUES

Recall: $P(x,y)\,dx + Q(x,y)\,dy = 0$

$$\Rightarrow P(x,y) + Q(x,y)\frac{dy}{dx} = 0$$

"Formally divide by dx or dy" $\Rightarrow P(x,y)\dfrac{dx}{dy} + Q(x,y) = 0.$

We don't know what is dependent/independent here.

But it works!

Similarly, $u = x/y \longrightarrow y = xu$

$dy = x\,du + u\,dx$

EXAMPLE 6 (continued): Consider $x' = Ax$ where

$$A = \begin{pmatrix} 3 & -1 \\ 4 & -1 \end{pmatrix}$$

$\Rightarrow \chi_A(\lambda) = \lambda^2 - 2\lambda + 1 \longrightarrow \lambda = 1$ eigenvalue w/ (algebraic) multiplicity 2.

Also, $v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector $\Rightarrow$ 1-dim space of solutions $x(t) = c e^{t}\begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $c \in \mathbb{R}$.

We seek another linearly independent solution *

$\Rightarrow$ Use generalized eigenvectors:

$$(A-\lambda I)w = v \in \ker(A-\lambda I); \quad \text{i.e., } (A-\lambda I)^2 w = 0.$$

E.g., $w = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$. Thus,

$$\left.\begin{cases} v = (1,2) \\ w = (0,-1) \end{cases}\right\} \text{ form a basis and } \quad \begin{array}{l} Av = \lambda v \\ Aw = \lambda w + v \end{array}$$

Now, we seek a solution of the following form

$$(\ast) \quad x(t) = \alpha(t)\, v + \beta(t)\, w, \quad \beta(t) \neq 0. \quad \text{(b/c already have such solution)}$$

If $x \in C^1$, then $\alpha, \beta \in C^1$. Plug in $(\ast)$ into ODE:

$$x'(t) = \alpha'(t) v + \beta'(t) w \quad \text{and}$$

$$Ax = \alpha(t) Av + \beta(t) Aw = \alpha(t)\lambda v + \beta(t)(\lambda w + v)$$

$$= \big(\alpha(t) + \lambda\beta(t)\big) v + \big(\beta(t)\lambda\big) w$$

Since (v, w) is a basis of $\mathbb{R}^2$,

$$\begin{cases} \alpha' = \alpha\lambda + \beta & (1) \\ \beta' = \lambda\beta & (2) \end{cases} \quad \text{(Triangular form)}$$

(2) $\Rightarrow$ $\beta(t) = ce^{\lambda t}$. $\longrightarrow$ Just want one solution w/ $\beta \neq 0$. So, take $\beta(t) = e^{\lambda t}$.

Solve (1): $(\lambda = 1)$

$$\alpha' - \alpha = e^t$$

Guess: $ae^t + bte^t$

Variation of constants: $\alpha' = \alpha \Rightarrow$ solution $=$ const $\cdot\, e^t$

So, for the inhomogeneous equation, look at

$$\alpha(t) = \underline{\underline{y(t)}}\, e^t \qquad \text{"Vary" the constant}$$

Plug into (1): $(\cancel{e^t} y + e^t y') - \cancel{e^t y} = e^t$

$$\cancel{e^t} y'(t) = \cancel{e^t} \Rightarrow y'(t) = 1 \Rightarrow y(t) = t + c$$

Since we only need one solution, take $y(t) = t$.

Summary: $\alpha(t) = te^t$ and $\beta(t) = ce^t$. Thus, the general solution of the original ODE $x' = Ax$, $A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$,

$$\begin{cases} x_1(t) = e^t v \\ x_2(t) = \alpha(t) v + \beta(t) w = te^t v + e^t w \end{cases}$$

$$x(t) = a_1 x_1(t) + a_2 x_2(t), \quad a_1, a_2 \in \mathbb{R}.$$

$$x(t) = a_1 e^t v + a_2 e^t (tv + w)$$
$$= a_1 \begin{pmatrix} e^t \\ 2et \end{pmatrix} + a_2 \begin{pmatrix} te^t \\ 2te^t - e^t \end{pmatrix}; \quad a_1, a_2 \in \mathbb{R}.$$

Phase Portrait:

$A = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}$

$\lambda = 1$ double

$(A - \lambda) v = 0$, $v = (1, 2)$

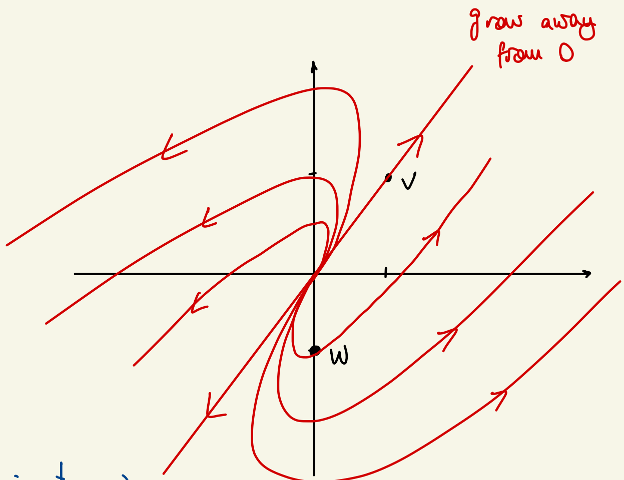
$(A - \lambda) w = v$, $w = (0, -1)$

General solution:

$x(t) = a_1 e^t v + a_2 e^t (tv + w)$

Same thing but w/ different sign for $t \searrow -\infty$

↳ v dominates w for $t \nearrow +\infty$, so solution gets closer to the v component.



EXAMPLE 7: (Double and Negative Eigenvalue) Consider $X' = BX$ w/

$$B = \begin{pmatrix} 0 & -1 \\ 4 & -4 \end{pmatrix}$$

$\chi_B(\lambda) = \lambda^2 + 4\lambda + 4$

$\Rightarrow \lambda = -2$ with algebraic multiplicity 2

Eigenvectors:

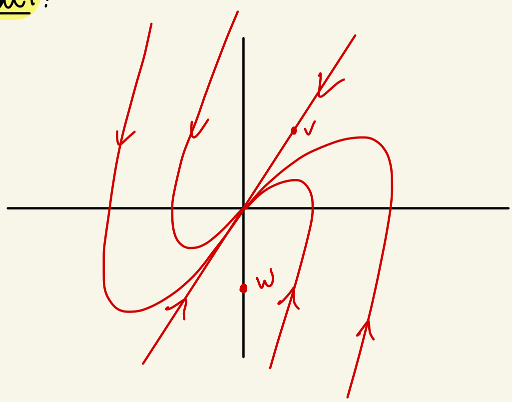
$v = (1, 2) \longrightarrow Av = -2v$

$w = (0, -1) \longrightarrow Aw = -2w + v$

General solution:

$$x(t) = a_1 e^{-2t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + a_2 e^{-2t}(tv + w)$$

Phase portrait:

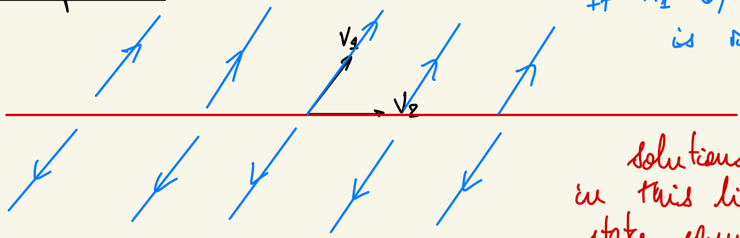


These examples 1–7 are the most "typical" cases. Now, some exceptional cases...

EXAMPLE 8: (0 eigenvalue) $X' = AX$, $A \in \mathbb{R}^{2\times 2}$ and $\lambda_1 > 0$ and $\lambda_2 = 0$. general solution is

$$x(t) = a_1 e^{\lambda_1 t} v_1 + a_2 v_2$$

Phase portrait:

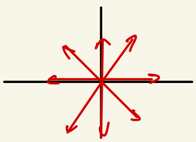


If $\lambda_1 < 0$, arrow's direction is reversed.

solutions don't move in this line $\Rightarrow$ steady state equilibria

EXAMPLE 9. (Double eigenvalue λ) 2 eigenvectors independent v_1, v_2. general solution: $x(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2$

Phase portrait:



$\lambda > 0$

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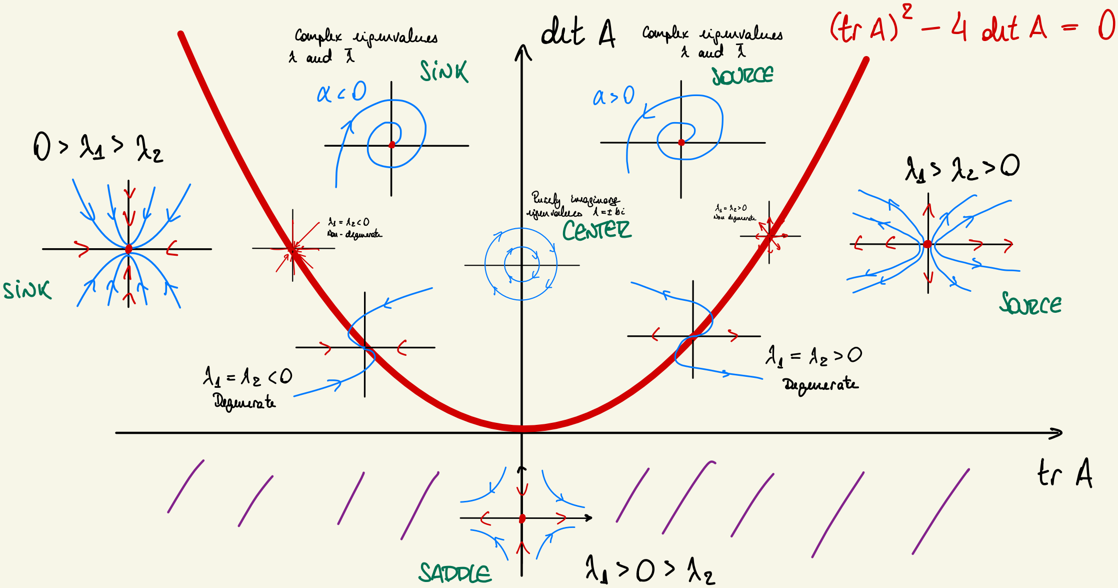
• CLASSIFICATION OF DYNAMICAL SYSTEMS $X' = AX$, $A \in \mathbb{R}^{2\times 2}$.

Recall: $\chi_A(\lambda) = \lambda^2 - \operatorname{tr} A + \det A$.

• Discriminant: $\Delta = (\operatorname{tr} A)^2 - 4 \det A$. The sign of Δ reveals the overall look of the eigenvalues.

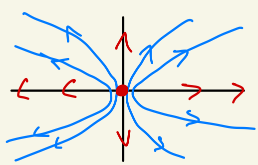
$$\begin{cases} \Delta = 0 \Rightarrow \text{repeated eigenvalues} \\ \Delta < 0 \Rightarrow \text{complex eigenvalues} \\ \Delta > 0 \Rightarrow \text{real and distinct eigenvalues} \end{cases}$$

Trace - Determinant Plane



→ **Case 1:** $\lambda_1, \lambda_2 > 0$ and $\lambda_1 > \lambda_2 > 0$. The Jordan Canonical Form is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Non-degenerate since $\lambda_1 \neq \lambda_2$.

E.g.: $\begin{pmatrix} 10 & 7 \\ 5 & 29 \end{pmatrix}$

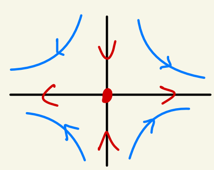


0 is an unstable node

Detect this case: $\Delta = (\mathrm{tr}\, A)^2 - 4 \det A > 0$

$$\Rightarrow \boxed{(\mathrm{tr}\, A)^2 > 4 \det A > 0.}$$

→ **Case 2:** $\lambda_1 > 0 > \lambda_2$. The Jordan Canonical Form is $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \leftrightarrow \begin{matrix} x' = \lambda_1 x \\ y' = \lambda_2 y \end{matrix}$

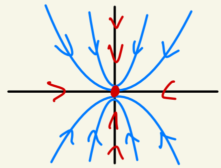


Unstable at 0.

Detect this case: $\det A < 0$. E.g., $\begin{pmatrix} 1 & 17 \\ 100 & 15 \end{pmatrix}$

→ **CASE 3:** $0 > \lambda_1 > \lambda_2$. Jordan Canonical Form: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

Phase portrait:



0 is stable node and non-degenerate ($\lambda_1 \neq \lambda_2$).

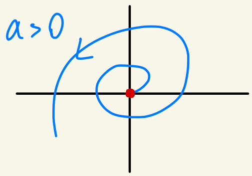
Detect this case: $\det A > 0$ and $\operatorname{tr} A < 0$

→ **CASE 4:** Complex eigenvalues

$$\lambda = \alpha + i\beta, \quad \bar{\lambda} = \alpha - i\beta, \quad \alpha > 0, \ \beta \neq 0.$$

Jordan Canonical Form

$$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$$

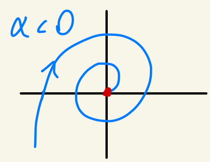


Unstable spiral

Detect this case: $\det A > 0$; $\operatorname{tr} A > 0$; $4 \det A > (\operatorname{tr} A)^2$.

→ **CASE 5:** Complex eigenvalues

$$\lambda = \alpha + i\beta, \quad \bar{\lambda} = \alpha - i\beta, \quad \alpha < 0, \ \beta \neq 0$$



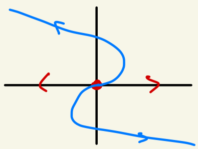
0 stable spiral

Detect this case: $4 \det A > (\operatorname{tr} A)^2$; $\det A > 0$; $\operatorname{tr} A < 0$.

- Now, the non typical cases...

→ **Case 6:** $\lambda_1 = \lambda_2 > 0$ Double eigenvalue, one eigenvector $\Rightarrow$ rank $(A - \lambda_1 I) = 1$. So, the Jordan Canonical Form is $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$

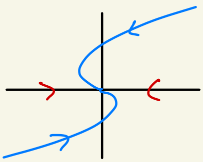
Detect this case: $(\operatorname{tr} A)^2 = 4 \det A$; $\det A > 0$; $\operatorname{tr} A > 0$
rank $(A - \lambda_1 I) = 1$



Unstable and degenerate ($\lambda_1 = \lambda_2$)

E.g.: Example 6

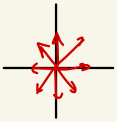
→ **Case 7:** $\lambda_1 = \lambda_2 < 0$ and rank $(A - \lambda_1 I) = 1$. $\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$



Detect this case $\det A > 0$; $\operatorname{tr} A < 0$;
$4 \det A = (\operatorname{tr} A)^2$.

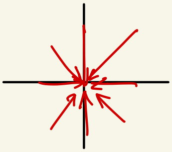
Stable and degenerate

→ **Case 8:** $0 < \lambda_1 = \lambda_2$ rank $(A - \lambda_1 I) = 2$

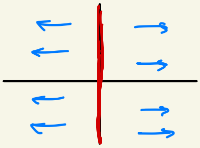


Non degenerate

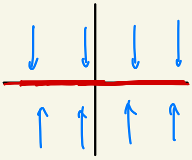
→ **Case 9:** Same as Case 8
but $\lambda_1 = \lambda_2 < 0$



→ **Case 10:**
$\lambda_1 = 0$, $\lambda_2 > 0$.



→ **Case 11:**
$\lambda_1 = 0$, $\lambda_2 < 0$.



→ **Case 12** $\lambda_1 = \lambda_2 = 0$

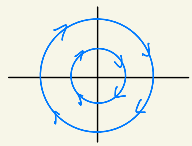
$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$ 1 eigenvector

→ **Case 13**

$\lambda_1 = \lambda_2 = 0$ $$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

2 eigenvectors

→ **Case 14**: Center w/ $\lambda = \pm i\beta$
(purely imaginary)



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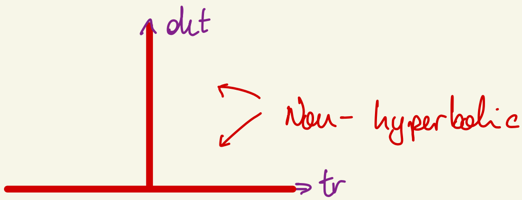
Lecture 7: Higher-Dimensional Linear Systems

Def: The origin is a **Hyperbolic Equilibrium** for the ODE $x' = Ax$ if all eigenvalues of A have nonzero real part.

In the tr-det plane, and $n = 2$, everything is hyperbolic except what is in red.



Def: A property (e.g., of matrices) is called **generic** if it is satisfied on any open, dense subset of $\mathbb{R}^{n \times n}$.

E.g.: $\det A \neq 0$ is generic and hyperbolicity is also generic.

Thm: The set of matrices in $\mathbb{R}^{n\times n}$ that have n distinct eigenvalues is generic (open and dense) in $\mathbb{R}^{n\times n}$.

FACT: For a polynomial $p_t(x) = \sum_{n=1}^{\tilde{n}} a_n x^n$ and the coeffs are some function of t: $a_n(t)$, then the roots of p_t are also continuous functions of t. $\square$

COROLLARY: (Cayley-Hamilton Theorem) Let $A \in \mathbb{R}^{n\times n}$ and $\chi_A(\lambda)$ be the char. poly. of A:

$$\chi_A(\lambda) = \sum_{m=0}^{n} a_m \lambda^m,$$

then $\chi_A(A) = \sum_{m=0}^{n} a_m A^m = 0$.

Pf: 1. Trivial if $A = D$ is a diagonal matrix.

2. The claim is invariant under changes of basis (i.e., $A = CDC^{-1}$, $\det C \neq 0$, then $\chi_A = \chi_D$ and $\chi_A(A) = 0 \Leftrightarrow \chi_D(D) = 0$ b/c $A^m = CD^m C^{-1}$).

3. Use that matrices w/ different eigenvalues are dense in $\mathbb{R}^{n\times n}$ and take lim. b/c coeffs. of $\chi_A(\lambda)$ depend continuously on entries of A, so $(*)$ is stable under limits. $\square$

"**Def: (BIFURCATION)**" As parameters change slowly, qualitative behaviour changes abruptly.

---

* CANONICAL FORMS: (Jordan)

Consider $x' = Ax$, $A \in \mathbb{R}^{n\times n}$ and analyze changes of variables. Set $x = Ty$, $T \in \mathbb{R}^{n\times n}$, y new variable and $\det T \neq 0$.

The system becomes

T has constant coeffs

$$y' = (T^{-1}x)' = T^{-1}x'$$

$$= T^{-1}(Ax) = (T^{-1}AT)y$$

$$\Rightarrow \boxed{y' = (T^{-1}AT)y.}$$

Goal, make the system matrix as simple as possible.

Ex. Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \Rightarrow$ $\lambda_1 = 1 \longrightarrow v_1 = (1,-1)$
$\lambda_2 = 3 \longrightarrow v_2 = (0, 1)$

Set

$$T = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$$

so that

$$T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}. \quad \text{So,}$$

$$y' = (T^{-1}AT)y = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} y \Rightarrow \text{simple to solve!}$$

$A \in \mathbb{R}^{n\times n}$ is diagonalizable $\Leftrightarrow$ A has n linearly independent eigenvectors $v_1, -, v_n$.

In that case,

$$A = T^{-1} D T, \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

and

$$T = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}.$$

Also, a <u>sufficient</u> condition for this to hold is $\lambda_1, -, \lambda_n$ are all distinct.

<u>NOTE</u>: $C := T^{-1} A T$ is called "CONJUGATION OF A BY T."

- Over $\mathbb{C}$ $\Rightarrow$ Jordan Canonical Form
- Over $\mathbb{R}$ $\Rightarrow$ Do $\mathbb{C}$ and get real solutions (using the real and imaginary parts of complex eigenvectors for λ). So, $\lambda = \alpha + i\beta$; $T = \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix}$ and $T^{-1} A T = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$

Now, over $\mathbb{C}$, $C = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix}$ (block diagonal)

where each block is

$$J_\ell = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix}_{\ell \times \ell}$$

<u>NOTE</u>: λ could be the same in different blocks.

SUMMARY: solve $X' = AX$

$$C = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$ eigenvalues

$$T = \begin{pmatrix} | & & | \\ v_1 & \cdots & v_n \\ | & & | \end{pmatrix}$$ eigenvectors

1. Find C s.t. $C = T^{-1}AT$;
2. solve $Y' = CY$;
3. Transform back: $x(t) = Ty(t)$.

LECTURE 8: HIGHER-DIMENSIONAL SYSTEMS

JORDAN CANONICAL FORM:

$$C = \begin{pmatrix} J_1 & & 0 \\ & \ddots & \\ 0 & & J_k \end{pmatrix},$$ Block diagonal

where

$$J_i = \begin{pmatrix} \lambda_i & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda_i \end{pmatrix}$$ $l_i \times l_i$ block.

same eigenvalues can appear in different blocks.

* $A = \underset{\sim}{T} C \underset{\sim}{T^{-1}}$ Change of variables/basis

STEP 1. Find the Canonical Form and T

• Suppose λ is an eigenvalue w/ algebraic multiplicity m.

Ex: $n = 5$. The possible Jordan blocks are

$$\begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} ; \quad \left(\begin{array}{c|cccc} \lambda & & & 0 & \\ \hline 0 & \lambda & 1 & & 0 \\ & & \lambda & 1 & \\ & & 0 & \lambda & 1 \\ & & & & \lambda \end{array}\right) ; \quad \left(\begin{array}{cc|ccc} \lambda & 1 & & 0 & \\ 0 & \lambda & & & \\ \hline & & \lambda & 1 & 0 \\ & 0 & & \lambda & 1 \\ & & 0 & & \lambda \end{array}\right)$$

1 block; $l = 5$

$\Rightarrow$ 1 eigenvector

$J_1 \to l_1 = 1$

$J_2 \to l_2 = 4$

$\Rightarrow$ 2 eigenvectors

$J_1 \to l_1 = 2$

$J_2 \to l_2 = 3$

$\Rightarrow$ 2 eigenvectors

$$\left(\begin{array}{c|c|ccc} \lambda & & & & \\ \hline & \lambda & & & \\ \hline & & \lambda & & \\ & & & \lambda & \\ & & & & \lambda \end{array}\right)$$

$J_1 \to l_1 = 1$

$J_2 \to l_2 = 1$

$J_3 \to l_3 = 3$

$\Rightarrow$ 3 eigenvectors

etc...

Moreover, we can write

$$J = \begin{pmatrix} \lambda & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{pmatrix} = \lambda \, Id + \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}}_{\text{Nilpotent}}$$

* **CONSEQUENCE**: we can distinguish which canonical form corresponding to A by looking at $\ker(A - \lambda)^m$.

- $\dim\left(\ker(A-\lambda)^{m-1}\right) = \#$ Jordan blocks for A of length $\leq (m-1)$.

$\to$ How to find a good basis for $\ker(A - \lambda I)^m$?

$\ker(A - \lambda I)^m \supset \ker(A - \lambda I)^{m-1} \supset \ker(A-\lambda I)^{m-2} \supset \dots \supset \ker(A - \lambda I) \supset \{0\}$.

* **Solve $y' = Jy$**: where $J = \lambda I + \underbrace{\begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}}_{l = \text{length}}{}^{= N}$.

To simplify $\underset{y(t)}{y} = e^{\lambda t} \underset{z(t)}{z} \Leftrightarrow z = e^{-\lambda t} y$.

So,

$$\begin{aligned} z' &= -\lambda e^{-t\lambda} y + e^{-t\lambda} y' \\ &= -\lambda e^{-t\lambda} y + e^{-t\lambda} [\overbrace{y\lambda + Ny}^{= Jy}] \\ &= Nz \end{aligned}$$

Now, we solve $z' = Nz$, $N = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}' = \begin{pmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 0 & & & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} z_2 \\ \vdots \\ z_n \\ 0 \end{pmatrix}$$

i.e.,

$$z_1' = z_2 \; ; \; z_2' = z_3 \; ; \; \text{—} , \; z_n' = 0$$

$\cdots \Leftarrow$ $\Leftarrow$ constant $c_n \in \mathbb{R}$

$z_{n-1}(t) = c_n t + c_{n-1}$

So,

$$z(t) = \begin{pmatrix} p(t) \\ p'(t) \\ \vdots \\ p^{(n-1)}(t) \end{pmatrix} \Rightarrow \boxed{y(t) = e^{\lambda t} z(t)}.$$

If y is complex, take $\mathrm{Re}(y)$ and $\mathrm{Im}(y)$.

if $\lambda = \alpha + i\beta$, $\beta \neq 0$.

2nd METHOD TO SOLVE $y' = Jy$: solution is $y(t) = e^{t(\lambda I + N)} y_0$

Initial condition

$$\Rightarrow y(t) = e^{t\lambda I} e^{tN} y_0$$

$$= e^{t\lambda} \Big(\underbrace{I + tN + \frac{t^2}{2} N^2 + \cdots + \frac{t^{l-1}}{(l-1)!} N^{l-1}}_{\text{finite sum}} \Big) y_0$$

$N^l = N^{l+1} = \cdots = 0$

Recall: $e^{tA} = \sum_{k \geq 0} \frac{t^k}{k!} A^k$

→ Assume claim in red: Had $y' = Cy$, $C =$ canonical form

Original equation $x' = Ax$; but $C = T^{-1}AT$.
So, we have to transform back: $x = Ty$. See text book for examples.

Def: (OPERATOR NORM) If $A \in \mathbb{R}^{n\times n}$, the norm of A is

$$\|A\| = \sup_{\substack{v\in\mathbb{R}^n \\ v\neq 0}} \frac{\|Av\|}{\|v\|} = \sup_{\substack{|u|=1 \\ u\in\mathbb{R}^n}} \|Au\|.$$

Note: $\|Av\| \leq \|A\|\,\|v\|$ by definition. Moreover,

$$\|A\cdot B\| \leq \|A\|\,\|B\|.$$

Matrix product

LECTURE 9: MATRIX EXPONENTIALS

As we saw before, e^{tA} is the solution for the system $x' = Ax$, $x(0) = y$. By definition, $e^{tA}y$ is the unique solution.

Think of it as a map:

$$e^{tA}: (t, y) \longmapsto e^{tA}y = x(t)$$

$$y \longmapsto x(\cdot) = e^{\cdot A}y$$

Initial value Solution

Equivalently $e^{0\cdot A} = Id$ and $\frac{d}{dt} e^{tA} = A e^{tA}$.

Many formulas for e^{tA}:

1) $$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}$$

2) If $A = \underbrace{TCT^{-1}}$, then $e^A = T e^C T^{-1}$

Canonical form

3) Can solve $x' = Ax$, $x(0) = e_i$ = ith unit vector

Call solutions $x_1, \ldots, x_n$

and $$e^{tA} = \begin{pmatrix} | & & | \\ x_1(t) & \cdots & x_n(t) \\ | & & | \end{pmatrix}$$

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A^2 = Id$. So, $A^k = \begin{cases} Id, & k = 4l \\ A, & k = 4l+1 \\ -Id, & k = 4l+2 \\ -A, & k = 4l+3 \end{cases}$

Then
$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!} = \sum_{j=0}^{\infty} \underbrace{\frac{t^{2j}}{(2j)!} (-1)^j I}_{} + \frac{t^{2j+1} (-1)^j}{(2j+1)!} A$$

$k = 2j \longrightarrow j$ even $\to k = 4l$

$\longrightarrow j$ odd $\to k = 4l+2$

$$= \cos(t) Id + \sin(t) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in SO(2)$$

So, e^{tA} in this case is a general element of the rotation group in the plane; i.e., $e^{tA} \in SO(2)$.

Ex: same, but in 3D: $A \in \mathbb{R}^{3\times 3}$ skew-symmetric: $A^T = -A$

$$A = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix}, \quad a,b,c \in \mathbb{R}.$$

What is e^{tA}? $\{e^{tA}\}_{t\in\mathbb{R}}$ is a subgroup of copy of $SO(2)$ in $SO(3)$ ← 3×3 orthogonal matrices with $\det = 1$ "rotations"

Normalization: $a^2+b^2+c^2 = 1$.

To compute e^{tA}, use eigenvalue / eigenvector

- $\lambda_1 = 0$ (by skew symmetry: $A^T = -A$)
- $\lambda_2 = -\lambda_3$ (skew symmetry: if λ is eigenvalue, so is $-\lambda$ b/c $\det(A-\lambda I) = \det(A-\lambda I)^T = (-1)\det(A+\lambda I)$)
- $\chi_A(\lambda) = -\lambda^3 - (a^2+b^2+c^2)\lambda = -\lambda(\lambda^2+1)$

$$\Rightarrow \lambda_1 = 0; \quad \underset{\downarrow\, v}{\lambda_2 = i}; \quad \underset{\downarrow\, \bar{v}}{\lambda_3 = -i}$$

→ Diagonalization Theorem for skewsymmetric matrices gives that $v \perp \bar{v}$ and $v, \bar{v} \perp \begin{pmatrix} c \\ b \\ a \end{pmatrix}$ in $\mathbb{C}^3$.

i.e., $(x,y) = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3$

$\|x\|^2 = x_1\bar{x}_1 + x_2\bar{x}_2 + x_3\bar{x}_3$

$\Rightarrow \mathrm{Re}(v) \perp \mathrm{Im}(v)$ in $\mathbb{R}^3$

so,

$\lambda_1 = 0$	$\lambda_2 = i$	$\lambda_3 = -i$
$\begin{pmatrix} c \\ b \\ a \end{pmatrix}$	v	$\bar{v}$

$\}$ orthogonal

Thus, the Jordan Canonical Form is

Complex $\begin{pmatrix} 0 & & 0 \\ & i & \\ 0 & & -i \end{pmatrix}$ $\qquad$ $\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right) =: C$ Real

A in 2D case

So,

Blocks "move" separately

$$e^{tC} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{array}\right).$$

Rotation by angle t about x-axis

e^{tA} from 2D example

Thus,

$$e^{tA} = T e^{tC} T^{-1}, \quad \text{where} \quad T = \begin{pmatrix} c & | & | \\ b & \mathrm{Re}(v) & \mathrm{Im}(v) \\ a & | & | \end{pmatrix}$$

Orthogonal

Therefore, e^{tA} is a rotation by angle t about axis defined by (c, b, a).

FACT: if $M \in SO(3)$, then $M = e^{tA}$ for some $t \in \mathbb{R}$ and some skew symmetric matrix A. So, any rotation in 3D has an axis and an angle.

Finally, $\{e^{tA}\}_{t\in\mathbb{R}}$ is a (1-parameter) subgroup of

Invertible

$$\{M \in \mathbb{R}^{n\times n} : \det M \neq 0\} =: GL(n),$$

Non compact manifold of dimension n^2 $\qquad$ "General Linear" group

and $SO(3)$ has dim. 3 (2 variables determine the axis and 1 determines the angle)

Solve Nonhomogeneous Systems: $x' = AX + \underbrace{f(t)}_{\text{Inhomogeneity}}$

The general solution is given by $x(t) = \underbrace{y(t)}_{\text{Particular solution}} + \underbrace{e^{tA}v}_{\text{general solution of homogeneous solution}}$

Goal: find a particular solution.

Lecture 10: Duhamel's Principle

solving inhomogeneous system of equations:

$X' = AX + f(x) \rightarrow$ general solution: $x(t) = \underbrace{y(t)}_{\substack{\text{Particular solution} \\ \text{to } X' = AX + f(t)}} + e^{tA}v$

Guess: Try $y(t) = e^{tA}v(t)$. "Variation of Constants"

$\boxed{X' = AX + f(t)}$

Plug in:

$$\begin{cases} y'(t) = A e^{tA} v(t) + e^{tA} v'(t) & \text{(LHS)} \\ Ay + f(t) = A e^{tA} v(t) + f(t) & \text{(RHS)} \end{cases}$$

ODE: (LHS) = (RHS) $\Rightarrow$ $e^{tA} v'(t) = f(t)$

$$\Rightarrow v'(t) = e^{-tA} f(t).$$

By FTC, get

$$v(t) = V_0 + \int_0^t e^{-sA} f(s)\, ds$$

say $V_0 = 0$
since we're looking for one particular solution

Transform back:

$$y(t) = e^{tA} v(t) = e^{tA} \int_0^t e^{-tA} f(s)\, ds$$

$$\Rightarrow \boxed{y(t) = \int_0^t e^{(t-s)A} f(s)\, ds}$$ DUHAMEL'S FORMULA

Note $t-s>0$.

Ex: (APPICATIONS)

1. PDE's: heat equation

Initial condition

$$u_t = \Delta u, \quad u(x,0) = u_0(x)$$

$u_t = \frac{\partial u}{\partial t}$; $\Delta u = \sum_{i=1}^{n} \partial_{x_i}^2 u$. Solution:

$$u(x,t) = (2\pi t)^{-n/2} \int_{\mathbb{R}^n} e^{\frac{-\|x-y\|^2}{4t}} u_0(y)\, dy$$

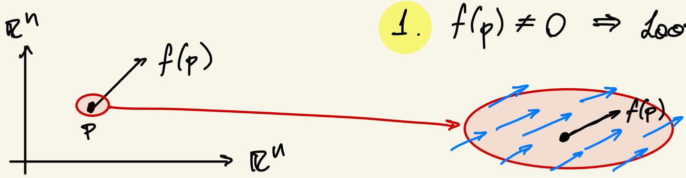
$$=: e^{t\Delta} u_0, \quad t>0$$

Duhamel: $v(t,\cdot) = e^{t\Delta} f(t,\cdot)$ then $v_t = \Delta v + f(x,t)$. □

LINEARIZATION (overview): constant coeffs are nice... so... we want constant coeffs. Approximate nonlinear to linear and nonlinear solutions are close to the linear solutions.

Autonomous system

Consider $x' = f(x)$, $f: \mathbb{R}^n \to \mathbb{R}^n$. Local behaviour near a point p: Two cases:



1. $f(p) \neq 0 \Rightarrow$ Look at neighborhood.

By continuity, $f(q) \approx f(p)$ if $q \approx p$.

$\Rightarrow$ Parallel flow solutions are lines

2. If $f(p) = 0 \Rightarrow x(t) = p$ is a solution (equilibrium/steady state). So,

Taylor $\rightarrow$ $f(x) = \underbrace{f(p)}_{\substack{\parallel \\ 0}} + \underbrace{Df(p)}_{A}(x-p) + \underbrace{\bar{o}(\|x-p\|)}_{\text{Error (can we neglect?)}}$

New variable: $y := x - p$. So, $y' = \underbrace{Df(p)}_{A}y + \bar{o}(\|y\|)$

Q: When is the behaviour determined by A?

• True system: $y' = Ay + \bar{o}(\|y\|)$, $A = Df(p)$

Non-linear

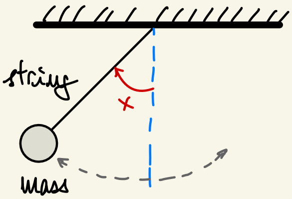
$\bar{o}(\|y\|)$ = remainder of Taylor.

Linearization $y' = Ay$

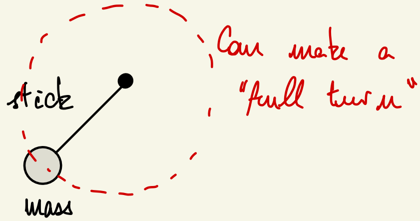
Q1: Suppose the lin. system has source/sink. Is the same true for (N)? (Yes)

Q2: Does phase portrait look "similar" for (L) and (N)? (Yes, if hyperbolic) $\rightarrow$ $Re(\lambda) \neq 0$ $\forall \lambda$ eigenvalue of A

Ex: (Mathematical Pendulum)



$\Longrightarrow$



The equation is: $m x'' + r x' = -\omega \sin(x)$

mass $\equiv 1$ — friction $r \geq 0$ — set $\equiv 1$

So,

$$x'' + r x' + \sin(x) = 0.$$

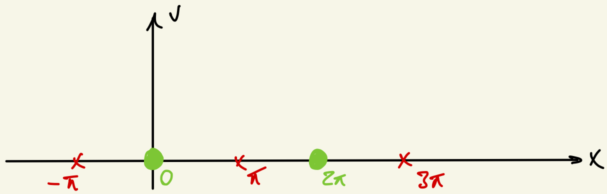
$x_1 \mapsto x$ (position)

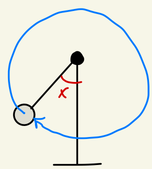
$x_2 = x_1' = x' = v$ (velocity)

$\Rightarrow$ 1st order system: $\begin{cases} x_1' = x_2 \\ x_2' = -r x_2 - \sin(x_1) \end{cases}$

* **Steady States:** $v = 0$

(Angles mod (2π)) $\quad \sin(x) = 0 \Rightarrow x = k\pi,\ k \in \mathbb{Z}$





- Stable $\rightarrow$ hanging down
- Unstable $\rightarrow$ standing up

* **Compute Linearization**

$f(x,v) = \begin{pmatrix} v \\ -rv - \sin(x) \end{pmatrix} \longrightarrow$ We're solving $\begin{pmatrix} x \\ v \end{pmatrix}' = f(x,v)$

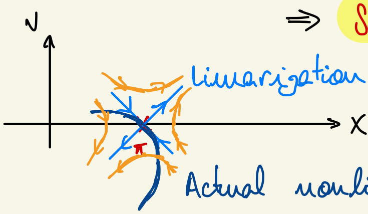
So,

$$Df(k\pi,0) = \begin{pmatrix} 0 & 1 \\ -\cos x & -r \end{pmatrix}\Bigg|_{x=k\pi} = \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix}, & k \text{ even (bottom)} \\ \begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix}, & k \text{ odd (top)} \end{cases}$$

1. Top: Linearization is $A = \begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix}$, $r \geq 0$.

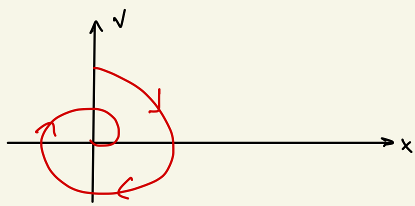
$\Rightarrow \det A = -1 \Rightarrow$ eigenvalues are real and opposite signs ($\operatorname{tr} A = -r \leq 0$)

$\Rightarrow$ **Saddle, unstable.**



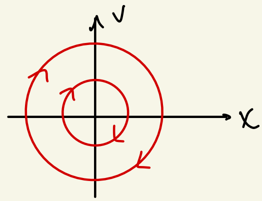
2. Bottom: Linearization is $A = \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix}$, $r \geq 0$.

→ First, $r > 0$: $\det A = 1 > 0$ and $\operatorname{tr} A = -r < 0$. So, linearization has a spiral (probably counterclockwise) (same for the nonlinear model)



→ If $r = 0$, then $\det A = 1 > 0$ and $\operatorname{tr} A = 0$.

$\Rightarrow$ Center (both linear and nonlinear)



SUMMARY:

1. find steady states
2. Linearize near the steady states
3. Tie all up in a big picture

— // —

* **DIGRESSION ON HOW TO SOLVE EQUATIONS:**

1) **EXACT DIFFERENTIAL EQUATIONS:** The ODE

$$P(x,y)\,dx + Q(x,y)\,dy = 0$$

is **EXACT** if $\exists f(x,y)$ s.t. $\dfrac{\partial f}{\partial x} = P(x,y)$ and $\dfrac{\partial f}{\partial y} = Q(x,y)$.

In that case, a 1-parameter family of solutions is

$$f(x,y) = C \leftarrow \text{arbitrary constant}$$

Ex: $y\,dx + x\,dy = 0$

$f(x,y) = xy \longrightarrow$ solution $x \cdot y = C$.

Thm: The ODE $P(x,y)\,dx + Q(x,y)\,dy = 0$. Assume P, Q, $\dfrac{\partial P}{\partial x}$, $\dfrac{\partial P}{\partial y}$, $\dfrac{\partial Q}{\partial x}$, $\dfrac{\partial Q}{\partial y}$ exist and are continuous on a simply connected region R (open and no holes). then

(a) ODE is exact if and only if

(b) $\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial x} Q(x,y)$.

Pf: (a) $\Rightarrow$ (b) Tautology. $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$

$$\Rightarrow \frac{\partial}{\partial y} P = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2}{\partial x \partial y} f = \frac{\partial}{\partial x} Q.$$

(b) $\Rightarrow$ (a) If such f exists, it has to satisfy

$$\frac{\partial f}{\partial x} = P(x,y) \xRightarrow{\text{FTC}} f(x,y) = \int_{x_0}^{x} P(x,y)\,dx + R(y)$$

Constant of $\int dx$ depends on y

Moreover,

$$\frac{\partial f}{\partial y} = Q(x,y) \Rightarrow \frac{\partial}{\partial y}\left(\int_{x_0}^{x} P(x,y)\,dx + R(y)\right) = \int_{x_0}^{x} \underbrace{\frac{\partial}{\partial y} P(x,y)}_{\overset{(b)}{=} \frac{\partial}{\partial x} Q(x,y)}\,dx + R'(y)$$

$$= Q(x,y) - Q(x_0,y) + R'(y)$$

$$\Rightarrow R'(y) = Q(x_0,y) \xrightarrow{\text{FTC}} R(y) = \int_{y_0}^{y} Q(x_0,y)\,dy$$

So,

$$\boxed{f(x,y) = \int_{x_0}^{x} P(x,y)\,dx + \int_{y_0}^{y} Q(x_0,y)\,dy},$$

where $(x_0, y_0) \in R$ = simply connected region and the line segments joining (x_0, y_0) and (x, y_0), (x_0, y_0) and (x_0, y) are confined in R.

Now, check $\frac{\partial f}{\partial x} = P(x,y)$ ✓ and $\frac{\partial f}{\partial y} = Q(x,y)$ ✓ $\Rightarrow$ ODE is exact.

□

Ex: $$(\underbrace{2x + y\cos x}_{P})\,dx + (\underbrace{2y + \sin x - \sin y}_{Q})\,dy = 0$$

Exact: $\frac{\partial P}{\partial y} = \cos x = \frac{\partial Q}{\partial x}$

$$f(x,y) = \int 2x + y\cos x \,dx = x^2 + y\sin x + R(y)$$

$$\frac{\partial f}{\partial y} = Q(x,y) = 2y + \cancel{\sin x} - \sin y \overset{!}{=} \cancel{\sin x} + R'(y)$$

$$\Rightarrow R'(y) = 2y - \sin y$$

$$R(y) = y^2 + \cos(y) + c$$

So, $\boxed{f(x,y) = x^2 + y\sin(x) + y^2 + \cos(y) = C}$.

2) INTEGRATING FACTORS

Def: A multiplying factor that will convert an inexact ODE into an exact ODE is called an INTEGRATING FACTOR.

Ex: $(\underbrace{t^2x - t}_{=P})\,dx + \underbrace{x}_{=Q}\,dt = 0$

$\frac{\partial P}{\partial t} = 2tx - 1 \neq \frac{\partial Q}{\partial x} = 1$

$\Rightarrow$ not exact!

- Let $P^*(t,x) = (t^2x - t)\,h(t)$

  $Q^*(t,x) = x\,h(t)$

- Want: $\frac{\partial P^*}{\partial t}\ (2tx - 1)h(t) + (t^2x - t)h'(t)$

  $= \frac{\partial Q^*}{\partial x} = h(t)$

$$\Rightarrow \frac{h'(t)}{h(t)} = \frac{2-2tx}{t^2x-t} = -\frac{2}{t}$$

CRUCIAL !

only depends on t =)

$$\Rightarrow \log|h(t)| = -2\log|t| \Rightarrow \boxed{h(t) = \frac{1}{t^2}}.$$

and now check and solve

- similar method if the integrating factor depends on $h(x)$; $h(t)$; $h(xt)$; $h\left(\frac{t}{x}\right)$; $h\left(\frac{x}{t}\right)$.

Good news: any ODE $x' = f(x,t)$ with $f \in C^1$ in a neighborhood of (x_0, t_0) admits an integrating factor.

Bad news: pretty useless in life.

LINEAR DIFFERENTIAL EQUATION OF ORDER 1:

$$\boxed{\frac{dy}{dx} + P(x)y = Q(x)}$$

Could use separation of variables and variation of constants.

Have same exponent one

2. Integrating factors: rewrite as $[P(x)y - Q(x)]dx + dy = 0$

Assume $\exists\, \mu(x)$ integrating factor. Want:

$$\frac{\partial}{\partial y}[\mu(x)P(x)y - \mu(x)Q(x)] = \frac{\partial}{\partial x}\mu(x)$$

$$\Rightarrow \mu(x)\cdot P(x) = \mu'(x) \rightsquigarrow \mu(x) = e^{\int P(x)\,dx}. \quad \text{so,}$$

$$P(x)e^{\int P(x)\,dx}y\,dx + e^{\int P(x)\,dx}dy = e^{\int P(x)\,dx}Q(x)\,dx$$

$$\Rightarrow \boxed{y = \left(e^{-\int P(x)\,dx}\right)\int\left(e^{\int P(x)\,dx}\right)Q(x) + Ce^{-\int P(x)\,dx}}$$

Lecture 11: Bernoulli's Equation

Bernoulli Equation:

$$(\dagger) \quad \frac{dy}{dx} + P(x)y = Q(x)y^n$$

If $n=1$, then equation is linear and separable

For $n \neq 1$, multiply $(\dagger)$ by $(1-n)y^{-n}$:

$$\underbrace{(1-n)y^{-n}\frac{dy}{dx}}_{du} + \underbrace{(1-n)P(x)}_{\tilde{P}(x)}\underbrace{y^{1-n}}_{u} = \underbrace{Q(x)(1-n)}_{\tilde{Q}(x)}$$

and this is linear in u.

Ex: $y' + xy = \frac{x}{y^3}$, $y \neq 0$. Multiply by $4y^3$ and get

$$\underbrace{4y^3 y'}_{\frac{d}{dx}(y^4)} + 4x(y^4) = 4x$$

For $u = y^4$, $\quad \frac{du}{dx} + \underbrace{4x}_{\tilde{P}(x)}u = \underbrace{4x}_{\tilde{Q}(x)}$

Integrating factor

$$e^{\int \tilde{P}(x)\,dx} = e^{4\int x\,dx} = e^{2x^2}$$

$$\Leftrightarrow \quad h(x)\frac{du}{dx} + h(x)\cdot 4xu = h(x)4x \Leftrightarrow h(x)du + \left(h(x)4xu - h(x)4x\right)dx = 0$$

To be exact, need $h'(x) = h(x)\cdot 4x \Rightarrow h(x) = e^{\int 4x\,dx} = e^{2x^2}$

So, get

$$e^{2x^2}du + \underbrace{\left(4xue^{2x^2} - 4xe^{2x^2}\right)}_{*}dx = 0.$$

$$f(x,u) = \int e^{2x^2}\,du \overset{*}{=} u e^{2x^2} + R(x). \quad \text{Now,}$$

$$\frac{\partial f}{\partial x} = 4xue^{2x^2} \overset{*}{+} R'(x) = 4xue^{2x^2} \overset{*}{-} 4xe^{2x^2}$$

$$\Rightarrow R(x) = -\int 4xe^{2x^2}dx = -e^{-2x^2} + C$$

Thus,

$$f(x,u) = ue^{2x^2} - e^{2x^2} = C'$$

$$\Leftrightarrow \quad y^4 e^{2x^2} - e^{2x^2} = C'$$

$$\Leftrightarrow \quad \boxed{y^4 = C'' e^{-2x^2} + 1.}$$

* **DYNAMICAL SYSTEMS:** $X' = AX$, $A \in \mathbb{R}^{n\times n} \to x(t) = e^{tA}v$, initial value $x(0) = v$.

2 points of view:

1. Fix $x(0) = v = x_0$. Then, the solution is

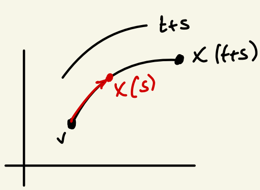
$$t \longmapsto x(t) = e^{tA}x_0 = e^{tA}v \quad \text{(trajectory / orbit)}$$

2. Fix t. $\quad v \longmapsto \underbrace{e^{tA}}_{\text{linear transformation}} v$

$\{e^{tA}\}_{t\in\mathbb{R}}$ family of isomorphisms w/ semigroup property:

$$e^{0\cdot A} = Id \quad \text{and} \quad e^{(t+s)A} \overset{(*)}{=} e^{tA}\cdot e^{sA}.$$

Proof of (*): $e^{(t+s)A}v \overset{\text{def}}{=}$ solution $x(t+s)$ where $x(0) = v$



$= y(t)$ where $y(0) = e^{sA} v$

Def: (DYNAMICAL SYSTEM) A <u>dynamical system</u> $\{\phi_t\}_{t\in\mathbb{R}}$ is a family of maps such that

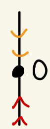
$$\phi_0(x) = x \quad \text{and} \quad \phi_{t+s} = \phi_t \circ \phi_s, \ \forall t,s \in \mathbb{R}.$$

So far $\phi_t = e^{tA}$ and we (in 267) assume that $\{\phi_t\}_{t\in\mathbb{R}}$ are all smooth.

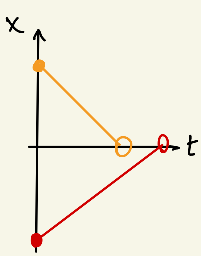
<u>E.g.</u>:
- Could have discrete times ($s,t \in \mathbb{Z}, \mathbb{N} \ldots$).
- Replace $\mathbb{R}^n \to$ manifolds
- "$\mathbb{R}$ acts on $\mathbb{R}^n$ by diffeomorphisms"
- $\mathbb{R}/t$ has a representation / action on $\mathbb{R}^n$ via diffeomorphisms ϕ_t.

Ex: (BAD EXAMPLES FOR EXISTENCE & UNIQUENESS)

1. On $\mathbb{R}$:
$$x' = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0 \end{cases}$$

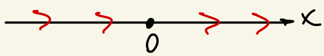


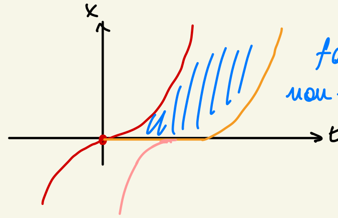
solutions flow down until $t = x_0$.



$\nexists$ solutions with $x(0) = 0$ b/c $x'(0) = -1$ so $\searrow$, $x < 0$ should $\nearrow$.

2. $x' = 3x^{2/3} \Rightarrow \int \frac{dx}{3x^{2/3}} = \int dt \Rightarrow x(t) = \left(x_0^{1/3} + t\right)^3.$





fan of non-uniqueness

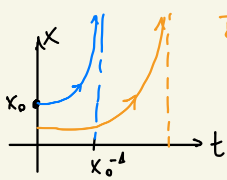
Moral: f cont. but not smooth $\Rightarrow$ Existence ✓
Uniqueness ✗

3. $x' = x^2 \Rightarrow$ separation of variables

$$\Rightarrow \int \frac{dx}{x^2} = \int dt \Rightarrow -\frac{1}{x} + \frac{1}{x_0} = t - t_0,$$

set $t_0 = 0$. So,
$$x(t) = \frac{x_0}{1 - x_0 t}.$$

- If $x_0 > 0$, $\lim_{t \to x_0^{-1}} x(t) = +\infty$



Blow up in finite time

Thm: (Local Picard's Theorem) Consider $x' = F(x)$, where the function $F: \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then, $\forall a \in \mathbb{R}^n$, there exists an interval $I = (-\alpha, \beta)$ containing the origin s.t. the ODE $x' = F(x)$ has a unique solution $x: I \to \mathbb{R}^n$ with $x(0) = a$.

Q1: Take $J \subset I$, $J \ni \vec{0}$. In $I \to$ take solution x_I unique on I
$J \to$ " " x_J " " J.

Do they agree?

Q2: If $J \ni \vec{0}$, another interval? Restrict to $I \cap J$.

Q3: Let $x(t)$ be unique solution for $x' = F(x)$ w/ $x(0) = a$ defined on $I \ni \vec{0}$. ¿$\underbrace{\exists \text{ max interval of existence for } x(t)}_{I_{max}}$?

Q4: If $I_{max} = (a, b)$, $b \neq \infty$, what can you say about $\lim_{t \to b} x(t)$?

Lecture 12: Existence and Uniqueness

Def: If $F = \{f_\alpha\}_{\alpha \in A}$ functions $E \subset \mathbb{R}^n \to \mathbb{R}^m$, then F is **uniformly bounded** if $\exists M$ s.t. $|f_\alpha(x)| < M$ $\forall x \in E$, $\forall \alpha \in A$.

Now, F is **equicontinuous** on E if $\forall \varepsilon > 0$ $\exists \delta > 0$ s.t. $\forall x, y \in E$ $\forall \alpha \in A$, $|x - y| < \delta \Rightarrow |f_\alpha(y) - f_\alpha(x)| < \varepsilon$.

Ex:
1) $\{f_\alpha(x) = \alpha x\}_{\alpha \in \mathbb{R}}$ is uniformly bounded on $[0, 1]$ but not equicontinuous.

2) $\{f_\alpha(x) = \alpha x\}_{\alpha \in [3,5]}$ is also equicontinuous.

3) $\{g_n(x) = x^n$, $x \in [0, 1]$ not equicontinuous.

4) $\{f_\alpha(x)\}_{\alpha \in A}$ are Lipschitz w/ contant L, then it's clearly equicontinuous.

$|f_\alpha(y) - f_\alpha(x)| < L|x - y|$

5) Any finite family of uniformly bounded functions is equicontinuous.

Notation: $f_n \rightrightarrows f$ if f_n converges uniformly to f
$\Leftrightarrow \|f_n - f\|_\infty \to 0$,

where $\|f\|_\infty := \sup\{|f(x)| : x \in E\}$.

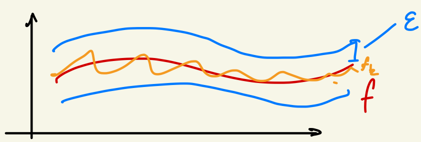
$\Rightarrow$ Continuous functions on $[0,1] = C[0,1]$ w/ $\|\cdot\|_\infty$ is a complete metric space. b/c $f_n \rightrightarrows f$, $f_n \in C[0,1]$ $\Rightarrow f \in C[0,1]$

Thm. Let $f_n : E \subset \mathbb{R}^n \to \mathbb{R}^m$ be a continuous collection. If E is compact, then $\{f_n\}$ is uniformly bounded and equicontinuous.

Pf: $f_k : E \to \mathbb{R}^m$ (E compact) $\Rightarrow |f_k(x)| \le M_k < \infty \quad \forall x \in E$

f_k cont., $f_k \rightrightarrows f \Rightarrow f$ cont. ($f : E \to \mathbb{R}^m$) $\Rightarrow f$ bdd: $|f(x)| \le M_0 < \infty \quad \forall x \in E$.

Uniformly Continuity:



$f_k \rightrightarrows f$

Equicontinuity: $f_n \rightrightarrows f$.

$$|f_n(x) - f_n(y)| \le \underbrace{|f_n(x) - f(x)|}_{<\varepsilon \text{ b/c } f_n \rightrightarrows f} + \underbrace{|f_n(y) - f(y)|}_{<\varepsilon \text{ b/c } f_n \rightrightarrows f} + \underbrace{|f(x) - f(y)|}_{<\varepsilon \text{ if } |x-y|<\delta}$$

$< 3\varepsilon$ if $|x-y| < \delta$.

Works if $n > N(\varepsilon)$ so $|f_n(x) - f(x)| < \varepsilon$ for $\{f_n\}_{n \le N(\varepsilon)}$ finitely many uniformly cont. functions, hence equicontinuous.

Thm: (ASCOLI – ARZELA' THEOREM) If $F = \{f_\alpha\}_{\alpha \in A}$ is an infinite family of functions that is uniformly bounded and equicontinuous, then $\exists \{f_n\}$, a sequence of different functions in F, that converges uniformly in E. Where $f_\alpha : E \subset \mathbb{R}^n \to \mathbb{R}^m$, E bounded.

"Pf:" Ideas: $\mathbb{Q}^n$ is dense and countable in $\mathbb{R}^n$. So, $\exists D \subset E$ dense and countable; $\forall d \in D$, $\{f_\alpha(d)\}_{\alpha \in A}$ is infinite and bounded in $\mathbb{R}^n$.

$\Rightarrow \exists$ converging subsequence.

Apply Cantor diagonal process $\Rightarrow \exists f_n \in F$ s.t.

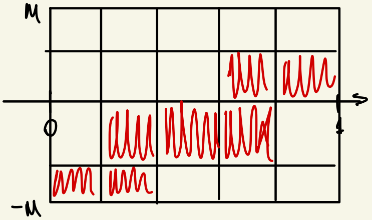
$$f_n(d) \to \text{something} =: f(d) \quad \forall d \in D.$$

Use equicontinuity to go from pointwise convergence in D to pointwise and uniform convergence in E.

Pf by picture.

Say $E = [0,1]$, $m = 1$.

$$|f_\alpha(x)| \le M < \infty$$



Def: A set B is relatively compact or a metric space if $\overline{B}$ is compact.

Cor: A set $A \subset C[K, \mathbb{R}^m]$ ($K \subset \mathbb{R}^m$ compact) is relatively compact in $\|\cdot\|_\infty \Leftrightarrow A$ is bounded and equicontinuous.

* FUNCTION APPROXIMATIONS:

Lemma: Let $f: \overline{B_r(0)} \subset \mathbb{R}^n \to \mathbb{R}^m$ be continuous. Then $F(x): \mathbb{R}^n \to \mathbb{R}^m$

$$F(x) = \begin{cases} f(x), & \|x\| \leq r \\ f\left(r \frac{x}{\|x\|}\right), & \|x\| > r \end{cases}$$

is continuous on $\mathbb{R}^n$ and $F|_{\overline{B_r(0)}} = f$.

* FIXED POINT THEOREMS:

→ BANACH CONTRACTION MAPPING THEOREM

Thm: (BROUWER'S THEOREM) Let $T: \overline{B(0,1)} \to \overline{B(0,1)}$ be continuous, then T has at least one fixed point (i.e., $T(z) = z$).

Thm: Let $E = (C([0,1], \mathbb{R}^n), \|\cdot\|_\infty)$. Let B = unit ball in E (i.e., $B = \{f \in E : \|f\|_\infty = 1\}$. Let $T: B \to B$ be continuous (w.r.t. $\|\cdot\|_\infty$), s.t. $T(B)$ is relatively compact. Then T has a fixed point.

Lecture 13: Existence and Uniqueness (continued)

Recall: $F = \{f_\alpha : E \subset \mathbb{R}^n \to \mathbb{R}^m\}$ is equicontinuous on E if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\forall x, y \in E\ \forall \alpha \in A$ with $\|x - y\| < \delta$ then $\|f_\alpha(x) - f_\alpha(y)\| < \varepsilon$.

Thm: (Arzelà-Ascoli) If F, as above, is uniformly bounded and equicontinuous, then $\exists (f_n) \subset F$ s.t. $f_n \rightrightarrows$ "something".

Thm: (Schauder - Tychonoff) If $E = (C([0,1], \mathbb{R}^n, \|\cdot\|_\infty))$ B is the unit ball $(= \{f \in E : \|f\|_\infty \leq 1\})$. $T : B \to B$ continuous and $T(B)$ relatively compact, then $\exists x \in B$ s.t. $Tx = x$.

$$\left\{ \begin{array}{l} x' = f(x) \\ x(0) = v \end{array} \right\} \text{IVP}$$

- $\exists$ at least one solution.
- uniqueness.
- Continuous dependence on parameters.

Cauchy's Problem (or IVP):

IVP: Given $f : [t_0, t_1] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ continuous, $v = \xi_0 \in \mathbb{R}^n$. Find, if possible, a function $x : [t_0, t_1] \to \mathbb{R}^n \in C^1([t_0, t_1], \mathbb{R}^n)$ such that $x'(t) = f(t, x(t))$ and $x(t_0) = \xi_0$.

Integral equation: Given f and ξ_0 as IVP, find a function $x \in C^1([t_0, t_1], \mathbb{R}^n)$ s.t. $\forall t \in [t_0, t_1]$

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

Claim: (IVP) $\Leftrightarrow$ (Integral equation)

"Pf:" FTC. $\square$

So, focus on the integral equation.

Define: $(Tx)(t) := (Ux)(t)$. Want $Tx = x$.

Thm: (CAUCHY – PEANO) Consider $x'(t) = f(t, x(t))$ on $[t_0, t_1]$; $x(t) \in \mathbb{R}^n$ and $x(t_0) = \xi_0$. Assume $f: [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and bounded (i.e., $\|f(t, x(t))\| \leq M$). Then (IVP) has at least one solution.

Pf 1: From the integral equation,

$$x(t) = \underbrace{\xi_0 + \int_{t_0}^{t} f(s, x(s))\, ds}_{=: (Tx)(t)}.$$

Consider the unit ball

$$B = \{x \in C([t_0, t_1], \mathbb{R}^n) : \|x(t)\| \leq 1,\ t \in [t_0, t_1]\}.$$

Now, note that

$$\|(Tx)(t)\| \leq \|\xi_0\| + \int_{t_0}^{t_1} \|f(s, x(s))\|\, ds \leq \|\xi_0\| + M(t_1 - t_0)$$

- Case 1: $\|\xi_0\| + M(t_1 - t_0) \leq 1$

  a) $T(B) \subset B$

b) if $x \in B$, $t, t' \in [t_0, t_1]$:

$$\|Tx(t) - Tx(t')\| \le M \|t - t'\|.$$

So, $T(B)$ is equicontinuous ($\delta = \varepsilon / M$)

c) Also, T must be continuous. So, assume $x_k, x \in B$ are such that $x_k \rightrightarrows x$ on $[t_0, t_1]$; i.e., $\|x - x_k\|_\infty \to 0$. Now,

$$\|Tx_k(t) - Tx(t)\|_\infty \xrightarrow{?} 0.$$

So,

$$\|Tx_k(t) - Tx(t)\| \le \int_{t_0}^{t} \|f(s, x_k(s)) - f(s, x(s))\|_\infty ds \longrightarrow 0$$

Because $x_k \rightrightarrows x$, f is cont. on $[t_0, t_1] \times [-1, 1]^n$. So, f is unif. cont.. So $\|f(s, x_k(s)) - f(s, x(s))\|_\infty \to 0$, $k \nearrow \infty$

Thus, by Schauder-Tychonoff, $\exists x \in B$ s.t. $Tx = x$. ✓

• <u>CASE 2</u>: if $|\xi_0| + M(t_1 - t_0) > 1$. Define $H := \|\xi_0\| + M(t_1 - t_0)$.

Define

$$g(t, x) = \frac{1}{H} f(t, Hx). \quad \text{Set } y_0 = \frac{\xi_0}{H}.$$

$$(IVP)_H = \begin{cases} y'(t) = g(t, y(t)) & \text{in } [t_0, t_1] \\ y(t_0) = y_0 \end{cases}$$

Falls into Case 1 $\Rightarrow$ Has solution $y(t)$. Take $x(t) = H y(t)$ to solve (IVP).

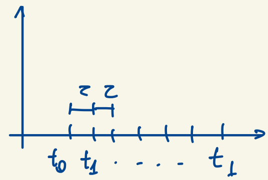
□

Aside: (EULER'S POLYGONAL) Want to solve

$$x'(t) = f(t, x(t)).$$

Approximate by

$$\frac{x_{j+1} - x_j}{t_{j+1} - t_j} = f(t_j, x_j).$$



E.g., take $t_j = j \cdot \tau$. So,

$$x_{j+1} = x_j + \tau f(j\tau, x_j).$$

Pf 2: WLOG, let $t_0 = 0$, $t_1 = 1$, $k = 1, 2, 3, \ldots$

$$x_k(t) = \begin{cases} \xi_0, & \text{if } 0 \le t \le \frac{1}{k}, \\ \xi_0 + \int_0^{t - 1/k} f(s, x_k(s))\, ds, & \text{if } \frac{j}{k} \le t \le \frac{j+1}{k}, \end{cases}$$

$$j = 1, \ldots, k-1.$$

Now, $\| f(t,x) \| \le M$. So,

$$\| x_k(t) - x_k(t') \| \le M |t - t'| \quad \text{(equicont. of } \{x_k\})$$

$$\Rightarrow \| x_k(t) \| \le \| \xi_0 \| + M \quad \text{(unif. bdd.)}$$

So, by Arzela-Ascoli, $\exists\ x_k'$ subsequence of x_k s.t. $x_k' \rightrightarrows x$. So,

$$\begin{array}{ccccccc} x_k'(t) & = & \xi_0 & + & \int_0^t f(s, x_k'(s))\, ds & - & \int_{t-\frac{1}{k}}^t f(s, x_k'(s))\, ds \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ x(t) & = & \xi_0 & + & \int_0^t f(s, x(s))\, ds & - & 0. \end{array}$$

□

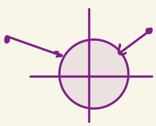
Lecture 14: Existence and Uniqueness (continued)

Recall:

$$(\text{IVP}): \begin{cases} x'(t) = \overbrace{f(t, x(t))}^{\text{on } [t_0, t_1]} \\ x(t_0) = \xi_0 \end{cases} \iff (\text{I.Eq.}) \quad x(t) = \xi_0 + \int_{t_0}^{t} f(s, x(s))\, ds$$

> **Thm:** (Cauchy–Peano) (IVP)
>
> $f: \overbrace{[t_0, t_1]}^{\text{local in time}} \times \mathbb{R}^n \to \mathbb{R}^n$ (global in space) continuous and bounded, then (IVP) has at least one solution.

Lemma: $f: \overline{B_r(0)} \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ continuous, then



$$F(x) = \begin{cases} f(x), & \|x\| \leq r \\ f\left(r \frac{x}{\|x\|}\right), & \|x\| > r \end{cases}, \quad F: \mathbb{R}^n \to \mathbb{R}^n \text{ continuous and } F\big|_{\overline{B_r(0)}} = f.$$

Use this to generalize time.

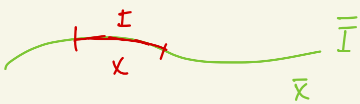
> **Thm:** (Banach Contraction) If E is a complete metric space, and $T: E \to E$ is s.t. $\exists\, 0 < q < 1$ with $d(T(x), T(y)) \leq q\, d(x,y)$ $\forall x, y \in E$. Then $\exists!$ fixed point $x^* \in E$ s.t. $T(x^*) = x^*$.

Now, consider $(IVP)_+ = \begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$.

Goal: find $I = [t_0, t_0+h]$, $h > 0$ with $x \in C^1(I, \mathbb{R}^n)$ such that x satisfies $(IVP)_+$ for $t \in I$.

- If $I = [t_0, t_0+h]$, then $x'(t_0+h) =$ left derivative.

Notation: If x solves $(IVP)_+$ defined on I, then we say that (x, I) solves $(IVP)_+$. Then, the solution can be continued to the right: if $\exists (\bar{x}, \bar{I})$ s.t. $\bar{I} \supset I$, $\bar{x}|_I = x$, then $(\bar{x}, \bar{I}) =$ continuation of (x, I).



Continuation is strict? $\bar{I} \supsetneq I$.

Def: Define partial order $(x_1, I_1) \succeq (x_2, I_2)$ if $I_1 \supset I_2$ and $x_1|_{I_2} = x_2$.

By Zorn's lemma, there exists a maximal element (x^*, I^*) continuation of (x, I): $I^* = \bigcup_{(\bar{x}, \bar{I}) \text{ cont. of } (x, I)} \bar{I}$

Thm: Let $(x_0, \xi_0) \in \mathbb{R} \times \mathbb{R}^n$ and, for $h > 0$ $a > 0$,

$$f: (t, \xi) \in A := [t_0 - h, t_0 + h] \times \overline{B_a(\xi_0)} \longrightarrow \mathbb{R}^n$$

be continuous. Let $M := \sup_{(t, \xi) \in A} \| f(t, \xi) \| < \infty$. Then, (IVP) $\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$

has at least one solution $x(t)$ defined on

$$I = \left[t_0 - \min\left(h, \frac{a}{M}\right),\ t_0 + \min\left(h, \frac{a}{M}\right) \right]$$

and any solution to (IVP) defined on $J \subset I$, J neighborhood of t_0, can be continued to I.

Pf: (1) Lemma $\Rightarrow$ extend f to $\bar{f}: [t_0 - h, t_0 + h] \times \mathbb{R}^n \to \mathbb{R}^n$ continuous and $\bar{f}|_A = f$, $\|\bar{f}(t, \xi)\| \leq M$. Apply Cauchy-Peano to

$$\overline{(IVP)} = \begin{cases} x'(t) = \bar{f}(t, x(t)), & t \in [t_0, t_0 + h] \\ x(t_0) = \xi_0 \end{cases}$$

$\Rightarrow$ get solution $\bar{x}$ to $\overline{(IVP)}$. Now, $\bar{x}$ is continuous, $\bar{x}(t_0) = \xi_0 \Rightarrow \exists j$, $0 < j \leq h$ s.t. if $j \in J = [t_0, t_0 + j]$, then $\|\bar{x}(t) - \xi_0\| \leq a$, so $\forall t \in J$ $(t, \bar{x}(t)) \in A$ so

$$\bar{f}(t, \bar{x}(t)) = f(x, \bar{x}(t)).$$

So, $x := \bar{x}|_J$ satisfies IVP for $t \in J$.

(2) Take $(x, J) \to$ continue to the right as much as possible to get $(\tilde{x}, \tilde{I})$ that cannot be strictly continued to the right. Let $t_1 = \tilde{t} =$ right endpoint of $\tilde{I} \leq t_0 + h$.

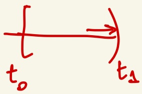
(2a) <u>Claim</u>: $\tilde{I} = [t_0, \tilde{t}] = [t_0, t_1]$; i.e., $t_1 \in \tilde{I}$.

Pf: If not, $\tilde{I} = [t_0, t_1)$, then $\forall t', t'' \in [t_0, t_1)$, $t' < t''$,

$$\| \tilde{x}(t'') - \tilde{x}(t') \| \leq \int_{t'}^{t''} \| f(s, \tilde{x}(s)) \| \, ds \leq M (t'' - t').$$

Cauchy

$\Rightarrow$ $\tilde{x}(t)$ converges as $t \uparrow t_1$. (call limit $\tilde{x}(t_1)$)



$\tilde{x}(t) \in \overline{B_a(\xi_0)}$ if $t \in [t_0, t_1)$

$\Rightarrow$ $\tilde{x}(t_1) \in \overline{B_a(\xi_0)}$

$$\tilde{x}(t_1) = \xi_0 + \lim_{t \uparrow t_1} \int_{t_0}^{t} f(s, \tilde{x}(s)) \, ds = \xi_0 + \int_{t_0}^{t} f(s, \tilde{x}(s)) \, ds$$

f cont. and bdd. on $[t_0, t_1]$

$\Rightarrow$ $\tilde{x}(t)$ satisfies (I.Eq) $\Leftrightarrow$ (IVP) at $t = t_1$. $\square$

(2b) <u>Claim</u>: $t_1 \geq t_0 + \min\left(h, \frac{a}{M}\right)$.

- If $t_1 = t_0 + h$ ✓
- WLOG, assume $t_1 < t_0 + h$. Then $\tilde{x}(t_1) \in \partial \overline{B_a(\xi_0)}$.

So,

$$a = \| \tilde{x}(t_1) - \xi_0 \| = \| \tilde{x}(t_1) - \tilde{x}(t_0) \|$$

$= \|f(t, x(t))\| \leq M$

$$\leq (t_1 - t_0) \sup_{t \in [t_0, t_1]} \| x'(t) \|$$

$$\leq M (t_1 - t_0)$$

$$\Rightarrow t_1 \geq t_0 + \frac{a}{M}.$$

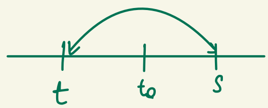
$\square$

Done for "right continuation".

(3) To the left of t_0, consider

$$(IVP)_* = \begin{cases} y'(t) = g(t, y(t)) \\ y(t_0) = \xi_0 \end{cases}$$

where $g(t, \xi) = - f(\underbrace{2t_0 - t}, \xi)$



So, $(IVP)_*$ falls under the hypothesis of continuation to the right.

$\Rightarrow$ $y(t)$ solves $(IVP)_*$ in $[t_0, t_0 + H]$

$\Rightarrow$ $x(t) := y(2t_0 - t)$ solves (IVP) in $[t_0 - H, t_0]$. $\square$

Thm: (EXISTENCE & UNIQUENESS) Consider the following (IVP):

$$y'(t) = f(x, y) \, , \quad y(x_0) = y_0 .$$

Let D be an open set in $\mathbb{R}^2$ (also works for $\mathbb{R} \times \mathbb{R}^n$) such that $f: D \to \mathbb{R}$ is continuous in x and Lipschitz in y with Lipschitz constant K. Then $\exists \, a > 0$ s.t. (IVP) has a solution on $(x_0 - a, x_0 + a)$ and the solution is unique.

Pf (via Banach Contraction). Choose rectangle $R' \subset D$ centered at (x_0, y_0):

$$R' := [x_0 - A, x_0 + A] \times [y_0 - L, y_0 + L].$$

Since f is cont., $\|f(x,y)\| \leq M \quad \forall (x,y) \in R'$. Now, let $a \leq \min\left(\frac{L}{M}, A, \frac{1}{K}\right)$. Now, let

$$R = [x_0 - a, x_0 + a] \times [y_0 - L, y_0 + L].$$

Define

$$X := \{ y \in C([x_0 - a, x_0 + a]) : \|y - y_0\|_\infty \leq L \}.$$

Now, if $y \in X$, then $\text{graph}(y) \subset R$

$$\Rightarrow \|f(x, y(x))\| \leq M \quad \forall x \in [x_0 - a, x_0 + a]$$

Define Picard's Map: $\Gamma : C([x_0 - a, x_0 + a]) \to C([x_0 - a, x_0 + a])$

$$(+) \quad \Gamma(y)(x) := y_0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

If we find $y \in C([x_0 - a, x_0 + a])$ s.t. $\Gamma y = y$, then y satisfies (I. Eq), hence y satisfies (IVP).

[Note: $y_1 \longrightarrow \Gamma y_1 = y_2 \longrightarrow \cdots$ = Picard Iteration]

(1) WTS: $\Gamma(X) = X$. Let $y \in X$, then $\underbrace{\Gamma y \in C([x_0 - a, x_0 + a])}_{\text{even is } C^1.}$

Note that

$$\| \Gamma(y)(x) - y_0 \| = \left\| \int_{x_0}^{x} f(s, y(s))\, ds \right\|$$

$$\leq \int_{\min(x,x_0)}^{\max(x,x_0)} \| f(s, y(s)) \|\, ds$$

Let $x_0 < x$ $\leq \int_{x_0}^{x} M\, ds = M \| x - x_0 \| \leq Ma \leq L$

$\Rightarrow \| \Gamma y - y_0 \|_\infty \leq L.$

(2) <u>WTS</u>: Γ is a contraction, i.e.,

$$\forall y, z \in X, \quad \| \Gamma y - \Gamma z \| \leq \underbrace{aK}_{<1} \| y - z \|.$$

So, $\forall x \in [x_0 - a, x_0 + a]$,

$$\| \Gamma(y)(x) - \Gamma(z)(x) \| = \left\| \int_{x_0}^{x} f(s, y(s)) - f(s, z(s))\, ds \right\|$$

WLOG, $x_0 < x$ $\leq \int_{x_0}^{x} \| f(s, y(s)) - f(s, z(s)) \|\, ds$

f Lipschitz on 2^{nd} variable w/ constant K $\leq \int_{x_0}^{x} K \| y(s) - z(s) \|\, ds$

$$\leq K \| y - z \|_\infty \| x - x_0 \| \leq (aK) \| y - z \|$$

(3) By Banach Contraction thm, we're done. $\square$

Pf (via Picard Iteration): Let

$$R' = [x_0 - A, x_0 + A] \times [y_0 - L, y_0 + L] \subset D$$

$\| f(x,y) \| \leq M \quad \forall x,y \in R'$. $\quad a := \min\left(\frac{L}{M}, A\right)$

$$R = [x_0 - a, x_0 + a] \times [y_0 - L, y_0 + L].$$

Picard Iterates:

$$y_1(t) \equiv y_0$$

$$y_2(t) = y_0 + \int_{x_0}^{x} f(s, y_1(s))\, ds$$

$$y_3(t) = y_0 + \int_{x_0}^{x} f(s, y_2(s))\, ds$$

$$\vdots$$

Claim: $y_n \rightrightarrows y(x)$ solves (I. Eq)

(a) $\forall n$, graph $(y_n) \in R$.

y_1 ✓

$$\| y_{n+1}(x) - y_0 \| \leq \int_{x_0}^{x} \| f(t, y_n(t)) \| \, dt \leq M \| x - x_0 \|$$

$$\leq Ma \leq L.$$

For $n \geq 2$,

$$\| y_{n+1}(x) - y_n(x) \| \leq \int_{x_0}^{x} \| f(t, y_n(t)) - f(t, y_{n-1}(t)) \| \, dt$$

$$\leq K \int_{x_0}^{x} \| y_n(t) - y_{n-1}(t) \| \, dt$$

By induction

$$\| y_{n+1}(x) - y_n(x) \| \leq K^{n-1} M \frac{\|x - x_0\|^n}{n!}$$

$$\leq K^{n-1} M \frac{a^n}{n!}$$

$$= \frac{M}{K} \frac{(Ka)^n}{n!} \quad \forall n \geq 1$$

$\forall x \in [x_0 - a, x_0 + a]$

$\Rightarrow$ so $(y_n(x))$ is Cauchy in $\|\cdot\|_\infty$.

(c) Now, WTS this solves (I. Eq). Recall

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt$$

$$\downdownarrows \qquad \downarrow \qquad \qquad \downarrow \text{ By Lipschitz}$$

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt$$

$$\left\| \int_{x_0}^{x} f(t, y_{n-1}(t)) - f(t, y(t))\, dt \right\|$$

$$\leq K a \| y_{n-1} - y \|_\infty \xrightarrow{n \uparrow \infty} 0.$$

$\square$

Lecture 15: Existence and Uniqueness (continued)

Recall: $y' = f(x,y)$; $y(x_0) = y_0$.

D open $\subset \mathbb{R} \times \mathbb{R}^n$. Take $\mathbb{R}^2$. $(x_0, y_0) \in D$. $f : D \to \mathbb{R}^n$ cont. in x and K Lipschitz in y. Then $\exists a > 0$, s.t. (IVP) has solution on $(x-a, x+a)$ and the solution is unique.

Pf: (a) Picard iteration

(b) Banach contraction map, $a = \min\left(\frac{L}{M}, A\right)$.

Yet another proof of Uniqueness: Suppose there is another solution z to the IVP on $[x_0 - a, x_0 + a]$ s.t. $z(x_0) = y_0$. graph of z is contained in

$$R = [x_0 - a, x_0 + a] \times [y_0 - L, y_0 + L]$$

$L \geq Ma$, $\quad |z(x) - z(x_0)| \leq \left| \int_x^{x_0} f(t, z(t))\, dt \right|$

$\leq |x - x_0| \cdot M \leq L.$

Now,

$$|y(x) - z(x)| = \left| \int_{x_0}^{x} f(t, y(t)) - f(t, z(t))\, dt \right|$$

WLOG $x_0 < x$ $\quad \leq \int_{x_0}^{x} |f(\cdots) - f(\cdots)|\, dt \leq K \int_{x_0}^{x} |y(t) - z(t)|\, dt$

$|y(t) - z(t)| \leq 2L$ $\quad \leq 2LK|x - x_0| \qquad (+)$

Iterate (†)

$$|y(x) - z(x)| \leq \int_{x_0}^{x} 2LK|t-x_0|\, dt$$

$$= \frac{2L(K|x-x_0|)^2}{2}$$

⋮

$$|y(x) - z(x)| \leq K \int_{x_0}^{x} \frac{2LK^{n-1}|t-x_0|^{n-1}}{(n-1)!}\, dt$$

$$= \frac{2L(K|x-x_0|)^n}{n!} \xrightarrow{n\uparrow\infty} 0$$

$\forall x \in [x_0 - a, x_0 + a]$.

<u>Another proof of Uniqueness</u>: get to (†) as before and define

$$u(x) = |y(x) - z(x)|, \quad \text{assume } x > x_0$$

$$u(x) \leq K \int_{x_0}^{x} u(t)\, dt.$$

Let $U(x) = \int_{x_0}^{x} u(t)\, dt$. Then

$$U'(x) \leq K\, U(x).$$

(For " = ", $U(x) = U(x_0)\, e^{K(x-x_0)}$, $U(x_0) = 0$.)

Consider

$$\left(\frac{U(x)}{e^{K(x-x_0)}}\right)' = \frac{U'(x) - K U(x)}{e^{K(x-x_0)}} \leq 0$$

$$\Rightarrow \frac{U(x)}{e^{K(x-x_0)}} \leq \frac{U(x_0)}{e^{K(x_0-x_0)}} \Rightarrow U(x) \leq U(x_0) e^{K(x-x_0)}$$

Now, $U(x_0) = 0 \Rightarrow 0 \leq U(x) \leq 0$

$\Rightarrow u = 0 \Rightarrow y = z$, if $x > x_0$.

Similar for $x < x_0$.

$\square$

Remarks:

- Have seen that if $f: \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$

$$\| f(t, x(t)) - f(t, y(t)) \| \leq K \| x(t) - y(t) \|$$

and $T = a$ s.t. $KT < 1$, then

$$U(x)(t) = \Gamma(x)(t) = v + \int_{t_0}^{t} f(s, x(s))\, ds$$ is a contraction in $C([0,T], \|\cdot\|_\infty)$ and, thus, $\exists!$ fixed point.

$y_0 = y(x_0)$

Q1: • How does this depend on v?

• How does this depend on some parameter?

Parameters are treated in the same way as initial values:

$\lambda \in \mathbb{R}^m$

$$(P_\lambda) \begin{cases} x'(t) = f(t, x(t), \lambda) \\ x(t_0) = v \end{cases}$$

$\Rightarrow$ Define $z(t) = \begin{pmatrix} x(t) \\ \lambda \end{pmatrix}$

$$z(t_0) = \begin{pmatrix} v \\ \lambda \end{pmatrix}$$

$$g(t, z(t)) = \begin{pmatrix} f(t, x(t), \lambda) \\ 0 \end{pmatrix}.$$

(P_λ) becomes $(\underline{P}) = \begin{cases} z'(t) = g(t, z(t)) \\ z(t_0) = w \end{cases}$

LECTURE 16: EXISTENCE AND UNIQUENESS (continued)

Osgood's Uniqueness theorem: Consider the following IVP:

$$y' = f(x, y) \; ; \; y(x_0) = y_0 .$$

Moreover, assume $D \subset \mathbb{R}^2$ (but could be $\mathbb{R}^n$), $(x_0, y_0) \in D$.

Assume $\forall\, (x, y_1), (x, y_2) \in D$,

$$\| f(x, y_1) - f(x, y_2) \| \leq \varphi(\| y_1 - y_2 \|),$$

$\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ continuous, $\varphi(0) = 0$, $\varphi(u) > 0$

for $u > 0$ and $\int_0^1 \frac{du}{\varphi(u)} = \infty$. Then, there is no more than one solution that passes through (x_0, y_0).

REMARKS: 1. We don't assume "f is continuous" so, we can't use Peano's theorem to prove existence.

2. An example of φ: $\varphi(x) = Kx \Rightarrow f$ is Lipschitz in y, then

$$\int_0^1 \frac{du}{Ku} = \frac{1}{K} \lim_{u \to 0} \log u = \infty,$$

so Osgoode applies.

3. If $\left\| \frac{\partial f}{\partial y} \right\| \leq K$ on D, then we have Lipschitz constant K as in (2) by MVT (for multiple variables)

Pf: Assume by contradiction that $\exists$ 2 solutions: $y_1(x)$ and $y_2(x)$ on $(\alpha, \beta) \ni x_0$ and $y_1(x_0) = y_2(x_0) = y_0$. Let $z(x) := y_1(x) - y_2(x)$. So, z satisfies

$$(IVP)_z = \begin{cases} \frac{dz}{dx} = f(x, y_1(x)) - f(x, y_2(x)) \\ z(x_0) = 0 \end{cases}$$

Note, if for some x, $z(x) \neq 0$, then

$$\frac{dz}{dx} = f(x, y_1(x)) - f(x, y_2(x)) \leq \varphi(\|z(x)\|) < 2\,\varphi(\|z(x)\|)$$

$z(x) \neq 0$.

Now, assume $y_1(x) \neq y_2(x)$, $\exists\, x_1 \in (\alpha, \beta)$, $y_1(x_1) \neq y_2(x_1)$

* <u>Case 1</u>: $x_1 > x_0$ and $y_1(x_1) > y_2(x_1)$: let v be the solution of

$$(IVP)_v = \begin{cases} \dfrac{dv}{dx} = 2\varphi(x) \\ v(x_1) = z(x_1) =: z_1 > 0 \end{cases}$$

separable

Solve: let

$$\Phi(v) := \int_v^{z_1} \frac{du}{\varphi(u)},$$

then

$$\Phi(v(x)) = \int_{\underbrace{v(x)}}^{z_1} \frac{dv}{\varphi(v)} = \int_x^{x_1} 2\,dx = 2(x_1 - x)$$

Well-defined b/c $\varphi > 0$

Now, this solution is defined in an interval containing x_1.

<u>Claim</u>: it is actually defined $\forall x$, $x < x_1$, φ cont., $\varphi > 0$ on $(0, \infty)$, which implies that $\Phi(z_1) = 0$:

$$\int_0^1 \frac{du}{\varphi(u)} = \infty \Rightarrow \Phi(v) \nearrow +\infty \text{ as } v \searrow 0.$$

$$\Rightarrow \Phi : (0, z_1] \to [0, +\infty), \quad \Phi'(v) = -\frac{1}{\varphi(v)} < 0$$

so $\Phi \searrow 0$.

$\Rightarrow \Phi$ is invertible with $\Phi^{-1} : [0, +\infty) \to (0, z_1]$.

So, $v(x) = \Phi^{-1}(2(x_1 - x))$, so $v(x)$ is defined $\forall x \leq x_1$.

So, v is increasing and $v>0$ on $(-\infty, x_1]$.

Now, compare $v(x)$ and $z(x)$: both pass through (x_1, z_1); by $(*)$, if there exists $\tilde{x}$ s.t. $0 < v(\tilde{x}) = z(\tilde{x})$, then $z'(\tilde{x}) < v'(\tilde{x})$ (b/c $1<2$). So, in a small neighborhood of $\tilde{x}$, z is above v to the left of $\tilde{x}$ and z is below v to the right of $\tilde{x}$. This means that the graphs of v and z cannot meet in (α, x_1) (they meet at x_1).

In conclusion, by construction,

$$z(x_1) = v(x_1),$$

so z is above v on $\{x < x_2\}$. But $z(x_0) = 0 < v(x_0)$.

$\hookrightarrow\!/\!/\!\hookleftarrow$

COMMENTS:

1. given φ, define $\bar{\varphi}(u) := \sup_{\tilde{u}\in[0,u]} \varphi(\tilde{u})$, then $\bar{\varphi}$ is nondecreasing. Now, if

$$\int_0^c \frac{du}{\varphi(u)} = \infty, \quad \text{then} \quad \int_0^c \frac{du}{\bar{\varphi}(u)} = \infty \quad \forall c\ ?$$

Not necessarily.

Recall: (Uniform Contraction Principle) If (X,d) is a complete metric space, $F: X \times I \longrightarrow X$, $I \subset \mathbb{R}$, s.t. $\forall \lambda \in I$, $F(\cdot, \lambda)$ is a (uniform) contraction; i.e., $\forall \lambda \in I$,

$\forall x, y \in X$, $d(F(x,\lambda), F(y,\lambda)) \le q\, d(x,y)$ for a fixed $0 \le q < 1$. Then $\forall \lambda$, $\exists!\ x^*(\lambda)$ fixed point; i.e., $F(x^*(\lambda), \lambda) = x^*(\lambda)$. Now, we have

$$\lambda_1 \longrightarrow \text{fixed point } x_1^*$$
$$\lambda_2 \longrightarrow \text{fixed point } x_2^*,$$

i.e., $F(x_i^*, \lambda_i) = x_i^*$

Then $d(x_1^*, x_2^*) \le \dfrac{1}{1-q}\, d(F(x_1^*, \lambda_1), F(x_2^*, \lambda_2))$.

(Maximum Time of Existence) $D \subset \mathbb{R}^n$ domain, $f: D \to \mathbb{R}^n$ locally Lipschitz continuous w/ constant L. Then $\exists v \in D$ $\exists T > 0$ time s.t.

$$(IVP) = \begin{cases} x' = f(x) \\ x(0) = v \end{cases}$$

has a unique solution $x: [-T, T] \to D$ (and x depends continuously on v).

<u>Lemma</u>: $D \subset \mathbb{R}^n$, $f: D \to \mathbb{R}^m$ continuously differentiable, then f is locally Lipschitz ($\forall K$ compact, $L := \sup_K \|Df(x)\|$ is the Lipschitz constant in K).

<u>Thm</u>: (GLOBAL EXISTENCE AND UNIQUENESS) If $D \subset \mathbb{R}^n$ is open and connected, $f: D \to \mathbb{R}^n$ is a locally Lipschitz vector space. Then, $\exists!$ maximal interval of excistence

$$I_{max} = (\underline{T}, \overline{T}) \ni 0$$

s.t.

i) (IVP) has a unique solution on I_{max}

ii) if $\overline{T} < \infty$, then

$$\lim_{\substack{t \to \overline{T} \\ t < \overline{T}}} \|x(t)\| + \frac{1}{\mathrm{dist}(x(t), \partial D)} = +\infty,$$

i.e., either $\|x(t)\| \nearrow +\infty$ or $x(t) \longrightarrow \partial D$.

Pf: By Contrapositive. Assume

(1) $x(t)$ exists up to time T (at least)

(2) $\|x(t)\| \not\longrightarrow \infty$ at T

$x(t) \not\longrightarrow \partial D$ at T

i.e., $\exists t_j \nearrow T$ s.t.

$$M := \sup_j \|x(t_j)\| < \infty \quad \text{and}$$

$$S := \inf_j \mathrm{dist}(x(t_j), \partial D) > 0$$

Let

$$K := \{z \in D : \|z\| \leq M, \ \mathrm{dist}(z, \partial D) \geq S\} \quad \text{Compact}$$

$\Rightarrow$ f is Lipschitz on K w/ constant $L = L(K)$.

$\Rightarrow$ $\exists$ fixed $\tau > 0$ s.t. $\forall w \in K$, (IVP) $\left| \begin{array}{l} x' = f(x) \\ x'(0) = w \end{array} \right.$

has a solution on $[-\tau, \tau]$.

Take $(IVP)_j$ $\left|\begin{array}{l} y' = f(y) \\ y(0) = x(t_j) = w_j \end{array}\right.$ $\longrightarrow$ solution $= y_j$.

Look at $\underbrace{y_j(t - t_j)}$ defined on $[t_j - \varepsilon, t_j + \varepsilon]$.

Agrees w/ $x(t)$ at t_j since $x(t)$ is a maximal solution $\Rightarrow x(t)$ is defined up to $t_j + \varepsilon$ if $j \nearrow \infty$, $t_j \to T$ so $t_j + \varepsilon > T \Rightarrow \bar{T} = T_{max} > T + \frac{\varepsilon}{2}$ so T can't be max. time of existence.

$\square$

* **Uniform Contraction Revisited:**

Given $F : D_1 \subset \mathbb{R}^n \times D_2 \subset \mathbb{R}^m \longrightarrow D_1 \subset \mathbb{R}^n$. Assume F is a uniform contraction

$0 \leq q < 1$ fixed

$$\| F(x_1, v) - F(x_2, v) \| \leq q \| x_1 - x_2 \| \quad \forall v \in D_2, \ \forall x_1, x_2 \in D_1$$

Assume F is C^1 in (x, v).

* **Want to solve:** $x = \underbrace{F(x, v)}$ (x fixed point),

$\underset{\text{C.M.T.}}{\Rightarrow} \exists x = x(v)$ solution.

Invertible

$\underbrace{x - F(x, v)}_{C^1} = 0 \underset{\text{IFT}}{\Longrightarrow}$ Check $D_x(x - F(x, v)) = \overbrace{Id - D_x F}$

know $\| D_x F \| \leq q < 1$.

Lemma: $A \in \mathbb{R}^{n\times n}$, $\|A\| = \rho < 1$ $\left(\|A\| = \sup_{x\neq 0} \frac{\|Ax\|}{\|x\|} \right)$

$\Rightarrow$ $Id - A$ is invertible

"Pf:" Von Neumann series

$$(Id - A)^{-1} = \sum_{k=0}^{\infty} A^k \quad \text{(absolute converges)}$$

$$\Rightarrow (Id - A)\left(\sum_{k=0}^{\infty} A^k\right) = Id$$

$$\left(\sum A^k\right)(Id - A) = Id.$$

Now, solving $x - F(x,v) = 0$, $D_x(x - F(x,v)) = Id - D_x F$ is invertible $\Rightarrow x(v) \in C^1$

Note: IFT says that if $F \in C^k$, solution is C^k.

Lecture 17: Continuous Dependence & Notes

Ex: $\begin{cases} x' = Ax, & x \in \mathbb{R}^n, A \in \mathbb{R}^{n\times n} \\ x(0) = v \end{cases}$ Find a condition on A such that $|x(t)| \to 0$ as $t \nearrow \infty$.

Change of basis/variables such that $A \to$ Jordan form.

$Y = TX$ so Y solves $Y' = JY$. So $|X(t)| \to 0$ iff $|Y(t)| \to 0$ for all solutions. All solutions are of the form

$$Y(t) = \sum_j P_j(t) e^{\lambda_j t} v_j$$

$$|Y(t)| \xrightarrow{t \nearrow \infty} 0 \iff \mathrm{Re}(\lambda_j) < 0 \quad \forall j$$
$\forall$ solution


1 day ago
Sorry, I don't give hints.

Thank you very much. Best regards,

Ignacio

good comment

$\square$

Note: In the proof Cauchy-Peano, $[0,1]$, $k = 1, 2, \ldots$

$$X_k(t) = \begin{cases} \xi_0, & \text{if } 0 \le t \le 1/k \\ \xi_0 + \int_0^{t-1/k} f(s, x(s))\, ds, & \text{if } \frac{j}{k} \le t \le \frac{j+1}{k} \end{cases}$$

$j = 1, 2, \ldots, k-1$.

Note.

$$(IVP)_v = \begin{cases} x' = f(t,x) \\ x(0) = v \end{cases}$$

$\longleftarrow$ solution $x_v(t)$ is known

if $v = v_k \to w$.

By continuous dependence, we know that $x_{v_k}(t) \longrightarrow x_w(t)$.

Thm: (GRÖNWALL'S LEMMA - GENERALIZED VERSION) given $f: [a,b] \to \mathbb{R}$ and $g: [a,b] \to \mathbb{R}_+$ continuous, and $y: [a,b] \to \mathbb{R}$ continuous and such that $\forall t \in [a,b]$

$$y(t) \le f(t) + \int_a^b g(s) y(s)\, ds. \quad (\dagger)$$

Then, for all $t \in [a,b]$,

$$y(t) \le f(t) + \int_a^b f(s)\, g(s) \left(e^{\int_s^t g(u)\,du} \right) ds .$$

In particular, if $f(t) = K$ (constant), then

$$y(t) \le K e^{\int_a^t g(s)\,ds} .$$

Plug $f(t) = K$ in (†) and get

$$(LHS) = K + K \int_a^b g(s) \left(e^{\int_s^t g(u)\,du} \right) ds \overset{?}{\le} K e^{\int_a^t g(s)\,ds} = (RHS)$$

$$(LHS)\big|_{t=a} = (RHS)\big|_{t=a} \qquad \left(e^{\int_a^t} \cdot e^{-\int_a^s} \right)$$

$$\Rightarrow \quad e^{-\int_a^t g(u)\,du} + \int_a^t g(s) \left(e^{-\int_a^s g(u)\,du} \right) ds = 1 \checkmark$$

Pf: (1) Assume $f(t) = K$. WLOG, $|y(t)| \le 1$ (divide by constant) $\forall t \in [a,b]$. So,

$$y(t) \le K + \int_a^t g(t_1) \left[K + \int_a^{t_1} g(t_2) \left(K + \int_a^{t_2} g(t_3)\, y(t_3)\, dt_3 \right) dt_2 \right] dt_1$$

$$+ \int_0^t g(t_1) \int_a^{t_1} g(t_2) \int_a^{t_2} g(t_3)\, y(t_3)\, dt_3\, dt_2\, dt_1 .$$

Let

$$I := \int_a^t g(t_1) \int_4^{t_1} g(t_2)\, dt_2\, dt_1 = \int_a^t M'(t_1)\, M(t_1)\, dt_1$$

$M(u) = \int_a^u g(s)\, ds$; $M'(u) = g(u)$, $M(a) = 0$

$$= \frac{1}{2}[M(t)]^2 = \frac{1}{2}\left[\int_a^t g(s)\,ds\right]^2.$$

Similarly, we get

$$y(t) \le K + K\frac{\int_a^t g(t_1)\,dt_1}{1!} + K\frac{\left(\int_a^t g(t_1)\,dt_1\right)^2}{2!} + \cdots$$

$$+ K\frac{\left(\int_a^t g(t_1)\,dt_1\right)^n}{n!} + \underbrace{\int_a^t g(t_1)\int_a^{t_1} g(t_2)\int_a^{t_2} g(t_3) \cdots \int_a^{t_n} g(t_{n+1})\,y(t_{n+1})\,dt_{n+1}\cdots dt_1}_{\mathcal{E} =:}$$

$$|\mathcal{E}| \le \frac{\left(\int_a^t g(t_1)\,dt_1\right)^{n+1}}{(n+1)!}$$

$g \ge 0$; $|y(t)| \le 1$

so,

$$y(t) \le K\left(e^{\int_a^t g(s)\,ds}\right) + \mathcal{E}$$

and $\mathcal{E} \to 0$ as $n \nearrow \infty$. $\square$

LECTURE 18: GENERALIZED GRÖNWALL & CONTINUOUS DEPENDENCE

Recall: Which condition on A s.t. the solution of $X' = AX$ is such that $\|X(t)\| \to 0^{\infty}$, $t \nearrow \infty$.

$\mathrm{Re}(\lambda_j) < 0$ $\quad$ $\mathrm{Re}(\lambda_j) > 0$ $\quad$ $\mathrm{Re}(\lambda_j) = 0$ and A diagonalizable.

bdd away

Thm: (Generalized Grönwall's Inequality) Let $f : [a,b]$ and $g : [a,b] \to \mathbb{R}^+$ be continuous and $y : [a,b] \to \mathbb{R}$ s.t.

$$y(t) \leq f(t) + \int_a^t g(s) y(s)\, ds \quad \forall t \in [a,b] \quad (H)$$

then

$$(\dagger) \quad y(t) \leq f(t) + \int_a^t f(s) g(s) \left(e^{\int_s^t g(u)\, du} \right) ds \quad \forall t \in [a,b]$$

Pf: $z(t) := \int_a^t g(s) y(s)\, ds$

$$(H) \Rightarrow z'(t) - g(t) z(t) \leq f(t) g(t).$$

Let $w(t) := z(t) e^{-\int_a^t g(s)\, ds}$ (would've been an "integrating factor" if had "="). Then

$$w'(t) \leq f(t) g(t) e^{-\int_a^t g(s) ds}$$

$w(a) = z(a) = 0$ and integrate from a to t.

$$w(t) \leq \int_a^t f(s_1) g(s_1) \left(e^{-\int_a^{s_1} g(s)\, ds} \right) ds_1$$

$$\Rightarrow z(t) \leq \int_a^t f(s_1) g(s_1) \left(e^{\int_{s_1}^t g(s)\, ds} \right) ds_1$$

$$\Rightarrow y(t) \leq f(t) + z(t) \text{ in } (\dagger).$$

$\square$

* **CONTINUOUS DEPENDENCE**

Thm: Let $x_1 : [a,b] \to \mathbb{R}^n$ and $x_2 : [a,b] \to \mathbb{R}^n$ be differentiable and such that

$$|x_1(a) - x_2(a)| \leq \delta$$

Let $f: [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be L-Lipschitz in the second variable: $\forall t \in [a,b]$, $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$|f(t,\xi_1) - f(t,\xi_2)| \leq L\,|\xi_1 - \xi_2|.$$

Assume that

$$\left.\begin{aligned} |x_1' - f(t,x_1(t))| &\leq \varepsilon_1 \\ |x_2' - f(t,x_2(t))| &\leq \varepsilon_2 \end{aligned}\right. \qquad \forall t \in [a,b]$$

Then

$$|x_1(t) - x_2(t)| \leq \delta e^{L(t-a)} + (\varepsilon_1 + \varepsilon_2)\,\frac{e^{L(t-a)} - 1}{L}$$

NOTE: $\varepsilon_1 = \varepsilon_2 = \delta = 0 \Rightarrow$ another proof of uniqueness.

Pf: (By Grönwall's Inequality) Let $\varepsilon := \varepsilon_1 + \varepsilon_2$ and let $g(t) = x_1(t) - x_2(t)$.

$$(\dagger) \quad |g'(t)| = |x_1'(t) - x_2'(t)| \overset{\Delta\text{-inequality}}{\leq} |f(t,x_1(t)) - f(t,x_2(t))| + \varepsilon$$

$$\overset{f\ L\text{-Lipschitz}}{\leq} L|g(t)| + \varepsilon.$$

So, integrating the above,

$$|g(t)| = \left| \int_a^t g'(s)\,ds + g(a) \right| \leq \int_a^t |g'(s)|\,ds + \underbrace{\delta}_{g(a)\leq\delta}$$

$$\overset{(+)}{\leq} \delta + \varepsilon(t-a) + \int_a^t L|g(s)|\,ds$$

By Grönwall's Inequality,

$$|g(t)| \leq \delta + \varepsilon(t-a) + \int_a^t L(\delta + \varepsilon(t-a))e^{L(t-s)}\,ds$$

Integrate $\;= \delta e^{L(t-a)} + \frac{\varepsilon}{L}\left(e^{L(t-a)} - 1\right).$

□

COROLLARY: Let $A(t) \in \mathbb{R}^{n\times n}$ be continuous in t.

$$(IVP): \begin{cases} x'(t) = A(t)X \\ x(t_0) = x_0 \end{cases}, \quad t_0 \in I, \quad x \in \mathbb{R}^n.$$

Then (IVP) has a unique solution on all of I.

$x(t) = x_0 e^{\int_{t_0}^t A(s)\,ds}$ is the (unique) solution

---

DIFFERENTIABILITY OF THE FLOW: Let $f \in C^1$, $x \in \mathbb{R}^n$

consider

$$x' = f(x), \quad x(t_0) = x_0 \qquad \text{(IVP)}$$

Solution: $\phi(t,x) = \phi_t(x)$ is C^1 in t and x. Then

$$\underbrace{\frac{d}{dt}}_{\text{time derivative}}(\phi(t,x)) = x'(t) - f(x(t)); \qquad \underbrace{D_x}_{\text{spatial derivative}}\phi_t(x)$$

Consider (IVP): $\begin{cases} x' = f(x) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases}$ $t \in J$ → closed and contains zero

For each $t \in J$, $\underbrace{A(t) = Df_{x(t)}}_{f \in C^1 \Rightarrow A(t) \text{ cont.}} = Df(x(t))$

$= $ Jacobian at $x(t)$.

Def: (VARIONAL EQUATION) $u' = A(t)\, u$; $u(0) = u_0$

By corollary have a unique solution on all J for all $u(0) = u_0$

(+) As in autonomous case, solutions of this system satisfy the linearity principle.

Meaning of this equation: if u_0 is small, then

$t \mapsto \underbrace{x(t)}_{\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}} + \underbrace{u(t)}_{\begin{cases} u' = A(t)u \\ u(0) = u_0 \end{cases};\ A(t) = Df(x(t))}$

is a good approximation to the solution of $\begin{cases} x' = f(x) \\ x(0) = x_0 + u_0 \end{cases}$

Prop: Let J be a closed interval containing zero.
Let $x(t)$ solve $\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$ for $t \in J$ and let

$u(t, \xi)$ solve $\begin{cases} u' = A(t)\, u \\ u(0, \xi) = \xi \end{cases}$, $A(t) = Df(x(t))$.

Assume ξ, $x_0 + \xi \in D = \mathrm{Dom}(f)$; $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ is C^1

Assume $y(t)$ solves $\begin{cases} y' = f(y) \\ y(0) = x_0 + \xi \end{cases}$.

Thm $$\lim_{\xi \to 0} \frac{|y(t,\xi) - x(t) - u(t,\xi)|}{|\xi|}$$

converges to 0 uniformly to $t \in J$ (i.e., $\forall \varepsilon > 0\ \exists \delta > 0$ s.t. $|\xi| \le \delta \Rightarrow |y(t,\xi) - x(t) - u(t,\xi)| \le \varepsilon |\xi|\ \forall t \in J$).

Note: $u(t,\xi)$ is linear in ξ (by (+))

Thm: $\phi(t,x)$ flow of $x' = f(x)$ is a C^1 function, i.e., $\frac{\partial \phi}{\partial t}$ and $\frac{\partial \phi}{\partial x}$ exist and are continuous in t and x

Pf: $\frac{\partial}{\partial t} \phi(t,x) = f(\phi_t(x)) = f(\phi(t,x))$ continuous ✓

$\frac{\partial \phi}{\partial x}$ if ξ is small: $\phi(t, x_0+\xi) - \phi(t, x_0) = y(t,\xi) - x(t)$

Now, prop. implies $\frac{\partial \phi(t,x)}{\partial x}$ is the linear map

$\xi \mapsto u(t,\xi)$ which is continuous in initial conditions and data.

$\square$

(IVP): $\begin{cases} \frac{\partial}{\partial t}(D\phi_t(x_0)) = Df(\phi_t(x)) \overbrace{D\phi_t(x_0)}^{= u(t,x_0)} \\ D\phi_0(x_0) = Id \end{cases}$

$u' = \frac{d}{dt} u(t,x_0) = A(t)\, u(t,x_0)$

(think of x_0 as a parameter $\phi_0(x) = x$)

!!

* **PARTICULAR CASE**: Assume $x(t) \equiv a$ (equilibrium solution of $x' = f(x)$). Then $A(t) = Df(x(t)) = \underbrace{Df(a)}_{\text{Doesn't depend on } t} =: A$.

So, we get

$$\begin{cases} \frac{d}{dt} D\phi_t(a) = A(D\phi_t(a)) \\ D\phi_0(a) = Id \end{cases}$$

The solution of this equation is $D_{\phi_t}(a) = e^{tA}$. So, in a neighborhood of an equilibrium point, the flow is approximately linear!

Pf (of Proposition) Integral equation

$$x(t) = x_0 + \int_0^t f(x(s))\, ds$$

$$y(t,\xi) = x_0 + \xi + \int_0^t f(y(s,\xi))\, ds$$

$$u(t,\xi) = \xi + \int_0^t [Df(x(s))][u(s,\xi)]\, ds$$

So,

$$|y(t,\xi) - x(t) - u(t,\xi)| \leq \int_0^t |f(y(s,\xi)) - f(x(s)) - Df_{x(s)}(u(s,\xi))|\, ds$$

Taylor

$$f(y) = f(z) + Df_z(y-z) + R(y, z-y); \quad \lim_{y\to z} \frac{R(z, y-z)}{|y-z|} = 0$$

unif. in y, if y in compact.

Thus,

$$g(t) := |y(t,\xi) - x(t) - u(t,\xi)| \leq \int_0^t |\overbrace{Df_{x(s)}}^{\in \mathbb{R}^{n\times n}} \overbrace{(y(s,\xi) - x(s) - u(s,\xi)}^{\in \mathbb{R}^n}| \, ds$$

$$+ \int_0^t |R(x(s), y(s,\xi) - x(s))| \, ds$$

$N := \max\{|Df_{x(s)}| : s \in J\} < \infty$ b/c J is closed. Then

$$g(t) \leq N \int_0^t g(s)\, ds + \int_0^t \underbrace{|R(x(s), y(s,\xi) - x(s))|}_{\leq \varepsilon |y(s,\xi) - x(s)|} \, ds.$$

Fix $\varepsilon > 0$ by Taylor
$\exists \delta_0 > 0 : \ |y(s,\xi) - x(s)| \leq \delta_0 \quad s \in J$

From thm $\underset{\varepsilon_1,\varepsilon_2,\delta}{}$ following from Grönwall, $\exists k \geq 0, \ \delta_1 > 0$
s.t. $|y(s,\xi) - x(s)| \leq |\xi| e^{ks} \leq \delta_0$ if $|\xi| \leq \delta_1$ and $s \in J$.

If $|\xi| \leq \delta_1, \ t \in J$

$$g(t) \leq N \int_0^t g(s)\, ds + \int_0^t \varepsilon |\xi| e^{ks}\, ds$$

$$= N \int_0^t g(s)\, ds + c\,\varepsilon |\xi|$$

$c = c(k, \text{length of } J) = \int_0^{\ell(J)} e^{ks}\, ds$

By Grönwall,

$$g(t) \leq c\,\varepsilon e^{Nt} |\xi| \quad \text{if } t \in J, \ |\xi| \leq \delta_1$$

$$\Rightarrow \frac{g(t)}{|\xi|} \to 0 \text{ unif. in } t \in J \text{ b/c } \varepsilon \text{ is arbitrary.}$$

$\square$

Lecture 19: Variational Equation

* Variational Equation

$$u(t) := D_v \Phi_t(v); \qquad \Phi_t(v) \text{ solves } \begin{cases} x' = f(x) \\ x(0) = v \end{cases}$$

$$\frac{d}{dt} u(t) = \frac{d}{dt} D_v \Phi_t(v) = D_v \left(\frac{d}{dt} \Phi_t(v) \right) = D_v(f(\Phi_t(v)))$$

Formally (need $\in C^2$)

Chain Rule

$$= Df(x(t)) \cdot u(t)$$

So, get matrix equation $\begin{cases} u' = Df(x(t)) \cdot u \\ u(0) = Id \end{cases}$, $u \in \mathbb{R}^{n \times n}$

Equivalently, linearize at x

$$\begin{cases} z'(t) = Df(x(t))\, z \\ z(0) = w \end{cases} \rightsquigarrow z(t) = u(t)\, v = \left(D_v \Phi_t(v)\right) w$$

So, $x(t) + z(t)$ solves $\begin{cases} y' = f(y) \\ y(0) = v + w \end{cases}$.

* If $x(t) \equiv a$ is a steady state, then

$$D f(x(t)) = Df(a) = \text{constant}$$

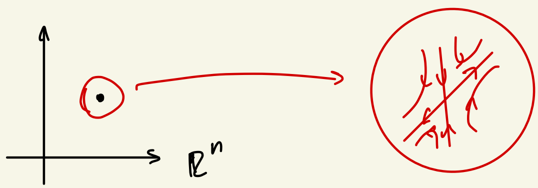
* Alternative way of getting linearization:

$$f(x) = f(a) + Df(a)(x-a) + \underbrace{o(|x-a|)}_{\text{error}}$$

Look at dynamics near steady state: Write the solution as $x(t) = a + y(t)$. Then, the ODE becomes

$$\text{ODE} \to 0 + y'(t) = f(x) = \underbrace{0}_{f(a)} + Ay + o(|y|)$$

Q: What does $y' = Ay$ tell us about the original problem?



We know that, if Φ_t is generated by $x' = f(x)$ then $D_v \Phi_t(a) = e^{tA}$ for t fixed

Initial value

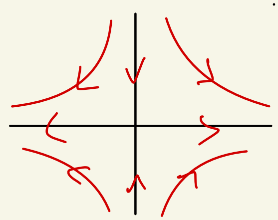
steady-state

Ex 1: $\begin{cases} x' = x + y^2 \\ y' = -y \end{cases} \quad \longleftrightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x + y^2 \\ -y \end{pmatrix} = f(x,y)$

$a = (0,0) =$ equilibrium

* Linearization:

$$DF_{(0,0)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A$$



$$\Rightarrow e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

* Solve non-linear system: $y(t) = y_0 e^{-t}$

$$\Rightarrow \quad x' = x + y_0^2 e^{-2t}$$

homogeneous   particular

$$\Rightarrow \text{ general solution: } \quad x(t) = x_h(t) + x_p(t)$$

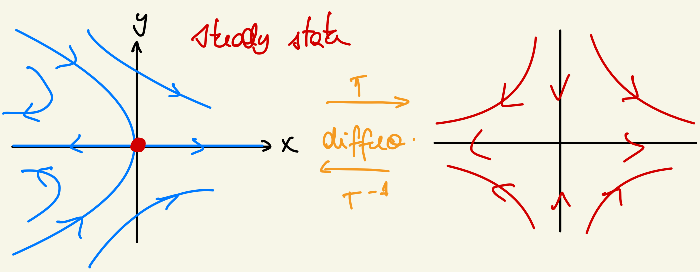
$x_h(t) = C e^t$ ($C = x_0 + \frac{y_0^2}{2}$ to get x_0 as initial value)

Particular: of the form $x_p(t) = b\, e^{-2t}$

$$\Rightarrow \quad -2b = b + y_0^2 \quad \Rightarrow \quad b = -\frac{y_0^2}{2}.$$

Thus,

$$\boxed{\begin{aligned} x(t) &= \left(x_0 + \frac{y_0^2}{2}\right) e^t - \frac{y_0^2}{2} e^{-2t} \\ y(t) &= y_0 e^{-t} \end{aligned}}$$



$$\begin{cases} u = x + \frac{1}{3} y^2 \\ v = y \end{cases}$$

$x_0 = \frac{1}{3} y_0^2$

$\Phi_t = T^{-1} e^{tA}\, T$

> The dynamics of the ODE (nonlinear) is conjugate to its linearization by a local diffeomorphism

CONJUGACY

- **For matrices**: A is conjugate to B if $\exists P$ s.t.

$$B = P^{-1}AP$$

- **For maps**: Φ is continuous to ψ if $\exists$ invertible map h s.t. $h \circ \Phi \circ h^{-1} = \psi$.

Ex 2:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2}(x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix}$$

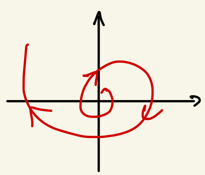
(0,0) is equilibrium

* linearization

$$Df_{(0,0)} = \begin{pmatrix} 1/2 & -1 \\ 1 & 1/2 \end{pmatrix} = A$$

Eigenvalues: $\frac{1}{2} \pm i$

$$e^{tA} = e^{t/2}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$



spiral source at (0,0)

LECTURE 20: LINEARIZATION (continued)

$x' = f(x)$; $f \in C^1$; $x \in \mathbb{R}^n$, $x(t) = a$ eq. pt.; i.e., $f(a) = 0$.

$$f(x) = f(a) + Df(a)(x-a) + o(|x-a|)$$

$\boxed{y' = DF(a)\,y}$ is the linearized system.

How close is it to $x' = f(x)$ near a?

* It's close if there exists a diffeomorphism between the systems, i.e., they are conjugate.

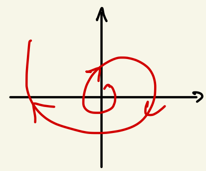
$\boxed{\text{Ex } 2}$:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{pmatrix} 1/2 & -1 \\ 1 & 1/2 \end{pmatrix}}_{A=} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2}(x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix}$$

$(0,0)$ = steady state

Linearization; eigenvalues $\frac{1}{2} \pm i$

$$e^{tA} = e^{t/2}\begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$



Non-linear system: $x = r\cos\theta$; $y = r\sin\theta$.

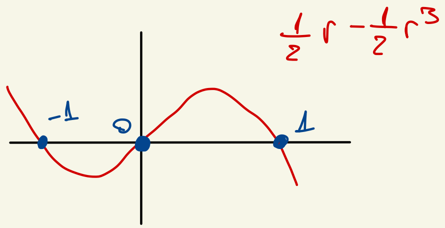
$$\begin{cases} x' = r'\cos\theta - r\sin\theta\,\theta' & (\mathrm{I}) \\ y' = r'\sin\theta + r\cos\theta\,\theta' & (\mathrm{II}) \end{cases}$$

$\Rightarrow$ $xx' + yy' = rr'$ and $-x'y + y'x = r^2\theta'$

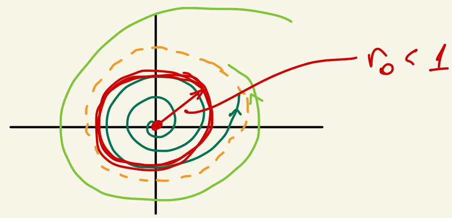
Now,

$$r' = \frac{1}{r}(xx' + yy') \overset{\text{ODE}}{=} \frac{1}{2}r - \frac{1}{2}r^3.$$

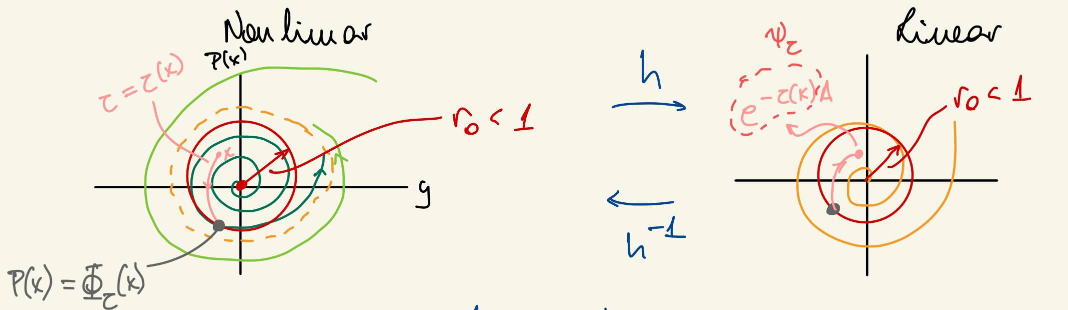
$$\theta' = \frac{1}{r^2}(-x'y + y'x) \overset{\text{ODE}}{=} 1$$



Nonlinear system



* Conjugacy between linear and nonlinear system is only possible locally ($r < 1$) b/c dynamics differ for $r \geq 1$.



Claim: $h \circ \psi_t \circ h^{-1} = \Phi_t(x)$

and h is a diffeo.

Ex 3:

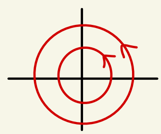
$$\begin{cases} x' = -y + \varepsilon x(x^2+y^2) \\ y' = x + \varepsilon y(x^2+y^2) \end{cases}$$

linearization:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

!! A eigenvalues $\pm i$, Real part $= 0$

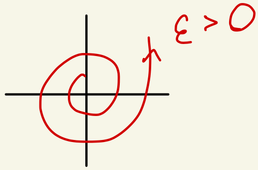
$$e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

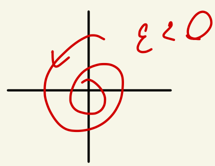


Polar coordinates: Nonlinear system

Not conjugate

$$\theta' = 1, \quad r' = \varepsilon r^3$$





Ex 5:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = f(x,y) \ ; \quad Df(0,0) = 0, \quad f(0,0) = 0$$

E.g.: $\begin{pmatrix} x \\ y \end{pmatrix}' = (x^2+y^2)\, g(x,y)$

$\rightarrow$ same geometry as $\begin{pmatrix} x \\ y \end{pmatrix}' = g(x,y)$

so anything can happen for nonlinear system.

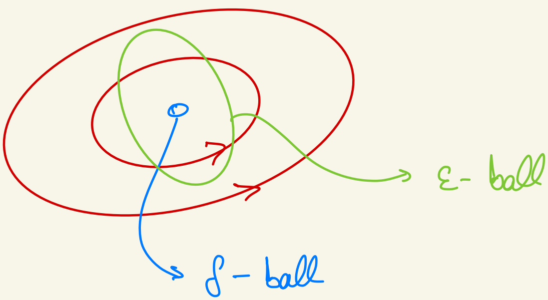
* **Stability** (Lyapunov)

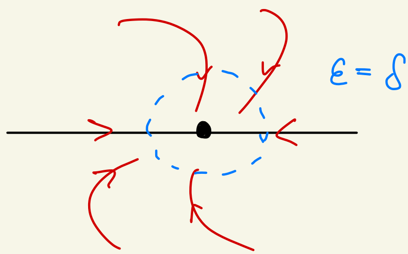
Def: $(\Phi_t)_{t\geq 0}$ is a dynamical system on $D \subset \mathbb{R}^n$ (i.e., $\Phi_0 = Id$; $\Phi_{s+t} = \Phi_s \circ \Phi_t$). Let a be a steady state / equilibrium / fixed point if $\Phi_t(a) = a$ for all t.

We say "a is <u>stable</u>" if $\forall \varepsilon > 0\ \exists \delta > 0$ s.t.

$$|x - a| < \delta \Rightarrow \sup_{t\geq 0} |\Phi_t(x) - a| < \varepsilon.$$

Ex:





Def: The equilibrium at a is <u>unstable</u> if $\exists\, \varepsilon > 0$ and a sequence $x_j \to a$ s.t.

$$\sup_{t\geq 0} |\Phi_t(x_j) - a| > \varepsilon \quad \forall j = 1, 2, \ldots$$

Negation of stability.

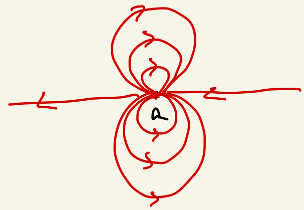
Def: The equilibrium at a is <u>asymptotically stable</u> if

→ Attractive

i) a is stable.

ii) $\exists\, \delta > 0$ s.t. $\lim_{t \nearrow \infty} |\Phi_t(x) - a| = 0 \quad \forall x$ with $|x-a| < \delta$.

Ex (ii) $\not\Rightarrow$ (i): on S^2

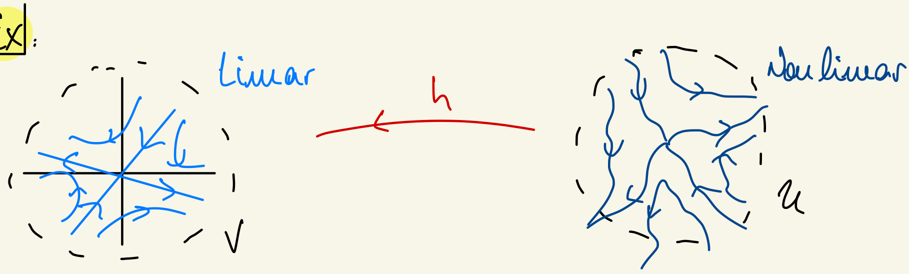


Thm: (GROBMAN – HARTMAN TOPOLOGICAL CONJUGACY) Consider $x' = f(x)$, $f \in C^1$, $f(a) = 0$ equilibrium, $A = Df(a)$. Assume A is hyperbolic (i.e., real part of eigenvalues of A are nonzero). Let Φ_t be the dynamical system for the ODE and $\varphi_t := e^{tA}$ the dynamical system for linearization. Then $\exists$ neighborhood $\mathcal{U} \ni a$, $V \ni 0$ and $h: \mathcal{U} \to V$ homeomorphism s.t.

$$\Phi_t = h^{-1} \circ \varphi_t \circ h$$

on $\mathcal{U}$ for t small enough.

Ex:



Q1: What if we want h to be a diffeo?

Need a non-resonant condition.

Resonance:

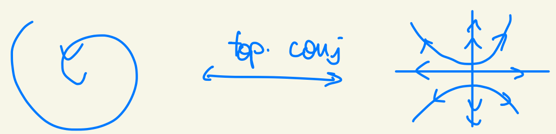
$$\lambda_j = \sum_{\substack{k=1 \\ k \neq j}}^{n} a_k \lambda_k$$

Integers

Q2: What does topological conjugacy means?

Not much... preserves stability / asymptotic stability.

All things are topologically conjugates



e.g., hyperbolicity

General principle (vague): if linear system is "structurally stable", then the non-linear system "locally" "looks like" its linearization.

topologically conjugate

Pf: (special case) $A = Df(a)$ w/ distinct real eigenvalues $\lambda_1, \ldots, \lambda_n < 0$. WLOG, $a = 0$

$$A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$$

$$\Rightarrow x' \overset{(*)}{=} \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} x + \underbrace{o(|x|)}_{\text{higher order terms}}$$

Lemma: if $\lambda_1, \ldots, \lambda_n$ are real and nonnegative then 0 is asymptotically stable for $(*)$; i.e., 0 is a sink.

Pf: Consider $L(x) = \frac{1}{2}|x|^2$ (Lyapunov function)

If $y(t)$ solves the linear system, then

$$\frac{d}{dt} L(y(t)) = \frac{d}{dt}\left(\frac{1}{2}|y|^2\right) = y_1 y_1' + \cdots + y_n y_n'$$

$$= y^T \cdot y' = y^T A y$$

$$= \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2 < 0$$

$$\leq -\min\{|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|\} |y|^2$$

$$< -c L y$$

$$\Rightarrow L(y(t)) \leq L(y(0)) e^{-ct}, \quad \forall t > 0. \text{ (by grönwall)}$$

For nonlinear: $x' = f(x)$. Just repeat

$$\frac{d}{dt} L(x(t)) = x^T x'$$

$$= x^T \left[\begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} x + o(|x|) \right]$$

$$\leq -c|x|^2 + o(|x|^2).$$

Choose a neighborhood $V \ni 0$ s.t.

$$\sup_{x \in V} \Big| \underbrace{f(x) - \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} x}_{\text{Taylor exp. of } f} \Big| < \frac{c|x|^2}{2}.$$

Then

$$\frac{d}{dt} L(x(t)) \leq -\frac{1}{2} c|x|^2 \quad \text{so long } x(t) \in V.$$

Choose $V = \{x : |x| < \rho\}$ → small enough.

→ for solutions of $x' = f(x)$ on V

Have $\frac{d}{dt} L(x(t)) \leq \tilde{c}\, L(x(t))$

$\Rightarrow$ on V, $L(x(t)) \leq L(x(0))\, e^{-\tilde{c}t}$

$L(x) = \frac{1}{2}|x|^2$ $\Rightarrow$ $|x(t)| \leq \sqrt{2\rho^{-\tilde{c}t}}\, |x(0)| \xrightarrow{t \nearrow \infty} 0$

$x(t) \in V$ $\forall t$ and $x(t) \to 0$, $t \to \infty$

$\square$

Def: (LYAPUNOV FUNCTION) A function $L: \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$ is called Lyapunov function (= energy) for a system $x' = f(x)$ on $\mathcal{U}$ neighborhood if L is nonincreasing along solutions:

<u>sufficient condition</u>: $\frac{d}{dt} L(x(t)) \le 0$

"strict" if $\frac{d}{dt} L(x(t)) < 0$.

LECTURE 21: TOPOLOGICAL CONJUGACY

Consider $x' = f(x)$ (*) in $\mathbb{R}^n$, equilibrium at a, linearize: $A := DF(a)$; Assume A has n distinct negative eigenvalues $\longrightarrow$ We proved that a is asymptotically stable for (*)

(i.e., $\exists\, \mathcal{U} \ni a$ s.t. if $x(0) \in \mathcal{U} \Rightarrow x(t) \in \mathcal{U}$ $\forall t > 0$ and $\lim_{t\to\infty} |x(t) - a| = 0$; i.e., a is a sink.)

Pf: Lyapunov function L s.t. $L(x(t))$ non-increasing

$L(x) = \frac{1}{2}|x-a|^2$.

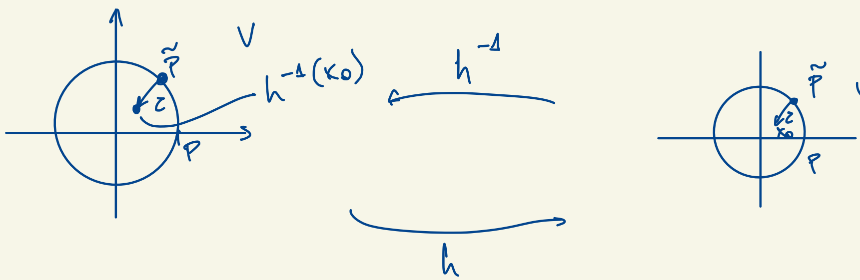
Prop (TOPOLOGICAL CONJUGACY) Under same hypothesis there exists $h : \underset{0}{\overset{\cup}{V}} \to \underset{a}{\overset{\cup}{U}}$ s.t.

$$\underbrace{\Phi_t}_{\text{flow of nonlinear system}} = h \circ \overbrace{e^{tA}}^{\text{linearization}} \circ h^{-1}$$

flow of nonlinear system $\Phi_t(x_0) = \underbrace{x(t)}_{\substack{\text{solution to} \\ x' = f(x),\ x(0) = x_0}}$

Pf: WLOG, $a = 0$ and A diagonal.

$U := B_\rho(0)$, ρ small s.t. if $x_0 \in B_\rho(0)$, then $x(t) \in B_\rho(0)$ $\forall t > 0$ and $|x(t)| \xrightarrow[t \uparrow \infty]{} 0$.



start at $x_0 \in U$, flow back τ time units to get to $\partial B_\rho(0)$. Hit it at $\tilde{p}$. Go to $\tilde{p}$ in V, flow forward by τ to get to $h^{-1}(x_0)$.

Algebraically, $\forall x \in B_\rho(0) = U$, let

$$T(x) := \sup_{t>0} \{\Phi_t(x) \notin U\}$$ = last time solution crossed $\{|x| = \rho\}$ before time 0.

<u>Claim:</u> T is differentiable for $x \in U \setminus \{0\}$.

Pf: $t \to \Phi_t(x)$ is a C^1 curve intersecting ∂U transversally (derivative vector of $\Phi_t(x)$ at point of intersection with ∂U is not tangent to the surface). Apply Implicit Function thm. ☐

Note, $T(x) \longrightarrow -\infty$ as $x \to 0$.

Define

$$h^{-1}(x) := e^{-T(x)A} \circ \Phi_{\underbrace{T(x)}_{T(x)<0}}(x)$$

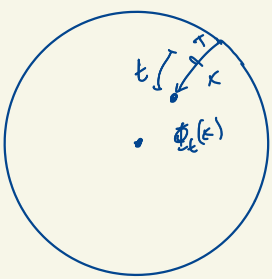
<u>WTS:</u>

$$h^{-1} \circ \Phi_t = e^{tA} \circ h^{-1} \quad \forall t > 0.$$

So,

$$h^{-1}(\Phi_t(x)) = e^{-T(\Phi_t(x))A} \, \Phi_{T(\Phi_t(x))}(\Phi_t(x))$$

Key Point: $T(\Phi_t(x)) = T(x) - t$



$$= e^{(-T(x)+t)A} \circ \underbrace{\Phi_{T(x)-t}(\Phi_t(x))}_{\Phi_{T(x)-t} \circ \Phi_t = \Phi_{T(x)-t+t} = \Phi_{T(x)}(x)}$$

$$= e^{tA} \circ h^{-1}(x).$$

Need to check $h^{-1} \in \mathrm{Hom}(U, V)$: only left to check cont. at zero.

$$\underbrace{h^{-1}(x)}_{} = y = \underbrace{e^{-T(x)A}}_{\to 0} \underbrace{\Phi_{T(x)}(x)}_{|\cdot| = p} \xrightarrow[x \to 0 \text{ exponentially}]{} 0$$

exp $\to$ 0 b/c $-T(x) > 0$ but A has negative eigenvalues

$\square$

In fact, it's enough to assume that the real parts of the eigenvalues of A are negative.

$$\lambda = -\alpha \pm i\beta, \quad \alpha > 0.$$

2×2 case: Jordan Canonical Form

$$\left(\begin{array}{c|cc} * & \multicolumn{2}{c}{0} \\ \hline 0 & -\alpha & -\beta \\ & \beta & -\alpha \end{array}\right)$$

ODE:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Same Lyapunov Function:

$$\frac{d}{dt}\left(\tfrac{1}{2}|x|^2\right) = (x_1 \ \ x_2)\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = (x_1 \ x_2)\begin{pmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$= -\alpha |x|^2 . \quad \square$$

Lecture 22: Nullclines

Thm: (Grosman–Hartman) $x' = f(x)$, $f \in C^1$, $x(t) \equiv a$ is the equilibrium, $A := Df(a)$. If A is hyperbolic, (i.e., $Re(\lambda) \neq 0$ $\forall \lambda$ eigenvalue of A). Then $\exists h: U \to V$ homeomorphism s.t.

$$\Phi_t = h \circ e^{tA} \circ h^{-1}.$$

- If $\forall \lambda$, $\mathbb{R} \ni \lambda < 0$ and A diagonalizable
- If $Re(\lambda) < 0$ and A diagonalizable, 2×2 ✓
- If Jordan block 2×2 $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, $\lambda < 0$.

$$y' = \begin{pmatrix} \lambda & \cancel{1}\,\varepsilon \\ 0 & \lambda \end{pmatrix} y$$

$$\frac{d}{dt}\left(\tfrac{1}{2}|y|^2\right) = (y_1 \quad y_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = (y_1, y_2)\begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

Lyapunov

$$= \lambda(y_1^2 + y_2^2) + \varepsilon y_1 y_2$$

Win if $\varepsilon < \frac{\lambda}{3}$, ~~say~~. $ab \leq \frac{a^2+b^2}{2}$

Claim: $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ is conjugate to $\begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix}$ $\forall \varepsilon$.

Pf: $$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1/\varepsilon \end{pmatrix}}_{T^{-1}}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix}}_{T} = \begin{pmatrix} \lambda & \varepsilon \\ 0 & \lambda \end{pmatrix}.$$

Upshot:

- If $\mathrm{Re}(\lambda) < 0 \quad \forall \lambda$ eigenvalue

  $\Downarrow$ ε-Jordan Form

  Lyapunov function $L(x) = \frac{1}{2} x^T M x$ decreases

  symmetric positive definite matrix

- Sink = asymptotically stable.

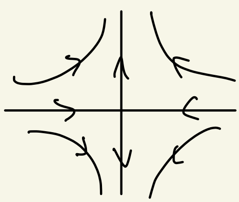
  $\Downarrow$

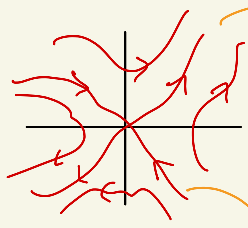
  Nonlinear is conjugate to linear system.

* SADDLE: $x' = f(x) = Ax + o(|x|)$.

If linear system has a saddle → Hyperbolic & Both positive & negative eigenvalues for A

NORMAL FORM





Unstable Manifold

Stable Manifold

these 2 manifolds are tangent at $x=a$ to corresponding linear manifolds in linearized system.

NULLCLINES

System $x' = f(x)$ on $D \subset \mathbb{R}^n$, equilibrium at $f(a) = 0$

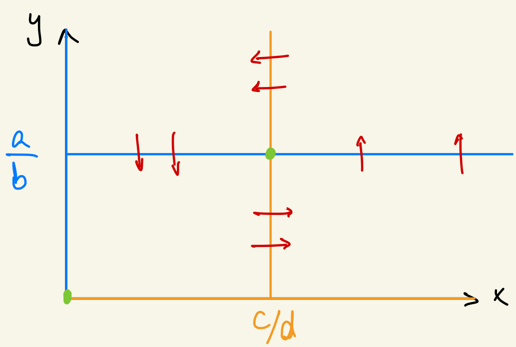
Def: (NULLCLINE) The j-th nullcline is

$$\{ x \in D : f_j(x) = 0 \}$$

Partition D into subdomains where $f_1, \ldots, f_n$ intersect at equilibria.

Ex: Lotka - Volterra

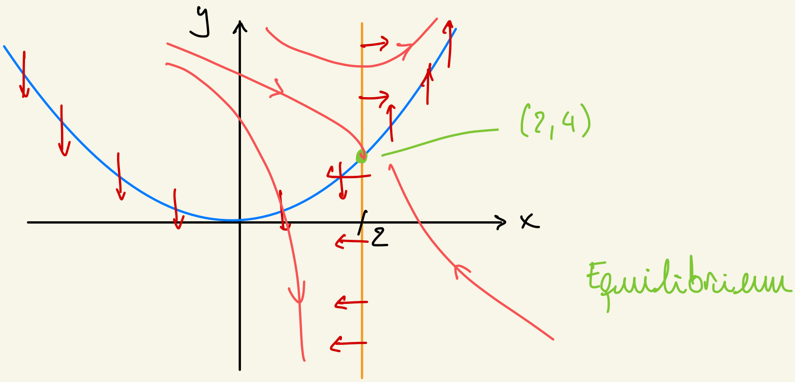
$$\begin{cases} x' = x(a - by) \rightsquigarrow x = 0;\ y = a/b \\ y' = y(-c + dx) \rightsquigarrow y = 0;\ x = c/d \end{cases}$$



Equilibria

Ex:

$$\begin{cases} x' = y - x^2 \rightsquigarrow y = x^2 \\ y' = x - 2 \rightsquigarrow x = 2 \end{cases}$$

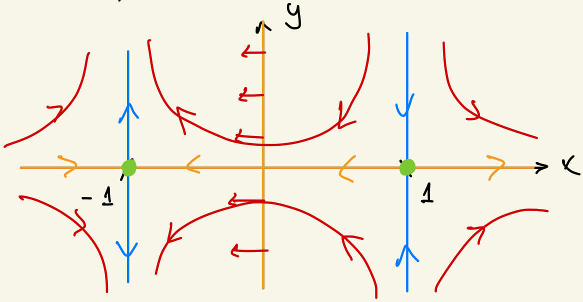


Linearize at $(2,4)$: $A = \begin{pmatrix} -4 & 1 \\ 1 & 0 \end{pmatrix}$; $\det A < 0$

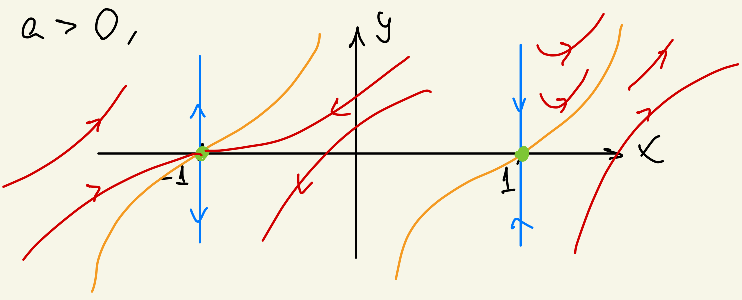
$\Rightarrow$ saddle.

Ex: $\begin{cases} x' = x^2 - 1 \rightsquigarrow x = \pm 1 \\ y' = -xy + a(x^2-1) \end{cases}$, a is $\mathbb{R}$-parameter.

If $a = 0$,



If $a > 0$,



Recall: Consequence from Grönwall's Inequality ($\varepsilon_1 = \varepsilon_2 = 0$)

x_1, x_2 solutions of $x' = f(t,x)$, $x_1 : [a,b] \to \mathbb{R}^n$.

$$|x_1(a) - x_2(b)| \leq \delta$$

if $|f(t,\xi_1) - f(t,\xi_2)| \leq L|\xi_1 - \xi_2|$

then $|x_1(t) - x_2(t)| \leq \delta e^{L(t-a)}$

Thm (Lyapunov): Let $D \subset \mathbb{R}^n$ be open, $L : D \to \mathbb{R}$ be differentiable that has isolated strict local minimum at $\tilde{x} \in D$

$$\frac{d}{dt} L(x(t)) \leq 0 \quad \forall x \text{ sol. to } x' \overset{(*)}{=} f(x) \text{ on } D$$

then $\tilde{x}$ is an stable equilibrium for (*).

If, in addition,

$$\frac{d}{dt} L(x(t)) < 0$$

for every non-constant solution on D then $\tilde{x}$ is a sink.

Pf: 1. Stability: given $\tilde{x}$, given $U \ni \tilde{x}$ neighborhood, want to find $V \subset U$ s.t. $x(0) \in V \Rightarrow x(t) \in U \ \forall t > 0$

shrink $U \rightsquigarrow$ WLOG, $\tilde{x}$ is a critical point in U.

$$L(\tilde{x}) := \min_{x \in U} L(x) \qquad (U \text{ bounded})$$

Let $m := \min_{x \in \partial U} L(x) > L(\tilde{x})$ ← compactness of $\bar{U}$ and continuity of L

$V := \{x : L(x) < m\}$

- if $x(0) \in V \Rightarrow x(t) < m \quad \forall t \Rightarrow x(t) \in V \subset U \quad \forall t.$

2. Asymptotic stability. Assume now $\frac{d}{dt} L(x(t)) < 0$ $\forall$ solution on D (except $x(t) \equiv \tilde{x}$).

Note that

$L(x(t))$ is decreasing & bounded below

$\Rightarrow \lim_{t \nearrow \infty} L(x(t)) \geq L(\tilde{x}).$

Fix $x(0) \in V \rightsquigarrow$ Let $x(t)$ be solution w/ $x(0) = x_0$. Now,

V compact $\Rightarrow \exists (t_j)_{j \geq 1} \nearrow \infty$ s.t.

$\lim_{j \to \infty} x(t_j) = y_0 \left(\Rightarrow L(x(t_j)) \to L(y_0) \right)$

If $y_0 = \tilde{x}$, then ✓.

If not, i.e., $y_0 \neq \tilde{x}$, denote $y(t)$ the solution w/ $y(0) = y_0$.

$\frac{d}{dt} L(y(t)) < 0 \quad \forall t \Rightarrow L(y(t)) < L(y_0)$

Fix $z > 0 \Rightarrow L(y(z)) < L(y_0).$

$\exists\, W_\tau$ neighborhood of $y(\tau)$ s.t. $\forall w \in W_\tau$,

$$L(w) < L(y_0).$$

By cont. dependence on initial parameters

$\Longrightarrow$ $\exists\, W_0 \ni y_0 = y(0)$ and $z(t)$, $z_0 \in W_0$ that solves $\begin{cases} z' = f(z) \\ z(0) = z_0 \end{cases}$

$\Rightarrow z(\tau) \in W_\tau$.

Take $\underset{W_0}{\overset{\cap}{z_0}} := x(t_j),\ j >>> 1$

$\Rightarrow x(t_j + \tau) \in W_\tau \Rightarrow L(x(t_j+\tau)) < L(y_0)$

L is decreasing

$\Rightarrow y_0 = \tilde{x}$.

$\square$

FACT: If $\tilde{x}$ is a sink for $x' = f(x)$, then there exists $L(x)$ Lyapunov function on an U-neighborhood of $\tilde{x}$ s.t. $\frac{d}{dt} L(x(t)) < 0$ $\forall x(t)$ non-constant solution on U s.t. $\tilde{x}$ is strict local min. of L.

* **GRADIENT SYSTEMS** $V: D \to \mathbb{R}$ is C^2, $x' = -\nabla V(x)$

$$L(x) = V; \quad \frac{d}{dt}\left(V(x(t))\right) = DV(x(t)) \cdot x'(t)$$
$$= \nabla V \circ (-\nabla V) = -|\nabla V|^2 < 0$$

Lecture 23: Limit Sets

Def: (Limit Sets) Suppose $x(t)$ is a solution to $x' = f(x)$, $\mathrm{Dom}(f) = D$. The ω-limit set of x is

$$\omega(x) = \left\{ y \in D : \exists t_j \nearrow \infty \text{ s.t. } \lim_{j\to\infty} x(t_j) = y \right\}_{\omega}$$

$$= \bigcap_{\substack{s>0 \\ s\in\mathbb{R} \\ s=1,2,\dots}} \overline{\{x(t) : t \geq s\}}$$

The α-limit set of x is:

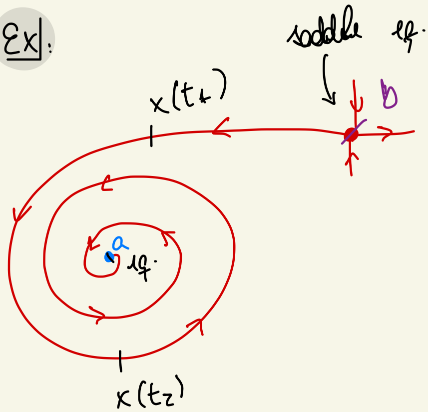
$$\alpha(x) = \left\{ y \in D : \exists t_j \searrow -\infty \text{ s.t. } \lim_{j\to\infty} x(t_j) = y \right\}_{\alpha}$$

Remark:

1) α, ω-limit sets could be $\emptyset$. E.g., $x' = 1$ so that $x(t) = t + C$, $\omega(x) = \emptyset$.

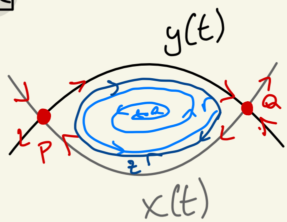
Typically: unbounded solutions and solutions that leave D need not have limit points.

Ex:



$$\omega(x) = \{a\}$$
$$\alpha(x) = \{b\}$$

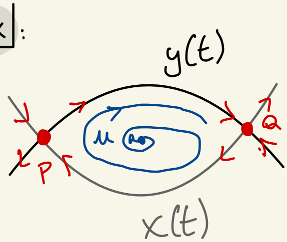
Ex:



$$\omega(x) = \{P\} \qquad \omega(y) = \{Q\}$$
$$\alpha(x) = \{Q\} \qquad \alpha(y) = \{P\}$$
$$\alpha(r) = \{a\} \qquad \omega(r) = \{z(t) : t \in \mathbb{R}\}$$
$$\omega(z) = \alpha(z) = \{z(t) : t \in \mathbb{R}\}.$$

Note: solutions connecting different points is a HETEROCLINIC and connecting the same point is a HOMOCLINIC.

Ex:



$$\alpha(u) = \{a\}$$
$$\omega(u) = \{P\} \cup \{Q\} \cup \{x(t) : t \in \mathbb{R}\} \cup \{y(t) : t \in \mathbb{R}\}.$$

↳ Namely, limit sets can be equilibria (a) orbits (x, y), cycles (z)

Assume now that $x(t)$ is a bounded solution s.t. $x(t) \in D \quad \forall t > 0$

$$\omega(x) = \bigcap_s \underbrace{\overline{\{x(t) : t \geq s\}}}_{=: A_s}$$

<u>Cantor Intersection Theorem</u>:

1) $K_1 \supset K_2 \supset K_3 \supset \cdots$ sequence of compact sets. If $\bigcap_j K_j \neq \emptyset$, then $\exists j_0$ s.t. $K_j \neq 0$

$\Updownarrow$

2) If $\{C_j\}_{j \geq 1}$ compact, then $\bigcap_j C_j \neq \emptyset$ if and only if $\exists N$ s.t.

$$C_1 \cap C_2 \cap \cdots \cap C_N \neq \emptyset$$

<u>PROPERTIES</u>:

a) $\omega(x) \neq \emptyset$ because $x(t) \in D$ and x is bounded. Apply Heine-Borel (or Cantor intersection)

b) If $y \in \omega(x)$, then the entire orbit of y is equal to $\{y(t) = \Phi_t(y) : t \in \mathbb{R}\} \subset \omega(x)$

c) ω is connected.

Pf: **(b)** given $y \in \omega(x)$, $t \in \mathbb{R}$,

<u>1st proof</u>: $\exists\, t_j \nearrow \infty$ s.t. $\lim_{j \nearrow \infty} x(t_j) = y$.

Then

$$\Phi_t(y) = \lim_{j \nearrow \infty} \Phi_t(x(t_j))$$

Φ_t cont.

$$= \lim_{j \nearrow \infty} x(t_j + t) \in \omega(x). \qquad \square$$

<u>2nd proof</u>: Take

$$\Phi_t(\omega(x)) \overset{\text{def}}{=} \Phi_t\Big(\bigcap_s \overline{\{x(r) : r \geq s\}}\Big)$$

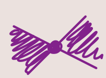
$$\subset \bigcap_s \Phi_t\big(\overline{\{x(r) : r \geq s\}}\big)$$

$$\subset \bigcap_s \overline{\Phi_t(\{x(r) : r \geq s\})}$$

$$= \bigcap_s \overline{\{x(t+r) : r \geq s\}}$$

$$= \bigcap_{t+s} \overline{\{x(p) : p \geq t+s\}} = \omega(x). \qquad \square$$

Def: A closed set $C \subset \mathbb{R}^n$ is <u>connected</u> if for any partition $C = C_1 \cup C_2$ with $C_1 \cap C_2 = \emptyset$ we have $C_1 = \emptyset$ or $C_2 = \emptyset$.

 not connected  connected

Def: Pathwise connected: $\forall x,y \in C \ \exists \gamma_{x,y}$ path connecting x and y (i.e., $\gamma_{x,y} : [0,1] \to C$ continuous s.t. $\gamma(0) = x$ and $\gamma(1) = y$).

Note: Pathwise connected $\Rightarrow$ Connected

E.g.: $\Gamma(\sin(1/x))$ is connected but not path connected

(c) Assume not: $\omega(k) = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$

$\neq \emptyset$, compact (closed and bdd)

$y_1 \in C_1$, $y_2 \in C_2$, $\exists d > 0 \ \forall z_1 \in C_1, z_2 \in C_2$:

$$|z_1 - z_2| \geq d > 0$$

By def. of $\omega(k)$ $\exists s_1 < t_1 < s_2 < t_2 < \cdots \to +\infty$ s.t. $y_1 = \lim_j x(s_j)$, $y_2 = \lim_j x(t_j)$.

By IVT

Lecture 24:

Gradient System:

$V : D \to \mathbb{R}$ ($D \subset \mathbb{R}^2$)

$x' = -\nabla V(x)$

$\underbrace{L(x)}_{\text{Lyapunov}} = V(x)$

$\frac{d}{dt} V(x(t)) = -|\nabla V(x)|^2 < 0$

except if $x = x^*$ is an equilibrium

Hamiltonian System:

$x'' = -\nabla V(x)$, $V =$ potential energy

System $\begin{cases} x' = y \\ y' = -\nabla V(x) \end{cases}$ $\Rightarrow$ $2n$ equations

$$L(x,y) = \frac{1}{2}|y|^2 + V(x) \quad \text{(Lyapunov)}$$

$$\frac{d}{dt} L(x(t), y(t)) = y y' + \nabla V(x) \underbrace{x'}_{= y}$$

$$= y y' - y' y = 0$$

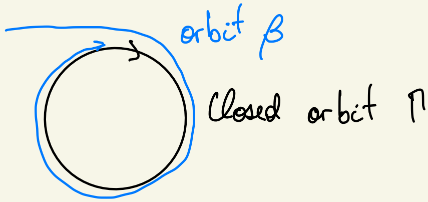
$\Rightarrow$ Energy is conserved!

Equivalently, $\exists H : \mathbb{R}^2 \to \mathbb{R}$ smooth

$$x' = \frac{\partial H}{\partial y}(x,y) \rightsquigarrow x_i' = \frac{\partial H}{\partial y_i}, \quad i = 1, \dots, n$$

$$y' = -\frac{\partial H}{\partial x}(x,y) \rightsquigarrow \cdots\cdots$$

* **COMMENT:** Assume Hamiltonian system



$$L|_\Gamma \equiv C_\Gamma$$

$$\underbrace{L}_{\text{Lyapunov}}|_\beta \equiv C_\beta$$

Constants.

LIMIT SETS: Consider $x' = f(x)$ on $D \subset \mathbb{R}^n$

$$\omega(x) = \{ y \in D : \exists t_j \nearrow +\infty \ \ x(t_j) \to y \}$$

$$\alpha(x) = \{ y \in D : \exists t_j \searrow -\infty \ \ x(t_j) \to y \}$$

If D is bounded,

1) $\omega(x) \neq \emptyset$, compact;

2) If $y \in \omega(x)$, $\{ y(t) : t \in \mathbb{R} \} \subset \omega(x)$;

3) $\omega(x)$ connected (Recall: Connected $\Leftarrow$ Path connected, $\cancel{\Rightarrow}$, $\Rightarrow$ only if set is open in $\mathbb{R}^n$)

Thm: (JORDAN CURVE THEOREM) Let γ be a simple closed curve on $\mathbb{R}^2$ (i.e., $\gamma : [a,b] \to \mathbb{R}^2$ continuous, $\gamma(a) = \gamma(b)$ and $\gamma(s) \neq \gamma(t)$ if $(s,t) \neq (a,b)$)

Then $\mathbb{R}^2 \setminus \gamma$ has 2 (connected) components:

- A bounded component homeomorphic to $\mathbb{D}$, $\mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \}$
- An unbounded homeomorphic to $\mathbb{R}^2 \setminus \overline{\mathbb{D}}$.

Def. $A \subset D$ domain; $x' = f(x)$ on $D \subset \mathbb{R}^n$. A is <u>positively invariant</u> if

$$\left(\Phi_t(A) \subset A \ \ \forall t > 0\right) \Leftrightarrow \left(x \in A \Rightarrow x(t) \in A \ \ \forall t > 0\right).$$

<u>negatively invariant</u> if

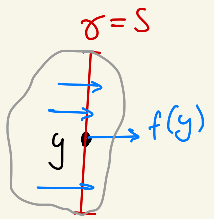
$$\left(\Phi_t(A) \subset A \ \ \forall t < 0\right) \Leftrightarrow \left(x \in A \Rightarrow x(t) \in A \ \ \forall t < 0\right).$$

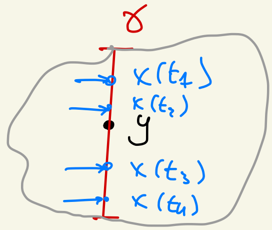
* A is <u>invariant</u> if it is positively or negatively invariant.
* We proved that limit sets $\begin{pmatrix} \omega(x) \\ \alpha(x) \end{pmatrix}$ are invariant.

<u>**Thm.**</u> **(POINCARÉ - BENDIXSON)** Let $x' = f(x)$ in $D \subset \mathbb{R}^2$ and $f \in C^1$. Let $x \in D$. Assume $\omega(x)$ contains no equilibrium points (and is contained in a compact set). <u>Then</u> $\omega(x)$ is a periodic orbit (i.e., there exists a periodic solution $y(t)$, w/ $y(t+T) = y(t)$, $T > 0$ period) s.t.

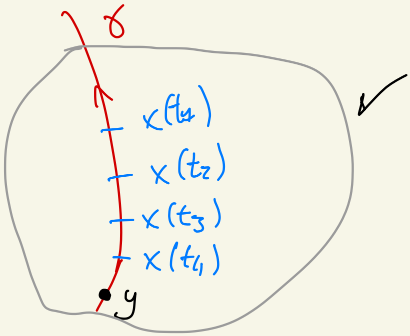
$$\omega(x) = \{y(t) : 0 \leq t \leq T\}).$$

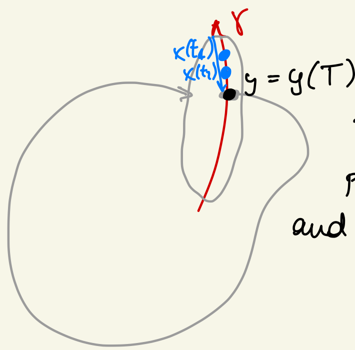
Proof: Sketch



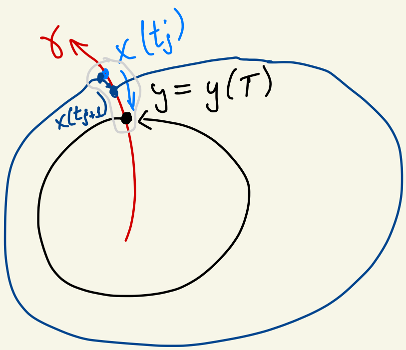


THIS CAN'T HAPPEN b/c $x(t_j) \nearrow y$ or $x(t_j) \searrow y$





$\Gamma = \{y(t)\}$ periodic and $\Gamma \in \omega(x)$

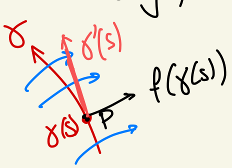


$$t_{j+1} - t_j \le \overline{T} < \infty$$

$$\operatorname{dist}(x(t), \Gamma) \xrightarrow[t \nearrow \infty]{} 0$$

$$\Rightarrow \omega(x) \subset \Gamma$$

Def: A section (γ) for the flow of $x' = f(x)$ is a curve segment s.t. it intersects the vector field transversally, i.e.,



$$\det(\gamma'(s), f(\gamma(s))) \neq 0 \quad \text{on } \gamma$$

(WLOG, > 0 by taking γ or $-\gamma$ and IVT)

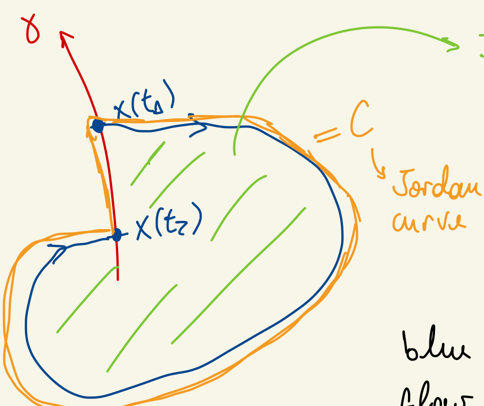
$\exists$ small neighborhood of γ on which flow goes through (almost) parallel to lines. By Implicit Function Theorem, $\exists$ sections near any x_0 s.t. $f(x_0) \neq 0$

KEY IDEA: Suppose a solution $x(t)$ intersects a section $\gamma(s)$ multiple times on

$$x(t_1) = \gamma(s_1)$$
$$x(t_2) = \gamma(s_2)$$
$$\vdots$$
$$x(t_j) = \gamma(s_j)$$

Claim: if $t_1 < t_2 < \cdots$ is nowhere monotone, then s_j is also monotone (i.e., $s_1 < s_2 < \cdots$ or $s_1 > s_2 > \cdots$)

Pf of Claim:

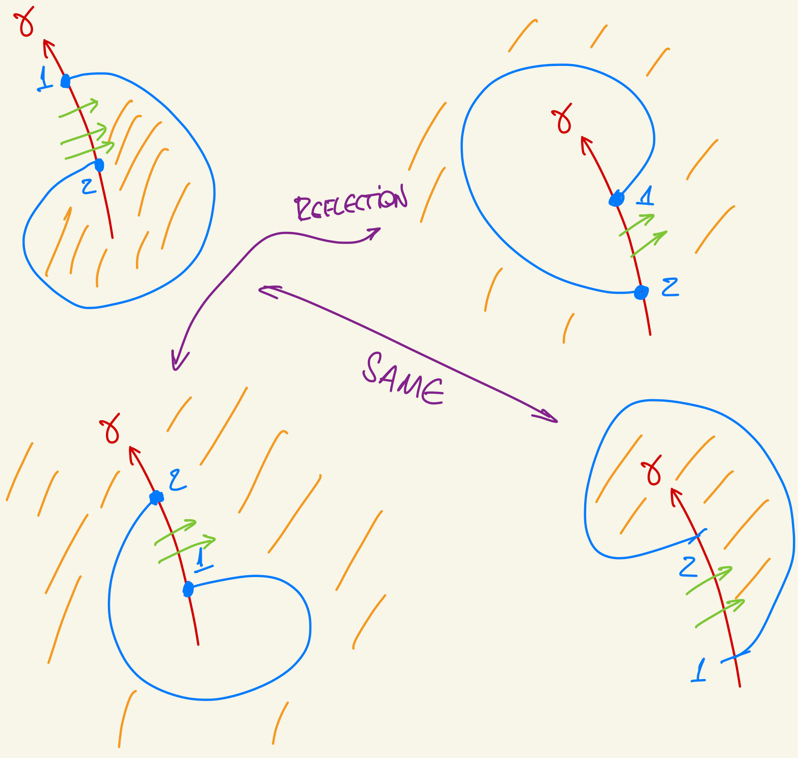


Inside region, in sense of Jordan curve theorem, is positively invariant for $x' = f(x)$.

Because can't cross blue (another solution) and flow in (orange $\cap$ red $= \gamma$) is inward

So, if $x(t_3) \in \gamma$, $t_3 > t_2 > t_1$, then $x(t_3)$ is inside C and so is monotone.

So, 4 possibilities:

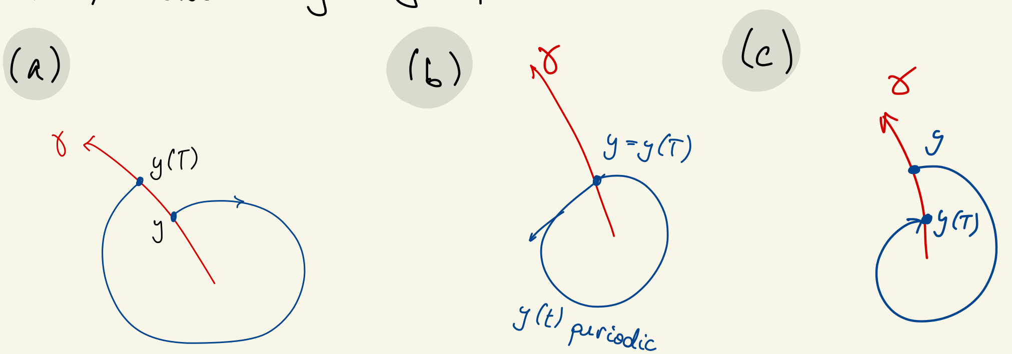


NOTE: Can take $x(t_j) \in \gamma$ b/c $y \in \omega(x)$

$$\Rightarrow \exists \underbrace{x(\tau_j)}_{\text{Need not } \in \gamma} \to y$$

By Implicit Function Theorem, if γ is smooth, $x \to x(t_1)$ is smooth.

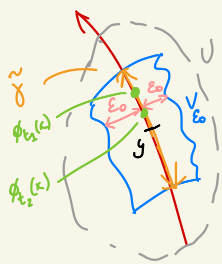
Now, evolve y by flow:



showed so far $y \in \omega(x) \Rightarrow \Gamma = \{y(t)\}$ is periodic and $\Gamma \in \omega(x)$

Claim: if Γ is closed orbit around $\omega(x)$, then $\omega(x) \subset \Gamma$ ($\Rightarrow \Gamma = \omega(x)$); it's enough to show

$$\text{dist}\,(\underbrace{x(t)}_{\phi_t}, \Gamma) \xrightarrow[t \nearrow \infty]{} 0$$



Claim: $\exists\, t_0 < t_1 < \cdots$ s.t.

1) $\phi_{t_n}(x) \in \tilde{\gamma}$

2) $\phi_{t_n}(x) \to y$

3) $\phi_t(x) \notin \tilde{\gamma}$ if $t_{n-1} < t < t_n$, $n = 1, 2, \ldots$

Fact: $\exists\, \bar{T} > 0 : t_{n+1} - t_n \leq \bar{T}$ for $n \geq n_0$

Pf: $\exists\, T > 0 : y(T) = y$

$$y(T) = \phi_T(y)$$

If $x_n = \phi_{t_n}(x)$ is close to y, then $\phi_T(x_n) \in V_{\varepsilon_0}$, then

$$\phi_{T+t}(x_n) \in \tilde{\gamma}, \text{ for some } t \in (-\varepsilon_0, \varepsilon_0)$$

$$\Rightarrow t_{n+1} - t_n \leq T + \varepsilon_0 = \bar{T}$$

Take $\beta > 0$ (β will $\to 0$). By cont. dependence on initial conditions, $\exists \delta > 0$ s.t. $|z - y| < \delta$

$$|t| \leq \bar{T} \Rightarrow |\Phi_t(z) - \Phi_t(y)| < \beta$$

If $n \geq n_0$, then $|\Phi_t(x_n) - \Phi_t(y)| < \beta$ if $n \geq n_0$, $|t| \leq \bar{T}$,

$$\operatorname{dist}(\phi_t(x), \Gamma) \leq |\Phi_t(x) - \Phi_t(y)|$$

$t_n \leq t \leq t_{n+1}$

$$= |\phi_{t-t_n}(x_n) - \phi_{t_n-t}(y)| < \beta$$

$|t - t_n| < \beta$.

$\square$

The End