SUMMARY

OUTLINE:

1) **SOLVING "SIMPLE" PDEs**: 1^{st} order, method of characteristics, classification of 2^{nd} order PDEs.

2) **CANONICAL EXAMPLES**: wave / diffusion on $\mathbb{R}$, $\mathbb{R}_+$ (Dirichlet / Neumann), Duhamel.

3) **FOURIER SERIES**: convergence & applications

4) **HARMONIC FUNCTIONS**: Laplace's equation on canonical domains, Poisson's formula, general domains!

5) **WAVE / DIFFUSION / SCHRÖDINGER IN $\mathbb{R}^3$**

6) **EIGENVALUE PROBLEMS ON GENERAL DOMAINS**: Diffusion on the disk (Bessel's equation), test functions, Rayleigh ratio! Minimum Principle! Completeness, Neumann problem, Sturm-Liouville Problem.

7) **DISTRIBUTIONS**: topology of $\mathcal{D}(\mathbb{R})$, examples, topology of $\mathcal{D}'(\mathbb{R})$, operations w/ distributions! source functions.

8)! **FOURIER TRANSFORM**: definition, convergence, examples, Fourier transform of distributions, Schwartz space & tempered distributions, Inverse Fourier Transform, Plancherel, Uncertainty Principle, Fourier & PDEs!!

9) **LAPLACE TRANSFORM**: definition & examples, wave & Laplace transform

10) **NONLINEAR PROBLEMS**: Schrödinger & KdV.

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1) Solving "simple" PDEs:

- **Classification of PDEs**: order, linearity, homogeneity.
- **1^{st} order**: $a u_x + b u_y + c u + d = 0$.

E.g.: $a u_x + b u_y = 0$ ⇝ Characteristics: $\frac{dy}{dx} = \frac{b}{a}$.

⇝ Change of variables:
$\tilde{x} = ax + by$, $\tilde{y} = bx - ay$.

E.g.: $u_{xy} + 3u_y = 0$ ⇝ $v := u_y$.

Important techniques: integrating factors, reduction of order, Chain Rule, etc...

E.g.: $u_{xx} + u = 0 \rightsquigarrow u(x,y) = f(y)\sin x + g(y)\cos x$.

- 2^{nd} order PDEs:

$$a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y + f u + g = 0$$

Classification:

$$b^2 - 4ac \begin{cases} < 0 & \Rightarrow \text{Hyperbolic} \rightsquigarrow \text{wave} \\ = 0 & \Rightarrow \text{Parabolic} \rightsquigarrow \text{diffusion} \\ > 0 & \Rightarrow \text{Elliptic} \rightsquigarrow \text{Laplace} \end{cases}$$

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2) CANONICAL EXAMPLES:

homogeneous

* WAVE EQUATION: $u_{tt} - c^2 u_{xx} = 0$

- General solution: $u(x,t) = f(x-ct) + g(x+ct)$

(i) WHOLE LINE:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & x \in \mathbb{R},\ t \in \mathbb{R}_+ \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

D'Alembert:

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

Energy: $E[u] = \frac{1}{2}\int_{\mathbb{R}} |u_t|^2 dx + \frac{c^2}{2}\int_{\mathbb{R}} |u_x|^2 dx$

$\rightarrow$ is conserved (proof: take time derivative and integrate by parts / FTC)

(ii) **HALF-LINE:** $x \in \mathbb{R}_+$, $t \in \mathbb{R}_+$

$$u_{tt} - c^2 u_{xx} = 0, \quad u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x)$$

- **Dirichlet:** $u(0,t) = 0 \rightsquigarrow$ **ODD EXTENSION**

$$\phi_{odd}(x) = \begin{cases} \phi(x), & x > 0 \\ 0, & x = 0 \\ -\phi(-x), & x < 0 \end{cases} ; \quad \psi_{odd}(x) = \begin{cases} \psi(x), & x > 0 \\ 0, & x = 0 \\ -\psi(-x), & x < 0 \end{cases}$$

Apply D'Alembert to these & go back.

- **Neumann:** $u_x(0,t) = 0 \rightsquigarrow$ **EVEN EXTENSION**

$$\phi_{even}(x) = \begin{cases} \phi(x), & x > 0 \\ \phi(-x), & x < 0 \end{cases} ; \quad \psi_{even}(x) = \begin{cases} \psi(x), & x > 0 \\ \psi(-x), & x < 0 \end{cases}$$

Apply D'Alembert to these & go back.

! IMPORTANT: if the boundary conditions are not homogeneous, define a new variable function to make them homogeneous. Apply the methods above and go back at the end.

(iii) INHOMOGENEOUS WAVE: $u_{tt} - c^2 u_{xx} = f(x,t)$,
$u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

Duhamel's Principle:

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) - \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\, ds$$
$$+ \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds .$$

Remark: if it is on the half-line (Dirichlet or Neumann), the terms from D'Alembert will change accordingly, but the term from Duhamel is the same as above.

* DIFFUSION EQUATION: $u_t - k u_{xx} = 0$, $u(x,0) = \phi(x)$

(i) WHOLE LINE: $x \in \mathbb{R}$, $t \in \mathbb{R}_+$

$$u(x,t) = (S * \phi)(x,t) = \int_{\mathbb{R}} S(x-y,t)\,\phi(y)\, dy$$

where $S(x,t)$ is the heat kernel:

$$s(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt} > 0 \quad \forall x,t.$$

Obs: • $S_t - kS_{xx} = 0, \quad S(x,0) = \delta_0.$

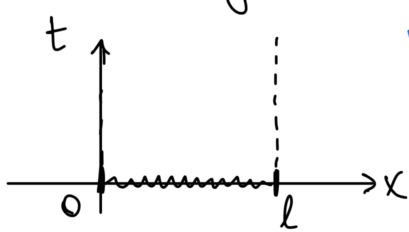
• $S \in C^\infty$

• S is <u>even</u>

• $\int_{\mathbb{R}} S(x,t)\,dx = 1$

MAX/MIN PRINCIPLE: the max. and min. of u is attained either initially or on the boundaries, <u>and nowhere else</u>. strong

• <u>Consequences</u>: Uniqueness of the solution to $u_t - ku_{xx} = F(x)$, $u(x,0) = \phi(x)$, $u(0,t) = f(t)$, $u(\ell,t) = g(t)$.



Pf: $w = u_1 - u_2$ satisfies the same but w/ all homog.. Max / min attained either initially or at $x=0$ or at $x=\ell$. But $w(x,0) = 0$ and $w(0,t) = w(\ell,t) = 0$. So $w \equiv 0$.

<u>Pf 2</u>: (Energy method) Compute $\int_0^\ell w(w_t - kw_{xx})\,dx = 0$ by parts & use that the energy $E(t) = \frac{1}{2}\int_0^\ell |w|^2 dx$ at most decreases (i.e., $\dot{E} \le 0 \Leftrightarrow E(t) \le E(0)$).

(ii) **HALF-LINE**: $u_t - k u_{xx} = 0$, $u(x,0) = \phi(x)$, $x \in \mathbb{R}_+$

- **Dirichlet**: $u(0,t) = 0 \rightsquigarrow$ ODD EXTENSION of ϕ

Solution: $$u(x,t) = \int_0^\infty [S(x-y,t) - S(x+y,t)]\phi(y)\,dy.$$

- **Neumann**: $u_x(0,t) = 0 \rightsquigarrow$ EVEN EXTENSION of ϕ

Solution: $$u(x,t) = \int_0^\infty [S(x-y,t) + S(x+y,t)]\phi(y)\,dy.$$

(iii) **INHOMOGENEOUS DIFFUSION**: $u_t - k u_{xx} = f(x,t)$, $u(x,0) = \phi(x)$ on $\mathbb{R} \times [0,\infty)$.

Duhamel's Principle:

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\phi(y)\,dy + \int_0^t \int_{\mathbb{R}} S(x-y,t-s) f(y,s)\,dy\,ds.$$

Remark: if it is on the half-line (Dirichlet or Neumann), the "homogeneous terms" will change accordingly, but the term from Duhamel is the same as above.

IMPORTANT: make sure the boundaries are homogeneous!

3) FOURIER SERIES

When using separation of variables to solve diffusion/wave on intervals, came across the following EIGENVALUE PROBLEM:

$$\begin{cases} X'' = -\lambda X, & x \in [a,b] \\ \text{(symmetric boundary conditions)} \end{cases}$$

→ Boundary condition is called symmetric if any two functions f and g satisfying said boundary conditions are such that $[f'g - fg']\Big|_{x=a}^{x=b} = 0$.

Obs: If the eigenvalue problem has SYMMETRIC BOUNDARY CONDITIONS, then we have orthogonality of eigenfunctions w.r.t. the L^2 inner product: $\langle X_n, X_m \rangle_{L^2} = \delta_{mn}$. (we can apply Gram-Schmidt to make them into an orthonormal basis of L^2, which is a Hilbert space).

Examples of symmetric bd. conditions: Dirichlet, Neumann, Robin, mixed (Dirichlet in one end & Neumann in the other).

Found from separation of variables (guessing if $\langle\,\rangle = 0$ ...)

All eigenvalues are real: $\begin{cases} X'' + \lambda X = 0 \ (\cdot \bar{X}) \\ \bar{X}'' + \bar{\lambda}\bar{X} = 0 \ \cdot (X) \end{cases}$ subtract and integrate.

Eigenvalues: $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n, \ldots \xrightarrow{n \nearrow \infty} +\infty$

Eigenfunctions: $X_1, X_2, X_3, \ldots, X_n, \ldots; \ \langle X_i, X_j \rangle = \delta_{ij}$.

Now, suppose $f : [a,b] \to \mathbb{R}$. The Fourier series of f is

$f \in L^2([a,b])$

$$f(x) = \sum_{n\in\mathbb{N}} \underbrace{A_n}_{\text{Fourier coeff's.}} \underbrace{X_n}_{\text{Eigenfunctions}}$$

- **Fourier Coefficients:**

$$A_n = \frac{\langle f, X_n\rangle_{L^2}}{\|X_n\|_{L^2}^2} = \frac{\int_a^b f(x)\,\overline{X_n(x)}\,dx}{\int_a^b |X_n(x)|^2\,dx}$$

- **Bessel's Inequality:**

$$\sum_{n=0}^{N} |A_n|^2 \|X_n\|_2^2 \le \|f\|_2^2$$

$$A_n = \frac{\langle f, X_n\rangle}{\langle X_n, X_n\rangle}$$

- **Parseval's Identity:** If $\sum_{n=0}^{N} A_n X_n \xrightarrow{N\nearrow\infty} f$ (very strong assumption), then

$$\sum_{n=0}^{\infty} |A_n|^2 \|X_n\|_2^2 = \|f\|_2^2 .$$

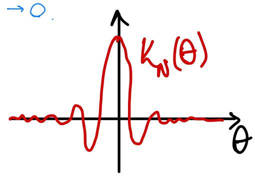
E.g.: (Parseval) $f = 1$ on $[0,\pi]$; $X'' + \lambda X = 0$, $X(0) = X(\pi) = 0 \Rightarrow \lambda_n = n^2$, $X_n(x) = \sin(nx)$, $n\in\mathbb{N}$. By Parseval,

$$\sum_{n\in\mathbb{N}} |A_n|^2 \|X_n\|_2^2 = \|f\|_2^2 \xrightarrow[|A_n|^2,\ \|X_n\|_2^2,\ \|f\|_2^2]{\text{compute each of the}} \sum_{n\text{ odd}} \frac{1}{n^2} = \frac{\pi^2}{8}.$$

- **DIRICHLET KERNEL**: used to show ptwise convergence of Fourier series in L^2.

  Say $X_n(x) = e^{-inx}$; $S_N(x) := \sum_{n=-N}^{N} A_n e^{-inx}$, A_n = Fourier coeffs. then WTS: $\forall x \in [-\pi, \pi]$, $S_N \to f$. $S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y) f(y)\, dy$. But $K_N \longrightarrow \delta_0$. So, $|S_N - f| \to 0$.

$$K_N(\theta) = \sum_{n=-N}^{N} e^{-in\theta} = \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin\left(\frac{\theta}{2}\right)}$$



- **L^2 CONVERGENCE**: The Fourier series converges in L^2 to f if $\|f\|_{L^2} < +\infty$ (i.e., if $f \in L^2$).

  Same as saying that $\left\| f - \sum_{n=-N}^{N} A_n X_n \right\|_2 \xrightarrow{N \uparrow \infty} 0$

- **POINTWISE CONVERGENCE**: The Fourier series converges ptwise to f provided that f is piecewise continuous and f' is continuous.

- **UNIFORM CONVERGENCE**: The Fourier series converges uniformly to f provided that f, f', f'' are continuous and that f satisfies the same boundary conditions as the eigenfunctions X_n.

- How to show this? Take the truncated series (i.e. from $n=-N$ to N), subtract from f and show the difference goes to zero (in norm). Use Bessel, Cauchy-Schwarz, LDCT, MCT, Minkowski, etc...

LDCT: If $\{f_n\}$, $f_n: \mathbb{R}^d \to \mathbb{R}$, is a sequence of measurable functions such that $f_n \to f$ a.e., and there exists $g \in L^p(\mathbb{R}^d)$, $1 \le p < +\infty$, such that $|f_n| \le g$ a.e. $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} \lim_{n\to\infty} f_n = \int_{\mathbb{R}^d} f$.

MCT: If $f_n: \mathbb{R}^d \to [0,\infty]$ is a seq. of measurable functions such that $f_n \nearrow f$ a.e., then $\int_{\mathbb{R}^d} f_n \nearrow \int_{\mathbb{R}^d} f$.

Fatou: f_n measurable, $\int \liminf f_n \le \liminf \int f_n$.

Reverse Fatou: f_n measurable, $\limsup \int f_n \le \int \limsup f_n$.

Cauchy-Schwarz: $|\langle u, v\rangle| \le \|u\| \|v\|$.

Hölder's inequality: if $\frac{1}{p} + \frac{1}{q} = 1$, $\|fg\|_1 \le \|f\|_p \|g\|_q$

Minkowski inequality: $\|f+g\|_p \le \|f\|_p + \|g\|_p$ (Δ-ineq. on L^p)

- **GIBBS PHENOMENON**: If f is discontinuous at x_0, the Dirichlet kernel gives a bad approximation around the jump discontinuity.

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4) HARMONIC FUNCTIONS

Def: $u \in C^2$ s.t. $\Delta u = 0$.

LAPLACE'S EQUATION

LAPLACE'S EQUATION:

- **Cartesian**: $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

- **Polar**: $$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

- **Spherical**:

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial u}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0.$$

MAX / MIN PRINCIPLE: If $\Omega \subset \mathbb{R}^n$ is open, bounded, and connected and $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic, then u attains its max/min on $\partial\Omega$.

Pf: Take $v(x,y) := u(x,y) + \varepsilon(x^2+y^2)$, $\varepsilon > 0$. Then $\Delta v = 4\varepsilon > 0$. Say v is max. at $(\tilde{x},\tilde{y}) \in \operatorname{int}\Omega$, then $v_{xx}(\tilde{x},\tilde{y}) \leq 0$ and $v_{yy}(\tilde{x},\tilde{y}) \leq 0$, but $\Delta v(\tilde{x},\tilde{y}) > 0$ ↯

UNIQUENESS OF SOLUTION OF LAPLACE'S EQUATION: $\Delta u = 0$ on Ω and $u|_{\partial\Omega} = g$. Pf: $w := u_1 - u_2$. Then $\Delta w = 0$ on Ω and $w|_{\partial\Omega} = 0$. So, by min/max, w is min/max. on $\partial\Omega$, but $w = 0$ there, so $w = 0$ on Ω.

FUNDAMENTAL SOLUTIONS!: Radially symmetric solutions to Laplace's equation in different dimensions.

$\mathbb{R}^2$
$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \xrightarrow[\text{symmetric}]{\text{radially}} u_{rr} + \frac{1}{r}u_r = 0$$

$$\Rightarrow \boxed{u(r) = C_1 \log r + C_2} \rightsquigarrow u(x,y) = C_1 \log\sqrt{x^2+y^2} + C_2$$

$\mathbb{R}^3$ Radially symmetric: $u_{rr} + \frac{2}{r}u_r = 0$

$$\Rightarrow \boxed{u(r) = \frac{C_1}{r} + C_2} \rightsquigarrow u(x,y) = \frac{C_1}{\sqrt{x^2+y^2}} + C_2.$$

Strategy to solve Laplace's eq. in "symmetric domains" is good old separation of variables.

$\rightsquigarrow$

LAPLACE'S EQUATION ON A DISK & POISSON'S FORMULA: Take the Dirichlet problem on a disk $D \subset \mathbb{R}^2$:

$$\begin{cases} \Delta u = 0 & \text{on } D = \{r < a\} \\ u(a, \theta) = f(\theta), & 0 \le \theta < 2\pi \end{cases}$$

$\leftsquigarrow$ $f: [0, 2\pi) \to \mathbb{R}$ is 2π-periodic & continuous

Solution: Polar coördinates $\to$ separation of variables by $u(r, \theta) = R(r)\Theta(\theta)$ $\to$ solve eigenvalue problems w/ homog. boundary $\to$ solve the other eigenvalue problem (Euler's eq.) $\to$ find coeffs $\to$ simplify expression.

Obtain that the solution to the Dirichlet problem on the disk $D = \{r < a\}$ is

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(a^2 - r^2)}{r^2 - 2ar\cos(\theta - \varphi) + a^2} f(\varphi)\, d\varphi$$

$$= \frac{1}{2\pi} \int_0^{2\pi} P(r, \theta - \varphi) f(\varphi)\, d\varphi$$

$$= \frac{1}{2\pi} (P * f)(r, \theta).$$

Obs: u harmonic in D, u continuous in $\bar{D}$, and u is the unique solution to the problem s.t. $u \to f$ as $r \to a$ (can show that $P \to \delta$ as $r \to a$).

Obs: Can write the disk solution in Cartesian coördinates:

$$u(\vec{x}) = \frac{a^2 - |\vec{x}|^2}{2\pi a} \int_{|\vec{x}'|=a} \frac{f(\vec{x}')}{|\vec{x} - \vec{x}'|^2} \, ds'$$

$ds' = a\,d\varphi$

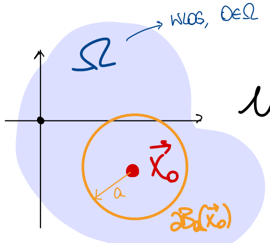
POISSON KERNEL: $P(r,\theta) := \dfrac{a^2 - r^2}{r^2 - 2ar\cos\theta + a^2}$

(i) $P(r,\theta) > 0 \quad \forall r < a$ and $0 < \theta < 2\pi$.

(ii) $\dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} P(r,\theta)\, d\theta = 1$

(iii) $\Delta P = 0$ in D.

MEAN VALUE PROPERTY: if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is s.t. $\Delta u = 0$ on Ω, then, for any $\vec{x}_0 \in \Omega$, take a ball $B_a(\vec{x}_0)$ of radius a centred at $\vec{x}_0$. We have that



$$u(\vec{x}_0) = \frac{1}{\operatorname{area}(\underbrace{\partial B_a(\vec{x}_0)}_{})} \int_{|\vec{x}_0 - \vec{x}| = a} u(\vec{x}) \, ds(\vec{x})$$

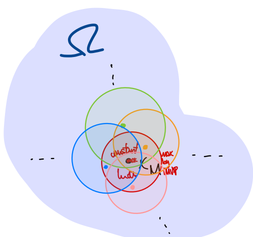
As long as $B_a(\vec{x}_0) \subset \Omega$ (but since Ω is open, can always find such a).

area element

That is, the value of u at $\vec{x}_0$ is the average value of u on the circle of radius a centred at $\vec{x}_0$. This is true for all pts. in Ω by translational invariance of Laplace's equation.

STRONG MIN/MAX. PRINCIPLE: if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic, then u attains its min./max. on $\partial\Omega$ and nowhere else, unless u is constant on Ω.

Pf: Apply MVP and translational invariance of $\Delta u = 0$.



Suppose by contradiction that $x_M \in \text{int}\,\Omega$ given max. of u. Then, take a circle and apply MVP. Repeat until you cover all of Ω. So, $u = $ const. on Ω. $\square$

SMOOTHING EFFECT: if $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is harmonic on Ω (i.e., $\Delta u = 0$ on Ω), then u is smooth on Ω (i.e., $u \in C^\infty(\Omega)$). Pf: WLOG, $0 \in \Omega$. Then take a disk inside Ω. Keep differentiating Poisson's formula. Apply translational invariance to cover all of Ω. (The radii of the disks can get arbitrarily small b/c Ω is assumed open).

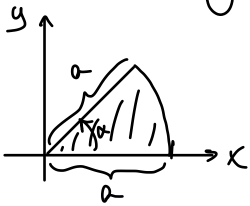
Upshot: Solving Laplace's equation on "symmetric domains" reduces to separation of variables / finding eigenvalues & eigenfunctions. Then finding coefficients by Fourier's method.

Recall: the general form of a harmonic function on the disk $D = \{r < a\}$ is

$$u(r,\theta) = \frac{1}{2}(C_0 + D_0 \log r) + \sum_{n=1}^{\infty} \Big[(C_n r^n + D_n r^{-n}) \cos(n\theta) + (A_n r^n + B_n r^{-n}) \sin(n\theta) \Big].$$

Finally, solutions must be <u>bounded</u> on the domain; e.g., if $\infty \in \Omega$, $C_n = A_n \overset{!}{=} 0$ $\forall n \in \mathbb{N}$, if $0 \in \Omega$, $D_0 = D_n = B_n \overset{!}{=} 0$.

E.g.: (Wedge) $\Delta u = 0$ on W and $u(r,0) = u(r,\beta) = 0$ and $u(a,\theta) = f(\theta)$. Separation of variables



$$u(r,\theta) = R(r)P(\theta) \rightsquigarrow \begin{cases} P'' + \lambda P = 0, \ P(0) = P(\alpha) = 0 \\ r^2 R'' + rR' - \lambda R = 0. \end{cases}$$

$n \in \mathbb{N}$

$$\lambda_n = \left(\frac{n\pi}{\alpha}\right)^2, \quad P_n(\theta) = \sin\left(\frac{n\pi\theta}{\alpha}\right).$$

$$r^2 R_n'' + r R_n' - \left(\frac{n\pi}{\alpha}\right)^2 R_n = 0 \ (\text{Euler eq.}) \Rightarrow R_n(r) = A r^{\sqrt{\lambda_n}} + \cancel{B r^{=\sqrt{\lambda_n}}} \to 0$$

$$\Rightarrow u(r,\theta) = \sum_{n\in\mathbb{N}} A_n r^{-n\pi/\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right); \quad u(a,\theta) = f(\theta) = \sum_{n\in\mathbb{N}} A_n a^{-n\pi/\alpha} \sin\left(\frac{n\pi\theta}{\alpha}\right)$$

$$\Rightarrow A_n = \frac{2a^{n\pi/\alpha}}{\alpha} \int_0^\alpha f(\theta) \sin\left(\frac{n\pi\theta}{\alpha}\right) d\theta.$$

LAPLACE'S EQUATION ON GENERAL DOMAINS !

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \subset \mathbb{R}^2, \mathbb{R}^3, \text{ bdd, connected, open} \\ u|_{\partial\Omega} = g. \end{cases}$$

Green's First Identity:

→ Shows uniqueness of solution

$$\iint_{\partial\Omega} u \underbrace{\frac{\partial v}{\partial n}}_{= \nabla v \cdot \hat{n}} \, ds = \iiint_{\Omega} \nabla u \cdot \nabla v \, d\vec{x} + \iiint_{\Omega} u \Delta v \, d\vec{x}$$

Green's Second Identity

$$\iiint_{\Omega} u \Delta v - v \Delta u \, d\vec{x} = \iint_{\partial\Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, ds.$$

Mean Value Property

For $u \in C^2(\Omega) \cap C(\overline{\Omega})$, $\Delta u = 0$ on Ω, we have that $\forall \vec{x}_0 \in \Omega$,

$$u(\vec{x}_0) = \frac{1}{\text{area}(\partial B_a(\vec{x}_0))} \int_{|\vec{x}_0 - \vec{x}| = a} u(\vec{x}) \, ds(\vec{x})$$

Dirichlet Principle:

(Minimization problem) Consider the set of functions

all fcts. that satisfy the Dirichlet bd. condition

$$\mathcal{A} := \{ w \in C^2(\Omega) \cap C(\overline{\Omega}) : w|_{\partial\Omega} = g \}.$$

Then, the (unique) function that minimizes the (energy) functional

$$E[w] := \frac{1}{2} \int_{\Omega} |\nabla w(\vec{x})|^2 \, d\vec{x}$$

is $u \in \mathcal{A}$ such that $\Delta u = 0$. Pf: Set $v := u - w$.

Then $\Delta v = 0$ on Ω and $v|_{\partial\Omega} = 0$. So, any $w \in \mathcal{A}$

$$E[w] = E[u-v] = \frac{1}{2}\int_\Omega |\nabla(u-v)|^2 \, d\vec{x}$$

$$= \frac{1}{2}\int_\Omega |\nabla u - \nabla v|^2 \, d\vec{x} = \frac{1}{2}\int_\Omega |\nabla u|^2 - 2\nabla u \nabla v + |\nabla v|^2 \, d\vec{x}$$

$$= E[u] + E[v] - \int_\Omega \nabla u \nabla v \, d\vec{x}$$

Green's 1st identity

$$= E[u] + E[v] - \underbrace{\iint_{\partial\Omega} \cancel{v} \frac{\partial u}{\partial n} \, ds}_{=0 \text{ b/c } v|_{\partial\Omega}=0} + \iiint_\Omega \underbrace{v \cancel{\Delta u} \, dx}_{0 \text{ b/c } u \text{ harmonic}}$$

$$= E[u] + E[v] \geq E[u] \quad \text{b/c } E[\bullet] \geq 0. \qquad \square$$

$\uparrow$ true $\forall w \in \mathcal{A}$.

REPRESENTATION FORMULA: Let $\Omega \subset \mathbb{R}^3$ be open, connected, and bounded. If $u \in C^2(\Omega) \cap C(\bar{\Omega})$ is s.t. $\Delta u = 0$ on Ω, then, $\forall \vec{x}_0 \in \Omega$,

$$u(\vec{x}_0) = \iint_{\partial\Omega} u(\vec{x}) \frac{\partial}{\partial n}\left(\frac{-1}{4\pi|\vec{x}-\vec{x}_0|}\right) - \frac{\partial u}{\partial n}(\vec{x})\left(\frac{-1}{4\pi|\vec{x}-\vec{x}_0|}\right) ds(\vec{x}).$$

Pf: Green's 2nd identity with $u \stackrel{!}{=} u$ and $v \stackrel{!}{=} -\frac{1}{4\pi|\vec{x}-\vec{x}_0|}$ over the domain $\Omega \setminus B_\varepsilon(\vec{x}_0) =: \Omega_\varepsilon$, $\varepsilon > 0$. Then $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon(\vec{x}_0)$.

So,

$$0 = \iiint_{\Omega_\varepsilon} \underbrace{u \cancel{\Delta v} - v \cancel{\Delta u}}_{\text{both harmonic on } \Omega_\varepsilon} \, d\vec{x} = \iint_{\partial\Omega_\varepsilon} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n} \, ds$$

$$= \underbrace{\iint_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, ds}_{\text{LHS of what we want}\cdot(-1)} + \underbrace{\iint_{\partial B_\varepsilon(\vec{x}_0)} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\, ds}_{}$$

as $\varepsilon \to 0$, one term goes to zero and the other goes to the average of u over $\partial B_\varepsilon(\vec{x}_0)$. Use MVP to conclude this is $u(\vec{x}_0)$.

Now, if $\Omega \subset \mathbb{R}^2$, open, bounded, and connected with $u \in C^2(\Omega) \cap C(\overline{\Omega})$ harmonic on Ω, then, $\forall \vec{x}_0 \in \Omega$,

$$u(\vec{x}_0) = \int_{\partial\Omega} u(\vec{x})\frac{\partial}{\partial n}\left(\frac{1}{2\pi}\log|\vec{x}-\vec{x}_0|\right) - \frac{\partial u}{\partial n}(\vec{x})\left(\frac{1}{2\pi}\log|\vec{x}-\vec{x}_0|\right) ds$$

Pf: Same as for the $\mathbb{R}^3$ case: Green's 2nd identity w/ $u \doteq u$ and $v \doteq \frac{1}{2\pi}\log r$, $r = |\vec{x}-\vec{x}_0|$, over $\Omega_\varepsilon := \Omega \setminus B_\varepsilon(\vec{x}_0)$

!

GREEN'S FUNCTIONS: If $u \in C^2(\Omega) \cap C(\Omega)$, where $\Omega \subset \mathbb{R}^n$ is open, connected and bounded, the solution to

$$\begin{cases} \Delta u = 0 & \text{on } \Omega \\ u|_{\partial\Omega} = g \end{cases}$$

at all $\vec{x}_0 \in \Omega$ is given by

G = "Green's function of Δ on Ω"

$$u(\vec{x}_0) = \int_{\partial\Omega} g(\vec{x})\frac{\partial G}{\partial n}(\vec{x},\vec{x}_0)\, dS.$$

"Green's 3rd Identity"

!

IMPORTANT: Green's fct. also allows us to solve POISSON'S EQUATION

$$\Delta u = f \text{ in } \Omega, \quad u|_{\partial\Omega} = h.$$

Then $\forall \vec{x}_0 \in \Omega$,

$$u(\vec{x}_0) = \iint_{\partial\Omega} h(\vec{x})\frac{\partial G}{\partial n}(\vec{x},\vec{x}_0)\,d\vec{x} + \iiint_{\Omega} f(\vec{x})G(\vec{x},\vec{x}_0)\,d\vec{x}.$$

Def: The Green function of Δ associated with domain $\Omega \subset \mathbb{R}^3$ for the Dirichlet problem is a function s.t. for all $\vec{x} \neq \vec{x}_0$,

$$G(\vec{x},\vec{x}_0) = -\frac{1}{4\pi|\vec{x}-\vec{x}_0|} + H(\vec{x},\vec{x}_0),$$

If $\Omega \subset \mathbb{R}^2$, change this term for $\frac{1}{2\pi}\log|\vec{x}-\vec{x}_0|$.

where $H \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\begin{cases} \Delta H = 0 & \text{on } \Omega \\ H(\vec{x},\vec{x}_0) = \dfrac{1}{4\pi|\vec{x}-\vec{x}_0|} & \text{on } \partial\Omega \end{cases}$$

If $\Omega \subset \mathbb{R}^2$, change this for $-\frac{1}{2\pi}\log|\vec{x}-\vec{x}_0|$

The following properties fully determine Green's function for Ω:

(i) $G \in C^2(\Omega \setminus \{\vec{x}_0\})$;

(ii) $\Delta G = 0$ on $\Omega \setminus \{x_0\}$;

(iii) $G(\vec{x},\vec{x}_0) = 0$ for all $\vec{x} \in \partial\Omega$.

!

Obs: $G(a,b) = G(b,a)$ $\forall a,b \in \Omega \subset \mathbb{R}^3$. Pf: Green's 2^{nd} identity with $u(x) \doteq G(x,a)$, $x \neq a$, and $v(x) \doteq G(x,b)$, $x \neq b$, over

$\Omega \setminus B_\varepsilon(a) \setminus B_\varepsilon(b) =: \Omega_\varepsilon$. Then

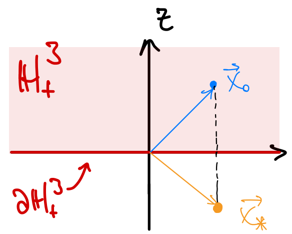
$$0 = \int_{\Omega_\varepsilon} u\cancel{\Delta v} - v\cancel{\Delta u}\, d\vec{x} = \int_{\partial\Omega_\varepsilon} = \cancel{\int_{\partial\Omega}} + \underbrace{\int_{\partial B_\varepsilon(a)} + \int_{\partial B_\varepsilon(b)}}_{\text{compute derivatives \& use MVP.}}$$

v, u are harmonic in Ω_ε by definition

$= 0$ b/c Green's fct. $= 0$ on the boundary

$\square$

EXAMPLE 1: $\Omega = \mathbb{H}^3_+ \subset \mathbb{R}^3$.



$\mathbb{H}^3_+ = \{(x,y,z) \in \mathbb{R}^3 : z > 0\}$.

For $\vec{x}_0 = (x_0, y_0, z_0) \in \mathbb{H}^3_+$, take the reflected pt. $\vec{x}_* := (x_0, y_0, -z_0)$.

Define
$$G(\vec{x}, \vec{x}_0) := -\frac{1}{4\pi|\vec{x}-\vec{x}_0|} + \frac{1}{4\pi|\vec{x}-\vec{x}_*|}.$$

Clearly, C^2, harmonic on $\mathbb{H}^3_+ \setminus \{x_0\}$ and $G(\vec{x}, \vec{x}_0) = 0$ for all $\vec{x} \in \partial\mathbb{H}^3_+$. So, G is Green's fct. for $\mathbb{H}^3_+$.

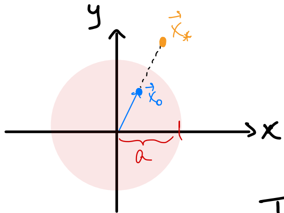
Finally, by Green's 3rd Identity, the solution to the Dirichlet problem $\Delta u = 0$ on $\mathbb{H}^3_+$, $u|_{\partial\mathbb{H}^3_+} = g$ is given by: $\forall \vec{x}_0 \in \mathbb{H}^3_+$,

$$u(\vec{x}_0) = \int_{\partial\mathbb{H}^3_+} g(\vec{x}) \frac{\partial G}{\partial n}(\vec{x}, \vec{x}_0)\, dS$$

$\iff$

$$u(x_0, y_0, z_0) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\left[-\frac{\partial G}{\partial z}\bigg|_{z=0}(\vec{x}, \vec{x}_0)\right] dx\,dy.$$

EXAMPLE 2: $D = \{r < a\} \subset \mathbb{R}^2$. Take $\mathbb{R}^2 \simeq \mathbb{C}$.



By reflection method: for $\vec{x}_0 = (x_0, y_0) = x_0 + i y_0 \in D$, take $\vec{x}_* := \dfrac{a^2 \vec{x}_0}{|\vec{x}_0|^2}$.

Thus, Green's function is:

$$G(\vec{x}, \vec{x}_0) = \frac{1}{2\pi} \log|\vec{x} - \vec{x}_0| - \frac{1}{2\pi} \log\left(\frac{|\vec{x}_0|}{a^2} |\vec{x} - \vec{x}_*| \right).$$

Finally, $\forall \vec{x}_0 \in D$, the solution of the Δ-Dirichlet problem on D is:

$$u(\vec{x}_0) = \int_{\partial D} g(\vec{x}) \frac{\partial G}{\partial n}(\vec{x}, \vec{x}_0)\, dS$$

Compute the derivatives

$$\Leftrightarrow \quad u(\vec{x}_0) = \frac{a^2 - |\vec{x}_0|^2}{2\pi a} \int_{|\vec{x}|=a} \frac{g(\vec{x})}{|\vec{x} - \vec{x}_0|^2}\, dS$$

Poisson's Formula from before

RIEMANN MAPPING THEOREM: Every open, simply-connected domain $\Omega \subset \mathbb{C} \simeq \mathbb{R}^2$ can be conformally mapped to the unit disk $D \subset \mathbb{C} \simeq \mathbb{R}^2$ via $W: \Omega \to D$. Let

$$W_{z_0}(z) := \frac{W(z) - W(z_0)}{1 - W(z)\overline{W(z_0)}} \in \mathrm{Aut}(D).$$

Thus, $|W_{z_0}(z)| = 1$ for $z \in \partial D$ and $W_{z_0}(z_0) = 0$. So, the Green's function for Ω is $G(z, z_0) = \frac{1}{2\pi} \log|W_{z_0}(z)|$.

5) WAVE / DIFFUSION / SCHRÖDINGER IN $\mathbb{R}^3$

(i) WAVE

$$\begin{cases} u_{tt} - c^2 \Delta u = 0, & \vec{x} \in \mathbb{R}^3, \ t > 0 \\ u(\vec{x}, 0) = \phi(\vec{x}) \\ u_t(\vec{x}, 0) = \psi(\vec{x}) \end{cases}$$

CAUSALITY: The value of $u(\vec{x}, t)$ only depends on the values of ϕ and ψ in the ball $\{|\vec{x} - \vec{x}_0| \leq ct_0\}$.

KIRCHHOFF'S FORMULA: $\forall (\vec{x}_0, t)$ in the light cone given by $S = \{\vec{x} \in \mathbb{R}^3 : |\vec{x} - \vec{x}_0| = ct_0\} \subset \mathbb{R}^3$, the solution to the wave equation in $\mathbb{R}^3$ is

$$u(\vec{x}_0, t_0) = \frac{1}{4\pi c^2 t_0} \int_S \psi(\vec{x})\, d\vec{x} + \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \int_S \phi(\vec{x})\, d\vec{x}\right].$$

HADAMARD'S DESCENT: To find the solution to the wave in $\mathbb{R}^2$, just compute $u(x, y, 0, t)$ (i.e., just ignore the third coördinate). Same strategy for higher dim.

(ii) DIFFUSION

$$\begin{cases} u_t - k \Delta u = 0, & \vec{x} \in \mathbb{R}^3, \ t > 0 \\ u(\vec{x}, 0) = \phi(\vec{x}). \end{cases}$$

Heat kernel in higher dimensions is obtained by just multiplying the 1-dim. heat kernel for each coörd.. So,

$$S_3(\vec{x},t) = S(x,t)\,S(y,t)\,S(z,t) = \frac{1}{(4\pi k t)^{3/2}} e^{-|\vec{x}|^2/4kt}.$$

SOLUTION: $\boxed{u(\vec{x},t) = (S_3 * \phi)(\vec{x},t).}$

(iii) SCHRÖDINGER

$$\begin{cases} \frac{1}{i} u_t - \frac{1}{2}\Delta u = 0, & \vec{x}\in\mathbb{R}^3,\ t>0 \\ u(\vec{x},0) = \phi(\vec{x}) \end{cases}.$$

KERNEL: $\boxed{S(x,t) = \frac{1}{\sqrt{2\pi i t}} e^{-x^2/2it}}$

SOLUTION: $\boxed{u(\vec{x},t) = (S * \phi)(\vec{x},t).}$

---

6) EIGENVALUE PROBLEMS ON GENERAL DOMAINS

When solving

$$u_{tt} - c^2\Delta u = 0 \qquad \text{or} \qquad u_t - k\Delta u = 0$$

on a general bounded/connected/open Ω by separation of variables, we find the following:

$u(x,y,z,t) := v(x,y,z)\,T(t)$ gives

$$\frac{T''}{c^2 T} = \frac{\Delta v}{v} = -\lambda \quad \text{or} \quad \frac{T'}{kT} = \frac{\Delta v}{v} = -\lambda .$$

Both yield this EIGENVALUE PROBLEM

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega \\ \text{(Dirichlet, Neumann or Robin)} & \text{on } \partial\Omega \end{cases}$$

$$\begin{cases} \text{Eigenvalues: } \lambda_n \geq 0 \quad \forall n \\ \text{Eigenfunctions: } v_n \quad \text{(real)} \end{cases}$$

Wave:

$$u(\vec{x},t) = \sum_n \left[A_n \cos(ct\sqrt{\lambda_n}) + B_n \sin(ct\sqrt{\lambda_n})\right] v_n(\vec{x})$$

Diffusion:

$$u(\vec{x},t) = \sum_n A_n e^{-k\lambda_n t} v_n(\vec{x}).$$

Coeffs. determined by initial conditions for both.

EXAMPLE: (Bessel fcts.) Diffusion on a disk

$$\begin{cases} u_t - \Delta u = 0 \\ u(a,\theta,t) = 0 \\ u(r,\theta,0) = \varphi(r,\theta) \end{cases}$$

2π-periodic

Separation of variables:

$$u(r,\theta,t) = v(r,\theta)\,T(t)$$

$$\frac{T'}{T} = \frac{\Delta v}{v} = -\lambda$$

$$\Rightarrow \begin{cases} -\Delta v = \lambda v \\ v|_{\partial D} = 0 \end{cases} \rightsquigarrow \text{Separation of variables}$$

$$v(r,\theta) = R(r)P(\theta).$$

$$\frac{P''}{P} = \frac{r^2}{R}\left(R'' + \frac{1}{r}R' + \lambda\right) = -\gamma$$

Solving for P: $\gamma_n = n^2, \quad n = 0, 1, 2, \ldots$

$$P_n(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

Solving for R: set $\rho = r\sqrt{\lambda_n}$

$$\Rightarrow R_{\rho\rho} + \frac{1}{\rho}R_\rho + \left(1 - \frac{n^2}{\rho^2}\right)R = 0 \leftarrow \text{Bessel}.$$

$\square$

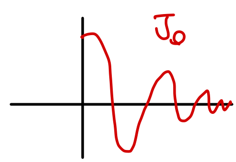
BESSEL EQUATION OF ORDER 0:

$$R_{rr} + \frac{1}{r}R_r + \alpha^2 R = 0$$

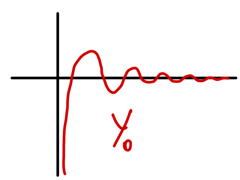
$\alpha > 0$

Solution: $R(r) = C_1 J_0(\alpha r) + C_2 Y_0(\alpha r).$

- J_0 = 0^{th} order Bessel fct. of 1^{st} kind



- Y_0 = 0^{th} order Bessel fct. of 2^{nd} kind



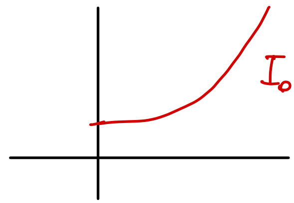
MODIFIED BESSEL EQUATION OF ORDER 0:

$$R_{rr} + \frac{1}{r} R_r - \alpha^2 R = 0$$

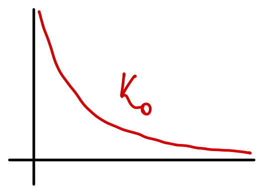
negative sign

Solution: $R(r) = C_1 I_0(\alpha r) + C_2 K_0(\alpha r)$.

I_0 = 0^{th}-order modified Bessel fct. of 1^{st} kind.



K_0 = 0^{th}-order modified Bessel fct. of 2^{nd} kind.



Obs: J_0, Y_0, I_0, K_0 are orthogonal in r w.r.t. L^2 $\langle \cdot , \cdot \rangle$.

EIGENVALUE PROBLEM w/ DIRICHLET BOUNDARY: !!

$$\begin{cases} -\Delta v = \lambda v, & x \in \Omega \subset \mathbb{R}^n \\ v|_{\partial\Omega} = 0 \end{cases}$$

bold...

GOAL: find eigenvalues $0 < \lambda_1 \le \lambda_2 \le \cdots$ and eigen-functions $v_1, v_2, \ldots$ w/ orthogonality condition $\langle v_i, v_j \rangle = \delta_{ij} \|v_i\|$

TEST FUNCTIONS: $\mathcal{A}_1 := \{ w \in C^2(\Omega) : w|_{\partial\Omega} = 0 \}$.

RAYLEIGH RATIO: $J(w) := \dfrac{\|\nabla w\|_2^2}{\|w\|_2^2}$.

("Potential energy")

Obs: $\|\nabla w\|_2 = \left(\int_\Omega \sum_i |\partial_i w(x)|^2 dx \right)^{1/2}$.

Goal: find $\min_{w \in \mathcal{A}_1} J(w)$.

Obs: We make the very strong assumption that $J(w)$ attains a minimum for some $w \in \mathcal{A}_1$.

Thm: (MINIMUM PRINCIPLE FOR 1st EIGENVALUE) ! If we have $m := \min_{w \in \mathcal{A}_1} J(w) = J(u)$ for some $u \in \mathcal{A}_1$, then $m = \lambda_1$ and $u = v_1$.

Pf: Say $u \in \mathcal{A}_1$ realizes this minimum. Take $v \in \mathcal{A}_1$ s.t. $w := u + \varepsilon v \in \mathcal{A}_1$. Define $f(\varepsilon) := J(w) = J(u + \varepsilon v)$. Note that $f'(0) = 0$ (bdd condition). At $\varepsilon = 0$, f reaches its minimum:

$$f(\varepsilon) = J(u + \varepsilon v) = \frac{\|\nabla(u + \varepsilon v)\|^2}{\|u + \varepsilon v\|^2} = \frac{\int_\Omega |\nabla(u + \varepsilon v)|^2 dx}{\int_\Omega |u + \varepsilon v|^2 dx}$$

$\Big\downarrow$ differentiate w.r.t. ε

$$f'(\varepsilon) = \frac{\left(\int_\Omega (2\nabla u \nabla v + 2\varepsilon|\nabla v|^2)\right)\left(\int_\Omega |u+\varepsilon v|^2\right) - \left(\int_\Omega |\nabla(u+\varepsilon v)|^2\right)\left(\int_\Omega 2uv + 2\varepsilon v^2\right)}{\left[\int_\Omega |u+\varepsilon v|^2\, dx\right]^2}$$

at $\varepsilon = 0$

$$0 \overset{\text{bdd condition}}{=} f'(0) = \frac{2\left(\int_\Omega \nabla u \nabla v\right)\left(\int_\Omega |u|^2\right) - \boxed{\left(\int_\Omega |\nabla u|^2\right)}\left(2\int_\Omega uv\right)}{\left(\int_\Omega |u|^2\right)\boxed{\left(\int_\Omega |u|^2\right)}}$$

$$= J(u) = \min_{w \in A_1} J(w) =: m$$

$$= \frac{2\int_\Omega \nabla u \nabla v}{\int_\Omega |u|^2} - \frac{2m \int_\Omega uv}{\int_\Omega |u|^2}$$

$$\Rightarrow \int_\Omega \nabla u \nabla v = m \int_\Omega uv \overset{\text{Green's 1st Identity}}{\Rightarrow} -\int_\Omega v \Delta u = m \int_\Omega uv$$

$$\Rightarrow \int_\Omega v(-\Delta u - mu) = 0 \underset{\text{Vanishing Thm}}{\Rightarrow} -\Delta u - mu = 0$$

$\Rightarrow$ u is an eigenfct. w/ eigenval. m.

Now, WTS: $m = \lambda_1$ and, thus, $u = v_1$. Since

$m \leq J(w) \quad \underline{\forall w \in A_1}, \quad m \leq J(v_j)$. But:

$$J(v_j) = \frac{\|\nabla v_j\|^2}{\|v_j\|^2} = \frac{\int_\Omega |\nabla v_j|^2}{\int_\Omega |v_j|^2} = \frac{\int_\Omega \nabla v_j \nabla v_j}{\int_\Omega |v_j|^2}$$

Green's 1st identity

$$\overset{\downarrow}{=} \frac{-\int_\Omega v_j \Delta v_j}{\int_\Omega v_j^2} = \frac{-\int_\Omega v_j(-\lambda_j v_j)}{\int_\Omega v_j^2} = \lambda_j.$$

Therefore $m = \lambda_1 \Rightarrow u = v_1$, as desired. $\circ$

! **INTUITION**: The first eigenvalue λ_1 is the minimum of the potential and the corresponding eigenfunction v_1 is the ground-state.

!!

Thm: (MINIMUM PRINCIPLE FOR THE nTH EIGENVALUE) Take the set of test functions

$$A_n := \{ w \in C^2(\Omega) : w|_{\partial\Omega} = 0 \ \underline{\text{and}} \ \langle w, v_1 \rangle = \cdots = \langle w, v_{n-1} \rangle = 0 \}.$$

Then, the minimum of the Rayleigh ratio over A_n is λ_n and it is attained at v_n; i.e.,

$$\lambda_n = \min_{w \in A_n} J(w) = J(v_n).$$

Obs: $\mathcal{A}_1 \supset \mathcal{A}_2 \supset \mathcal{A}_3 \supset \mathcal{A}_4 \supset \cdots$ b/c there are more constraints as n grows. This means that

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_4 \leq \cdots .$$

Pf: As before, suppose $u \in \mathcal{A}_n$ is s.t. $J(u) = \min_{w \in \mathcal{A}_n} J(w) =: m$. As before, take $v \in \mathcal{A}_n$ so that $w := u + \varepsilon v \in \mathcal{A}_n$, $\varepsilon > 0$. Set $f(\varepsilon) := J(u+\varepsilon v)$. As before, $f'(0) = 0$ and f is minimum at $\varepsilon = 0$. So, computing $f'(\varepsilon)$ and evaluating at $\varepsilon = 0$, we find (same calculation as before) that

$$\int_\Omega v(-\Delta u - mu) = 0 \qquad \forall v \in \mathcal{A}_n \ \underline{\&}\ \langle v, v_j \rangle = 0$$
$$j = 1, \ldots, n-1.$$

vanishing thm

$$\Rightarrow -\Delta u - mu = 0 \Rightarrow -\Delta u = mu$$

$\Rightarrow m$ is an eigenval. w/ eigenfct. u.

WTS: $m = \lambda_n$ and, hence, $u = v_n$. For that, set $v \overset{!}{=} v_j$, $j = 1, \ldots, n-1$: $\int_\Omega v_j(-\Delta u - mu) = 0$

Green's 2^{nd} identity

$$\int_\Omega (v_j \Delta u + m u v_j) = 0$$

$$\underbrace{\int_\Omega v_j \Delta u}_{} + m \int_\Omega u v_j = 0$$

$\Longrightarrow$ Green's 2nd identity (b/c of bdd condition)

$$\int_\Omega u \Delta v_j + m \int_\Omega u v_j = 0$$

$$\int_\Omega u(-\lambda_j v_j) + m \int_\Omega u v_j = 0$$

$$(m - \lambda_j) \underbrace{\int_\Omega v_j u}_{=0 \text{ b/c } \langle u, v_j \rangle = 0 \ \forall j = 1, \dots, n-1.} = 0$$

...

$\square$

Remark:

$$J(v_2) = \frac{\|\nabla v_2\|^2}{\|v_2\|^2} = \frac{\int |\nabla v_2|^2}{\int |v_2|^2} = \frac{\int \nabla v_2 \nabla v_2}{\int |v_2|^2}$$

$$= \frac{-\int v_2 \Delta v_2}{\int |v_2|^2} = \lambda_2 \frac{\int |v_2|^2}{\int |v_2|^2} = \lambda_2 .$$

INTUITION: This gives the spectrum of an operator. The λ_n are the various values of energy that give "stationary solutions" and v_n are the corresponding eigen-functions that are orthogonal between themselves.

COMPLETENESS OF $L^2(\Omega)$: Let $f \in L^2(\Omega)$. Set

$$c_n := \frac{\langle f, v_n \rangle}{\langle v_n, v_n \rangle} .$$

Then

$$\Big\| f - \sum_{n=1}^{N} c_n v_n \Big\|^2_{L^2(\Omega)} \xrightarrow{N \nearrow \infty} 0 .$$

Pf: Show that $r_N(x) := f(x) - \sum_{n=1}^{N} c_n v_n$ is s.t. $r_N \in \mathcal{A}_{N+1}$ (i.e., $r_N|_{\partial\Omega} = 0$ and $\langle r_N, v_1 \rangle = \dots = \langle r_N, v_N \rangle = 0$). Use that $\lambda_{N+1} \leq \frac{\|\nabla r_N\|^2}{\|r_N\|^2}$ to conclude that

$$\|r_N\|^2 \leq \frac{\|\nabla r_N\|^2}{\lambda_N} = \frac{\|\nabla f\|^2 - \sum_{n=1}^{N} \lambda_n \frac{\langle f, v_n \rangle}{\|v_n\|^2}}{\lambda_N}$$

$$= \frac{\|\nabla f\|^2}{\lambda_N} \xrightarrow[\substack{\text{b/c } \lambda_N \nearrow \infty \\ \text{as } N \nearrow \infty}]{N \nearrow \infty} 0 . \qquad \square$$

IMPORTANT: For the Neumann problem

$$-\Delta v = \lambda v \text{ on } \Omega , \qquad \frac{\partial v}{\partial n}\Big|_{\partial\Omega} = 0,$$

do the same, but the space of test functions have less constraints. Namely, they only need to be $C^2(\Omega)$ and be orthogonal to the $(n-1)$ first eigenfunctions:

Obs: Neumann boundary condition automatically satisfied if the fct. is in $C^2(\Omega)$.

$$\tilde{\mathcal{A}}_1 := \{ w \in C^2(\Omega) \} ;$$

$$\tilde{\mathcal{A}}_n := \{ w \in C^2(\Omega) : \langle w, v_1 \rangle = \dots = \langle w, v_{n-1} \rangle = 0 \} .$$

STURM-LIOUVILLE PROBLEMS: In 1 dimension,

$$\begin{cases} -(p(x)\, u')' + q(x)\, u = \lambda\, m(x)\, u \\ u(0) = u(\ell) = 0 \end{cases}$$

where $p \in C^1([0,\ell])$ and $q, m \in C^0([0,\ell])$ and $p, m > 0$ on $[0,\ell]$.

There are ∞-many eigenval.

$$\lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$
$$v_1, \quad v_2, \quad v_3, \quad \ldots$$

Weighted orthogonality: $\displaystyle\int_0^\ell v_i(x)\, v_j(x)\, m(x)\, dx = 0$ whenever $\lambda_i \neq \lambda_j$.

7)! DISTRIBUTIONS

Space of smooth test functions with compact support:

$$\mathcal{D}(\mathbb{R}) := \{ \varphi \in C^\infty(\mathbb{R}) : \operatorname{supp} \varphi \text{ compact} \}.$$

e.g., bump functions

$$\varphi(x) = \begin{cases} e^{-1/(1-x^2)}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}.$$

DISTRIBUTION: A distribution is a functional

$$T : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$\varphi \longmapsto (T, \varphi)$$

$(T, \alpha\varphi_1 + \beta\varphi_2) = \alpha(T, \varphi_1) + \beta(T, \varphi_2)$

that is linear and continuous. That is, a distribution is an element of the dual space: $T \in \mathcal{D}'(\mathbb{R})$.

TOPOLOGY OF $\mathcal{D}(\mathbb{R})$: A sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of fcts $\varphi_n \in \mathcal{D}(\mathbb{R})$ is said to converge to $\varphi \in \mathcal{D}(\mathbb{R})$, i.e., $\varphi_n \longrightarrow \varphi$, in the $\mathcal{D}(\mathbb{R})$ topology provided that

(i) The supports of all φ_n are contained in a finite interval $K \subset \mathbb{R}$,

(ii) $\forall m \in \mathbb{N}$, $\partial_x^m \varphi_n \rightrightarrows \partial_x^m \varphi$ (very strong).

Under the above conditions, if $\varphi_n \longrightarrow \varphi$, then $(T, \varphi_n) \longrightarrow (T, \varphi)$.

EXAMPLES:

1) DIRAC DISTRIBUTION: associates a test fct. $\varphi \in \mathcal{D}(\mathbb{R})$ with the value $\varphi(a)$ for some $a \in \operatorname{supp} \varphi$.

$$\begin{aligned} \delta_a : \mathcal{D}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \varphi(a) =: (\delta_a, \varphi) \end{aligned}$$

Clearly linear and continuous.

2) Let $f \in L^1(\mathbb{R})$. Then, the following functional

$$\begin{aligned} T_f : \mathcal{D}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto \int_{\mathbb{R}} f(x) \varphi(x) \, dx =: (T_f, \varphi) \end{aligned}$$

is a distribution.

Well-defined:

supp φ compact

$$\left| \int_{\mathbb{R}} f(x) \varphi(x) \, dx \right| \leq K \int_{\mathbb{R}} |f(x)| \, dx < \infty$$

Now, take a sequence $\varphi_n \to \varphi$ in $\mathcal{D}(\mathbb{R})$. Then

$$\left| \int_{\mathbb{R}} f(x)\left(\varphi_n(x) - \varphi(x)\right) dx \right|$$

$$\leq \sup_x |\varphi_n(x) - \varphi(x)| \int_{\mathbb{R}} |f(x)|\, dx$$

$$\Rightarrow (T_f, \varphi_n) \longrightarrow (T_f, \varphi)$$

Same works if $f \in L^1_{loc}(\mathbb{R})$ (i.e., $\forall K \subset \mathbb{R}$ compact, $\int_K |f(x)|\, dx < \infty$).

3) NON-EXAMPLE: If we take $f(x) = \frac{1}{x}$ $\notin L^1_{loc}(\mathbb{R})$, then $T_{1/x} : \mathcal{D}(\mathbb{R}) \to \mathbb{R}$ given by

$$(T_{1/x}, \varphi) = \int_{\mathbb{R}} \frac{1}{x} \varphi(x)\, dx$$

is <u>not</u> a distribution because not well-defined around 0.

4) PRINCIPAL VALUE. Define

$$\begin{aligned} pv\left(\tfrac{1}{x}\right) : \mathcal{D}(\mathbb{R}) &\longrightarrow \mathbb{R} \\ \varphi &\longmapsto pv\left(\int_{\mathbb{R}} \frac{1}{x} \varphi(x)\, dx\right) = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{1}{x} \varphi(x)\, dx. \end{aligned}$$

This is a distribution. Clearly linear & cont..
Need to check if this is well-defined: $\int_{|x|>\varepsilon} \frac{1}{x}\varphi < \infty$.

$$\lim_{\varepsilon\to0}\int_{|x|>\varepsilon}\frac{1}{x}\varphi(x)\,dx = \lim_{\varepsilon\to0}\int_{|x|>\varepsilon}\frac{1}{x}\big(\varphi(x)-\varphi(0)+\varphi(0)\big)dx$$

$$= \lim_{\varepsilon\to0}\int_{|x|>\varepsilon}\frac{\varphi(x)-\varphi(0)}{x}\,dx$$

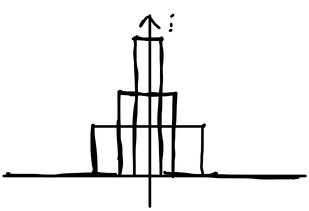
$$+\ \varphi(0)\cancel{\lim_{\varepsilon\to0}\int_{|x|>\varepsilon}\frac{1}{x}\,dx} \quad = 0 \text{ by symmetry}$$

$$< C.$$

TOPOLOGY OF $\mathcal{D}'(\mathbb{R})$: (weak topology) We say that a sequence of distributions $\{T_n\}$ converges to a distribution T in the weak/distribution sense if for all $\varphi \in \mathcal{D}(\mathbb{R})$, $(T_n, \varphi) \longrightarrow (T, \varphi)$. We then write the weak convergence as $T_n \rightharpoonup T$.

EXAMPLES:

1) **HAT FONCTION**



$$f_n(x) = \begin{cases} n/2, & |x| \leq \frac{1}{n} \\ 0, & \text{else} \end{cases}$$

Take $T_{f_n} =: T_n : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$, where

$$(T_n, \varphi) = \int_{-1/n}^{1/n} \frac{n}{2} \varphi(x)\, dx$$

$\nwarrow$ $f_n \equiv 0$ outside here anyways...

Given $\varphi \in \mathcal{D}(\mathbb{R})$,

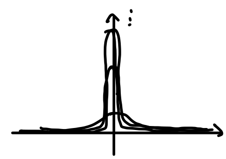
$$(T_n, \varphi) = \int_{-1/n}^{1/n} \frac{n}{2} \varphi(x)\, dx$$

$$= \frac{1}{2/n} \int_{-1/n}^{1/n} \varphi(x)\, dx = \text{mean value of } \varphi \text{ on } \left[-\tfrac{1}{n}, \tfrac{1}{n}\right].$$

$$= \varphi(0) \Rightarrow \boxed{T_n \to \delta_0}.$$

2) HEAT KERNEL: let

$$S_n(x, t_n) = \frac{1}{\sqrt{4\pi k t_n}}\, e^{-x^2/4kt_n},$$



where t_n is a seq. of positive real numbers that $t_n \to 0$ as $n \nearrow \infty$. Consider

$$T_{S_n} : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$\varphi \longmapsto \int_{\mathbb{R}} S_n(x, t_n)\, \varphi(x)\, dx.$$

Now, recall that we showed that

$$\lim_{t_n \to 0} \int_{\mathbb{R}} S_n(x,t_n)\,\varphi(x)\,dx = \varphi(0)$$

$u_t - k u_{xx} = 0$, $u|_{t=0} = \varphi(x)$,
as $t \to 0$, $u|_{x=0} \to \varphi(0)$

Thus, $\boxed{S_n \to \delta_0}$.

3) DIRICHLET KERNEL

$$K_N(\theta) = \sum_{n=-N}^{N} e^{-in\theta} = \frac{\sin\left(\left(N+\frac{1}{2}\right)\theta\right)}{\sin(\theta/2)}.$$

Now,

$$\int_{-\pi}^{\pi} K_N(\theta)\,\varphi(\theta)\,d\theta \xrightarrow{N \nearrow \infty} 2\pi\,\varphi(0)$$

$\Rightarrow \boxed{K_N \to 2\pi\,\delta_0}$.

Remarks: (i) All locally integrable fcts (i.e., $f \in L^1_{loc}$) are identified with the distribution

$$T_f : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$$
$$\varphi \longmapsto (T_f, \varphi) = \int_{\mathbb{R}} f(x)\,\varphi(x)\,dx$$

(obs: $f \in L^1(\mathbb{R}) \Rightarrow f \in L^1_{loc}$). For example, $f(x) = \frac{1}{\sqrt{|x|}}$

is a distribution because $\frac{1}{\sqrt{|x|}} \in L^1_{loc}$. However, $f(x) = \frac{1}{|x|}$ is not a distribution.

DERIVATIVE OF DISTRIBUTIONS: Let $T \in \mathcal{D}'(\mathbb{R})$. Then, we define T', the derivative of T, as:

$$(T', \varphi) := -(T, \varphi').$$

That is, $T' : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{R}$

$$\varphi \mapsto (T', \varphi) := -(T, \varphi').$$

Note: If $f \in C^1$, this identifies to

$$(T_f', \varphi) = -(T_f, \varphi').$$

But, integrating by parts,

$$(T_f', \varphi) = -\int_{\mathbb{R}} f(x)\varphi'(x)\,dx = \int_{\mathbb{R}} f'(x)\varphi(x)\,dx$$
$$= (T_{f'}, \varphi).$$

EXAMPLES

1) DERIVATIVE OF δ_0: By definition,

$$(\delta_0', \varphi) = -(\delta_0, \varphi') = -\varphi'(0)$$

$$(\delta_0'', \varphi) = (-1)^2 (\delta_0, \varphi'') = \varphi''(0)$$

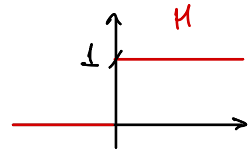
$$\vdots \qquad\qquad \vdots$$

Well-def. b/c φ smooth.

$$\Rightarrow \boxed{(\delta^{(n)}, \varphi) = (-1)^n \varphi^{(n)}(0)}$$

2) HEAVISIDE FUNCTION

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$



Clearly $H \in L^1_{loc}$ $\Rightarrow$ identify H w/ distribution T_H.

Then, by definition

$$(H', \varphi) = -(H, \varphi')$$

$$= - \int_{\mathbb{R}} H(x)\, \varphi'(x)\, dx$$

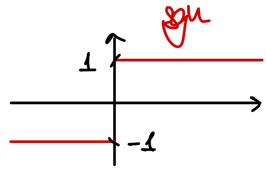
$$= - \int_0^\infty \varphi'(x)\, dx$$

$$\overset{\text{FTC}}{=} - \left[\varphi(x)\right]_{x=0}^{x=\infty} = \varphi(0) = (\delta_0, \varphi)$$

$$\Rightarrow \boxed{H' = \delta_0}.$$

3) Sign Function

$$\mathrm{sgn}(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$



Also, $\mathrm{sgn} \in L^1_{loc}$. So,

$$(\mathrm{sgn}', \varphi) = -(\mathrm{sgn}, \varphi')$$

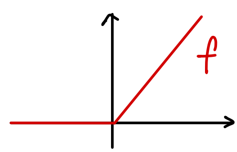
$$= -\int_{\mathbb{R}} \mathrm{sgn}(x)\, \varphi'(x)\, dx$$

$$= -\int_0^\infty \varphi'(x)\, dx - \int_{-\infty}^0 \varphi'(x)\, dx$$

$$= \varphi(0) + \varphi(0) = 2\varphi(0) = 2(\delta_0, \varphi)$$

$$\Rightarrow \boxed{\mathrm{sgn}' = 2\delta_0}.$$

4) $f(x) = \begin{cases} \alpha x, & x > 0 \\ 0, & x < 0 \end{cases}$



$f \in L^1_{loc}$. So,

$$(f', \varphi) = -(f, \varphi') = -\int_0^\infty \alpha x\, \varphi'(x)\, dx$$

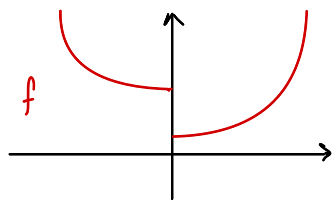
$$\overset{\text{by parts}}{=} -\cancelto{0}{\left[\alpha x\varphi(x)\right]_{x=0}^{x=\infty}} + \alpha\int_0^\infty \varphi(x)\,dx$$

$$= \alpha \int_{\mathbb{R}} H(x)\,\varphi(x)\,dx$$

$$= \alpha\,(H,\varphi) \Rightarrow \boxed{f' = \alpha H}$$

Obs: "Classical derivative" of f is $\begin{cases} \alpha, & x>0 \\ 0, & x<0 \end{cases}$.
But $f' = \alpha H$ is more general b/c it captures the behavior at the discontinuity.

5) $$\boxed{f(x) = \begin{cases} x^2+1, & x>0 \\ x^2+4, & x<0 \end{cases}}$$



$f \in L^1_{loc}$. So,

$$(f',\varphi) = -(f,\varphi') = -\int_{\mathbb{R}} f(x)\,\varphi'(x)\,dx$$

$$= -\int_0^\infty (x^2+1)\,\varphi'(x)\,dx - \int_{-\infty}^0 (x^2+4)\,\varphi'(x)\,dx$$

$$\overset{\text{by parts}}{=} -\left[(x^2+1)\varphi(x)\right]_{x=0}^{x=\infty} + 2\int_0^\infty x\,\varphi(x)\,dx$$

$$-\left[(x^2+4)\varphi(x)\right]_{x=-\infty}^{x=0} + 2\int_{-\infty}^0 x\,\varphi(x)\,dx$$

$$= \varphi(0) - 4\varphi(0) + \int_{-\infty}^{\infty} 2x\,\varphi(x)\,dx$$

$$= \int_{\mathbb{R}} \underbrace{2x}_{\text{classical derivative away from the jump.}} \varphi(x)\,dx - 3\varphi(0)$$

$$\Rightarrow \boxed{f' = \underbrace{\{f'\}}_{\text{Classical derivative away from zero}} - \underbrace{3\delta_0}_{\text{captures the behavior at the jump}}}$$

REMARKS ON δ:

(i) If $f_n(x) \longrightarrow \delta(x)$, then

$$f_n(2x) \longrightarrow \delta(2x) = \frac{1}{2}\delta(x).$$

$$(f_n(2x), \varphi) = \int_{\mathbb{R}} f_n(2x)\varphi(x)\,dx \underset{\substack{2x = y \\ 2dx = dy}}{=} \frac{1}{2}\int_{\mathbb{R}} f_n(y)\,\varphi\left(\frac{y}{2}\right)dy \xrightarrow{n\nearrow\infty} \frac{1}{2}\varphi(0).$$

(ii) Suppose $f_n(x) \to \delta(x)$, then

$$f_n((x-a)(x-b)) \longrightarrow \delta((x-a)(x-b)) = \frac{1}{|b-a|}\left[\delta(x-a) + \delta(x-b)\right]$$

$$\int_{\mathbb{R}} f_n((x-a)(x-b))\,\varphi(x)\,dx \overset{y=(x-a)(x-b)}{=} \underbrace{\int_{-\infty}^{c}}_{\downarrow \; -\frac{\varphi(a)}{a-b}} + \underbrace{\int_{c}^{\infty}}_{\downarrow \; \frac{\varphi(b)}{b-a}},$$ where c is max/min of the parabola $(x-a)(x-b)$

MULTIPLY SMOOTH FUNCTIONS TO DISTRIBUTIONS: for $\alpha \in C^\infty(\mathbb{R})$, $\alpha\varphi \in \mathcal{D}(\mathbb{R})$ $\forall \varphi \in \mathcal{D}(\mathbb{R})$. So, for $T \in \mathcal{D}'(\mathbb{R})$, we define

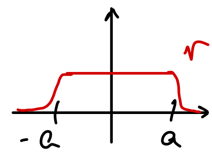
$$(\alpha T, \varphi) := (T, \alpha\varphi).$$

!

EXAMPLE: Find all distributions T s.t. $xT = 0$.

Claim: $T = C\delta_0$, $C \in \mathbb{R}$

That is, $\forall \varphi \in \mathcal{D}(\mathbb{R})$, $(xT, \varphi) = (T, x\varphi) = 0$. Fix $r \in \mathcal{D}(\mathbb{R})$ s.t. $r(x) = 1$ for $|x| \leq a$.



Then $\forall \psi \in \mathcal{D}(\mathbb{R})$, WTS: $(T, \psi) = C\psi(0)$.

Note that

$$\psi(x) = \psi(0)r(x) + [\psi(x) - \psi(0)r(x)]$$

$$= \psi(0)r(x) + \int_0^x \psi'(y)\,dy + \cancel{\psi(0)}$$

$$- \psi(0)\int_0^x r'(y)\,dy - \cancel{\psi(0)\,\underbrace{r(0)}_{=1}}$$

$$= \psi(0)r(x) + \int_0^x [\psi'(y) - \psi(0)r'(y)]\,dy$$

Set $\psi(x) := x\varphi(x)$ $\forall \varphi \in \mathcal{D}(\mathbb{R})$. Then

$$\psi(x) = \int_0^x \psi'(y) - \psi(0)r'(y)\,dy$$

$y = xz$
$dy = x\,dz$

$$\overset{?}{=} \quad x \int_0^1 \psi'(xz) - \psi(0)\, r'(xz)\, dz$$

$$= x\,\varphi(x) \Rightarrow (T, \tilde{\psi}) = (T, x\varphi) = 0$$

$$\Rightarrow \boxed{T = C\delta}.$$

SOURCE FUNCTIONS !

(i) **HEAT EQUATION**: $u_t - k u_{xx} = 0$, $u(x,0) = \phi(x)$

Heat kernel $S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-x^2/4kt}$. Moreover,

$$\begin{cases} S_t - k S_{xx} = 0 \\ S(x,0) = \delta(x) \end{cases}$$

So, the solution to the heat equation is

$$u(x,t) = (S * \phi)(x,t)$$

(ii) **SCHRÖDINGER'S EQUATION**: $\frac{1}{i} u_t - \frac{1}{2} u_{xx} = 0$, $u(x,0) = \phi(x)$.

Kernel: $S(x,t) = \frac{1}{\sqrt{2\pi i t}} e^{-x^2/2it}$ that satisfies

$$\frac{1}{i} S_t - \frac{1}{2} S_{xx} = 0, \quad S(x,0) = \delta(x)$$

Then, $u(x,t) = (S * \phi)(x,t)$.

(iii) WAVE EQUATION: $u_{tt} - c^2 u_{xx} = 0$, $u(x,0) = f(x)$, $u_t(x,0) = g(x)$. Want to find a source fct. for the wave equation w/ $f(x) \equiv 0$: by D'Alembert,

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} g(s)\, ds$$

$$\overset{!}{=} \int_{\mathbb{R}} S(x-y,t)\, g(y)\, dy$$

$$\Rightarrow S(x-y,t) = \begin{cases} \frac{1}{2c}, & x-ct < y < x+ct \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow S(x,t) = \begin{cases} \frac{1}{2c}, & -ct < x < ct \\ 0, & \text{else} \end{cases}$$

$$\Rightarrow \boxed{S(x,t) = \frac{1}{2c} H(c^2t^2 - x^2)}.$$

Obs: $S_{tt} - c^2 S_{xx} = 0$, $S(x,0) = 0$, $S_t(x,0) = \delta(x)$

Thus, $u(x,t) = (S * g)(x,t)$.

(iv) LAPLACE'S EQUATION:

ASIDE: δ in 3-dim is defined as

$$\delta(\vec{x}-\vec{x}_0) := \delta_{\vec{x}_0} : \mathcal{D}(\mathbb{R}^3) \ni \varphi(\vec{x}) \longmapsto \varphi(\vec{x}_0) \in \mathbb{R},$$

and $(\partial_{x_0} T, \varphi) = -(T, \partial_{x_0}\varphi)$.

From HW and TT2, for $\varphi \in C^\infty(\mathbb{R}^3 \setminus \{0\})$,

$$\varphi(0) = \int_{\mathbb{R}} \Delta\varphi(x)\left(-\frac{1}{4\pi|\vec{x}|}\right) d\vec{x}$$

Is $L^1_{loc}(\mathbb{R}^3\setminus\{0\})$ b/c w/out the origin.

$\partial_{xx}\varphi + \partial_{yy}\varphi + \partial_{zz}\varphi$

$$= \left(-\frac{1}{4\pi|\vec{x}|}, \Delta\varphi\right)$$

$$= \left(\Delta\left(-\frac{1}{4\pi|\vec{x}|}\right), \varphi\right)$$

$$\Rightarrow \boxed{\Delta\left(-\frac{1}{4\pi|\vec{x}|}\right) = \delta(x)}$$

Now, if we have $\Delta u = f$ on Ω and $u|_{\partial\Omega} = 0$ then $\forall \vec{x}_0 \in \Omega$

see pg. 47

$$u(\vec{x}_0) = \iiint_\Omega f(\vec{x})\, G(\vec{x},\vec{x}_0)\, d\vec{x},$$

where G is the Green fct. for Δ on Ω. Recall

$\Delta G = 0$ on $\Omega\setminus\{\vec{x}_0\}$, $G(\vec{x},\vec{x}_0) = -\dfrac{1}{4\pi|\vec{x}-\vec{x}_0|} + H(\vec{x},\vec{x}_0)$, H is harmonic on Ω, $G(\vec{x},\vec{x}_0) = 0$ if $\vec{x} \in \partial\Omega$.

So, on the entire

$$\begin{cases} \Delta G(\vec{x},\vec{x}_0) = \delta(\vec{x}-\vec{x}_0) & \text{in } \Omega \\ G(\vec{x},\vec{x}_0) = 0 & \text{on } \partial\Omega \end{cases}$$

$\Rightarrow$ The Green's fct $G(\vec{x},\vec{x}_0)$ is the source function of Laplace's Equation.

---

8) **FOURIER TRANSFORM**: For $f \in L^1(\mathbb{R})$, define

$$\hat{f}(k) = (\mathcal{F}f)(k) := \int_{\mathbb{R}} f(x)\, e^{-ikx}\, dx$$

Obs: $\|\hat{f}\|_1 = \int_{\mathbb{R}} |f|\cdot|e^{-ikx}|\, dx = \int_{\mathbb{R}} |f| < \infty$.

To calculate the Fourier transform just apply the definition and recall some useful identities:

$$\sin x = \frac{e^{ix}-e^{-ix}}{2i}, \quad \cos x = \frac{e^{ix}+e^{-ix}}{2}$$

$$\sinh x = \frac{e^{x}-e^{-x}}{2}, \quad \cosh x = \frac{e^{x}+e^{-x}}{2},$$

and integrate by parts.

In 2 dimensions:

$$f(x_1,x_2) \xrightarrow{\text{Fourier}} \hat{f}(k_1,k_2)$$

$$\hat{f}(\vec{k}) := \int_{\mathbb{R}^2} f(\vec{x})\, e^{-i\vec{k}\cdot\vec{x}}\, d\vec{x}$$

FUNCTION	FOURIER TRANSFORM
$f(x)$	$\hat{f}(k)$
$f'(x)$	$(ik)\hat{f}(k)$
$f^{(n)}(x)$	$(ik)^n \hat{f}(k)$
$x f(x)$	$i \hat{f}'(k)$
$e^{iax} f(x)$	$\hat{f}(k-a)$
$f(x-a)$	$e^{-iak} \hat{f}(k)$
$f(ax)$	$\frac{1}{\lvert a\rvert} \hat{f}\left(\frac{k}{a}\right)$
$(f * g)(x)$	$\hat{f}(k) \cdot \hat{g}(k)$
$f(x) \cdot g(x)$	$(\hat{f} * \hat{g})(k)$
$a f(x) + b g(x)$	$a \hat{f}(k) + b \hat{g}(k)$

<u>Obs</u>: $\mathcal{F}$ is a linear operator.

<u>IMPORTANT</u>:

$$\widehat{\Delta f}(k) = -|k|^2 \hat{f}(k).$$

NOTE: Want to define Fourier transf. of distributions as $(\hat{T}, \varphi) = (T, \hat{\varphi})$. <u>But</u> there is no reason guaranteeing that for $\varphi \in \mathcal{D}(\mathbb{R})$, its Fourier transform $\hat{\varphi}$ is also in $\mathcal{D}(\mathbb{R})$. ← In general not true.

So, to solve this issue, the "largest space" in which $\mathcal{F}$ "preserves nice properties" is:

SCHWARTZ SPACE: defined as

$$S(\mathbb{R}^n) := \left\{ \varphi \in C^\infty(\mathbb{R}^n) : \sup_{x \in \mathbb{R}^n} |x^\alpha D^\beta \varphi| < \infty \right\}.$$

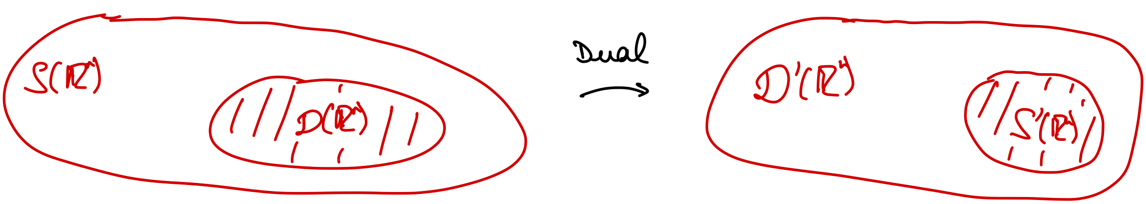
↪ Loosely speaking, space of smooth fcts. that, w/ their derivatives, decay faster than any polynomial.

<u>e.g.</u>: Gaussian (e^{-x^2}); bump fcts.; $\operatorname{sech}(x)$; fcts w/ compact support, etc.

TEMPERED (OR SCHWARTZ) DISTRIBUTIONS: Linear and continuous functionals on $S(\mathbb{R}^n)$; i.e.,

$$T : S(\mathbb{R}^n) \longrightarrow \mathbb{R}.$$

The set of tempered distributions is denoted $S'(\mathbb{R}^n)$.



FACT: "By construction", the Fourier transform

$$\mathcal{F} : S(\mathbb{R}^n) \longrightarrow S(\mathbb{R}^n)$$

is a <u>linear</u> <u>isomorphism</u> (i.e., $\varphi \in S(\mathbb{R}^n) \Leftrightarrow \hat{\varphi} \in S(\mathbb{R}^n)$).

FACTS:
- $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.
- $\mathcal{D}(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$, $1 \leq p < \infty$.
- If $f \in L^p$ and $g \in L^1$, $(f * g) \in L^p$ b/c $\|f * g\|_p \leq \|f\|_p \|g\|_1$.
- <u>Riemann - Lebesgue Lemma</u>: if $f \in L^1$, then $|\hat{f}(k)| \longrightarrow 0$ as $|k| \to \infty$.

Pf: Since cont., compactly supp. fcts are dense in L^1, $\forall \varepsilon > 0$, take $g \in C_c^0$ s.t. $\|f - g\|_1 < \varepsilon$. Then

$$\limsup_{|k| \to \infty} |\hat{f}(k)| = \limsup_{|k| \to \infty} \left| \int f(x) e^{-ikx} dx \right|$$

$$= \limsup_{|k| \to \infty} \left| \int [f(x) - g(x) + g(x)] e^{-ikx} dx \right|$$

$$\leq \limsup_{|k| \to \infty} \left| \int [f(x) - g(x)] e^{-ikx} dx \right|$$

$$+ \cancel{\limsup_{|k| \to \infty} \left| \int g(x) e^{-ikx} dx \right|} = 0 \text{ b/c supp } g \text{ is compact.}$$

$< \varepsilon. \Rightarrow |\hat{f}(k)| \to 0$ as $|k| \to \infty$. $\square$

Claim: $\varphi \in S(\mathbb{R}) \Rightarrow \hat{\varphi} \in S(\mathbb{R})$.

Pf: $\varphi \xrightarrow{\mathcal{F}} \hat{\varphi}$

$$x\varphi \longrightarrow i\hat{\varphi}'$$

$$\frac{1}{i}x\varphi \longrightarrow \hat{\varphi}'$$

$$\frac{1}{i^n}x^n\varphi \longrightarrow \hat{\varphi}^{(n)} \quad \text{and} \quad \frac{1}{i^l}\varphi^{(l)} \longrightarrow k^l\hat{\varphi}^{(l)}$$

Conclusion follows by Riemann - Lebesgue Lemma. $\square$

↳ all derivatives of φ are L^1.

Fourier Transform of Distributions: If $T \in S'(\mathbb{R})$,

$$(\hat{T}, \varphi) = (T, \hat{\varphi}) \quad \forall \varphi \in S(\mathbb{R}).$$

e.g.: $\mathcal{F}\delta(x) = 1$; $\mathcal{F}\delta(x-a) = e^{-ika}$; $\underbrace{\mathcal{F}1 = 2\pi\delta(k)}$

$f(x) := 1$. $(\hat{f}, \varphi) = (f, \hat{\varphi}) \ \forall \varphi \in S(\mathbb{R})$.

$f'(x) = 0 \Rightarrow \widehat{f'}(k) = 0 = ik\hat{f}(k)$

$\Rightarrow ik\hat{f}(k) = 0$

From before $\rightarrow$ $\Rightarrow \hat{f}(k) \overset{*}{=} C\delta(k)$. <u>WTF</u>: C

Take $\varphi(x) = e^{-x^2/2} \in S(\mathbb{R})$.

$$\varphi(x) = e^{-x^2/2} \longrightarrow \sqrt{2\pi}\, e^{-k^2/2} = \hat{\varphi}(k)$$

So,

$$(C\delta, \varphi) \overset{*}{=} (\hat{f}, \varphi) = (f, \hat{\varphi})$$

$$\| \qquad\qquad = \int_{\mathbb{R}} 1 \cdot \hat{\varphi}(k)\, dk$$

$$(\delta, C\varphi)$$

$$\| \qquad\qquad = \sqrt{2\pi} \underbrace{\int_{\mathbb{R}} e^{-k^2/2}\, dk}_{=\sqrt{2\pi}}$$

$$C\varphi(0) = C \qquad\qquad = 2\pi$$

$$\Rightarrow C = 2\pi \Rightarrow \boxed{\hat{f}(k) = 2\pi\delta(k).}$$

$$\mathcal{F}(\mathrm{sgn}(x)) = \frac{2}{i}\, \mathrm{pv}\left(\frac{1}{k}\right)$$

$$\hookrightarrow f(x) = e^{-a|x|} \longrightarrow \hat{f}(k) = \frac{2a}{a^2+k^2}$$

Now, $$f'(x) = \begin{cases} a e^{ax}, & x<0 \\ -a e^{-ax}, & x>0 \end{cases} = -a\,\mathrm{sgn}(x)\, e^{-a|x|}$$

$$f'(x) \longrightarrow (ik)\hat{f}(k)$$

$$\|$$

$$-a\,\mathrm{sgn}(x)\, e^{-a|x|} \longrightarrow \frac{2aik}{a^2+k^2}$$

$$a\,\mathrm{sgn}(x)\,e^{-a|x|} \longrightarrow -\frac{2aik}{a^2+k^2}$$

$$\mathrm{sgn}(x)\,e^{-a|x|} \longrightarrow -\frac{2ik}{a^2+k^2}$$

Take $a \to 0$

$$\mathrm{sgn}(x) \longrightarrow -\frac{2ik}{k^2} = \frac{2}{i}\,\mathrm{pv}\left(\frac{1}{k}\right).$$

$$\mathcal{F}(H(x)) = \frac{1}{i}\,\mathrm{pv}\left(\frac{1}{k}\right) + \pi\,\delta(k).$$

$$H(x) = \frac{1}{2}\left[\mathrm{sgn}(x) + 1\right] = \frac{1}{2}\,\mathrm{sgn}(x) + \frac{1}{2}\cdot 1$$

$$\begin{aligned}\mathcal{F}(H(x)) &= \mathcal{F}\left(\frac{1}{2}\,\mathrm{sgn}(x) + \frac{1}{2}\cdot 1\right) \\ &= \frac{1}{2}\,\mathcal{F}(\mathrm{sgn}(x)) + \frac{1}{2}\,\mathcal{F}(1) \\ &= \frac{1}{2}\cdot\frac{2}{i}\,\mathrm{pv}\left(\frac{1}{k}\right) + \frac{1}{2}\cdot 2\pi\,\delta(k).\end{aligned}$$

INVERSE FOURIER TRANSFORM: for $f \in L^1$,

$$f(x) = \mathcal{F}^{-1}(\hat{f}(k)) = \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(k)\,e^{ikx}\,dk.$$

Pf: $e^{ika}\,\hat{f}(k) = \hat{g}(k)$. Integrate w.r.t. k:

$g(y) =: f(y+a)$

$$\int_{\mathbb{R}} e^{ika}\hat{f}(k)\,dk = \int_{\mathbb{R}} 1\cdot\hat{g}(k)\,dk$$
$$= (1,\hat{g}) = (\hat{1}, g)$$
$$= (2\pi\delta, g) = 2\pi\, g(0)$$
$$= 2\pi f(a) \quad \forall a. \qquad \square$$

e.g.: PULSE FUNCTION $f(x) = H(a-|x|)$.

$$\hat{f}(k) = \frac{2\sin(ak)}{k} \Rightarrow H(a) = f(0) = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{2\sin(ak)}{k}\,dk$$
$$\Rightarrow \int_0^\infty \frac{\sin k}{k}\,dk = \frac{\pi}{2}.$$

PLANCHEREL THEOREM: for $f \in L^2$,

$$\|f\|_{L^2}^2 = \frac{1}{2\pi}\|\hat{f}\|_{L^2}^2 .$$

Pf: $\|f\|_2^2 = \int |f(x)|^2\,dx = \int f(x)\overline{f(x)}\,dx$

$$= \int f(x)\,\overline{\left(\frac{1}{2\pi}\int F(k)e^{ikx}\,dk\right)}\,dx$$

$$= \frac{1}{2\pi}\int f(x)\left(\int \overline{F(k)}\, e^{-ikx}\,dk\right)dx$$

$$= \frac{1}{2\pi} \int \left(\int f(x) \overline{F(k)} e^{-ikx} dx \right) dk$$

$$= \frac{1}{2\pi} \int \left(\int f(x) e^{-ikx} dx \, \overline{F(k)} \right) dk$$

$$= \frac{1}{2\pi} \int F(k) \overline{F(k)} \, dk = \frac{1}{2\pi} \| F \|_2^2 \qquad \square$$

UNCERTAINTY PRINCIPLE: Suppose $f, f' \in L^2$ and $\| x f(x) \|_2 < \infty$ ("momentum condition"). Then

$$\| f(x) \|_2 \le \sqrt{\frac{2}{\pi}} \, \| x f(x) \|_2 \cdot \| k \hat{f}(k) \|_2 .$$

Pf: Consider the following:

$$\left| \int x f(x) f'(x) \, dx \right| = \left| \langle x f(x), f'(x) \rangle_{L^2} \right| \overset{\text{Cauchy Schwarz}}{\le} \| x f(x) \|_2 \, \| f' \|_2$$

$$= \left(\int |x|^2 |f(x)|^2 dx \right)^{1/2} \left(\int |f'(x)|^2 dx \right)^{1/2}$$

$$\overset{\text{Plancherel}}{=} \left(\int |x|^2 |f(x)|^2 dx \right)^{1/2} \left(\frac{1}{2\pi} \int |ik \hat{f}(k)|^2 dk \right)^{1/2} .$$

Integrate LHS by parts: $\int x f(x) f'(x) \, dx = -\frac{1}{2} \int |f|^2 dx$

$\square$

FOURIER & PDEs

! : Needs to be a linear PDE and the variable needs to "run w/ boundaries" (e.g., $x \in \mathbb{R}$).

1) DIFFUSION EQUATION

$$(A) \begin{cases} u_t - a\, u_{xx} = 0, & x \in \mathbb{R},\ t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

Recall that the source fct. is

$$S(x,t) = \frac{1}{\sqrt{4\pi a t}}\, e^{-x^2/4at},$$

and it satisfies the following:

$$(B) \begin{cases} S_t - a\, S_{xx} = 0 \\ S(x,0) = \delta(x) \end{cases}$$

So, apply Fourier transform to (A) w.r.t. x:
set $\hat{u}(k,t) := U(k,t)$, then

$$\widehat{u_t - a u_{xx}} = \hat{0}$$

$$\widehat{(u_t)} - a\,\widehat{(u_{xx})} = 0$$

$$U_t - a(ik)^2 U = 0$$

(ODE 1) $\begin{cases} U_t = -a k^2 U \\ U(k,0) = \hat{u}_0(k) \end{cases}$

Do the same for (B):

(ODE 2) $\begin{cases} \hat{S}_t + a k^2 \hat{S} = 0 \\ \hat{S}(k,0) = \hat{\delta}(k) = 1 \end{cases}$

Solve (ODE 2): $\hat{S}(k,t) = A(k) e^{-ak^2 t}$

$\hat{S}(k,0) = 1 = A(k) \Rightarrow \boxed{\hat{S}(k,t) = e^{-ak^2 t}}$

Thus, the solution to (ODE 1) is:

$$U(k,t) = \hat{u}_0(k) e^{-ak^2 t}$$

$\hat f \cdot \hat g = \widehat{(f * g)}$

$$= \hat{u}_0(k) \cdot \hat{S}(k,t)$$

$\Rightarrow u(x,t) = (S * u_0)(x,t)$. So, we need to compute the Inverse Fourier Transform of $\hat{S}(k,t) = e^{-ak^2 t}$:

$$e^{-x^2/2} \longrightarrow \sqrt{2\pi}\, e^{-k^2/2}$$

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \longrightarrow e^{-k^2/2}$$

$$\frac{1}{\sqrt{2\pi}} e^{-(x/\sqrt{2at})^2/2} \longrightarrow \sqrt{2at}\, e^{-(k\cdot\sqrt{2at})^2/2}$$

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/4at} \longrightarrow \sqrt{2at}\, e^{-atk^2}$$

$$S(x,t) = \frac{1}{\sqrt{4\pi a t}} e^{-x^2/4at} \longrightarrow e^{-atk^2} = \hat{S}(k,t)$$

Therefore, $u(x,t) = (S * u_0)(x,t)$.

2) LAPLACE'S EQUATION

$$\begin{cases} \Delta u = 0 & \text{on } \{(x,y) : x > 0\} \\ u(0,y) = f(y) \end{cases}$$

right half-plane

Fourier transform in y: $\hat{u}(x,k) := U(x,k)$.

Obtain:

$$\widehat{\Delta u} = 0$$

$$\widehat{u_{xx}} + \widehat{u_{yy}} = 0$$

$$U_{xx} + (ik)^2 U = 0$$

$$(\text{ODE}) \quad \begin{cases} U_{xx} - |k|^2 U = 0 \\ U(0,k) = \hat{f}(k) \end{cases}$$

0 b/c needs to be bdd as $x \to \infty$

$$U(x,k) = \cancel{C_1(k)\, e^{|k|x}} + C_2(k)\, e^{-|k|x}$$

$$U(0,k) = \hat{f}(k) = C_2(k)$$

$$\Rightarrow \boxed{U(x,k) = \hat{f}(k)\, e^{-|k|x}}$$

Apply Inverse Fourier Transform:

$$\mathcal{F}^{-1}(U(x,k)) = \mathcal{F}^{-1}\left(\hat{f}(k) \cdot e^{-|k|x}\right)$$

$$u(x,y) = f * \mathcal{F}^{-1}\left(e^{-|k|x}\right)$$

Need to compute $\mathcal{F}^{-1}\left(e^{-|k|x}\right)$:

$$\mathcal{F}^{-1}\left(e^{-|k|x}\right) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-|k|x}\, e^{iky}\, dk$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{-kx}\, e^{iky}\, dk + \frac{1}{2\pi}\int_{-\infty}^{0} e^{kx}\, e^{iky}\, dk$$

$$= \frac{1}{2\pi}\int_{0}^{\infty} e^{(-x+iy)k}\, dk + \frac{1}{2\pi}\int_{-\infty}^{0} e^{(x+iy)k}\, dk$$

$$= \frac{1}{2\pi}\left[\frac{1}{-x+iy}\, e^{-(x-iy)k}\right]_{k=0}^{k=\infty} + \frac{1}{2\pi}\left[\frac{1}{x+iy}\, e^{(x+iy)k}\right]_{k=-\infty}^{k=0}$$

$$= \frac{1}{2\pi}\left(\frac{1}{x-iy} + \frac{1}{x+iy}\right)$$

$$= \frac{1}{2\pi}\,\frac{x+\cancel{iy}+x-\cancel{iy}}{x^2+y^2} = \frac{1}{2\pi}\cdot\frac{2x}{x^2+y^2} = \frac{x}{\pi(x^2+y^2)}.$$

Therefore,

$$\boxed{u(x,y) = \int_{-\infty}^{\infty} \frac{f(y')\, x}{\pi\left[x^2+(y-y')^2\right]}\, dy'}$$

$\blacksquare$