LECTURE 1

(ch 2)

Mar 24th, 2026

DEF: (GENERAL PDE) $n \geq 1$, $k \geq 1$, $U \subset \mathbb{R}^n$ open,

$$F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times U \to \mathbb{R}$$

$\rightsquigarrow$ (scalar) k-th order PDE determined by F:

$$F\left(D^k u(x), D^{k-1} u(x), \dots, Du(x), u(x), x\right) = 0, \quad x \in U,$$

with unknown function $u : U \to \mathbb{R}$.

sometimes too strong...

Remark: Classical solⁿ of kth order PDE $\rightsquigarrow$ $u \in C^k$

Well - Posedness:
(i) Existence
(ii) Uniqueness
(iii) Continuous dependence on data

SPECIAL CASES:

1) LINEAR EQUATIONS: $\displaystyle\sum_{|\alpha| \leq k} a_\alpha(x) D^\alpha u(x) = 0$

$$\left(\begin{array}{l} \alpha : (\alpha_1, \dots, \alpha_n) \quad \text{MULTI-INDEX} \\ |\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n \\ D^\alpha u := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u \end{array}\right)$$

• EXAMPLES:

$u_t + b\, Du = 0$, $Du = (\partial_{x_1} u, \dots, \partial_{x_n} u)$ $(= \nabla u)$ (Transport Equation)

$-\Delta u = 0$ $(= f)$ (Laplace (Poisson) Equation) Elliptic

$$u_t - \Delta u = 0$$ (Heat Equation) Parabolic

$$u_{tt} - \Delta u = 0$$ (Wave Equation) Hyperbolic

2) SEMI-LINEAR EQUATIONS:

$$F: \sum_{|\alpha| = k} a_\alpha(x) D^\alpha u + F_0 \left(D^{k-1} u, \ldots, Du, u, x \right)$$

• EXAMPLES:

$$-\Delta u = f(u)$$ (Semilinear Elliptic)

$$u_{tt} - \Delta u = \pm |u|^p u$$ (Nonlinear Equations)

3) QUASI-LINEAR EQUATIONS

$$F: \sum_{|\alpha| = k} F_\alpha \left(D^{k-1} u, \ldots, Du, u, x \right) D^\alpha u + F_0 \left(D^{k-1} u, \ldots, Du, u, x \right)$$

(Coeffs. of highest order terms may depend on lower derivatives)

EXAMPLES: Burgers' Equation $u_t + u u_x = 0$

4) FULLY NONLINEAR EQUATIONS

EXAMPLES: Monge-Ampere $\det(D^2 u) = f$ (e.g.: $n = 2$, $u_{xx} u_{yy} - u_{xy}^2 = f$)

* **Systems of PDEs** : unknown $u : U \to \mathbb{R}^n$

Example : Euler $u : \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^d$
$$\begin{cases} u_t + (u \cdot \nabla) u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{cases}$$

Navier-Stokes
$$\begin{cases} u_t + (u \cdot \nabla) u + \nabla p = \Delta u \\ \operatorname{div} u = 0 \end{cases}$$

Linear Transport : Fix $b \in \mathbb{R}^n$ and consider

$$u_t + b \cdot Du = 0 \qquad (*)$$

on $\mathbb{R}^n \times (0,\infty)$ with unknown for $u = u(x,t)$, $Du = (u_{x_1}, \dots, u_{x_n})$ is the gradient of u w.r.t. x

Assume: We have sol$^{\underline{n}}$ $u = u(x,t)$ which is smooth

Fix $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and, for $s \in \mathbb{R}$,

$$z(s) = u(x+sb,\, t+s)$$

Then

$$\frac{dz}{ds} = Du(x+sb,\, t+s) \cdot b + u_t(x+sb,\, t+s)$$

Since u solves $(*)$, $Du \cdot b + u_t = 0$ evaluated everywhere.

So, $\dfrac{dz}{ds} = 0 \quad \forall s \implies \boxed{z \text{ is constant in } s}$.

Upshot: $\forall (x,t)$, any smooth sol$^{\underline{n}}$ u is constant along the line through (x,t) which goes in the direction $(b,1) \in \mathbb{R}^{n+1}$

(IVP) $\left\{ \begin{array}{ll} u_t + b \cdot Du = 0, & (x,t) \in \mathbb{R}^n \times (0,\infty) \\ u(x,0) = g(x), & (x,t) \in \mathbb{R}^n \times \{0\} \end{array} \right.$

↑
Initial Val. Problem
(Cauchy Problem)

REMARK: The line through a pt. (x,t) w/ direction $(b,1)$ is

$$\{(x+bs, t+s) : s \in \mathbb{R}\}$$

This line intersects the plane $\mathbb{R}^n \times \{0\}$ (where the smooth sol$^{\underline{n}}$ u should satisfy the initial cond$^{\underline{n}}$). $s=-t$, i.e., $(x-tb, 0)$

PROPOSITION: If u is a smooth sol$^{\underline{n}}$ to (IVP) then

$$u(x,t) = g(x-tb), \quad x \in \mathbb{R}^n, \ t \geq 0.$$

$\blacktriangleleft$

* NON-HOMOGENEOUS PROBLEM

(**) $\left\{ \begin{array}{ll} u_t + b \cdot Du = \boxed{f}, & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(0) = g, & \text{on } \mathbb{R}^n \end{array} \right.$

- Fix $(x,t) \in \mathbb{R}^{n+1}$ and define $z(s) = u(x+sb, t+s)$ for $s \in \mathbb{R}$.
- Now, if u solves (**) and is smooth

$$\frac{dz}{ds} = (Du)(x+sb, t+s) \cdot b + u_t(x+sb, t+s)$$

$$\overset{(**)}{=} f(x+sb, t+s)$$

- So, $z(-t) = u(x-tb, 0) \overset{!}{=} g(x-tb)$. So, by the FTC, we have

$$z(0) - z(-t) = \int_{-t}^{0} \frac{dz}{ds}\, ds$$

$$= \int_{-t}^{0} f(x+sb, t+s)\, ds$$

$s \mapsto s-t$

$$= \int_{0}^{t} f(x+(s-t)b, s)\, ds$$

Upshot:

$$u(x,t) = z(0) = z(-t) + \int_0^t f(x+(s-t)b, s)\, ds$$

$$\Rightarrow \boxed{u(x,t) = g(x-tb) + \int_0^t f(x+(s-t)b, s)\, ds}$$

Laplace & Poisson Equations

$n \geq 1$, $U \subset \mathbb{R}^n$ open, $u : \overline{U} \to \mathbb{R}$, $f : U \to \mathbb{R}$

(L) $\Delta u = 0$ (Laplace)

$$\left(\Delta = \sum_{i=1}^{n} \partial_{x_i x_i} \right)$$

(P) $-\Delta u = f$ (Poisson)

More general: $\mathrm{div}(A(x) Du(x))$

$$\Delta u = \mathrm{div}(Du) = \mathrm{tr}(D^2 u)$$

DEF: $u \in C^2$ and $\Delta u = 0$, we say u is HARMONIC.

PROPOSITION: (Mean Value Property) If $U \subset \mathbb{R}^n$ open, and $u \in C^2(U)$ is harmonic then $\forall x \in \mathbb{R}^n$, $r > 0$,

$$u(x) = \fint_{\partial B_x(r)} u \, dS$$

$$\fint_A f \, d\mu := \frac{1}{\mu(A)} \int_A f \, d\mu$$

(AVERAGE)

$$= \frac{1}{\mathrm{vol}_{n-1}(\partial B_x(r))} \int_{\partial B_x(r)} u(y) \, dS(y)$$

(Harmonic $\Rightarrow$ satisfies MVP)

LECTURE 2

(ch 2)

Mar 25th 2026

LAPLACE's EQUATION (ctd.)

$$\Delta u = 0 \qquad u \equiv \begin{cases} \text{- chemical concentration} \\ \text{- temperature} \\ \text{- electrostatic potential} \end{cases} \Rightarrow \{\Delta u = 0\} \Leftrightarrow \begin{cases} \text{- Fick's law} \\ \text{- Fourier law} \\ \text{- Ohm's law} \end{cases}$$

$u : U \subset \mathbb{R}^n \to \mathbb{R}$

$\Delta u = \operatorname{div}(Du) = \operatorname{tr}(D^2 u)$

* FUNDAMENTAL SOLUTION

- NOTE: Laplace's eq. is rotationally invariant $\Rightarrow$ seek sol$^{\text{ns}}$ in terms of $r := |x|$.

(Say $U = \mathbb{R}^n$, $u(x) = u(r) \equiv u(|x|)$)

$$\frac{\partial r}{\partial x_i} = \frac{\partial}{\partial x_i}\left((x_1^2+\dots+x_n^2)^{1/2}\right) = \frac{x_i}{r} \quad (x \neq 0)$$

$\boxed{u(x) \equiv v(r)}$

$$\Rightarrow \begin{cases} u_{x_i} = \dfrac{\partial u}{\partial x_i} = \dfrac{\partial r}{\partial x_i}\dfrac{\partial u}{\partial r} = \dfrac{x_i}{r} v' \\[2ex] u_{x_i x_i} = \dfrac{\partial}{\partial x_i}\left(\dfrac{x_i}{r} v'\right) = v'\left(\dfrac{1}{r} - \dfrac{x_i^2}{r^3}\right) + \dfrac{x_i^2}{r^2} v'' \end{cases}$$

i.e.,

$$\Delta u = v''(r) + \frac{n-1}{r} v'(r) = 0 \iff \boxed{v'' + \frac{n-1}{r} v' = 0}$$

$$\Rightarrow \quad v(r) = \begin{cases} B \log r + C, & n = 2 \\ \dfrac{B}{r^{n-2}} + C, & n \geq 3 \end{cases}$$

DEF: (Fundamental Solution of Laplace's Eq) $x \in \mathbb{R}^n$, $x \neq 0$,

$$\Phi(x) := \begin{cases} -\dfrac{1}{2\pi} \log|x|, & n = 2 \\[2ex] \dfrac{1}{n(n-2)\alpha(n)} \dfrac{1}{|x|^{n-2}}, & n \geq 3 \end{cases}.$$

$\alpha(n) := vol_n(B_1(0))$

$|D\Phi(x)| \leq \frac{C}{|x|^{n-1}}$

$|D^2\Phi(x)| \leq \frac{C}{|x|^n}$ $(x \neq 0)$

Poisson's Equation

$$(-\Delta u = f)$$

THM: Let

$$u(x) := \int_{\mathbb{R}^n} \Phi(x-y) f(y)\, dy .$$

Fundamental sol^{n} of Laplace

Then

(i) $u \in C^2(\mathbb{R}^n)$

(ii) $-\Delta u = f$ on $\mathbb{R}^n$.

Pf: $u(x) = \int \Phi(x-y) f(y)\, dy = \int \Phi(y) f(x-y)\, dy$

$$\Rightarrow \frac{u(x+he_i) - u(x)}{h} = \int \Phi(y) \left[\frac{f(x+he_i-y) - f(x-y)}{h} \right] dy$$

uniformly $\downarrow$ $h \downarrow 0$

f_{x_i}

$$\Rightarrow u_{x_i}(x) = \int_{\mathbb{R}^n} \Phi(y) f_{x_i}(x-y)\, dy, \quad i = 1, \dots, n$$

$$u_{x_i x_j}(x) = \underbrace{\int_{\mathbb{R}^n} \Phi(y) f_{x_i x_j}(x-y)\, dy}_{\text{continuous on } x \Rightarrow u \in C^2(\mathbb{R}^n)}, \quad i,j = 1, \dots, n.$$

Note: Φ blows up at 0. Fix $\varepsilon > 0$.

$$\Delta u(x) = \underbrace{\int_{B_\varepsilon(0)} \Phi(y) \Delta_x f(x-y)\, dy}_{=: I_\varepsilon} + \underbrace{\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \Phi(y) \Delta_x f(x-y)\, dy}_{=: J_\varepsilon}$$

$$|I_\varepsilon| \le C \|D^2 f\|_\infty \int_{B_\varepsilon(0)} |\Phi(y)|\, dy \le \begin{cases} C\varepsilon^2 |\log \varepsilon|, & n = 2 \\ C\varepsilon^2, & n \ge 3 \end{cases}$$

by parts

$$J_\varepsilon = \underbrace{-\int_{\mathbb{R}^n \setminus B_\varepsilon(0)} D\Phi(y) \cdot D_y f(x-y)\, dy}_{=: K_\varepsilon} + \underbrace{\int_{\partial B_\varepsilon(0)} \Phi(y) \frac{\partial f}{\partial \nu}(x-y)\, dS(y)}_{=: L_\varepsilon}$$

$$|L_\varepsilon| \le \|Df\|_\infty \int_{\partial B_\varepsilon(0)} |\Phi(y)|\, dS(y) \le \begin{cases} C\varepsilon |\log \varepsilon|, & n = 2 \\ C\varepsilon, & n \ge 3 \end{cases}.$$

by parts

$$K_\varepsilon = \int_{\mathbb{R}^n \setminus B_\varepsilon(0)} \underbrace{\Delta \Phi(y)}_{=0} f(x-y)\, dy - \int_{\partial B_\varepsilon(0)} \frac{\partial \Phi}{\partial \nu}(y) f(x-y)\, dS(y)$$

$$= -\int_{\partial B_\varepsilon(0)} \frac{\partial \Phi}{\partial \nu}(y)\, f(x-y)\, dS(y)$$

$$D\Phi(y) = -\frac{1}{n\alpha(n)} \frac{y}{|y|^n}, \quad y \neq 0 \quad \text{and} \quad \nu = \frac{-y}{|y|} = \frac{-y}{\varepsilon} \quad \text{on } \partial B_\varepsilon(0)$$

$$\Rightarrow \frac{\partial \Phi}{\partial \nu}(y) = \nu \cdot D\Phi(y) = \frac{1}{n\alpha(n)\varepsilon^{n-1}} \quad \text{on } \partial B_\varepsilon(0)$$

$n\alpha(n)\varepsilon^{n-1}$ is the surface area of $\partial B_\varepsilon(0)$

$$\Rightarrow K_\varepsilon = -\frac{1}{n\alpha(n)\varepsilon^{n-1}} \int_{\partial B_\varepsilon(0)} f(x-y)\, dS(y)$$

$$= ⨍_{\partial B_\varepsilon(x)} f(y)\, dS(y) \longrightarrow -f(x) \quad \text{as } \varepsilon \to 0.$$

<u>Upshot</u>: As $\varepsilon \to 0$, $-\Delta u = f$.

$\square$

---

* <u>MEAN VALUE FORMULAS</u>

<u>THM</u>: (Mean Value Formula for $\Delta u = 0$) If $u \in C^2(U)$ and $\Delta u = 0$ on U (i.e., u is harmonic on U), then

$$u(x) = ⨍_{\partial B_r(x)} u\, dS = ⨍_{B_r(x)} u\, dy$$

for each ball $B_r(x) \subset U$.

Pf: Set $\phi(r) := \fint_{\partial B_r(x)} u(y)\, dS(y) = \fint_{\partial B_1(0)} u(x+rz)\, dS(z)$

$$\Rightarrow \phi'(r) = \fint_{\partial B_1(0)} Du(x+rz)\cdot z \; dS(z)$$

(Green's Formulas)

$$= \fint_{\partial B_r(x)} Du(y)\cdot \frac{y-x}{r}\, dS(y)$$

$$= \fint_{\partial B_r(x)} \frac{\partial u}{\partial \nu}\, dS(y)$$

$$= \frac{r}{n} \fint_{B_r(x)} \Delta u(y)\, dy$$

$$\overset{\Delta u = 0}{=} 0 \implies \phi \text{ is constant.}$$

i.e., $\phi(r) = \lim_{t\to 0} \phi(t) = \lim_{t\to 0} \fint_{\partial B_t(x)} u(y)\, dS(y) = u(x)$.

Next, in polar coordinates, $\int_{B_r(x)} u\, dy = \int_0^r \left(\int_{\partial B_s(x)} u\, dS \right) ds$

$$= u(x) \int_0^r n\alpha(n) s^{n-1}\, ds$$

$$= \alpha(n) r^n u(x)$$

$\square$

ASIDE: (Distributions) $\mathcal{D}(U) := \{ f \in C^\infty(U) : \operatorname{supp} f \text{ compact} \}$, where $\operatorname{supp} f = \overline{\{x \in U : f(x) \neq 0\}} \subset U$

equipped with the following convergence

$\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{D}(U)$, $f \in \mathcal{D}(U)$

$f_n \xrightarrow{n\to\infty} f$ in $\mathcal{D}(U)$ $\iff$ $\exists K \subset\subset U$ s.t. $\operatorname{supp} f_n \subset K\ \forall n$ and $\|f_n - f\|_{C^k(U)} \longrightarrow 0\ \forall k$

$\left(\|f\|_{C^k(U)} := \sum_{|\alpha|\le k} \|D^\alpha f\|_{L^\infty(U)} \right)$

THM: (Converse to Mean Value Property) If $u \in C^2(U)$ satisfies

$$u(x) = \fint_{\partial B_r(x)} u \, dS \quad \text{for each } B_r(x) \subset U \qquad (\text{MVP})$$

then u is harmonic on U (i.e., $\Delta u = 0$)

Pf: Contradiction $\rightarrow$ $\Delta u \not\equiv 0 \Rightarrow \exists B_r(x) \subset U$ s.t. $\Delta u > 0$ (WLOG)

Then for $\phi(r) = \fint_{\partial B_r(x)} u(y) \, dS(y)$,

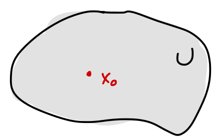
$$0 = \phi'(r) = \frac{r}{n} \fint_{B_r(x)} \Delta u(y) \, dy > 0$$

By MVP, and continuity, $u(x) = \phi(r)$ for suff. small $r > 0$

$\hookrightarrow$ // $\hookleftarrow$

$\square$

THM: (Strong Maximum Principle) Suppose $u \in C^2(U) \cap C(\bar{U})$ is harmonic in U. Then



(1) $\max_{\bar{U}} u = \max_{\partial U} u$ (maximum is attained at ∂U)

Weak Maximum Principle

(2) If U is connected and $\exists x_0 \in U$ s.t. $u(x_0) = \max_{\bar{U}} u$, then u is constant in U. (max attained in interior $\Rightarrow$ constant)

Strong Maximum Principle

(min $\leftrightarrow$ max $\Leftarrow$ $u \leftrightarrow -u$)

Pf: Say $u(x_0) = M := \max_{\bar{U}} u$. Then, for $0 < r < \text{dist}(x_0, \partial U)$, by the MVP,

$$M = u(x_0) = \fint_{B_r(x_0)} u \, dy \leq M \Rightarrow u(y) \equiv M \quad \forall y \in B_r(x_0)$$

So, $\{x \in U : u(x) = M\}$ is open and relatively closed in U, hence it equals U if U is connected. (1) follows from (2). $\square$

> **THM:** (POSITIVITY) If $U \subset \mathbb{R}^n$ is open and connected, and $u \in C^2(U) \cap C(\bar{U})$ satisfies
> $$\begin{cases} \Delta u = 0 & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$$
> then $u > 0$ everywhere in U if g is positive somewhere on ∂U.

Pf: SMP to $-u$. $\square$

> **THM:** (UNIQUENESS) Let $g \in C(\partial U)$, $f \in C(U)$. Then, there exists at most one sol^{n} $u \in C^2(U) \cap C(\bar{U})$ for
> $$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}.$$

Pf: u and $\tilde{u}$ satisfy then SMP to $w := u - \tilde{u}$.

$$\Rightarrow \begin{cases} \Delta w = 0 & \text{in } U \\ w \overset{(*)}{=} 0 & \text{on } \partial U \end{cases} \quad \overset{\text{SMP}}{\leadsto} \quad \max_{\bar{U}} w = \max_{\partial U} w \overset{(*)}{=} 0$$

$$\max_{\bar{U}} -w = \max_{\partial U} -w \overset{(*)}{=} 0$$

i.e., $w = 0$ on $\bar{U}$. $\square$

THM: (SMOOTHNESS) If $u \in C(U)$ satisfies MVP for each ball $B_r(x) \subset U$, then $u \in C^\infty(U)$

($u \in C^2$ harmonic $\Downarrow$ u smooth) ↑ actually, analytic...

↖ only interior, no guarantee on the bdry

NOTE: if u harmonic in U, then

$$|D^\alpha u(x_0)| \leq \frac{C_k}{r^{n+k}} \|u\|_{L^1(B_r(x_0))},$$

$\forall B_r(x_0) \subset U$ and each multiindex α, $|\alpha| = k$.

THM: (LIOUVILLE) Suppose $u: \mathbb{R}^n \to \mathbb{R}$ is harmonic and bounded. Then u is constant.

THM. (HARNACK'S INEQUALITY) For each connected open set $V \subset\subset U$ (i.e., $V \subset \bar{V} \subset U$ w/ $\bar{V}$ compact), $\exists\, C_V > 0$, depending only on V, such that

$$\sup_V u \leq C \inf_V u$$

for all nonnegative harmonic functions u in U.

In particular,

$$\frac{1}{C} u(y) \leq u(x) \leq C u(y) \quad \forall x, y \in V.$$

i.e., values of a nonnegative harmonic fct in V are comparable: u cannot be very small (or very large) at any pt. in V unless u is very small (or very large) everywhere in V.

* ENERGY METHODS (energy $\rightsquigarrow$ L^2 norm)

THM: (Uniqueness – again) Before: SMP $\exists!$ sol$^{\underline{n}}$ $u \in C^2(\bar{U})$ of

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}.$$

∂U: assume C^1 (i.e., locally parametrized by C^1 functions)

Pf: Say $u, \tilde{u}$ are sol$^{\underline{ns}}$. Set $w := u - \tilde{u}$. Then

$$\begin{cases} \Delta w = 0 & \text{in } U \\ w = 0 & \text{on } \partial U \end{cases} \implies 0 = -\int_U w \Delta w \, dx \overset{\text{by parts}}{=} \int_U |Dw|^2 \, dx$$

i.e., $Dw \equiv 0$ in U, and, since $w \equiv 0$ on ∂U, we have $w = u - \tilde{u} \equiv 0$ in U.

$\square$

DIRICHLET'S PRINCIPLE: Define

⚠ $$E[w] := \int_U \frac{1}{2}|Dw|^2 - wf \, dx \qquad \text{(ENERGY FUNCTIONAL)}$$

for $w \in \mathcal{A} := \{ v \in C^2(\bar{U}) : v = g \text{ on } \partial U \}$.

Suppose $u \in C^2(\bar{U})$ solves $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$. Then

$$E[u] = \min_{w \in \mathcal{A}} E[w].$$

Conversely, if $u \in \mathcal{A}$ is s.t. $E[u] = \min_{w \in \mathcal{A}} E[w]$, then u solves $\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases}$.

(i.e., provided that $u \in \mathcal{A}$, the PDE $-\Delta u = f$ is equivalent to the statement that u minimizes the energy $E[\cdot]$ among functions in $\mathcal{A}$
$\rightsquigarrow$ 1st variation $\frac{d}{d\varepsilon}\left[E[u+\varepsilon v]\right]\Big|_{\varepsilon=0} = 0$ for each $v \in C_c^\infty(U)$.

LECTURE 3

Mar 31st 2026

$$\mathcal{D}(U) := \{ f \in C^\infty(U) : \text{supp } f \text{ compact} \}$$

FACTS ABOUT DISTRIBUTIONS

(1) $\mathcal{D}(U)$ is not metrizable

(2) $T : \mathcal{D}(U) \to \mathbb{R}$ linear is sequentially continuous $\overset{\text{HW02}}{\iff}$ $\forall K \subset\subset U$ $\exists n \geq 0$ $C > 0$ s.t. $\forall \phi \in \mathcal{D}_K := \{ \phi \in \mathcal{D} : \text{supp } \phi \subset K \}$ one has $|T\phi| \leq C \|\phi\|_{C^n(U)}$

(3)

$$f \in L^1_{loc}(U) = \{ f : U \to \mathbb{R} : f \text{ measurable } \& \ \forall K \subset\subset U, \int_K |f| \, dx < \infty \}$$

$T_f : \mathcal{D}(U) \to \mathbb{R}$

$$T_f(\phi) = \int_U \phi f \, dx$$

is sequentially continuous linear functional on $\mathcal{D}$; i.e., an element of the dual $\mathcal{D}' := [\mathcal{D}(U)]'$

space of all sequentially cont. lin. functionals on $\mathcal{D}$.

$f \mapsto T_f$ is injective from $L^1_{loc} \hookrightarrow \mathcal{D}'$ $\iff$ $\boxed{\mathcal{D}' \text{ contains a copy of } L^1_{loc}}$.

We call elements of $\mathcal{D}'$ DISTRIBUTIONS ("gen. fcts")

$$\left(\int \phi T_f = T_f(\phi) \right)$$

Ex: $\underset{\mathcal{D}'}{\delta_0 \in} : \mathcal{D} \to \mathbb{R}$ defined $\delta_0(\phi) = \phi(0)$ is a distribution on $\mathbb{R}^n$.

$$\left[\int \delta_0(x) \phi(x) \, dx = \phi(0) \right]$$

DERIVATIVE OF DISTRIBUTIONS: for $f, g \in C_c^\infty(U)$

$$\int_U (D^\alpha f)\, g \, dx = (-1)^{|\alpha|} \int f \,(D^\alpha g)\, dx .$$

FACT: (1) $\forall\, \alpha$ multiindices, and all $T \in \mathcal{D}'$,

$$\phi \xmapsto{D^\alpha T} (-1)^{|\alpha|}\, T(D^\alpha \phi)$$

defines a distribution $D^\alpha T \in \mathcal{D}'$.

(2) $k \geq 0$, $u \in C^k(U)$, α multi-index with $|\alpha| \leq k$, if $T_u : \phi \mapsto \int u \phi \, dx$ is the distribution defined by u, then $D^\alpha T_u = T_{D^\alpha u}$; i.e., $\underbrace{(D^\alpha T_u)(\phi)}_{= (-1)^{|\alpha|} \int u (D^\alpha \phi)} = \underbrace{T_{D^\alpha u}(\phi)}_{= \int (D^\alpha u)\phi}$

Pf: for $\phi \in \mathcal{D}$,

$$(D^\alpha T_u)(\phi) = (-1)^{|\alpha|}\, T_u(D^\alpha \phi)$$

$$= (-1)^{|\alpha|} \int u \, D^\alpha \phi$$

$$= \int (D^\alpha u)\, \phi$$

$$= (T_{D^\alpha u})(\phi) . \qquad \square$$

REMARKS: $\mathcal{D}(U)$ & convergence in $\mathcal{D}(U)$.

$U \subset \mathbb{R}^n$ open

$$\mathcal{D}(U) := C_c^\infty(U) = \{\phi \in C^\infty(U) : \operatorname{supp}\phi \subset U \text{ compact}\}$$

$\phi_j \to 0$ in $\mathcal{D}(U)$ $\iff$ $\exists$ compact $K \subset U$ s.t. $\operatorname{supp}\phi_j \subset K$ $\forall j$
<u>and</u> $\forall$ multiindex α,
$D^\alpha \phi_j \to 0$ uniformly on K

$$\|\phi\|_{C^n(K)} = \sup_{|\alpha| \le n} \sup_{x \in K} |D^\alpha \phi(x)| .$$

$\phi_j \to \phi$ in $\mathcal{D}(U)$ $\iff$ $\exists$ compact $K \subset U$ s.t. $\operatorname{supp}\phi_j \subset K$ $\forall j$
and $\forall \alpha$,
$D^\alpha \phi_j \to D^\alpha \phi$ unif. on K.

$$\mathcal{D}'(U) = \{T : \mathcal{D}(U) \to \mathbb{R} : T \text{ linear } \& \text{ sequentially continuous}\}$$

If $\phi_j \to \phi$ in $\mathcal{D}(U)$, then $T\phi_j \to T\phi$ in $\mathbb{R}$.

<u>NOTE</u>: After showing $T : \mathcal{D}(U) \to \mathbb{R}$ is linear, WLOG can show seq. continuity only at zero (b/c can translate that around): $\phi_j \xrightarrow{\text{in } \mathcal{D}(U)} 0 \Rightarrow T\phi_j \xrightarrow{\text{in } \mathbb{R}} 0$.

* GREEN'S FUNCTIONS → Solving Laplace in domains $U \subset \mathbb{R}^n$

$$\begin{cases} U \subset \mathbb{R}^n \text{ open and bdd, } \partial U \text{ is } C^1 \\ -\Delta u = f \quad \text{in } U \\ u = g \quad \text{on } \partial U \end{cases}$$

Memorize Green's Identities

IMPORTANT (memorize)

DEF: Green's function for the region $U \subset \mathbb{R}^n$ is

$$G : U \times U \setminus \{(x,x) : x \in U\} \to \mathbb{R}$$

$$G(x,y) := \Phi(y-x) - \phi^x(y), \quad x, y \in U, \quad x \neq y,$$

where Φ is the fundamental solution to Laplace in $\mathbb{R}^n$ and ϕ^x is s.t., for each $x \in U$, $y \mapsto \phi^x(y)$ solves

"CORRECTOR" FUNCTION

$$\begin{cases} \Delta \phi^x(y) = 0, & y \in U \\ \phi^x(y) = \Phi(y-x), & y \in \partial U \end{cases}.$$

THM: (SOLUTIONS IN TERMS OF $G(x,y)$) If $u \in C^2(U)$ solves

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U \end{cases},$$

then, for each $x \in U$,

$$u(x) = -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x,y)\, dS(y) + \int_U f(y)\, G(x,y)\, dy$$

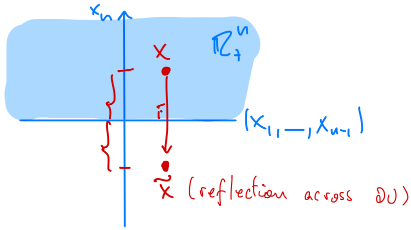
$$\begin{cases} -\Delta_y G(x,y) = \delta_x, & y \in U \\ G(x,y) = 0, & y \in \partial U \end{cases}$$

! IMPORTANT !

FACTS : (1) $\forall x, y \in U,\ x \neq y$, we have $G(x,y) = G(y,x)$.

(2) HALF - SPACE $\mathbb{R}^n_+ := \{ x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0 \}$

↖ unbounded, so computations are done formally



$$(c) \begin{cases} \Delta \phi^x = 0 & \text{in } U = \mathbb{R}^n_+ \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial U \end{cases}$$

$$\phi^x(y) := \Phi(y - \tilde{x}), \quad \text{where } \tilde{x} := (x_1, \dots, x_{n-1}, -x_n)$$

This solves (c). Thus, Green's fct for $\mathbb{R}^n_+$ is:

$$\Rightarrow G(x,y) = \Phi(y-x) - \Phi(y-\tilde{x}), \quad \begin{matrix} x, y \in \mathbb{R}^n_+ \\ x \neq y \end{matrix}.$$

Note: $G_{y_n}(x,y) = \Phi_{y_n}(y-x) - \Phi_{y_n}(y-\tilde{x})$

$$= -\frac{1}{n\alpha(n)} \left[\frac{y_n - x_n}{|y-x|^n} - \frac{y_n + x_n}{|y-\tilde{x}|^n} \right]$$

$\Rightarrow$ On $\partial \mathbb{R}^n_+$,

$$\frac{\partial G}{\partial \nu}(x,y) = -G_{y_n}(x,y) = \underbrace{-\frac{2x_n}{n\alpha(n)} \frac{1}{|x-y|^n}}_{} =: K(x,y)$$

POISSON KERNEL FOR $\mathbb{R}^n_+$.

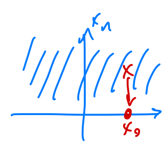
$\Rightarrow$ Assume $g \in C(\mathbb{R}^{n-1}) \cap L^\infty(\mathbb{R}^{n-1})$ and set

$$\boxed{u(x) = \frac{2x_n}{n\alpha(n)} \int_{\partial \mathbb{R}^n_+} \frac{g(y)}{|x-y|^n}\, dy, \quad x \in \mathbb{R}^n_+;}$$

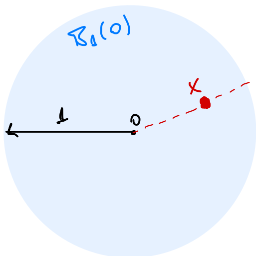
↳ $\Delta u = 0 \equiv f$
⇓
$\int_{\mathbb{R}^n_+} G(x,y) f(y)\, dy = 0$

then

- $u \in C^\infty(\mathbb{R}^n_+) \cap L^\infty(\mathbb{R}^n_+)$
- $\Delta u = 0$ in $\mathbb{R}^n_+$
- $\lim\limits_{\substack{x \to x_0 \\ x \in \mathbb{R}^n_+}} u(x_0) = g(x_0)$ for each $x_0 \in \partial \mathbb{R}^n_+$



(3) Unit Ball $B_1(0) \subset \mathbb{R}^n$



$\tilde{x} = x/|x|^2 \leftarrow$ dual w.r.t. $\partial B_1(0) = S^{n-1}$

$x \mapsto \tilde{x} \equiv$ inversion through S^{n-1}

Fix $x \in B_1(0)$. Need to find corrector

$$\begin{cases} \Delta \phi^x = 0 & \text{in } B_1(0) \\ \phi^x = \Phi(y-x) & \text{on } \partial B_1(0) = S^{n-1} \end{cases}$$

Green's Function: $G(x,y) = \Phi(y-x) - \phi^x(y)$.

$(n \geq 3)$ $y \mapsto \Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$

$\Rightarrow y \mapsto |x|^{2-n} \Phi(y-\tilde{x})$ is harmonic for $y \neq \tilde{x}$

$\Rightarrow \phi^x(y) := \Phi(|x|(y-\tilde{x}))$ is harmonic on $B_1(0)$

and if $y \in S^{n-1}$, $x \neq 0$,

$$|x|^2|y-\tilde{x}|^2 = |x|^2\left(|y|^2 - \frac{2y\cdot x}{|x|^2} + \frac{1}{|x|^2}\right)$$
$$= |x|^2 - 2y\cdot x + 1$$
$$= |x-y|^2$$

Thus, $(|x|\,|y-\tilde{x}|)^{-(n-2)} = |x-y|^{-(n-2)}$

i.e., $\phi^x(y) = \Phi(y-x)$, $y \in S^{n-1}$, as needed.

$\Rightarrow$ Green's Function for $B_1(0)$ is:

$$G(x,y) := \Phi(y-x) - \Phi(|x|(y-\tilde{x})), \quad x,y \in B_1(0),\ x \neq y.$$

(also valid for $n=2$)

$\rightsquigarrow$ Want to solve Laplace $\begin{cases} \Delta u = 0 & \text{on } B_1(0) \\ u = g & \text{on } \partial B_1(0) = S^{n-1} \end{cases}$

$$\rightsquigarrow u(x) = -\int_{\partial B_1(0)} g(y) \frac{\partial G}{\partial \nu}(x,y) \, dS(y)$$

$$\frac{\partial G}{\partial \nu}(x,y) = \sum_{i=1}^{n} y_i \, G_{y_i}(x,y)$$

for $y_i \in \partial B_1(0)$

$$G_{y_i}(x,y) = \Phi_{y_i}(y-x) - \left[\Phi(|x|(y-\tilde{x}))\right]_{y_i}$$

$$= \frac{1}{n\alpha(n)} \frac{x_i - y_i}{|x-y|^n} - \left[-\frac{1}{n\alpha(n)} \frac{y_i|x|^2 - x_i}{|x-y|^n} \right]$$

$$= -\frac{(1-|x|^2)}{n\alpha(n)|x-y|^n}.$$

$$\Rightarrow \boxed{u(x) = \frac{1-|x|^2}{n\alpha(n)} \int_{\partial B_1(0)} \frac{g(y)}{|x-y|^n} \, dy}$$

More generally, for $\begin{cases} \Delta u = 0 & \text{in } B_r(0), \quad r > 0 \\ u = g & \text{on } \partial B_r(0) \end{cases}$,

$\tilde{u}(x) = u(rx)$ with $\tilde{g}(x) = g(rx)$.

Poisson kernel for $B_r(0)$: $K(x,y) = \dfrac{r^2 - |x|^2}{n\alpha(n) r} \dfrac{1}{|x-y|^n}$, $\quad x \in B_r(0)$, $y \in \partial B_r(0)$

$$\rightsquigarrow u(x) = \frac{r^2 - |x|^2}{n\alpha(n) r} \int_{\partial B_r(0)} \frac{g(y)}{|x-y|^n} \, dS(y)$$

If $g \in C(\partial B_r(0))$,
- $u \in C^\infty(B_r(0))$
- $\Delta u = 0$ in $B_r(0)$
- $\lim_{\substack{x \to x_0 \\ x \in B_r(0)}} u(x) = u(x_0) \quad \forall x_0 \in \partial B_r(0)$.

LECTURE 4

Apr 1st 2026

HEAT EQUATION (Evans 2.3)

$t>0$, $x \in U \underset{\text{open}}{\subset} \mathbb{R}^n$,
$u : \bar{U} \times [0,\infty) \to \mathbb{R}$
$f : \bar{U} \times [0,\infty) \to \mathbb{R}$.

$$u_t - \Delta u = 0 \qquad \qquad u_t - \Delta u = f$$

(homogeneous) (non-homogeneous)

Physical Interpretation: Diffusion of some density u (e.g., chemical concentration, heat, etc). Rate of change out of $V \subset U$:

$$\frac{d}{dt}\int_V u\, dx = -\int_{\partial V} F \cdot \nu \, dS, \qquad F \equiv \text{flux density}.$$

i.e., $u_t = -\operatorname{div} F$.

- FUNDAMENTAL SOLUTION: If $(x,t) \mapsto u(x,t)$ solves (HEAT) then so does $(x,t) \mapsto u(\lambda x, \lambda^2 t)$ $\forall \lambda \in \mathbb{R}$.

$\Rightarrow$ Ansatz: $u(x,t) = v\left(\frac{r^2}{t}\right) \equiv v\left(\frac{|x|^2}{t}\right)$.

$(\lambda = 1/t)$ Works but tedious. Faster approach is to consider $u(x,t) = \frac{1}{t^\alpha} v\left(\frac{x}{t^\beta}\right)$

$\Rightarrow$ $u(x,t) = \lambda^\alpha u(\lambda^\beta x, \lambda t)$ $\forall \lambda > 0$, $t>0$

$\Rightarrow$ $\alpha v + \frac{1}{2} y \cdot Dv + \Delta v = 0$

$\Rightarrow$ If $v(y) = w(|y|)$ is radial,

$$\alpha w + \frac{1}{2} r w' + w'' + \frac{n-1}{r} w' = 0 .$$

$$\vdots$$

$$w = b e^{-r^2/4} .$$

MEMORIZE. YOUR BRAIN IS TOO SLOW.

IMPORTANT

DEFINITION: Fundamental solⁿ of the Heat Equation

$$\Phi(x,t) := \begin{cases} \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} , & x \in \mathbb{R}^n ,\ t > 0 \\ 0 , & x \in \mathbb{R}^n ,\ t < 0 \end{cases} .$$

LEMMA: $\int_{\mathbb{R}^n} \Phi(x,t)\, dx = 1$

Pf:

$$\int_{\mathbb{R}^n} \Phi(x,t)\, dx = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}} e^{-|x|^2/4t}\, dx$$

$z := 2\sqrt{t}\, x \longrightarrow$
$$= \frac{1}{\pi^{n/2}} \int_{\mathbb{R}^n} e^{-|z|^2}\, dz$$

$$= \frac{1}{\pi^{n/2}} \prod_{i=1}^{n} \int_{\mathbb{R}} e^{-z_i^2}\, dz_i = 1 .$$

$\square$

! IMPORTANT ! MEMORIZE. YOU ARE SLOW.

- <u>INITIAL VALUE PROBLEM</u>: Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, and consider

$$\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}.$$

Define

$$u(x,t) := (\Phi * g)(x,t) = \int_{\mathbb{R}^n} \Phi(x-y,t)\, g(y)\, dy$$

$$= \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/4t}\, g(y)\, dy.$$

Then

(i) $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$

(ii) $u_t(x,t) - \Delta u(x,t) = 0$ for $x \in \mathbb{R}^n$, $t > 0$

(iii) $\displaystyle\lim_{\substack{(x,t)\to(x_0,0) \\ x\in\mathbb{R}^n,\ t>0}} u(x,t) = g(x_0)$ for each $x_0 \in \mathbb{R}^n$.

<u>REMARK</u>:
$$\begin{cases} \Phi_t - \Delta\Phi = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ \Phi = \delta_0 & \text{on } \mathbb{R}^n \times \{0\} \end{cases}.$$

LECTURE 5

Apr 7, 2026

HEAT EQUATION (ctd)

$u : \mathbb{R}_+ \times U \to \mathbb{R}$, $\quad U \subset \mathbb{R}^n$ open

$$\partial_t u - \Delta u = 0 \,, \quad x \in \mathbb{R}^n, \ t > 0 .$$

<u>Fundamental Solution</u>: $\displaystyle \Phi(t,x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t} \, \mathbb{1}_{\{t>0\}}$.

<u>Initial Value Problem</u>: $\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^n \\ u = g & \text{on } \{0\} \times \mathbb{R}^n . \end{cases}$

<u>THM</u>: Let $g \in C(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ then, for $t > 0$, $x \in \mathbb{R}^n$,

$$u(t,x) := (\Phi * g)(t,x) = \int_{\mathbb{R}^n} \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} g(y)\, dy$$

is a $C^\infty(\mathbb{R}_+ \times \mathbb{R}^n)$ solution to $\begin{cases} u_t - \Delta u = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^n \\ g = 0 & \text{on } \{0\} \times \mathbb{R}^n \end{cases}$

such that

$$\lim_{\substack{(t,x) \to (0,x_0) \\ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n}} u(t,x) = g(x_0) \quad \forall x_0 \in \mathbb{R}^n .$$

<u>Pf</u>: smoothness of u and solution to (HEAT) is clear.

Main question is initial condition limit.

Idea: Use continuity of g at each x_0.

$$\left| u(t,x) - 1\cdot g(x_0) \right| = \left| \int_{\mathbb{R}^n} \Phi(t, x-y)\left[g(y) - g(x_0)\right] dy \right|$$

$$\int_{\mathbb{R}} \Phi(t,x)\, dx = 1$$

split into region near x_0 and another one far from x_0.

- Contribution of $g(y) - g(x_0)$ is bdd.

$\square$

INHOMOGENEOUS INITIAL VALUE PROBLEM

$$(*) \begin{cases} u_t - \Delta u = f, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n \\ u(0,x) = 0, & x \in \mathbb{R}^n \end{cases}.$$

Strategy: Fund. sol$^{\underline{n}}$ and convolutions $\longrightarrow$ DUHAMEL

STEP 1: Set $w(x,t,s) = \int_{\mathbb{R}^n} \Phi(t-s, x-y)\, f(s,y)\, dy$.

Note. w solves, for each $s \in \mathbb{R}$

$$\begin{cases} \partial_t w - \Delta w = 0 & \text{in } (s,\infty) \times \mathbb{R}^n \\ u = f(s,\cdot) & \text{on } \{s\} \times \mathbb{R}^n \end{cases}.$$

$$u(t,x) = \int_0^t w(x,t,s)\, ds$$

from before $\rightarrow$ $= \displaystyle\int_0^t \frac{1}{[4\pi(t-s)]^{n/2}} \int_{\mathbb{R}^n} e^{-|x-y|^2/(4(t-s))} f(s,y)\, dy\, ds$

$$\Longrightarrow \boxed{u(t,x) = \int_0^t (\Phi * f)(t-s, y)\, ds .}$$

Thm 2 Evans pg 50 for proof on how to differentiate under the integral sign

Formally: using $\dfrac{d}{dt}\left[\int_0^t G(t,s)\, ds\right] = G(t,t) + \int_0^t \partial_t G(t,s)\, ds.$

we have (omit x-dependency)

$$u_t = w(t,t) + \int_0^t \partial_t w(t,s)\, ds = f(t) + \int_0^t \Delta_x w(t,s)\, ds$$

$$= f(t) + \Delta_x \int_0^t w(t,s)\, ds$$

$$= f(t) + \Delta u \longrightarrow u_t - \Delta u = f.$$

UPSHOT: Solution to **! IMPORTANT !**

$$\begin{cases} u_t - \Delta u = f & \text{in } \mathbb{R}_+ \times \mathbb{R}^n \\ u = g & \text{on } \{0\} \times \mathbb{R}^n \end{cases}$$

$$u(t,x) = (\Phi(t) * g)(x) + \int_0^t \big(\Phi(t-s) * f(s)\big)(x)\, ds$$

$$= \int_{\mathbb{R}} \Phi(t, x-y)\, g(y)\, dy$$

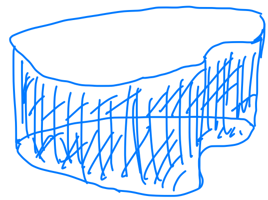
$$+ \int_0^t \int_{\mathbb{R}} \Phi(t-s, x-y)\, f(s,y)\, dy\, ds$$

MEAN VALUE FORMULAS FOR HEAT EQUATION :

- $U \subset \mathbb{R}^n$ open and bounded

$U_T := U \times (0, T]$, $T > 0$ } PARABOLIC CYLINDER

$\Gamma_T := \overline{U}_T \setminus U_T$ } PARABOLIC BOUNDARY =



← Γ_T is the sides & bottom BUT NOT THE TOP

Motivations : MVP for harmonic fcts ⟷ average over balls

$\Delta u = 0$ ⇝ radially symmetric

{ level sets of fundamental sol^n of $\Delta u = 0$

Look at level sets of the fund. sol^n to heat eq. $\Phi(t-s, x-y)$

* Level sets for fund. sol^n. of heat equation : $r > 0$,

$$E(x,t,r) = \left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} : s \leq t, \ \Phi(x-y, t-s) \geq \frac{1}{r^n} \right\}.$$

2 derivatives in space

PROPOSITION : If $u \in C^2_1(U_T) := \{ u \in C^0(U_T) : D_x u, D_x^2 u, u_t \in C^0(U_T) \}$

(Evans Ch 2)

1 derivative in time

then

$$u(t,x) = \frac{1}{4r^n} \iint_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds .$$

$\forall (x,t,r)$ w/ $E(x,t,r) \subset U_T$.

Sketch Pf: look at $\phi(r) := \frac{1}{r^n} \iint_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds$

and differentiate

$\square$

COROLLARIES: Let $U \subset \mathbb{R}^n$ open & bdd.

⚠ IMPORTANT ⚠

MAXIMUM PRINCIPLE FOR U_T (HEAT): Let $u \in C^2_1(U_T) \cap C(\overline{U_T})$ solve $u_t - \Delta u = 0$. Then

1. $\max_{\overline{U_T}} u = \max_{\Gamma_T} u$

$U \times (0,T] = \overline{U_T}$, $\Gamma_T \overset{\text{def}}{=} \overline{U_T} \setminus U_T$

and

2. If U is connected and $\exists (t_0, x_0) \in U_T$ s.t. $u(t_0, x_0) = \max_{\overline{U_T}} u$, then u is constant in $\overline{U_T}$.

PROPOSITION: U connected, $u \in C^2_1(U_T) \cap C(\overline{U_T})$ solves

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = 0 & \text{on } \partial U \times [0,T] \\ u = g & \text{on } \partial U \times \{0\} \end{cases}$$

with $g \geq 0$, then, if $g \not\equiv 0$, we have $u > 0$ on all of U_T.

PROPOSITION: (Uniqueness of sol$^{\underline{n}}$ to heat eq.) Let $g \in C(\Gamma_T)$, $f \in C(U_T)$. Then, there is <u>at</u> <u>most</u> <u>one</u> <u>sol^n</u> $u \in C^2_1(U_T) \cap C(\overline{U_T})$ to

$$\begin{cases} u_t - \Delta u = f & \text{on } U_T \\ u = g & \text{in } \Gamma_T \end{cases}.$$

QUESTION: How to extend these results to unbounded domains, say $U = \mathbb{R}^n$?

MAX. PRINCIPLE FOR HEAT EQUATION ON $\mathbb{R}^n$: Let $u \in C^2_1(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$ be a sol$^{\underline{n}}$ to

$$\begin{cases} u_t - \Delta u = 0 & \text{on } \mathbb{R}^n \times (0,T] \\ u = g & \text{in } \mathbb{R}^n \times \{0\} \end{cases}.$$

<u>If</u> $\exists\, a, A > 0$ s.t. $\underbrace{u(t,x) \leq A e^{a|x|^2}}_{\text{growth condition}}$ for all $x \in \mathbb{R}^n$, $t \in [0,T]$, <u>then</u>

$$\sup_{\substack{x \in \mathbb{R}^n \\ 0 \leq t \leq T}} u(t,x) = \sup_{x \in \mathbb{R}^n} g(x).$$

Pf: Suppose $T < \frac{1}{4a}$. Choose $\varepsilon > 0$ s.t. $T < \frac{1}{4a} - \varepsilon$. Fix $y \in \mathbb{R}^n$, $\mu > 0$, and define

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{1/4}} e^{|x-y|^2/[4(T+\varepsilon-t)]}$$

Clearly: $v_t - \Delta v = 0$ on $\mathbb{R}^n \times (0,T]$. Apply Max Princ. to v.

Fix $r > 0$ and set $U := \operatorname{int}(B_r(y))$, $U_T := U \times (0,T]$, $\Gamma_T := (\partial B_r(y) \times [0,T)) \cup (\overline{B_r(y)} \times \{0\})$. By the max. principle for heat eq. on U_T,

$$\max_{\overline{U}_T} v = \max_{\Gamma_T} v$$

But, $T + \varepsilon < \frac{1}{4a}$. So, $\frac{1}{4(T+\varepsilon)} > a$ and set $\gamma := \frac{1}{4(T+\varepsilon)} - a > 0$

Then $\forall x$ s.t. $|x-y| = r$, $0 \leq t \leq T$, we have

$$v(x,t) = u(x,t) - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{|x-y|^2/(4(T+\varepsilon-t))}$$

$$\leq A e^{a|x|^2} - \frac{\mu}{(T+\varepsilon-t)^{n/2}} e^{r^2/(4(T+\varepsilon-t))}$$

$$\leq A e^{a(|y|+r)^2} - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{r^2/(4(T+\varepsilon))}$$

$$\leq \underbrace{A e^{a(|y|+r)^2} - \mu \left(4(\gamma+a)\right)^{n/2} e^{(\gamma+a) r^2}}_{\text{gets small as } r \text{ grows}}$$

Fix $M = \sup_{x \in \mathbb{R}^n} g(x)$ and let r be large enough so that

$$v(x,t) \leq M \quad \text{for } |x-y| = r, \ 0 \leq t \leq T.$$

Then,
$$\begin{aligned}
v(y,t) &\leq \max_{\overline{U}_T} v \\
&= \max_{\Gamma_T} v \\
&= \max \left\{ \max_{\substack{|x-y|=r \\ 0 \leq t \leq T}} v(t,x), \ \max_{|x-y|<r} v(0,x) \right\} \\
&\leq \max \left\{ M, \ \max_x v(x,0) \right\}
\end{aligned}$$

Now, for $x \in \mathbb{R}^n$, by definition,

$$v(x,0) = u(x,0) - \frac{\mu}{(T+\varepsilon)^{n/2}} e^{|x-y|^2/(4(T+\varepsilon))}$$

$$\leq u(x,0) = g(x).$$

$$= \max_x g(x).$$

μ arbitrary, take $\mu \to 0$ and so

$$u(x,t) \le \sup_x g(x) \quad \forall x \in \mathbb{R}^n, \; 0 \le t \le T, \; T < 1/4a.$$

Then, repeat for all times $[0,T_1], [T_1, 2T_1], \ldots$ w/ $T_1 := \frac{1}{8a}$.

$\square$

LECTURE 6

Apr 9, 2026

<u>RECALL</u>: MVP & Max. Principle for $u_t - \Delta u = 0$ (Heat Eq.)

* <u>MVP</u>: $$u(x,t) = \frac{1}{4r^n} \iint_{E(x,t,r)} u(y,s) \frac{|x-y|^2}{(t-s)^2} \, dy \, ds.$$

HEAT BALL: $E(x,t,r) := \left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} : s \le t, \; \Phi(x-y, t-s) \ge \frac{1}{r^n} \right\}$

(level sets of fund. sol$^{\underline{n}}$ of Heat eq.)

$\forall x, t, r$ s.t. $E(x,t,r) \subset U_T$, $U \subset \mathbb{R}^n$ open & bdd

$U_T := U \times (0,T]$, $\underline{\Gamma_T} := \overline{U_T} \setminus U_T$

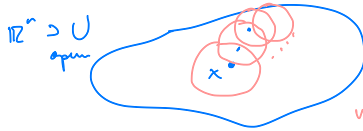
(parabolic boundary)

* <u>MAX. PRINCIPLE FOR HEAT</u>

(i) <u>BOUNDED</u> <u>DOMAIN</u> $U \subset \mathbb{R}^n$: Suppose $\exists (t,x) \in U_T$ s.t.

$u(t,x) = \max_{\overline{U_T}} u$, then u is constant on $\overline{U_T}$.

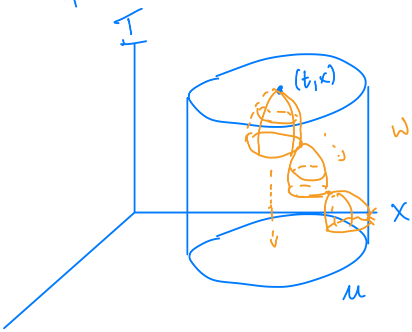
Analogy w/ $\Delta u = 0$: MVP $\quad u(x) = \fint_{B_r(x)} u(y)\,dy$



Walk balls towards ∂U

$\max_{\bar{U}} u = \max_{\partial U} u$

Heat eq. $u_t - \Delta u = 0$: MVP



walk heat balls towards the parabolic bdry $\Longrightarrow$ Max. Principle

$\max_{\bar{U}_T} u = \max_{\Gamma_T} u$

COROLLARY: (Unique solⁿ to IVP in $\mathbb{R}^n$) Let
(Evans Thm 7)

$g \in C(\mathbb{R}^n)$ and $f \in C(\mathbb{R}^n \times [0,T])$, then we have at most one solution $u \in C^2(\mathbb{R}^n \times (0,T]) \cap C(\mathbb{R}^n \times [0,T])$

$$(\mathrm{IVP}) \begin{cases} u_t - \Delta u = f & \text{on } \mathbb{R}^n \times (0,T] \\ u = g & \text{on } \mathbb{R}^n \times \{0\} \end{cases}$$

such that $|u(x,t)| \le A e^{a|x|^2}$ for some $A, a > 0$.

growth condⁿ needed for the Max Principle in all of $\mathbb{R}^n$.] Without this, no more uniqueness (e.g., non-physical solⁿˢ)

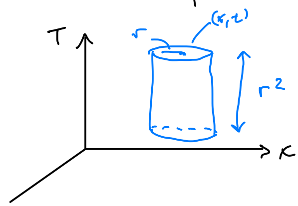
Pf: u_1, u_2 solⁿˢ, then by max principle in $\mathbb{R}^n$ to $w_1 = u_1 - u_2$ and $w_2 = -(u_1 - u_2)$ we get uniqueness. $\square$

Thm: If $u \in C_1^2(U_T)$ solves $u_t - \Delta u = 0$ in U_T, then $u \in C^\infty(U_T)$. ← even if u not diff'able on Γ_T

$\Leftrightarrow$ HEAT (diffusion) EQUATION is SMOOTHING
(even smoothes discontinuous initial conditions)

Pf: IDEA: Smoothness of convolutions and Hadamard finite part integrals for cutoffs and uniqueness.

Define cylinders $C(x,t,r)$ in space-time with center-top point (x,t), radius r, and height r^2:

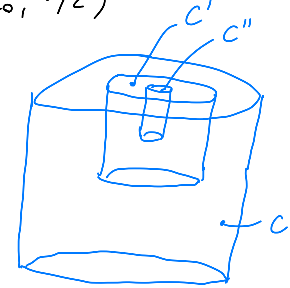


$$C(x,t,r) := \left\{ (y,s) \in \mathbb{R}^n \times \mathbb{R} : \begin{array}{l} |y-x| \leq r, \\ t - r^2 \leq s \leq t \end{array} \right\}$$

Step 1: local representation formula for $u \in C^\infty(U_T)$.

Fix $(x_0,t_0) \in U_T$. Choose $r > 0$ s.t. $C := C(x_0,t_0,r) \subset U_T$

Define $C' := C(x_0,t_0,3r/4)$, $C'' := C(x_0,t_0,r/2)$



Let ξ be a $C_c^\infty(U_T)$ cutoff fct

- $\xi \equiv 1$ on C'
- $0 \leq \xi(x,t) \leq 1 \quad \forall x,t$
- $\xi \equiv 0$ outside C in strip $\mathbb{R}^n \times [0,t_0]$ and $\equiv 0$ near parabolic bdry of C.

Set $v(x,t) := \xi(x,t)\, u(x,t)$, $(x,t) \in \mathbb{R}^n \times [0,t_0]$.

$\Rightarrow$ $v(x,0) = 0$ $\forall x \in \mathbb{R}^n$ and

- $v_t = \xi_t u + \xi u_t = \xi_t u + \xi \Delta u$
- $\Delta v = \sum_i \partial_i (\xi u) = \sum_i \partial_i (\xi_i u + \xi u_i)$

$$= \sum_i \xi_{ii} u + 2 \xi_i u_i + \xi u_{ii}$$

$$= (\Delta \xi) u + 2 D\xi \cdot Du + \xi \Delta u .$$

i.e., $v_t - \Delta v = \xi_t u - 2 D\xi \cdot Du - u \Delta \xi =: \tilde{f}$

Use uniqueness to find a formula for v: set

$$\tilde{v}(x,t) := \int_0^t \int_{\mathbb{R}^n} \Phi(x-y, t-s)\, \tilde{f}(y-s)\, dy\, ds$$

$\Rightarrow$ $\tilde{v}$ solves $\begin{cases} \tilde{v}_t - \Delta \tilde{v} = \tilde{f} \\ \tilde{v}|_{t=0} = 0 \end{cases}$.

Moreover, v and $\tilde{v}$ are s.t. $\underline{|v| \le A,\ |\tilde{v}| \le A}$ for some $A>0$

$\Rightarrow$ $v = \tilde{v}$ by Max Principle's uniqueness (in $\mathbb{R}^n \times [0,t_0]$). the cylinders are <u>bounded</u> domains

Thus, by the support of ξ,

$$u(x,t) = \iint_C \Phi(x-y, t-s) \left[(\xi_s(y,s) - \Delta \xi(y,s))\, u(y,s) \right.$$

$$- 2 D\xi(y,s) \cdot Du(y,s) \Big] \, dy \, ds$$

by parts

$$= \iint_C \Big[\Phi(x-y, t-s)\big(\xi_s(y,s) + \Delta\xi(y,s)\big) + 2 D_y \Phi(x-y, t-s) \cdot D\xi(y,s) \Big] u(y,s) \, dy \, ds$$

STEP 2: This formula remains valid for $(x,t) \in C''$ even if u is only as in assumptions. To see this, let η_ε be the usual seq. of mollifiers to find a formula for $u_\varepsilon := \eta_\varepsilon * u$ and take $\varepsilon \to 0$.

STEP 3: So, for $(x,t) \in C''$,

$$u(x,t) = \iint_C K(x,t,y,s) \, u(y,s) \, dy \, ds,$$

for some kernel K s.t. $K(x,t,y,s) = 0 \;\; \forall (y,s) \in C'$ $\left(\xi = 1 \text{ on } C'\right)$ and $(y,s) \mapsto K(x,t,y,s)$ is smooth in $C \setminus C'$. Thus, can differentiate using LDCT to get $u \in C^\infty(C'')$.

$\square$

THM: $\forall \, k, l \in \mathbb{N} \cup \{0\}$, $\exists \, C_{k,l} > 0$ s.t.

(Evans pg 62)

$$\max_{(y,s) \in C(x,t,r/2)} \left| D_x^k D_t^l u(y,s) \right| \le \frac{C_{k,l}}{r^{k+2l+n+2}} \|u\|_{L^1(C(x,t,r))}$$

for all cylinders $C(x,t,r/2) \subset C(x,t,r) \subset U_T$ and all sol$^{\underline{ns}}$ to heat equation u in U_T.

ENERGY METHODS FOR HEAT EQUATION

(Integration by parts instead)

$\mapsto$ without maximum principle :)

THM: (Uniqueness Revisited) There is at most one sol$^{\underline{n}}$ to

$$(\text{IVP}) \begin{cases} u_t - \Delta u = f & \text{in } U_T \\ u = g & \text{in } \Gamma_T \end{cases}$$

with $u \in C_1^2(\overline{U_T})$, $U \subset \mathbb{R}^n$ bounded open and $\partial U \in C^1$ and $T > 0$.

Pf: Let $u, \tilde{u}$ be sol$^{\underline{ns}}$ to (IVP). Then $w := u - \tilde{u}$ solves

$$\begin{cases} w_t - \Delta w = 0 & \text{in } U_T \\ w = 0 & \text{on } \Gamma_T \end{cases}$$

Set $e(t) := \int_U w^2 \, dx$, then, formally,

$$\frac{d}{dt} e(t) = 2 \int_U w w_t \, dx = 2 \int_U w \Delta w \, dx$$

Green's Identity $\longrightarrow$

$$= -2 \int_U Dw \cdot Dw \, dx$$

$$= -2 \int_U \underbrace{|Dw|^2}_{\geq 0} \, dx$$

≤ 0; i.e., $t \mapsto e(t)$ decreases on $[0, T]$.

i.e., $0 \leq e(t) \leq e(0) = 0$ for $t \in [0, T]$

i.e., $e(t) = 0$ on $[0,T]$

i.e., $w \equiv 0$ on U_T

i.e., $u = \tilde{u}$ on U_T.

$\square$

THM: (Backwards Uniqueness) Suppose $u, \tilde{u} \in C_1^2(\overline{U_T})$ are two sol^{ns} to

$$\begin{cases} u_t - \Delta u = 0 & \text{in } U_T \\ u = g & \text{on } \partial U \times [0,T] \end{cases}$$

for some $T > 0$. Then, if $u(x,T) = \tilde{u}(x,T)$ $\forall x \in U$ then $u = \tilde{u}$ in U_T. (i.e., if sol^{ns} agree at some point in spacetime, then they must be the same on all space for the past)

Pf: $w = u - \tilde{u}$, $e(t) := \int_U w(x,t)^2 \, dx$.

$$\frac{d}{dt} e(t) = -2 \int_U |Dw|^2 \, dx \leq 0 \quad \text{as before,}$$

$$\frac{d^2}{dt^2} e(t) = -4 \int_U Dw_t \cdot Dw \, dx$$

$$= 4 \int_U (\Delta w) w_t \, dx = 4 \int_U (\Delta w)^2 \, dx \geq 0$$

i.e.,

$$e'(t) \overset{\text{Cauchy-Schwarz}}{\leq} (e''(t))^{1/2} (e(t))^{1/2}$$

i.e., $\boxed{\dot{e}^2 \leq \ddot{e}\, e}$ $\Longrightarrow$ $e(t) \equiv 0$ $\forall\, 0 \leq t \leq T$.

$\square$

LECTURE 7

Apr 14, 2026

→ different from Lapace on $\mathbb{R}^n$ b/c the first summand of Laplacian is now negative!

$u_{00} + \sum_{i=1}^{n-1} u_{ii}$

WAVE EQUATION

$$u_{tt} - \Delta u = 0 \qquad \text{(hyperbolic)}$$

Solve for odd dimensions then get the other ones by descending.

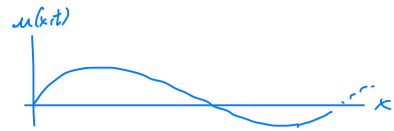
$$u(t,x) : [0,\infty) \times \overline{U} \longrightarrow \mathbb{R}, \quad U \subset \mathbb{R}^n \text{ open bdd}$$

$$\Box u := u_{tt} - \Delta u \qquad \text{(D'Alembertian)}$$

1 – DIMENSION $(n=1)$

$$u_{tt} - u_{xx} = 0$$

→ vibrating string
$u(x,t)$ = displacement of x at time t



SOLUTION TO (Initial value problem)

$$\begin{cases} u_{tt} - u_{xx} = 0 & \text{for } x \in \mathbb{R},\ t>0 \\ u = g & x \in \mathbb{R},\ t=0 \\ u_t = h & x \in \mathbb{R},\ t=0 \end{cases}$$

- Factorize

$$0 = u_{tt} - u_{xx} = (\partial_t + \partial_x)(\partial_t - \partial_x) u$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2}\left[g(x+t) - g(x-t) \right] + \frac{1}{2}\int_{x-t}^{x+t} h(y)\, dy$$

(D'ALEMBERT'S FORMULA)

$\Rightarrow$ THM: If $g \in C^2(\mathbb{R})$, $h \in C^1(\mathbb{R})$, then the solution to the IVP wave equation above is u given by d'Alembert, and

1) $u \in C^2([0,\infty) \times \mathbb{R})$

2) $u_{tt} - u_{xx} = 0$ on $(0,\infty) \times \mathbb{R}$

3) $\forall x_0 \in \mathbb{R}$, $\displaystyle\lim_{\substack{(x,t) \to (x_0,0) \\ t>0}} u(t,x) = g(x_0)$ and $\displaystyle\lim_{\substack{(x,t)\to(x_0,0) \\ t>0}} u_t(t,x) = h(x_0)$.

REMARK: If $h \equiv 0$, then

$$u(t,x) = \frac{1}{2}\left[g(x+t) + g(x-t)\right]$$

superposition of two waves w/ speeds ± 1.

3-DIMENSIONAL ($n=3$)

$$\begin{cases} w_{tt} - \Delta w = 0 & \text{in } (0,\infty) \times \mathbb{R}^3 \\ w\big|_{t=0} = g, \quad w_t\big|_{t=0} = h & \text{on } \mathbb{R}^3 \end{cases}$$

IDEA: Use $n=1$ case for radially symmetric solutions.

- For $n \geq 2$, $x \in \mathbb{R}^n$, $t > 0$, $r > 0$,

$$U(x,r,t) := \fint_{\partial B_r(x)} u(t,y)\, dS(y) \qquad \text{(SPHERICAL AVERAGES)}$$

$$G(x,r) := \fint_{\partial B_r(x)} g(y)\, dS(y)$$

$$H(x,r) := \fint_{\partial B_r(x)} h(y)\, dS(y)$$

- Note that these averages satisfy:

radially symmetric part of $-\Delta$

$$\begin{cases} U_{tt} - U_{rr} - \left(\frac{n-1}{r}\right) U_r = 0 & \text{on } (0,\infty) \times \mathbb{R}_+ \\ U|_{t=0} = G, \quad U_t|_{t=0} = H & \text{on } \mathbb{R}_+ \end{cases}$$

Idea: Compute $\partial_r U = \partial_r \left(\fint_{\partial B_1(0)} u(x+rz,t)\, dS(z) \right)$

$$= \fint_{\partial B_r(x)} (Du)(y) \cdot \nu(y)\, dS(y)$$

Green's 1st Identity $\rightsquigarrow$ $= \frac{r}{n} \fint_{B_r(x)} \Delta u \, dy$

- For $n = 3$, set $\tilde{U} := rU$, $\tilde{G} := rG$, $\tilde{H} := rH$. Then

$$\begin{cases} \tilde{U}_{tt} - \tilde{U}_{rr} = 0 & \text{on } (0,\infty) \times \mathbb{R}_+ \\ \tilde{U}|_{t=0} = \tilde{G}, \quad \tilde{U}_t|_{t=0} = \tilde{H} & \text{on } \mathbb{R}_+ \end{cases}$$

$\leftarrow$ Not full line as before

$\Downarrow$

odd reflect

- Moreover, $\tilde{G}_{rr}\big|_{x=0}$ -
- So, using extension by oddness,

$$\tilde{U}(x,t,r) = \frac{1}{2}\left[\tilde{G}(r+t) - \tilde{G}(t-r)\right] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\,dy \quad \text{for } 0 \le r \le t.$$

- So, going back definitions, $U(x,t,r) \xrightarrow{r\downarrow 0} u(x,t)$ at least if u is smooth enough.

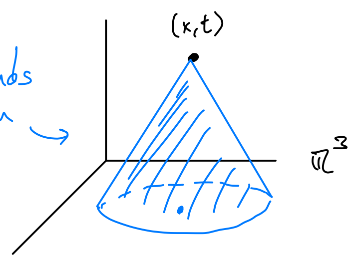
- Upshot: KIRCHOFF'S FORMULA (3-d wave)

$$u(t,x) = \fint_{\partial B_t(x)} t h(y) + g(y) + (Dg)(y)\cdot(y-x)\, dS(y)$$

REMARKS:

- Kirchoff's formula only depends on g, h evaluated on $B_t(x)$ in $\mathbb{R}^3$.

finite speed of propagation $\Leftarrow$ solution only depends on the conditions in this cone



- Same approach doesn't work in $\mathbb{R}^2$. We need to construct sol$^{\underline{ns}}$ on $\mathbb{R}^2$ by finding "x_3 = free" sol$^{\underline{n}}$ in $\mathbb{R}^3$.

LECTURE 8

Apr 21 2026

WAVE IN DIM = 2 (using $n=3$'s Kirchoff's)

dim = 2

THM: If u is smooth sol^{n} to $\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}^{2} \times (0,\infty) \\ u|_{t=0} = g, \ u_t|_{t=0} = h & \text{on } \mathbb{R}^2 \end{cases}$

then

$$u(x,t) = \frac{1}{2} \fint_{B_t(x)} \frac{t g(y) + t^2 h(y) + t (Dg)(y) \cdot (y-x)}{\left[t^2 - |y-x|^2 \right]^{1/2}} \, dy$$

depends on interior of ball

REMARK: Idea is always to go from odd-dims. to 1 lower even dimension:

dim $3 \longrightarrow 2$

dim $5 \longrightarrow 4$

dim $7 \longrightarrow 6$

$\vdots$

INHOMOGENEOUS PROBLEM: Duhamel's formula

$$\begin{cases} u_{tt} - \Delta u = f, & \mathbb{R}^2 \times (0,\infty) \\ u|_{t=0} = g, \ u_t|_{t=0} = h, & \mathbb{R}^2 \end{cases}$$

$$u(x,t) = \int_0^t \tilde{u}(x,t,s) \, ds, \quad \text{where } \tilde{u} \text{ sol}^{n} \text{ to: } \begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} = 0, & t>0 \\ \tilde{u}|_{t=0} = 0, \ \tilde{u}_t|_{t=0} = f(\cdot, s) \end{cases}$$

__FOURIER TRANSFORM__: $u \in L^1(\mathbb{R}^n)$, $n \geq 1$,

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} u(x)\, dx$$

$$\check{u}(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} u(\xi)\, d\xi .$$

PARSEVAL IDENTITY

$$u \in L^1 \cap L^2 \Rightarrow u \in L^2 \text{ w/ } \boxed{\|\hat{u}\|_{L^2} = \|u\|_{L^2}} .$$

__REMARK__: In the Schwarz space, $\mathcal{F}$ is isomorphism.

__REMARK__: $u, v \in L^2(\mathbb{R}^n)$,

1) $\widehat{(D^\alpha u)}(\xi) = (i\xi)^\alpha \hat{u}(\xi)$

2) $\displaystyle\int_{\mathbb{R}^n} u \bar{v}\, dx = \int_{\mathbb{R}} \hat{u}\, \overline{\hat{v}}\, d\xi \iff \langle u, v \rangle = \langle \hat{u}, \hat{v} \rangle$

3) $u, v \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$,

$$\widehat{(u * v)}(\xi) = \hat{u}(\xi)\, \hat{v}(\xi)$$

4) $u = (\hat{u})^\vee$.

5) $\int f = \hat{f}(0)$

6) As distributions, $\langle f, g \rangle = \langle \hat{f}, \check{g} \rangle$.

WAVE EQ. & FOURIER :

$$\text{(WAVE)} \begin{cases} u_{tt} - \Delta u = 0 \\ u|_{t=0} = g, \ u_t|_{t=0} = h \end{cases}$$

Fourier transform of u w.r.t. x :

$$\hat{u}(t,\xi) := \mathcal{F}_{x\to\xi}\left[u(t,x)\right](\xi)$$

Then, $\hat{u}$ solves:

$$\mathcal{F}_{x\to\xi}\left[\text{(WAVE)}\right] : \begin{cases} \hat{u}_{tt} + |\xi|^2 \hat{u} = 0 \\ \hat{u}|_{t=0} = \hat{g}, \ \hat{u}_t|_{t=0} = \hat{h}. \end{cases}$$

ODE in t ⚠

converted PDE in space into ODE in Fourier dom.

$$\Longrightarrow \quad \hat{u}(t,\xi) = \hat{g}(\xi)\cos(t|\xi|) + \frac{\hat{h}(\xi)}{|\xi|}\sin(t|\xi|).$$

- Inverse Fourier Transform.

$$u(t,x) = \cos(t|\nabla|)\,g + \frac{\sin(t|\nabla|)}{|\nabla|}\,h, \quad \text{where}$$

$$\cos(t|\nabla|)\,f := \left(\cos(t|\xi|)\,\hat{f}(\xi)\right)^{\vee},$$

$$\frac{\sin(t|\nabla|)}{|\nabla|}\,f := \left(\frac{\sin(t|\xi|)}{|\xi|}\,\hat{f}(\xi)\right)^{\vee}$$

Upshot: INHOMOGENEOUS WAVE:

$$\begin{cases} u_{tt} - \Delta u = f \\ u|_{t=0} = g, \quad u_t|_{t=0} = h \end{cases} \rightsquigarrow u(t,x) = \cos(t|\nabla|)\, g + \frac{\sin(t|\nabla|)}{|\nabla|}\, h + \int_0^t \frac{\sin((t-s)|\nabla|)}{|\nabla|} f(s)\, ds.$$

ENERGY METHODS: $E(t) := \frac{1}{2} \int_U w_t^2 + |Dw|^2 \, dx$.

- Can show uniqueness of solⁿ $u \in C^2(\bar{U}_T)$ for the inhomog. problem (using Green's identities)

$U_T = U \times (0,T]$

LECTURE 9

Apr 23, 2026

THM: Suppose $u \in C^2(\mathbb{R}^n)$ solves $\begin{cases} u_{tt} - \Delta u = 0 & \text{on } \mathbb{R}^n \times (0,\infty) \\ u(x,0) = u_t(x,0) = 0 & \forall x \in B_{t_0}(x_0) \subset \mathbb{R}^n. \end{cases}$

for some $(x_0, t_0) \in \mathbb{R}^n \times (0,\infty)$.

Then $u(x,t) \equiv 0$

$$\forall (x,t) \in \underbrace{K(t_0,x_0) := \{(x,t) : 0 \le t \le t_0,\ |x - x_0| \le t_0 - t\}}_{\text{BACKWARDS WAVE CONE}}.$$

Pf: Define $e(t) := \frac{1}{2}\int_{B_{t_0-t}(x_0)} u_t^2(x,t) + |Du(x,t)|^2\,dx$,

$0 \leq t \leq t_0$ and compute

$$\frac{d}{dt}e(t) = \int_{B_{t_0-t}(x_0)} u_t u_{tt} + Du \cdot Du_t - \frac{1}{2}\int_{\partial B_{t_0-t}(x_0)} u_t^2 + |Du|^2\,dS.$$

WTS: $e'(t) \leq 0$ and $e(0) = 0$ and $e(t) \geq 0$ $\forall t$.

$\square$

(Ch. 3 of Evans)

1st ORDER NONLINEAR PDEs – METHOD OF CHARACTERISTICS

$$\text{(PDE)} \begin{cases} F(Du, u, x) = 0 & \text{in } U \\ u = g & \text{on } \Gamma \end{cases}, \quad \Gamma \in \partial U,\ U \subset \mathbb{R}^n \text{ open}, \quad F, g \text{ smooth};\ g: \Gamma \to \mathbb{R}.$$

GOAL: • try to formulate an "equivalent" system of ODEs.

• Try to follow some "paths" (characteristics) $x(s) = (x_1(s), \_\_, x_n(s))$, $s \in I \subset \mathbb{R}$ s.t. we can "follow the solⁿ" along $s \mapsto x(s)$ starting at initial point $x_0 = x(0) \in \Gamma$.

- For $s \in I \subset \mathbb{R}$, assuming that u is a C^2 solution to (PDE), define

$$\begin{cases} z(s) := u(x(s)) \\ p(s) := (Du)(x(s)) \leftarrow \left(p_i(s) = u_{x_i}(x(s)) \right) \end{cases}$$

- Make suitable choice of $s \mapsto x(s)$.
- Compute $\frac{d}{ds}$ of these quantities: CHAIN RULE RULES THE LAND

$$\begin{cases} \dot{p}_i(s) = \sum_{j=1}^{n} u_{ij}(x(s))\, \dot{x}_j(s) \\ \left(\sum_{j=1}^{n} F_{p_j}(p,z,x)\, u_{ij} \right) + F_z(p,z,x)\, u_i + F_{x_i}(p,z,x) = 0, \quad i=1,\dots,n \end{cases}$$

$$\Rightarrow \boxed{\text{Choose } \dot{x}_j(s) := F_{p_j}(p,z,x)}$$

Then,

SYSTEM OF COUPLED NONLINEAR ODEs

$$\begin{cases} \dot{p}_i(s) = -F_z(p,z,x)\, p_i - F_{x_i}(p,z,x) \\ \dot{x}_i(s) = F_{p_i}(p,z,x) \\ \dot{z}(s) = \sum_{j=1}^{n} \underbrace{u_{x_j}(x(s))}_{=p_j(s)}\, \dot{x}_j(s) = p(s) \cdot (D_p F)(p,z,x) \end{cases}$$

i.e.,

$$\boxed{\begin{cases} \dot{p} = -D_x F(p,z,x) - D_z F(p,z,x)\, p \\ \dot{z} = D_p F(p,z,x) \cdot p \\ \dot{x} = D_p F(p,z,x) \end{cases}}$$

REMARKS:

1) $\underline{F(p,z,x) = b(x)\cdot p + c(x)\, z}$!

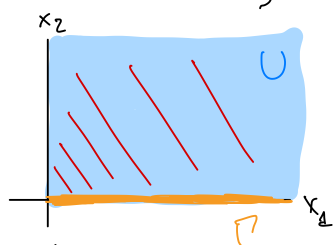
$$\Longleftrightarrow \quad b(x)\cdot (Du)(x) + c(x)\, u(x) = 0 \quad \Big] \text{ (PDE)}$$

$$D_p F = b \rightsquigarrow \begin{cases} \dot{x}(s) = b(x(s)) \\ \dot{z}(s) = b(x(s)) \cdot p(s) = -c(x(s))\, z(s) \end{cases}$$

(Linear ODE for z once we solve for $x(s)$.)

EXAMPLE 1:

Take $\begin{cases} b(x) := \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} & \text{on } U = \{(x_1,x_2)\in\mathbb{R}^2 : x_1, x_2 > 0\} \\ c(x) = -1 \end{cases}$



$$\Rightarrow \begin{cases} x_1 u_{x_2} - x_2 u_{x_1} = u & \text{in } U \\ u = g & \text{on } \Gamma := \{(x_1, 0) : x_1 > 0\}. \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \\ \dot{z} = z \end{cases}, \quad (x_1,x_2)\Big|_{s=0} = (x^0, 0) \in \Gamma$$

$\Rightarrow$ Solve system of ODEs for x_1, x_2, z and get

$$x_1(s) = x^0 \cos(s), \quad x_2(s) = x^0 \sin(s),$$

$$z(s) = z_0 e^s = g(x^0)\, e^s$$

- Now, for fixed $x = (x_1, x_2) \in U$, choose $s > 0$ and $x^0 > 0$ s.t.

$$(x_1, x_2) = (x^0 \cos(s), x^0 \sin(s))$$

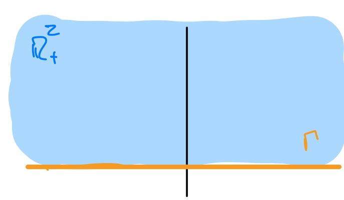
$$\rightsquigarrow \quad x^0 = \sqrt{x_1^2 + x_2^2}, \quad s = \arctan\left(\frac{x_2}{x_1}\right)$$

$$\Rightarrow \boxed{\begin{aligned} u(x_1, x_2) &= z(s) = g(x^0)\, e^s \\ &= g\left(\sqrt{x_1^2 + x_2^2}\right) e^{\arctan(x_2/x_1)} \end{aligned}}.$$

EXAMPLE 2:

nonlinear PDE

$$\begin{cases} u_{x_1} + u_{x_2} = u^2 & \text{in } \mathbb{R}^2_+ := \{(x_1, x_2) : x_2 > 0\} \\ u = g & \text{on } \Gamma := \partial \mathbb{R}^2_+ \end{cases}$$



Note: This is just $F(\underbrace{Du}_{p}, \underbrace{u}_{z}, x) = (1,1) \cdot \underbrace{Du}_{p} - \underbrace{u^2}_{z^2} = 0$

$$F = (1,1) \cdot p - z^2.$$

i.e., this is
$$\begin{cases} \dot{x} = D_p F = (1,1) \\ \dot{z} = D_p F \cdot p = (1,1)\cdot (Du)(s) = u^2 = z^2 \end{cases}$$

- Given initial / bdry pt. $(x_0, 0) \in \partial\mathbb{R}^2_+$, then

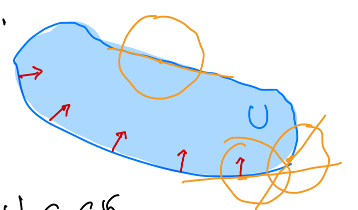
$$(x_1, x_2) = (x_0 + s, s) \quad , \quad \text{and}$$

$$z(s) = \frac{z_0}{1 - s z_0} = \frac{g(x_0)}{1 - s g(x_0)}$$

- Then, fix $(x,y) \in U$ and trace back to find a suitable $(x,0)$ and s: $x_0 = x - y$ and $s = y$

$$\Longrightarrow \boxed{u(x,y) = \frac{g(x-y)}{1 - y\, g(x-y)}}$$

GOAL: Write sol^{ns} near a "nice enough" ∂U.



- Common tool: Flattening the boundary

- Let $U \subset \mathbb{R}^n$ open and bounded with $\partial U \in C^k$.

i.e., $\forall x_0 \in \partial U$, $\exists r > 0$ and $\gamma \in C^k(\mathbb{R}^{n-1})$ s.t. (up to a change in bdry orientation),
$U \cap B_r(x_0) = \{ x \in B_r(x_0) : x_n > \gamma(x_1, \dots, x_{n-1}) \}$

- Set

$$\begin{cases} \Phi_i(x) := x_i, \quad i = 1, \dots, n-1 \\ \Phi_n(x) := x_n - \gamma(x_1, \dots, x_{n-1}) \end{cases}$$

- Next set $y := \Phi(x)$ and define

$$\begin{cases} \Psi_i(y) := y_i\,, & i = 1, \dots, n-1 \\ \Psi_n(y) := y_n + \gamma(y_1, \dots, y_{n-1}) \end{cases}$$

i.e., $\Phi = \Psi^{-1}$ and $x \mapsto \Phi(x)$ "straightens out"

$\Rightarrow$ ∂U near x_0 w/ $\det D\Phi = \det D\Psi = 1$

- Given $u : U \to \mathbb{R}$, $V := \Phi(U)$, set $v := u \circ \Psi$ on V so that $u = v \circ \Phi$ on V. If u solves

$$\begin{cases} F(Du, u, x) = 0 & \text{on } U \\ u = g & \text{in } \Gamma \end{cases}$$

$\Rightarrow$ Chain Rule gives that v solves

$$F\big(Dv(y)\, D\Phi(\Psi(y)),\ v(y),\ \Psi(y)\big) = 0$$

w/ suitable bdry cond$^{\underline{n}}$ $v = g(\Psi(y))$ on $\Phi(\Gamma)$

So, WLOG, assume $x_0 \in \Gamma$, Γ is flat near x_0.

<u>GOAL</u>: Find $K(p^0, z^0, x^0)$ to solve characteristic path and produce sol$^{\underline{ns}}$ near x^0

<u>Need</u>: 1) $s \mapsto x(s)$ need to pass through x_0

2) $u(x_1, \dots, x_{n-1}, 0) = g(x_1, \dots, x_n)$ near x_0

3) $(p^0)_i = ?$ $\leftarrow$ Need $u_{x_i}(x_0) = g_{x_i}(x_0)$, $i = 1, \dots, n-1$

$$\longrightarrow \begin{cases} (p^0)_i := g_{x_i}(x_0), & i = 1, \dots, n-1 \\ F(p^0, z^0, x^0) = 0 \end{cases}$$

$F(p_0, z_0, x_0) = 0$

If (p^0, z^0, x^0) satisfies this, we call the pt. ADMISSIBLE.

Step 3: If we have an admissible pt, can we perturb it?

Set $y = (y_1, \dots, y_n, 0) \in \Gamma$, $p_0 := q(y)$, look for $(q(y), g(y), y)$ admissible s.t.

$$\begin{cases} q_i(y) = g_{x_i}(y) \\ F(q(y), g(y), y) = 0 \end{cases} \quad \forall y \text{ close to } x_0.$$

Claim: $\exists!$ sol$^{\underline{n}}$ to
$$\begin{cases} q_0(x_0) = p_0 \\ q_i(y) = g_{x_i}(y) \\ F(q(y), g(y), y) = 0 \end{cases}$$

provided that $F_{p_n}(p_0, z_0, x_0) \neq 0$ $\leftarrow$ called non-characteristic pt.

Step 4: Patch ODE paths together to construct a sol$^{\underline{n}}$ in the descred region.

Fix q, $y = (y_0, \dots, y_n, 0)$ s.t. $(q(y), g(y), y)$ is admissible close to x_0. Use nondegeneracy to patch the sol$^{\underline{ns}}$ to the ODE together and show that the result (implicit) formula solves the ODE.

PG 108 of Evans $\rightarrow \square$

LECTURE 10

Apr 28, 2026

* SOBOLEV SPACES & WEAK SOLUTIONS (Ch 5 of Evans)

IDEA: For more general problems, we don't hope to have sol$^{\underline{n}}$ formulas. Instead, we separate "existence of solutions" from their "regularities":

(i) Formulate notion of "weak sol$^{\underline{ns}}$"

(ii) Show existence of weak sol$^{\underline{ns}}$

(iii) Show that weak sol$^{\underline{ns}}$ are smooth

$\rightsquigarrow$ Functional analysis perspective: view PDEs as operators between Banach spaces.

POISSON

$$(-\Delta u)(x) = f(x) \quad \forall x$$

$Au = f$, $(u \overset{A}{\longmapsto} -\Delta u)$

$A : X \to Y$, $f \in Y$, $u \in X$
linear

What is X, Y?

HEAT

$$\begin{cases} u_t - \Delta u = 0 \\ u(0) = u_0 \end{cases} \rightsquigarrow \frac{du}{dt} = Au(t), \quad \underset{\text{linear}}{A : X \to Y}$$

$u \in C^1((0,\infty); Y) \cap C^0([0,\infty), X)$

with $u(0) = u_0$.

Possible choices of X, Y:

1) HÖLDER SPACES

- $U \subset \mathbb{R}^n$ open & bdd, $0 < \gamma \leq 1$, $u : U \to \mathbb{R}$

<u>DEF</u>: u is continuous bounded on U iff

$$\|u\|_{C(\bar{U})} = \sup_{x \in \bar{U}} |u(x)| < \infty .$$

<u>DEF</u>: If u is s.t., for some constant $C>0$ (indep. of x, y),

$$|u(x) - u(y)| \leq C|x-y|^\gamma ,$$

we say u is Hölder continuous with exponent γ.

$\iff u \in C^{0,\gamma}(U)$. Set

$$[u]_{C^{0,\gamma}(\bar{U})} := \sup_{x,y \in U} \frac{|u(x) - u(y)|}{|x-y|^\gamma} .$$

<u>Hölder Norm</u>: $\|u\|_{C^{0,\gamma}(\bar{U})} := \|u\|_{C^0(\bar{U})} + [u]_{C^{0,\gamma}(\bar{U})}$.

<u>DEF</u>: $k \in \mathbb{N}$, $0 < \gamma < 1$,

$$C^{k,\gamma}(\bar{U}) := \left\{ u \in C^k(\bar{U}) : \|u\|_{C^{k,\gamma}(\bar{U})} < \infty \right\},$$

C^k fcts whose derivatives are γ-Hölder continuous

(is Banach w.r.t. $\|\cdot\|_{C^{k,\gamma}(\bar{U})}$)

where

$$\|u\|_{C^{k,\gamma}(\bar{U})} := \sum_{|\alpha| \leq k} \|D^\alpha u\|_{C(\bar{U})} + \sum_{|\alpha| = k} [D^\alpha u]_{C^{0,\gamma}(\bar{U})}$$

2) SOBOLEV SPACES (more compatible w/ integration)

- $U \subset \mathbb{R}^n$ measurable, $1 \leq p \leq \infty$,

$$L^p(U) := \{ [f : U \to \mathbb{R}] : f \text{ measurable and } \|f\|_{L^p(U)} < \infty \}$$

$$\|f\|_{L^p(U)} := \left(\int_U |f(x)|^p \, dx \right)^{1/p}, \quad 1 \leq p < \infty$$

$$\|f\|_{L^\infty(U)} := \operatorname*{ess\,sup}_{x \in U} |f(x)|, \qquad p = \infty$$

MINKOWSKI: $\|f+g\|_p \leq \|f\|_p + \|g\|_p$

HÖLDER: $\|fg\|_1 \leq \|f\|_p \|g\|_q$, $\frac{1}{p} + \frac{1}{q} = 1$, $1 \leq p, q \leq \infty$

* WEAK DERIVATIVES

$U \subset \mathbb{R}^n$ open bdd, $u \in C^1(U)$, $\varphi \in C_c^\infty(U)$, have integration by parts

$$\int_U u \, \varphi_{x_i} = - \int_U u_{x_i} \varphi .$$

$\hookrightarrow$ differentiation via integration by parts identity

DEF: (WEAK DERIVATIVE) $u, v \in L^1_{loc}(U)$ and α a multi-index. Then v is the α-weak derivative of u

$$D^\alpha u = v \iff \int_U u \, D^\alpha \varphi = (-1)^{|\alpha|} \int_U v \varphi .$$

$v \rightsquigarrow$ distributional derivative $\qquad = (-1)^{|\alpha|} \int_U \underbrace{(D^\alpha u)}_{= v} \varphi$

PROPOSITION: If α-weak derivative exists, then it is unique up to a set of measure zero (i.e., a.e.).

↑ b/c we used integral to define

KEY: Choose mollifier $\eta_\varepsilon(x) := \frac{1}{\varepsilon}\eta\left(\frac{x}{\varepsilon}\right)$,

$$\eta(x) = \begin{cases} c\, e^{-1/|x|^2-1}, & |x| \le 1 \\ 0, & \text{else} \end{cases} \quad \text{w/ } c \text{ s.t. } \int \eta = 1$$

Then, for $f \in L^1_{loc}$, $(\eta_\varepsilon * f)(x) = \int \eta_\varepsilon(x-y) f(y)\, dy \xrightarrow[\text{a.e. in } \mathbb{R}^n]{\varepsilon \to 0} f(x)$

Pf: By Analysis, have Lebesgue points: a.e. $x \in \mathbb{R}^n$,

$$⨍_{B_\varepsilon(x)} |f(y) - f(x)| \xrightarrow{\varepsilon \to 0} 0 \quad \leftarrow \text{pts where this is true are called Lebesgue points}$$

• FACT: for $f \in L^1_{loc}$, almost all points in dom f are Lebesgue points.

For $x \in \mathbb{R}^n$ Lebesgue pt.,

$$\left| (\eta_\varepsilon * f)(x) - f(x) \right| = \left| \int_{B_\varepsilon(x)} \eta_\varepsilon(x-y)\left[f(y) - f(x)\right] dy \right|$$

$$\le \frac{C(n)}{\mathrm{vol}_n^{Leb}(B_\varepsilon(x))} \int_{B_\varepsilon(x)} \eta\left(\frac{x-y}{\varepsilon}\right) |f(x) - f(y)|\, dy$$

$\|\eta\|_\infty < \infty$

$$= \tilde{C}(n) ⨍_{B_\varepsilon(x)} |f(x) - f(y)|\, dy \longrightarrow 0$$

Pf: (Proposition) say $v, \tilde{v}$ are both weak ders. of u. Then $\forall \varphi \in C_c^\infty(U)$,

$$(-1)^{|\alpha|} \int_U \tilde{v}\, \varphi = \int_U u\, D^\alpha \varphi\, dx = (-1)^{|\alpha|} \int_U v\, \varphi$$

Thus, $\displaystyle\int_U (\tilde{v} - v)\varphi = 0 \quad \forall \varphi \in C_c^\infty(U)$

For $x \in \mathbb{R}^n$ Lebesgue pt. for $\tilde{v} - v$, choose $\varphi = \eta_\varepsilon(x - \cdot)$ and let $\varepsilon \to 0$ to conclude that $v = \tilde{v}$. $\square$

DEF: (SOBOLEV SPACE) $U \subset \mathbb{R}^n$ open, $1 \le p \le \infty$, $k \in \mathbb{N} \cup \{0\}$, the <u>SOBOLEV SPACE</u> $W^{k,p}(U)$ consists of all functions $u \in L^1_{loc}(U)$ such that, for each multi-index α with $|\alpha| \le k$, the α-weak partial derivatives of u exist and are in $L^p(U)$.

$$\|u\|_{W^{k,p}(U)} := \begin{cases} \left(\displaystyle\sum_{|\alpha| \le k} \int_U |D^\alpha u|^p \right)^{1/p}, & 1 \le p < \infty \\ \displaystyle\sum_{|\alpha| \le k} \operatorname{ess\,sup}_U |D^\alpha u|, & p = \infty \end{cases}$$

[$(W^{k,p}(U), \|\cdot\|_{W^{k,p}(U)})$ is Banach, $1 \le p \le \infty$]

i.e.,

$$\|u\|_{W^{k,p}} = \left(\sum_{|\alpha| \le k} \|D^\alpha u\|_{L^p}^p \right)^{1/p}, \quad 1 \le p < \infty$$

$$\|u\|_{W^{k,\infty}} = \sum_{|\alpha| \le k} \|D^\alpha u\|_{L^\infty} = \sum_{|\alpha| \le k} \operatorname{ess\,sup} |D^\alpha u|, \quad p = \infty$$

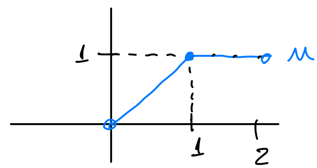
LECTURE 11

Apr 30, 2026

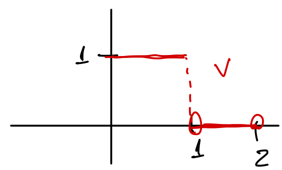
EXAMPLES: WEAK DERIVATIVES IN $W^{k,p}(U)$

1) $U = (0,2) \subset \mathbb{R}$

$$u(x) := \begin{cases} x, & 0 < x \leq 1 \\ 1, & 1 \leq x < 2 \end{cases}$$



$$L^1((0,2)) \ni v(x) := \begin{cases} 1, & 0 < x < 1 \\ 0, & 1 \leq x < 2 \end{cases}$$



Claim: v is the weak derivative of u.

Pf: WTS: $\forall \varphi \in C_c^\infty((0,2))$, $\int_0^2 u \varphi' = -\int_0^2 v \varphi$.

Calculate:

$$\int_0^2 u \varphi' \, dx = \int_0^1 x \varphi' \, dx + \underbrace{\int_1^2 1 \cdot \varphi' \, dx}_{\text{FTC}}$$

by parts

$$= [x\varphi] \Big|_{x=0}^{x=1} - \int_0^1 1 \cdot \varphi \, dx + \underline{\varphi(2) - \varphi(1)}$$

$$= \varphi(1) - 0 - \int_0^1 \varphi \, dx + 0 - \varphi(1)$$

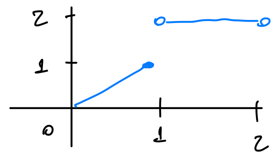
$$= -\int_0^1 \varphi \, dx$$

$$= -\int_0^2 v\varphi \, dx, \quad \text{as desired.}$$

$\square$

2) $U = (0,2)$

$$u(x) = \begin{cases} x, & 0 < x \leq 1 \\ 2, & 1 < x \leq 2 \end{cases}$$



<u>Claim</u>: u does not have any weak derivatives in $L^1((0,2))$.

<u>Pf</u>: If $v \in L^1_{loc}((0,2))$ is s.t., by contradiction,

$$\int_0^2 u\varphi' = -\int_0^2 v\varphi \qquad \forall \varphi \in C_c^\infty((0,2)),$$

then,

$$\text{LHS} = \int_0^2 u\varphi' = \int_0^1 x\varphi' + \int_1^2 2\cdot\varphi'$$

$$= [x\varphi]\Big|_{x=0}^{x=1} - \int_0^1 \varphi + 2\varphi(2) - 2\varphi(1)$$

$$= \varphi(1) - \int_0^1 \varphi - 2\varphi(1)$$

$$= -\int_0^1 \varphi - \varphi(1)$$

$$\text{RHS} = -\int_0^2 v\varphi$$

take $\varphi := \varphi_\varepsilon$ s.t. $0 \le \varphi_\varepsilon \le 1$ $\forall x$, $\varphi_\varepsilon(x) \to 0$ for $x \neq 1$, $\varphi_\varepsilon(1) = 1$ $\forall \varepsilon$. Then LHS $\to 0 - 1 = -1$

RHS $\to 0$ $\not=$

$\hookrightarrow\!\!/\!/\!\!\hookleftarrow$

$\square$

PROPERTIES OF SOBOLEV SPACES: IMPORTANT ⚠

1) $W^{k,p}(U) := \{ u \in L^1_{loc}(U) : D^\alpha u \in L^p(U) \ \forall |\alpha| \le k \}$

where $D^\alpha u$ is the distributional derivative of u.

2) $u, v \in W^{k,p}(U)$, $|\alpha| \le k$, then

(i) $D^\alpha u \in W^{k-|\alpha|,p}(U)$

(ii) $D^\beta(D^\alpha u) = D^\alpha(D^\beta u) = D^{\alpha+\beta} u$

(iii) Weak derivative is linear: $\forall \lambda, \mu \in \mathbb{R}$,

$$D^\alpha(\lambda u + \mu v) = \lambda D^\alpha u + \mu D^\alpha v .$$

(iv) $V \subset U$ open, then $u \in W^{k,p}(V)$

(v) LEIBNIZ RULE: $\forall \psi \in C_c^\infty(U)$, $\psi u \in W^{k,p}(U)$,

and

$$D^\alpha(\psi u) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} (D^\beta \psi)(D^{\alpha-\beta} u) .$$

$\binom{\alpha}{\beta} := \frac{\alpha!}{\beta!(\alpha-\beta)!}$

$\alpha! = (\alpha_1!)(\alpha_2!)\cdots(\alpha_n!)$

Pf: (Sketch) Induction on $|\alpha|$.

$|\alpha| = 1$: $\forall \phi \in C_c^\infty(U)$,

$$\int_U \psi u D^\alpha \phi = \int_U u [D^\alpha(\psi\phi) - \phi D^\alpha \psi] = \int_U u D^\alpha(\psi\phi) - \int_U u \phi D^\alpha \psi$$
$$= -\int_U (D^\alpha u)(\psi \phi) - \int_U u \phi D^\alpha \psi$$

$\Rightarrow D^\alpha(\psi u) = \psi D^\alpha u + u D^\alpha \psi$, as desired. $|\alpha| = k$ induction look at Evans. $\square$

! 3) $\| D^\alpha u \|_{L^p} \leq \| u \|_{W^{k,p}}$

$\forall \alpha$ with $|\alpha| \leq k$.

$$\| u \|_{W^{k,p}} = \left(\sum_{|\alpha| \leq k} \| D^\alpha u \|_{L^p}^p \right)^{1/p}$$

$\| D^\alpha u \|_{L^p}$, $|\alpha| \leq k$, is only one of the terms of this summation.

! 4) Denote $W^{k,2}(U) =: H^k(U)$, $k = 0, 1, 2, \dots$

e.g.: $W^{0,2}(U) = H^0(U) = L^2(U)$

Hilbert space

APPROXIMATING FUNCTIONS

Proposition (Local Approximation by smooth functions) ⚠

$U \subset \mathbb{R}^n$ open, $u \in W^{k,p}(U)$, $1 \leq p < \infty$, set

$$u_\varepsilon := \eta_\varepsilon * u \quad \text{on} \quad U_\varepsilon := \{ x \in U : \text{dist}(x, \partial U) > \varepsilon \}.$$

Then, $u_\varepsilon \in C^\infty(U_\varepsilon)$ $\forall \varepsilon > 0$, and

$$u_\varepsilon \xrightarrow{\varepsilon \to 0} u \quad \text{in} \quad W^{k,p}_{loc}(U)$$

i.e., in $W^{k,p}(V)$, $\forall V \subset\subset U$.

Pf: Argue as for $u_\varepsilon \to u$ in L^p applied to each $D^\alpha u_\varepsilon$, $|\alpha| \le k$.

$\square$

NEXT: We can find smooth functions which approximate in $W^{k,p}(U)$ and not just $W^{k,p}_{loc}(U)$. Note that we were no assumptions on smoothness of ∂U.

THM: (GLOBAL APPROXIMATION BY SMOOTH FUNCTION: MEYERS - SERRIN)

For $U \subset \mathbb{R}^n$ open and bounded, then

$$C^\infty(U) \cap W^{k,p}(U) \text{ is dense in } W^{k,p}(U)$$

i.e., $\forall u \in W^{k,p}(U)$, $1 \le p < \infty$, $\exists \{u_m\}_{m \in \mathbb{N}}$ s.t.

$$u_m \xrightarrow{m \to \infty} u \text{ in } W^{k,p}(U).$$

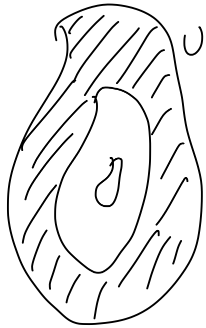
NOTE: smoothness only in interior of U (not bdry)

Pf: Write $U = \bigcup_i U_i$, $U_i = U_{\varepsilon_i}$, $\varepsilon_i := 1/i$ $\forall i \in \mathbb{N}$.

Set $V_i := U_{i+3} \setminus \overline{U_{i+1}}$.

Choose $\{\xi_i\} \subset C_c^\infty(V_i)$
w/ $0 \le \xi_i \le 1$ $\forall x \in V_i$
Set $\sum_{i=0}^\infty \xi_i = 1$ on U

PARTITION OF UNITY (POU)

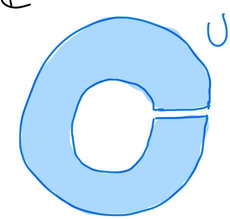


Then, apply local approximation result to $\xi_i u$, $i \in \mathbb{N}$.

$\square$

EXAMPLE: (DANGER OF EXTENDING RESULT TO ∂U)

Let $U := \{\text{annulus with a slit}\} \subset \mathbb{R}^2 \cong \mathbb{C}$
$= \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$
$\setminus \{(x_1, 0) : 1 < x_1 < 2\}$



$$\boxed{\Rightarrow C^1(\bar{U}) \text{ is not dense in } W^{1,1}(U)}$$

Pf: $u(r,\theta) := \theta$ (in polar coordinates)
$x = (x_1, x_2) = (r\cos\theta, r\sin\theta)$, $r > 0$.
Then, $u \in C^1(U) \cap W^{1,1}(U)$; $\frac{\partial u}{\partial \theta} = 1$, thus

$$\int_U \frac{\partial u}{\partial \theta} = \mathrm{vol}(U) = \pi(2^2 - 1^2) = 3\pi .$$

<u>Claim</u>: $\nexists \{u_m\}$ seq. in $C^1(\bar{U})$ with $u_m \to u$ in $W^{1,1}(U)$

<u>Idea</u>: if $u_m \in C^1(\bar{U})$, then

$$\int_U \frac{\partial u_m}{\partial \theta} = \int_1^2 \int_0^{2\pi} \frac{\partial u_m}{\partial \theta} \, r \, d\theta \, dr$$

$$= \int_1^2 u_m(r, 2\pi) - u_m(r, 0) \, dr$$

$$= 0 \text{ by symmetry.}$$

Suppose $u_m \longrightarrow u$ in $W^{1,1}(U)$, by integration by parts:

$$\frac{\partial u}{\partial \theta} = \nabla u \cdot \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}, \quad \frac{\partial u_m}{\partial \theta} = \nabla u_m \cdot \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$

$$\Rightarrow 3\pi = \left| \int_U \frac{\partial u}{\partial \theta} - \frac{\partial u_m}{\partial \theta} \right| \leq \int_U \left| \nabla u \cdot \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} - \nabla u_m \cdot \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix} \right|$$

$$\leq 2 \int_U |\nabla u - \nabla u_m| \longrightarrow 0$$

$$\hookrightarrow\!\!/\!\!/\!\!\leftarrow$$

$\square$

<u>RESOLUTION</u>: Can recover density of $C^\infty(\bar{U}) \cap W^{k,p}(U)$ in $W^{k,p}(U)$ by imposing assumptions on U (all <u>away</u> if $\partial U \in C^1$) $\longrightarrow$ Evans Thm 3.

REMARK: When $\partial U \in C^1$, can use other extension theorems to extend $u \in W^{1,p}(U)$ to $\tilde{u} \in W^{1,p}(\mathbb{R}^n)$ w/ $\|\tilde{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(U)}$.

THM: For $1 \le p \le \infty$, $U \subset \mathbb{R}^n$ open bdd, $\partial U \in C^1$. Let V open bdd s.t. $U \subset\subset V$. Then, $\exists$ bdd linear $E : W^{1,p}(U) \longrightarrow W^{1,p}(\mathbb{R}^n)$ and $C = C(p, U, V) > 0$ s.t.

(1 derivative)

1) $(Eu)(x) = u(x)$ a.e. on U

2) $\operatorname{supp} Eu \subset V$

3) $\|Ev\|_{W^{1,p}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(U)}$

EXTENSION THEOREM

$W^{1,p}(U) \rightsquigarrow W^{1,p}(\mathbb{R}^n)$

REMARK: Need to be careful when extending $W^{1,p}(U)$ to $W^{1,p}(\mathbb{R}^n)$ because, if you're not, the function might have such a bad discontinuity along ∂U that it no longer has a weak 1st partial derivative (e.g., usually very bad idea to just set fct to be zero on $\mathbb{R}^n \setminus U$).

NOTE: Much harder for $W^{k,p}(U) \rightsquigarrow W^{k,p}(\mathbb{R}^n)$ (Calderón & Stein) w/ $k > 1$.

Lecture 12

May 5, 2026

<u>Question</u>: If $f \in W^{1,p}(U)$, does this mean f is in other spaces we know? $\quad U \subset \mathbb{R}^n$

<u>A</u>: Yes! 3 cases (dim $U = n$)

1. $1 \leq p < n$
2. $n < p \leq \infty$
3. $p = n$ (borderline – more annoying)

<u>Case 1</u>: Assume, for now, $\boxed{1 \leq p < n}$

⚠ **IMPORTANT** ⚠

<u>Thm</u>: (Sobolev – Gagliardo – Nirenberg)

If $1 \leq p < n$, $\exists\, C = C(n,p)$ s.t., if $\frac{1}{q} + \frac{1}{n} = \frac{1}{p}$,

then
$$\|u\|_{L^q(\mathbb{R}^n)} \leq C \|Du\|_{L^p(\mathbb{R}^n)} \qquad \left(\text{i.e., } q = \frac{np}{n-p}\right)$$

for all $u \in C_c^1(\mathbb{R}^n)$

i.e., $W^{1,p}(U) \xrightarrow{\text{embeds}} L^q(U)$

<u>Pf</u>: See Evans pg 277.

$\square$

⚠ IMPORTANT ⚠

COROLLARY: Let $U \subset \mathbb{R}^n$ be open and bounded with $\partial U \in C^1$, $1 \le p < n$ and let $u \in W^{1,p}(U)$. Then $u \in L^q(U)$ with

$$\|u\|_{L^q(U)} \le C \|u\|_{W^{1,p}(U)}, \quad C = C(p,n,U), \quad \frac{1}{q} + \frac{1}{n} = \frac{1}{p}.$$

i.e., $W^{1,p}(U) \overset{\text{embeds}}{\hookrightarrow} L^q(U)$

Pf: $\partial U \in C^1 \Rightarrow \exists$ extension $Eu = \bar{u} \in W^{1,p}(\mathbb{R}^n)$ s.t.

supp $\bar{u} \subset \mathbb{R}^n$ compact, $\bar{u}|_U = u$, $\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} \le C \|u\|_{W^{1,p}(U)}$ (*)

$\Downarrow$

$\exists\, u_m \in C_c^\infty(\mathbb{R}^n)$, $m = 1, 2, \ldots$
s.t. $u_m \to \bar{u}$ in $W^{1,p}(\mathbb{R}^n)$

By Sobolev – Gagliardo – Nirenberg,

$$\|u_m - u_l\|_{L^q(\mathbb{R}^n)} \le C \|Du_m - Du_l\|_{L^p(\mathbb{R}^n)} \quad \forall\, l, m \ge 1.$$

Thus, $u_m \to \bar{u}$ in $L^q(\mathbb{R}^n)$ b/c u_m is Cauchy (and $L^q(\mathbb{R}^n)$ is Banach ✓)

Since, by Sobolev – Gagliardo – Nirenberg, we also have

$\|u_m\|_{L^q(\mathbb{R}^n)} \le C \|Du_m\|_{L^p(\mathbb{R}^n)}$, we also have

$$\|\bar{u}\|_{L^q(\mathbb{R}^n)} \le C \|D\bar{u}\|_{L^p(\mathbb{R}^n)} \le C \|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)}$$

By definition,

$$\|\bar{u}\|_{W^{1,p}(\mathbb{R}^n)} = \left(\sum_{|\alpha| \le 1} \|D^\alpha \bar{u}\|^p_{L^p(\mathbb{R}^n)} \right)^{1/p} = \left(\underbrace{\|\bar{u}\|^p_{L^p}}_{\ge 0} + \|D\bar{u}\|^p_{L^p} \right)^{1/p} \ge \|D\bar{u}\|_{L^p(\mathbb{R}^n)}$$

$$\overset{(*)}{\le} C \|u\|_{W^{1,p}(U)}$$

□

DEF: $W_0^{1,p}(U) := \overline{C_c^\infty(U)}^{\,\text{in } W^{1,p}(U)}$, $U \subset \mathbb{R}^n$ open bounded.

IMPORTANT

COROLLARY: For $U \subset \mathbb{R}^n$ open and bounded, and $u \in W_0^{1,p}(U)$, $1 \leq p < n$, then

$$\|u\|_{L^r(U)} \leq C \|Du\|_{L^p(U)},$$

for each $r \in [1, q]$, $\frac{1}{q} + \frac{1}{n} = \frac{1}{p}$, $C = C(p, q, n, U)$.

In particular, for all $1 \leq p \leq \infty$,

$$\|u\|_{L^p(U)} \leq C \|Du\|_{L^p(U)}.$$

(POINCARÉ'S INEQUALITY)

CASE 2: $\boxed{n < p \leq \infty}$

THM: (MORREY'S INEQUALITY) Let $n < p \leq \infty$, then $\exists\, C = C(p,n) > 0$ s.t., $\forall u \in C^1(U)$,

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}, \quad \gamma := 1 - \frac{n}{p}.$$

Stronger! If $p > n$ and $u \in W^{1,p}(\mathbb{R}^n)$, then u is actually Hölder continuous (after possibly being redefined in a measure zero set)

LECTURE 13

May 7, 2026

CASE 2: $\boxed{n < p \leq \infty}$ (ctd)

MORREY'S INEQUALITY: $n < p \leq \infty$. Then $\exists C = C(n,p) > 0$ such that

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leq C \|u\|_{W^{1,p}(\mathbb{R}^n)}$$

for all $u \in C^1(\mathbb{R}^n)$, where $\gamma := 1 - \frac{n}{p}$.

i.e., $n < p \leq \infty$
$W^{1,p}(\mathbb{R}^n) \hookrightarrow C^{0,\gamma}(\mathbb{R}^n)$
$\gamma = 1 - \frac{n}{p}$

Stronger! If $p > n$ and $u \in W^{1,p}(\mathbb{R}^n)$, then u is actually Hölder continuous (after possibly being redefined in a measure zero set)

COROLLARY: $\Omega \subset \mathbb{R}^n$ open and bdd, $\partial\Omega \in C^1$, $u \in W^{1,p}(\Omega)$, $p > n$, then, after changing u on set of measure zero, $u \in C^{0,\gamma}(\overline{\Omega})$, $\gamma := 1 - n/p$. In particular, u is continuous.

$\Rightarrow$ Always identify $u \in W^{1,p}(\Omega)$, $p > n$, with its continuous version.

$\tilde{u}$ version of $u \Leftrightarrow \tilde{u} = u$ a.e.

(EMBEDDINGS : $X \hookrightarrow Y \iff X$ continuously embedded on Y
$\iff X \subset Y$ and $\|f\|_Y \leq C \|f\|_X$)

GENERAL SOBOLEV INEQUALITIES : Let $U \subset \mathbb{R}^n$ open and bdd, $\partial U \in C^1$.

Let $u \in W^{k,p}(U)$.

(1) If $k < \frac{n}{p}$, then $u \in L^q(U)$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$,

and $\|u\|_{L^q(U)} \leq C(k,p,n,U) \, \|u\|_{W^{k,p}(U)}$.

i.e., if $k < n/p$, $W^{k,p}(U) \hookrightarrow L^q(U)$.

(2) If $k > \frac{n}{p}$, then $u \in C^{k - \lfloor \frac{n}{p} \rfloor - 1, \gamma}(\bar{U})$, where

$$\gamma := \begin{cases} \lfloor \frac{n}{p} \rfloor + 1 - \frac{n}{p}, & \text{if } \frac{n}{p} \notin \mathbb{N} \\ \text{any element of } (0,1), & \text{if } \frac{n}{p} \in \mathbb{N} \end{cases},$$

and $\|u\|_{C^{k-\lfloor n/p \rfloor - 1, \gamma}(\bar{U})} \leq C(k,p,n,\gamma,U) \, \|u\|_{W^{k,p}(U)}$

i.e., $k > \frac{n}{p} \Rightarrow W^{k,p}(U) \hookrightarrow C^{k - \lfloor n/p \rfloor - 1, \gamma}(\bar{U})$.

* COMPACTNESS & SOBOLEV : $W^{1,p}(U)$ is compactly embedded in $L^q(U)$, in fact, for $1 \leq q < pn/n-p$.

THM: (RELLICH - KONDRACHOV) $U \subset \mathbb{R}^n$ open bdd with $\partial U \in C^1$.
Then, for all $1 \leq q \leq \dfrac{np}{n-p}$,

$$W^{1,p}(U) \text{ is compactly embedded in } L^q(U)$$

in the sense that $W^{1,p}(U) \hookrightarrow L^q(U)$ and every bdd seq. in $W^{1,p}(U)$ has a subseq. which converges in $L^q(U)$ to a limit in $L^q(U)$.

Application

POINCARÉ'S INEQUALITY: Let $U \subset \mathbb{R}^n$ open bdd w/ $\partial U \in C^1$, $1 \leq p \leq \infty$, then $\exists C = C(n, p, U)$ s.t.

$$\| u - f_U u \|_{L^p(U)} \leq C \| Du \|_{L^p(U)}$$

$\forall u \in W^{1,p}(U)$.

—— // ——

SECOND ORDER ELLIPTIC PDEs (Chapter 6 of Evans)

(BVP) $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$,

$U \subset \mathbb{R}^n$ open bdd
$f: U \to \mathbb{R}$ given
$L \equiv 2^{nd}$ order partial differential operator

$$Lu = -\sum_{i,j=1}^{n} \left[a_{ij}(x)\, u_{x_i}\right]_{x_j} + \sum_{i=1}^{n} b_i(x)\, u_{x_i} + c(x)\, u$$

(DIVERGENCE FORM)

with $a_{ij}(x) = a_{ji}(x)$. Setting $A(x) := (a_{ij}(x))_{ij} : U \to \mathbb{R}^{n\times n}$,

$$\boxed{Lu = -\operatorname{div}(A\nabla u) + b\cdot\nabla u + cu}$$

⚠

DEF: L is (uniformly) ELLIPTIC if $\exists\, \theta > 0$ s.t.

$$\sum_{i,j=1}^{n} a_{ij}(x)\,\xi_i\,\xi_j \geq \theta\,|\xi|^2$$

for a.e. $x \in U$, and all $\xi \in \mathbb{R}^n$

i.e., for each pt. $x \in U$, the symmetric $n\times n$ matrix $A(x) = (a_{ij}(x))_{ij} = A(x)^T$ is also positive definite with smallest eigenvalue $\geq \theta$.

REMARK: Assume $a_{ij}, b_i, c \in L^\infty(U)$ $\forall\, i,j$

$f \in L^2(U)$.

DEF: (IMPORTANT)

1. The BILINEAR FORM $B[\cdot,\cdot]$ associated w/ the divergence form elliptic operator L (as above) is defined as

$$B[u,v] := \int_U \sum_{i,j=1}^n a_{ij}(x)\, u_{x_i} v_{x_j} + \sum_{i=1}^n b_i(x)\, u_{x_i} v + c u v \, dx$$

for $u, v \in H_0^1(U)$ $\equiv$ completion of $C_c^\infty(U)$ in the $H^1(U) = W^{1,2}(U)$ norm

2. $u \in H_0^1(U)$ is a WEAK SOLUTION of (BVP) if

$$B[u,v] = \langle f, v \rangle_{L^2(U)} \quad \forall v \in H_0^1(U).$$

called VARIATIONAL FORMULATION of (BVP)

LECTURE 14

May 17th 2026

Last time:
$$\begin{cases} Lu = f & \text{in } U \subset \mathbb{R}^n \text{ open bdd} \\ u = 0 & \text{on } \partial U \end{cases}$$

$$Lu = -\operatorname{div}(A \nabla u) + b \cdot \nabla u + c u, \quad A^T = A > 0, \quad b \in \mathbb{R}^n, \; c \in \mathbb{R}$$

L Elliptic $\iff \exists\, \theta > 0$ s.t. $\sum_{i,j} a_{ij}(x)\, \xi_i \xi_j \geq \theta |\xi|^2$.

$u \in H_0^1(U)$ is WEAK SOLUTION $\iff B[u,v] = \langle f, v \rangle_{L^2(U)} \quad \forall v \in H_0^1(U)$ where

$$B[u,v] := \int_U \Big(\sum_{i,j} a_{ij}\, u_{x_i} v_{x_j} + \sum_i b_i\, u_{x_i} + c\, u v \Big)\, dx.$$

IMPORTANT

THM: (LAX-MILGRAM) $\mathcal{H}$ Hilbert space (over $\mathbb{R}$). Let

$$B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$$

be a bilinear map such that $\forall u, v \in \mathcal{H}$,

1. $\exists\, \alpha > 0$ s.t. $|B[u,v]| \leq \alpha \|u\|_{\mathcal{H}} \|v\|_{\mathcal{H}}$ (B is continuous)
2. $\exists\, \beta > 0$ s.t. $B[u,u] \geq \beta \|u\|^2_{\mathcal{H}}$ (B is coercive).

If $f : \mathcal{H} \to \mathbb{R}$ ($f \in \mathcal{H}^* \overset{\text{Riesz}}{\cong} \mathcal{H}$) is a bounded linear functional, then $\exists!\, u_f \in \mathcal{H}$ s.t.

$$B[u_f, v] = \langle f, v \rangle_{L^2} \quad \forall v \in \mathcal{H}.$$

⚠

WARNING: Here, $B[\cdot,\cdot]$ does not define an inner product to directly apply Riesz Representation! (unless it's symmetric).

- Consider the Elliptic PDE case: $\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$

$$B[u,v] := \int_U \left(\sum_{i,j} a_{ij} u_{x_i} v_{x_j} + \sum_i b_i u_{x_i} + cu \right) dx$$

LEMMA: (ENERGY ESTIMATE) $\exists\, \alpha, \beta > 0$ and $\gamma \geq 0$ s.t.

1. $|B[u,v]| \leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$
2. $\beta \|u\|^2_{H_0^1(U)} \leq B[u,u] + \gamma \|u\|^2_{L^2(U)}$

$\forall\, u, v \in H_0^1(U)$.

Not exactly what L-M says...

IMPORTANT

THM: (EXISTENCE OF WEAK SOLUTIONS I) $\exists\, \gamma \geq 0$ s.t. $\forall \mu \geq \gamma$, and $\forall f \in L^2(U)$, the boundary value problem

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique solution $u \in H_0^1(U)$.

<u>Pf</u>: Set

$$B_\mu[u,v] := B[u,v] + \mu \langle u,v\rangle_{L^2}, \quad u,v \in H_0^1(U).$$

Then, given $f \in L^2(U)$, set $f(v) := \langle f, v\rangle_{L^2(U)}$ bdd linear functional on L^2, hence H_0^1

Then

$$|B_\mu[u,v]| \leq |B[u,v]| + \mu\, |\langle u,v\rangle_{L^2}|$$

$$\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} + \mu\, |\langle u,v\rangle_{L^2}|$$

Cauchy-Schwarz $\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} + \mu \|u\|_{L^2} \|v\|_{L^2}$

$$\leq \alpha \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)} + \mu \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$

$$= (\alpha+\mu) \|u\|_{H_0^1(U)} \|v\|_{H_0^1(U)}$$ (1) from L-M (continuous) ✓

$$B_\mu[u,u] = B[u,u] + \mu \|u\|_{L^2}^2$$

$$\geq \underbrace{\beta}_{>0} \|u\|_{H_0^1}^2 - \gamma \|u\|_{L^2}^2 + \mu \|u\|_{L^2}^2$$

$\mu \geq \gamma$ $\geq (\mu+\gamma) \|u\|_{L^2}^2$ (2) of L-M (coercive) ✓

Thus: Lax-Milgram $\Longrightarrow$ $\exists !\ u_* \in H_0^1$ with

$$B_\mu[u_*, v] = \langle f, v\rangle_{L^2} \quad \forall v \in H_0^1$$

i.e., $u_* \in H_0^1$ is unique solution to BVP !

$\square$

FREDHOLM ALTERNATIVE

DEF:

1. L^* is the FORMAL ADJOINT of L:

$$L^* v := -\sum_{i,j} \left(a_{ij}(x)\, v_{x_j}\right)_{x_i} - \sum_i b_i v_{x_i} + \left(c - \sum_i (b_i)_{x_i}\right) v$$

provided $b_i \in C^1(U)$, $i = 1, \dots, n$.

2. Adjoint of bilinear form: $B^*: H_0^1(U) \times H_0^1(U) \to \mathbb{R}$

$$B^*[v,u] := B[u,v] \qquad \forall u, v \in H_0^1(U)$$

3. $v \in H_0^1(U)$ is weak solution of the ADJOINT PROBLEM

$$\begin{cases} L^* v = f & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases} \quad \text{iff} \quad B^*[v,u] = \langle f, u\rangle_{L^2} \quad \forall u \in H_0^1(U).$$

MEMORIZE (YOU ARE SLOW)

THM: (EXISTENCE OF WEAK SOLUTIONS II)

(1) Precisely one of the following statements holds:

Fredholm Alternative

(a) $\forall f \in L^2(U)$ $\exists!$ weak solution $u \in H_0^1(U)$ to

$$(A) \begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

or else

(b) $\exists$ weak solution $u \not\equiv 0$ of

$$(B) \begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}.$$

2. If (b) holds, dimension of the subspace $N \subset H_0^1(U)$ of weak solutions of (B) is finite and equals the dimension of the subspace $N^* \subset H_0^1(U)$ of weak solutions to its adjoint problem $(B^*) \begin{cases} L^* v = 0 & \text{in } U \\ v = 0 & \text{on } \partial U \end{cases}.$

3. (A) has a weak solution $\iff$ $\langle f, v \rangle = 0$ $\forall v \in N^*$.

THM: (EXISTENCE OF WEAK SOLUTIONS III)

(1) $\exists\, \Sigma \subset \mathbb{R}$ at most countable (AKA (REAL) SPECTRUM OF L) such that

$$\begin{cases} Lu = \lambda u + f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique weak solution for each $f \in L^2(U)$ if and only if $\lambda \notin \Sigma$. (i.e., $(L - \lambda \mathrm{Id})$ is invertible $\Leftrightarrow \lambda \notin \Sigma$)

(2) If $\Sigma \subset \mathbb{R}$ is infinite, then $\Sigma = \{\lambda_k\}_{k=1}^{\infty}$ s.t.

$\lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots$ and $\lambda_k \nearrow \infty$ as $k \nearrow \infty$.

REMARK: $\begin{cases} Lu = \lambda u & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$ has a nontrivial solution $w \not\equiv 0$ $\Longleftrightarrow \lambda \in \Sigma$

For $L = -\Delta$, this is called HELMHOLTZ EQUATION

In this case, $\lambda \rightsquigarrow$ eigenvalue of L

$w \rightsquigarrow$ eigenfunction of L

—— // ——

REGULARITY OF WEAK SOLUTIONS

GOAL: if we have $u \in H_0^1(U)$ weak sol^{n} to $Lu = f$, can we say u is in fact a "classical solution"?

$\rightsquigarrow$ METHOD OF DIFFERENCE QUOTIENTS

DEF: For $u \in L^1_{loc}(U)$, $V \subset\subset U$,

$$D_i^h u(x) := \frac{u(x+he_i) - u(x)}{h}, \qquad \begin{array}{l} i = 1, \dots, n \\ x \in V \\ 0 < |h| < \operatorname{dist}(V, \partial U) \end{array}$$

$$D^h u(x) := \left(D_1^h u(x), \dots, D_n^h u(x) \right).$$

PROPOSITION:

(1) Let $1 \leq p < \infty$, $u \in W^{1,p}(U)$. $\forall V \subset\subset U$,

$$\| D^h u \|_{L^p(V)} \leq C \| Du \|_{L^p(U)} \quad \text{for } 0 < |h| < \tfrac{1}{2} \operatorname{dist}(V, \partial U)$$

(2) Let $1 < p < \infty$, $u \in L^p(U)$,

FALSE if $p = 1$

$$\sup_{|h| \in \left(0, \frac{1}{2}\operatorname{dist}(V, \partial U)\right)} \| D^h u \|_{L^p(V)} < \infty \implies u \in W^{1,p}(U)$$

LECTURE 15

May 17, 2026

$$Lu = - \sum_{i,j} \left(a_{ij}(x)\, u_{x_j} \right)_{x_i} + \sum_i b_i(x)\, u_{x_i} + c(x)\, u .$$

THM: (INTERIOR H^2 REGULARITY) Assume

$$a_{ij}(x) \in C^1(U), \quad b_i \in L^\infty(U), \quad c \in L^\infty(U), \quad i,j = 1, \dots, n.$$

and $f \in L^2(U)$. Let $u \in H^1(U)$ be a weak solution to the elliptic PDE

$$Lu = f \text{ in } U .$$

Then $u \in H^2_{loc}(U)$ and, $\forall\, V \subset\subset U$ open,

$$\| u \|_{H^2(V)} \leq C \left(\| f \|_{L^2(U)} + \| u \|_{L^2(U)} \right)$$

REMARKS: 1) No need for $u \overset{!}{\in} H^1_0(U)$ because we don't assume the boundary condition $u = 0$ on ∂U.

2) $u \in H^2_{loc}(U) \Rightarrow Lu = f$ a.e. in U i.e., u is an a.e. solution to PDE

Pf: $\forall v \in C_c^\infty(U)$, $B[u,v] = \langle f, v \rangle$

$u \in H^2_{loc}(U) \Rightarrow$ Integrate by parts to get:

$$B[u,v] = \langle Lu, v \rangle$$

i.e., $\langle Lu - f, v \rangle = 0 \quad \forall v \in C_c^\infty(U)$

i.e., $Lu = f$ a.e. by Fundamental Lemma of Calculus of Variations.
