LECTURE 1

Def: Let I be a countable set (state space)

$i \in I$ is state

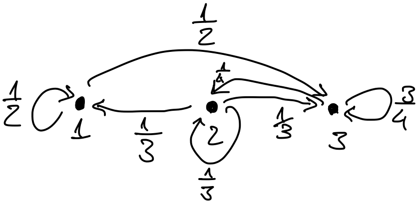
Let $\underset{\substack{\parallel \\ (\lambda_i : i\in I)}}{\lambda}$ be a probability measure on I (i.e., $0 \le \lambda_i < \infty \ \forall i$), if, in addition, the total mass $\sum_{i\in I} \lambda_i = 1$ we call λ a distribution.

Let $P = (p_{ij})_{i,j\in I}$ be a stochastic matrix, i.e.,

$$p_{ij} \ge 0, \quad \sum_j p_{ij} = 1 \quad \forall i .$$

p_{ij} represents moving from state i to state j.

$$\begin{pmatrix} 1/2 & 0 & 1/2 \\ 1/3 & 1/3 & 1/3 \\ 0 & 1/4 & 3/4 \end{pmatrix}$$



Def: (MARKOV CHAIN) $(X_n)_{\overset{\text{discrete time}}{n\ge 0}}$ is a Markov Chain with initial distribution λ and transition matrix P iff:

(1) X_0 has distribution λ (i.e., $\mathbb{P}(X_0 = i) = \lambda_i$)

(2) $\forall n \ge 0$, conditional on $\{X_n = i\}$, X_{n+1} has distribution $(p_{ij})_{j\in I}$ independent of $X_0, \ldots, X_n$.

Equivalently, these conditions can be written as

(1) $\mathbb{P}(X_0 = i) = \lambda_i$

(2) $\mathbb{P}(X_{n+1} = j \mid X_n = i) = p_{ij}$.

Thm: A discrete time random process $(X_n)_{n\geq 0}$ is Markov(λ, P) if and only if $\forall\, i_0, i_1, \dots, i_N \in I$ we have

$$\mathbb{P}(X_0 = i_0, X_1 = i_1, \dots, X_N = i_N) = \lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{N-1}, i_N}.$$

Pf: $(\Longrightarrow)$ Since $(X_n)_{n\geq 0}$ is Markov(λ, P)

$$\mathbb{P}(X_0 = i_0, \dots, X_N = i_N) = \mathbb{P}(X_0 = i_0)\, \mathbb{P}(X_1 = i_1, \dots, X_N = i_N \mid X_0 = i_0)$$

$$\vdots$$

$$= \mathbb{P}(X_0 = i_0)\, \mathbb{P}(X_1 = i_1 \mid X_0 = i_0) \cdot \mathbb{P}(X_N = i_N \mid X_0 = i_0, X_1 = i_1, \dots, X_{N-1} = i_{N-1})$$

Markov

$$= \mathbb{P}(X_0 = i_0)\, \mathbb{P}(X_1 = i_1 \mid X_0 = i_0) \cdots \mathbb{P}(X_N = i_N \mid X_{N-1} = i_{N-1})$$

$$= \lambda_{i_0} p_{i_0 i_1} \cdots p_{i_{N-1}, i_N}.$$

$(\Longleftarrow)$ If it holds for N, then we sum both sides over $i_N \in I$ and, by $\sum_{j\in I} p_{ij} = 1$, we have that it holds for $N-1$. So, by induction, $\mathbb{P}(X_0=i_0,\dots,X_n=i_n)=\lambda_{i_0}p_{i_0i_1}\cdots p_{i_{n-1},i_n}$ holds $\forall n=1,\dots,N$. In particular, $\mathbb{P}(X_0=i_0)=\lambda_{i_0}$. For $n=0,1,\dots,N-1$,

$$\mathbb{P}(X_{n+1}=i_{n+1}\mid X_n=i_n,\dots,X_0=i_0)$$

$$=\frac{\mathbb{P}(X_0=i_0,\dots,X_n=i_n,X_{n+1}=i_{n+1})}{\mathbb{P}(X_0=i_0,\dots,X_n=i_n)}$$

$$=p_{i_n,i_{n+1}}.$$

$\blacksquare$

WEAK MARKOV PROPERTY (i.e., Markov chains have no memory)

Write $\delta_i=(\delta_{ij})_{j\in I}$ for unit mass at i, where

$$\delta_{ij}=\begin{cases}1, & i=j\\ 0, & i\neq j\end{cases}.$$

Thm: (Weak Markov Property) Let $(X_n)_{n\geq 0}$ be Markov(λ,P). Then conditional on $\{X_m=i\}$, $(X_{m+n})_{n\geq 0}$ is Markov(δ_i,P) and is independent of $X_0,\dots,X_m$.

n-STATE PROBABILITY

Goal: Compute probability of Markov chain being in a given state after n steps. (reduces to computing powers of P)

λ — row vector indexed by I $\qquad (\lambda P)_j := \sum_{i\in I} \lambda_i P_{ij}$.

P — matrix indexed by $I \times I$ $\qquad (P^2)_{ik} := \sum_{j \in I} P_{ij} P_{jk}$.

$Id = (Id_{ij})_{i,j\in I} = \delta_{ij}$.

- If $\lambda_i > 0$, write $\mathbb{P}_i(A) := \mathbb{P}(A \mid X_0 = i)$

n-STATE PROBABILITY COMPUTATION

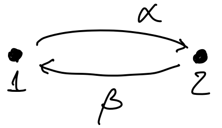
Thm: Let $(X_n)_{n\geq 0}$ be Markov(λ, P). Then $\forall n, m \geq 0$,

(1) $\mathbb{P}(X_n = j) = (\lambda P^n)_j$

(2) $\mathbb{P}_i(X_n = j) = \mathbb{P}(X_{n+m} = j \mid X_m = i) = p_{ij}^{(n)}$ → $p_{ij}^{(n)} := (P^n)_{ij}$ (i,j)-entry of P^n

e.g.: $$P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} \qquad P^2 = \begin{pmatrix} P_{11}^2 + P_{12}P_{21} & \cdots \\ \cdots & \cdots \end{pmatrix} .$$

EXAMPLE:



$$P = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

$$P^{n+1} = P^n P \rightsquigarrow p_{11}^{(n+1)} = p_{12}^{(n)} \beta + p_{11}^{(n)} (1-\alpha)$$

$$p_{11}^{(n)} + p_{12}^{(n)} = \mathbb{P}_1(X_n = 1 \text{ or } 2) = 1$$

So, can rewrite the recurrence as:

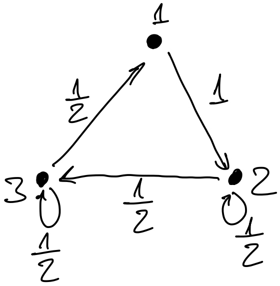
$$p_{11}^{(n+1)} = (1-\alpha-\beta)\,p_{11}^{(n)} + \beta\,, \quad p_{11}^{(0)} = 1\,.$$

Solution to recurrence

$$p_{11}^{(n)} = \begin{cases} \dfrac{\beta}{\alpha+\beta} + \dfrac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n\,, & \alpha+\beta>0 \\ 1\,, & \alpha+\beta=0 \end{cases} .$$

◢

EXAMPLE:



$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix}$$

Find $\mathbb{P}(X_n = 1 \mid X_0 = 1) = \mathbb{P}_1(X_n = 1) = p_{11}^{(n)}$.

To do this, diagonalize $P = UDU^{-1} \Longrightarrow P^n = UD^nU^{-1}$

Note: $\lambda = 1$ always eigenvalue since $\vec{1}$ always eigenvector

since P is stochastic

$$\det(P - \lambda\,\mathrm{Id}) = \cdots = (-\lambda+1)\left(\lambda^2 + \frac{1}{4}\right) \rightsquigarrow \begin{array}{l} \lambda = 1 \\ \lambda = \pm\frac{i}{2} \end{array}$$

So,

$$P = U \begin{pmatrix} 1 & 0 & 0 \\ 0 & i/2 & 0 \\ 0 & 0 & -i/2 \end{pmatrix} U^{-1}$$

$$P^n = U \begin{pmatrix} 1^n & 0 & 0 \\ 0 & (i/2)^n & 0 \\ 0 & 0 & (-i/2)^n \end{pmatrix} U^{-1}$$

hence $p_{11}^{(n)} = a \cdot 1^n + b \cdot \left(\frac{i}{2}\right)^n + c\left(-\frac{i}{2}\right)^n$, a, b, c constants

$$\left(\pm\frac{i}{2}\right)^n = \left(\frac{1}{2}\right)^n e^{\pm in\pi/2} = \left(\frac{1}{2}\right)^n \left(\cos\frac{n\pi}{2} \pm i \sin\frac{n\pi}{2}\right)$$

$$\wr\downarrow$$

$$p_{11}^{(n)} = \alpha + \left(\frac{1}{2}\right)^n \left(\beta \cos\frac{n\pi}{2} + \gamma \sin\frac{n\pi}{2}\right),$$

α, β, γ constants.

$$\begin{aligned} &\underline{n=0}: \quad 1 = p_{11}^{(0)} = \alpha + \beta && && \alpha = \frac{1}{5} \\ &\underline{n=1}: \quad 0 = p_{11}^{(1)} = \alpha + \frac{1}{2}\gamma && \xrightarrow[\text{system}]{\text{solve}} && \beta = \frac{4}{5} \\ &\underline{n=2}: \quad 0 = p_{11}^{(n)} = \alpha - \frac{1}{4}\beta && && \gamma = -\frac{2}{5} \end{aligned}$$

Thus: $$p_{11}^{(n)} = \frac{1}{5} + \left(\frac{1}{2}\right)^n \left(\frac{4}{5}\cos\frac{n\pi}{2} - \frac{2}{5}\sin\frac{n\pi}{2}\right).$$

◢

CLASS STRUCTURE (break Markov chains into smaller pieces)

Def: State i *leads to* j ($i \to j$) if

$$\mathbb{P}_i(X_n = j \text{ for some } n \geq 0) > 0.$$

We say i *communicates with* j ($i \leftrightarrow j$) if we have both $i \to j$, $j \to i$.

Def: Markov chain with 1 communicating class is *irreducible*

Def: Class C is *closed* if $i \to j \Rightarrow j \in C$.

Def: i is *absorbing* if $\{i\}$ is a closed class.

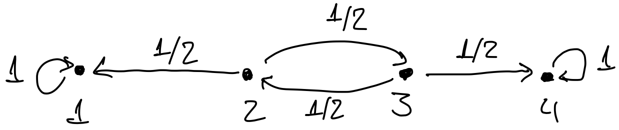
Def: $A \subset I$, the r.v. $H^A := \inf\{n \geq 0 : X_n \in A\}$ is called the *hitting time* of A. ($H^A = \infty$ if $X_n \notin A \ \forall n$)

Def: $h_i^A := \mathbb{P}_i(H^A < \infty)$ is *absorbing/hitting probability*, where

$$\mathbb{P}_i(E) = \mathbb{P}(E \mid X_0 = i)$$

$$k_i^A := \mathbb{E}_i(H^A) = \sum_n n\,\mathbb{P}(H^A = n) .$$

EXAMPLE: $h_i = \mathbb{P}_i(\text{hit } 4)$ $\quad k_i = \mathbb{E}_i(\text{time to hit } \{1,4\})$.



$h_1 = 0 \qquad k_1 = 0$

$h_4 = 1 \qquad k_4 = 0$

$$h_2 = \frac{1}{2} h_1 + \frac{1}{2} h_3 \qquad k_2 = \underbrace{\boxed{1}}_{\text{TIME FOR } 1^{st} \text{ STEP}} + \frac{1}{2} k_1 + \frac{1}{2} k_3$$

$$h_3 = \frac{1}{2} h_4 + \frac{1}{2} h_2 \qquad k_3 = \boxed{1} + \frac{1}{2} k_4 + \frac{1}{2} k_2$$

Thus $h_2 = \frac{1}{2} h_3 = \frac{1}{4} h_2 + \frac{1}{4} \longrightarrow h_2 = \frac{1}{3}$ = prob. of hitting 4 starting at 2

$k_2 = 1 + \frac{1}{2}\left(1 + \frac{1}{2} k_2\right) \longrightarrow k_2 = 2$ = mean time for absorption

Thm: The vector of hitting probabilities $h^A = (h_i^A)_{i \in I}$ is the minimal nonnegative solution to the system of linear eqs.:

$$\begin{cases} h_i^A = 1 & \text{for } i \in A \\ h_i^A = \sum_{j \in I} p_{ij} h_j^A & \text{for } i \notin A \end{cases}$$

Minimality means that if $x = (x_i)_{i \in I}$ is another solution w/ $x_i \geq 0$ $\forall i$, then $x_i \geq h_i$ $\forall i$.

Thm: The vector of mean hitting times $k^A = (k_i^A)_{i \in I}$ is the minimal nonnegative solution to the system of lin. eqs.:

$$\begin{cases} k_i^A = 0 & \text{for } i \in A \\ k_i^A = 1 + \sum_{j \notin A} p_{ij} h_j^A & \text{for } i \notin A \end{cases}$$

LECTURE 2

DISCRETE TIME MARKOV CHAIN

$(X_n)_{n\geq 0}$

I state space (finite or countable)

λ initial distribution on I (tells us how our system starts)

P transition matrix

$$p_{ij}^{(n)} := (P^n)_{ij} = \mathbb{P}(X_n = j \mid X_0 = i) = \mathbb{P}_i(X_n = j).$$

MARKOV PROPERTY:

- WEAK VERSION: Let (X_n) be Markov(λ, P) then, conditional on $\{X_m = i\}$, $(X_{n+m})_{n\geq 0}$ is Markov (δ_i, P) and independent of $(X_0, X_1, \ldots, X_{m-1})$.

Def: (Stopping Time) A random variable $T : \Omega \to \mathbb{N} \cup \{0, \infty\}$ s.t. the event $\{T = n\}$ "only depends on $X_0, X_1, \ldots, X_n$" or, more precisely, $\{T = n\} \in \sigma(X_0, \ldots, X_n)$ $\forall n$.

Def: (First Passage Time) $T_j = \inf\{n \geq 1 : X_n = j\}$.

Note: $\{T_j = m\} = \{X_0 \neq j, X_1 \neq j, \ldots, X_{n-1} \neq j, X_m = j\}$ so clearly $\{T_j = m\} \in \sigma(X_0, \ldots, X_m)$.

Def: (Hitting Time of A) Set $A \subset I$,

$$H^A = \inf_n \{n \geq 0 : X_n \in A\}$$

Obs: Last Exit Time from A is a r.v. BUT NOT A STOPPING TIME

Thm: (STRONG MARKOV PROPERTY) Let (X_n) be Markov (λ, P) and T be stopping time. Then, conditional on $T<\infty$ and $\{X_T = i\}$, we have $(X_{T+n})_{n\geq 0}$ is Markov (δ_i, P) and independent of $X_0, \dots, X_{T-1}$.

Pf: Let B be an event determined by $X_0, X_1, \dots, X_T$.

$$\mathbb{P}(\{X_T = j_0, X_{T+1} = j_1, \dots, X_{T+n} = j_n\} \cap B \mid T<\infty, X_T = i)$$

Since T is stopping time (and finite)

$$= \sum_{k\in\mathbb{N}} \mathbb{P}(\{X_T = j_0, \dots, X_{T+n} = j_n\} \cap B \cap \{T=k\} \mid T<\infty, X_T = i)$$

But $B \cap \{T=k\}$ is determined by $X_0, \dots, X_k$. So, we can apply the mark property to each $k \in \mathbb{N}$:

$$= \sum_{k\in\mathbb{N}} \mathbb{P}(\{X_k = j_0, \dots, X_{k+n} = j_n\} \cap B \cap \{T=k\} \mid T<\infty, X_T = i)$$

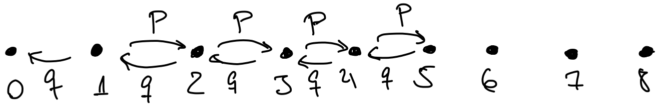
$$\overset{\text{Weak MP}}{=} \sum_{k\in\mathbb{N}} \mathbb{P}_i(\{X_0=j_0,\dots,X_n=j_n\})\,\mathbb{P}(B\cap\{T=k\}\mid T<\infty, X_T=i)$$

$$= \mathbb{P}_i(\{X_0=j_0,\dots,X_n=j_n\})\,\mathbb{P}(B\mid T<\infty, X_T=i).$$

■

EXAMPLE: $I = \mathbb{N}\cup\{0\}$

Prob. going up $= p$; prob going down $= q$. $\quad 0<p=1-q<1$.



$H_j := \inf\{n\geq 0 : X_n = j\}$ (hitting time of j)

$h_i := \mathbb{P}_i(\text{hitting } 0)$ (i.e., prob. of hitting 0 starting at i)

$$\begin{cases} h_0 = 1 \\ h_i = p h_{i+1} + q h_{i-1}, \quad i \geq 1. \end{cases}$$

By working out the recurrence above, we have

$$h_i = A + B\left(\frac{q}{p}\right)^i, \quad p\neq q.$$

(i) if $p<q$, then $B=0$ so $h_i=1$ $\forall i$.

(ii) if $p=q$, then $h_i = A + iB$ so $B=0$ so $h_i = 1$ $\forall i$.

↖ Called "GAMBLER'S RUIN"

(iii) if $p>q$, then $1 = h_0 = A+B$ so

$$h_i = \left(\frac{q}{p}\right)^i + A\left(1-\left(\frac{q}{p}\right)^i\right) \geq 0.$$

hence $A \geq 0$. But there are ∞-many solution. From last time, the solution is the minimum nonnegative solution; i.e, when $A=0$. So,

$$h_i = \left(\frac{q}{p}\right)^i.$$

→ Exponentially decaying.

Can use Markov property to compute distribution of stopping time. Define

Generating Function of H_0:

$$\phi(s) := \mathbb{E}_1[s^{H_0}] = \sum_{n\in\mathbb{N}} s^n \, \mathbb{P}_1(H_0 = n)$$

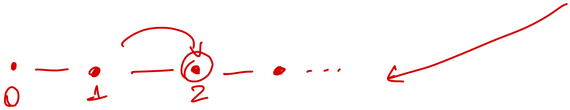
Under $\mathbb{P}_2$: $H_0 = H_1 + \tilde{H}_0$ has same distribution of H_1 and is indep of H_1 b/c we can go right a lot before going one step to the left of starting pt.

$$\Rightarrow \mathbb{E}_2(s^{H_0}) = \mathbb{E}_2(s^{H_1+\tilde{H}_0}) \overset{\text{indep}}{=} \mathbb{E}_2(s^{H_1})\,\mathbb{E}_2(s^{\tilde{H}_0})$$

$\tilde{H}_0$ & H_1 have same distrib.

$$= \left[\mathbb{E}(s^{H_1})\right]^2 = \phi(s)^2.$$

Now, $\phi(s) = \mathbb{E}_1(s^{H_0}) = p\,\underbrace{\mathbb{E}_1(s^{H_0} \mid X_1 = 2)} + q\,\mathbb{E}_1(s^{H_0} \mid X_1 = 0)$ $\Rightarrow H_1 = 1$



$$\underbrace{\qquad\qquad\qquad}$$
$$q\,\overset{\parallel}{\mathbb{E}_1}(s) = qs$$

$H_0 = 1 + \overline{H_0}$,

$\overline{H_0} \sim H_0$ under $\mathbb{P}_2$

by Markov property.

$$= ps\, \mathbb{E}_1\left(s^{\overline{H_0}} \mid X_1 = 2\right) + qs$$

$$= ps\,\phi(s)^2 + qs$$

Thus: $\boxed{\phi(s) = ps\,\phi(s)^2 + qs}$ (Functional equational)

$$\rightsquigarrow \phi(s) = \frac{1 \pm \sqrt{1-4pqs^2}}{2ps}$$

but "+" gives an unbounded solution near $s \sim 0$.

Upshot: $\phi(s) \overset{(A)}{=} \dfrac{1-\sqrt{1-4pqs^2}}{2ps}$.

Note: $\phi(s) \overset{\text{def}}{=} s\,\boxed{\mathbb{P}_1(H_0 = 1)} + s^2\,\boxed{\mathbb{P}_1(H_0 = 2)} + \dots$

so can expand (A) in a power series w/ powers of s and match coeffs.:

$$\phi(s) = \boxed{q}\,s + \boxed{0}\cdot s^2 + (pq)^2 \cdot s^3 + \dots$$

Finally,

cl: $\mathbb{E}_1(H_0) = \lim\limits_{s \to 1^-} \phi'(s) = \dots = \dfrac{1}{q-p}$.

LECTURE 3

Def: Let $(X_n)_{n\geq 0}$ be Markov (λ, P). We say that state i is:

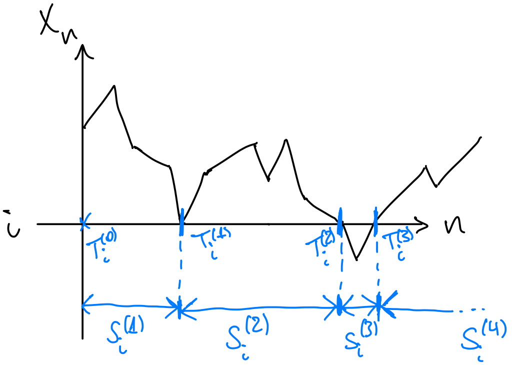
- recurrent if $\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 1$;
  ↳ keep coming back to this state
- transient if $\mathbb{P}_i(X_n = i \text{ for infinitely many } n) = 0$.
  ↳ leave state forever

Def: (First Passage Time) $T_i := \inf\{n \geq 0 : X_n = i\}$. $\inf \emptyset = \infty$

Def: (r-th Passage Time) $T_i^{(0)} = 0$, $T_i^{(1)} = T_i$,
and $T_i^{(r+1)} = \inf\{n \geq T_i^{(r)} + 1 : X_n = i\}$, $r = 0, 1, 2, \dots$.

Def: length of r-th excursion to i defined as

$$S_i^{(r)} := \begin{cases} T_i^{(r)} - T_i^{(r-1)}, & \text{if } T_i^{(r-1)} < \infty \\ 0, & \text{otherwise} \end{cases}.$$



Key: Analysis of recurrence/transience relies on finding joint distributions of these excursion lengths.

Lemma: For $r = 2, 3, \ldots$, conditional on $T_i^{(r-1)} < \infty$, we have $S_i^{(r)}$ is independent of $\{X_m : m \le T_i^{(r-1)}\}$ and

$$\mathbb{P}\left(S_i^{(r)} = n \mid T_i^{(r-1)} < \infty\right) = \mathbb{P}_i(\underset{\nwarrow \text{ 1st passage time}}{T_i} = n)$$

Pf: Markov property on $T := T_i^{(r-1)}$. Then get that $X_T = i$ on $T < \infty$. So, conditional on $T < \infty$, $(X_{T+n})_{n \ge 0}$ is Markov(δ_i, P) and indep. of $X_0, \ldots, X_T$. But

$$S_i^{(r)} = \inf\{n \ge 1 : X_{T+n} = i\} \implies$$

$S_i^{(r)}$ is the first passage time of $(X_{T+n})_{n \ge 0}$ to state i.

■

Def: **Number of Visits** V_i to i as $V_i := \sum_{n=0}^{\infty} \mathbb{1}\{X_n = i\}$.

Note: $$\mathbb{E}_i(V_i) = \mathbb{E}_i \sum_{n=0}^{\infty} \mathbb{1}_{\{X_n = i\}} = \sum_{n=0}^{\infty} \mathbb{E}_i(\mathbb{1}_{\{X_n = i\}})$$

$$= \sum_{n=0}^{\infty} \mathbb{P}_i(X_n = i) = \sum_{n=0}^{\infty} P_{ii}^{(n)}.$$

<u>Def</u>: <u>Return Probability</u> $f_i := \mathbb{P}_i(T_i < \infty)$

<u>Lemma</u>: Compute the distribution of V_i using the return probability: for $r = 0, 1, 2, \ldots$, we have $\mathbb{P}_i(V_i > r) = f_i^r$.

Pf: Induction.

<u>Note</u>: Useful formula: $\sum_{r=0}^{\infty} \mathbb{P}(V > r) = \sum_{v=1}^{\infty} v\,\mathbb{P}(V = v) = \mathbb{E}(V)$.

DETERMINE TRANSIENT VS. RECURRENT

<u>Thm</u>: The following dichotomy holds:

- if $\mathbb{P}_i(T_i < \infty) = 1$, then i is recurrent and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$
- if $\mathbb{P}_i(T_i < \infty) < 1$, then i is transient and $\sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$

In particular, every state is either transient or recurrent.

Pf: If $\mathbb{P}_i(T_i < \infty) = 1$ then by lemma above

$$\mathbb{P}_i(V_i = \infty) = \lim_{r \to \infty} \mathbb{P}_i(V_i > r) = 1$$

So i is recurrent and $\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i(V_i) = \infty$.

If $\mathbb{P}_i(T_i<\infty) < 1$, i.e., if $f_i < 1$, then by the same lemma above,

$$\sum_{n=0}^{\infty} p_{ii}^{(n)} = \mathbb{E}_i(V_i) = \sum_{r=0}^{\infty} \mathbb{P}_i(V_i > r) \overset{\text{lemma}}{=} \sum_{r=0}^{\infty} f_i^{\,r}$$

$$\overset{f_i<1}{=} \frac{1}{1-f_i} < \infty .$$

So, $\mathbb{P}_i(V_i=\infty) = \lim_{r\to\infty} \mathbb{P}_i(V_i>r) = 0$, hence i is transient.

■

Thm: Let C be a communicating class. Then either all states are transient or recurrent.

Pf: Take any states $i,j\in C$. Suppose i is transient. Then $\exists\, n,m\geq 0$ s.t. $p_{ij}^{(n)} > 0$ and $p_{ji}^{(m)} > 0$ (since C communicating class). Moreover $\forall r \geq 0$,

$$p_{ii}^{(n+m+r)} \geq p_{ij}^{(n)}\, p_{ji}^{(m)}\, p_{jj}^{(r)} \Longrightarrow \sum_{r=0}^{\infty} p_{jj}^{(r)} \leq \frac{1}{p_{ij}^{(n)}\, p_{ji}^{(m)}} \sum_{r=0}^{\infty} p_{ii}^{(n+r+m)} \overset{\text{Thm above}}{<\infty}$$

Thus: j transient by Thm above.

■

Thm: Every recurrent class is closed.

Pf: Let C be not closed class. Then $\exists i \in C, j \notin C$ and $m \geq 1$ s.t. $\mathbb{P}_i(X_m = j) > 0$. Since

$$\mathbb{P}_i\left(\{X_m = j\} \cap \{X_n = i \text{ for infinitely many } n\}\right) = 0,$$

we have:

$$\mathbb{P}_i(X_n = i \text{ for infinitely many } n) < \infty$$

so i is not recurrent and so neither is C. ■

Thm: Every finite closed class is recurrent.

Pf: Say C is finite & closed and $(X_n)_{n\geq 0}$ starts in C. Then

$$0 < \mathbb{P}(X_n = i \text{ for infinitely many } n)$$

strong Markov property

$$= \mathbb{P}(X_n = i \text{ for some } n)\, \mathbb{P}_i(X_n = i \text{ for infinitely many } n)$$

$\Longrightarrow$ i is <u>not</u> transient i.e. C is recurrent. ■

Upshot:

$$\begin{tikzcd} \boxed{\text{Recurrent}} \arrow[r, bend left] & \boxed{\text{Closed}} \arrow[l, bend left, "\text{(only if } C \text{ is FINITE)}"] \end{tikzcd}$$

chain can get from one state to the other w/ positive prob.

Thm: Suppose P is irreducible and recurrent. Then, for all $j \in I$, we have $\mathbb{P}(T_j < \infty) = 1$.

Pf: By the Markov property:

$$\mathbb{P}(T_j < \infty) = \sum_{i \in I} \mathbb{P}(X_0 = i)\, \mathbb{P}_i(T_j < \infty)$$

$\Rightarrow$ suffices to show $\mathbb{P}_i(T_j < \infty) = 1 \quad \forall i \in I$.

Choose $m \geq 0$ s.t. $p_{ji}^{(m)} > 0$. Then

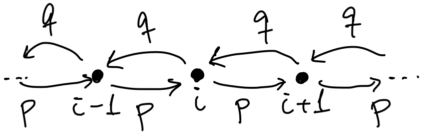
$$\begin{aligned}
1 &= \mathbb{P}_j(X_n = j \text{ for infinitely many } n) \\
&= \mathbb{P}_j(X_n = j \text{ for some } n \geq m+1) \\
&= \sum_{k \in I} \mathbb{P}_j(X_n = j \text{ for some } n \geq m+1 \mid X_m = k)\, \mathbb{P}_j(X_m = k) \\
\text{Markov property} \rightarrow &= \sum_{k \in I} \mathbb{P}_k(T_j < \infty)\, p_{jk}^{(m)} .
\end{aligned}$$

But $\sum_{k \in I} p_{jk}^{(m)} = 1$ b/c P stochastic; i.e. $\mathbb{P}_k(T_j < \infty)\ \forall k \in I$. ■

RECURRENCE & TRANSIENCE OF RANDOM WALKS

As seen before, finite-state classes are "boring" because every recurrent class is closed, and every finite and closed class is recurrent. ⟶ Interesting things happen for classes w/ infinite state spaces (e.g.: random walks)

EXAMPLE: (Simple Random Walk on $\mathbb{Z}$)



$$0 < p = 1-q < 1$$

Start at 0: cannot return to zero after an odd number of steps

$$\Longrightarrow P_{00}^{(2n+1)} = 0 \quad \forall n.$$

Any given seq. of $2n$ steps from 0 to 0 occurs w/ probability $p^n q^n$ (n steps out and n steps in). There are $\binom{2n}{n}$ such possible choices of steps. Thus:

$$P_{00}^{(2n)} = \binom{2n}{n} p^n q^n = \frac{(2n)!}{(n!)^2}(pq)^n \underset{\uparrow}{\sim} \frac{(4pq)^n}{A\sqrt{n/2}}$$

Stirling's formula: $n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$ as $n \to \infty$.

- $p = q = \frac{1}{2}$: then $4pq = 1$ so for some N and all $n \geq N$,

$$P_{00}^{(2n)} \geq \frac{1}{2A\sqrt{n}} \Longrightarrow \sum_{n=N}^{\infty} P_{00}^{(2n)} \geq \frac{1}{2A}\sum_{n=N}^{\infty}\frac{1}{\sqrt{n}} = \infty$$

Thus: the random walk is recurrent in this case.

- $p \neq q$: then $4pq := r < 1$, so, for some N,

$$\sum_{n=N}^{\infty} P_{00}^{(n)} \leq \frac{1}{A} \sum_{n=N}^{\infty} r^n < \infty$$

Thus: the random walk is transient in this case. ◢

INVARIANT DISTRIBUTIONS (long-term behavior of Markov chains)

row vector

Def: We say a measure $\lambda = (\lambda_i)_{i \in I}$ is invariant iff $\lambda P = \lambda$.

Synonym: stationary

Thm: Let $(X_n)_{n \geq 0}$ be Markov(λ, P) and suppose that λ is invariant for P. Then $(X_{m+n})_{n \geq 0}$ is also Markov(λ, P).

Pf: $P(X_m = i) = (\lambda P^m)_i = \lambda_i$ $\forall i$ b/c invariant.
Conditional on $\{X_{m+n} = i\}$, X_{m+n+1} is indep. of $X_m, \dots, X_{m+n}$ and has distrib. $(p_{ij})_{i,j \in I}$. ∎

Thm: Let I be finite. Suppose for some $i \in I$ that $p_{ij}^{(n)} \xrightarrow{n \to \infty} \pi_j$ $\forall j \in I$. Then $\pi_j = (\pi_j)_{j \in I}$ is an invariant distrib.

Def: (Expected Return Time) $m_i := \mathbb{E}_i(T_i)$

Def: We say the state i is

- POSITIVE RECURRENT if $m_i < \infty$
- NULL RECURRENT if $m_i = \infty$

Thm: Let T be irreducible (i.e., it has only one class; i.e., you can go from every state to every state w/ positive prob.).
TFAE:

(i) Every state is positive recurrent

(ii) Some state is positive recurrent

(iii) P has invariant distribution (of mass 1) π

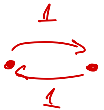
Moreover, if (iii) holds, then $m_i = \frac{1}{\pi_i}$ for all i.

LECTURE 4

CONVERGENCE TO EQUILIBRIUM

Given (X_n) and (λ, P), $\lim_n p_{ij}^{(n)} = ?$

WARNING: limit does not always exist $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



$(d=2 \text{ here})$

Def: P is irreducible if $\forall i,j \ \exists n$ s.t. $p_{ij}^{(n)} > 0$.

Def: Period of P: $d := \gcd\{n : p_{ii}^{(n)} > 0\}$

Def: P is aperiodic if $d = 1$.

Note: i is aperiodic iff $\exists n_0 : p_{ii}^{(n)} > 0 \quad \forall n \geq n_0$

$$\iff \gcd\{n : p_{ii}^{(n)} > 0\} = 1.$$

Thm: (Convergence to Equilibrium) Let P be irreducible and aperiodic. Suppose that P has some invariant distribution π. Let λ be any distribution. Then, if $(X_n)_{n\geq 0}$ is Markov(λ, P), we have that

$$p_{ij}^{(n)} \xrightarrow{n\to\infty} \pi_j \quad \forall i,j \in I.$$

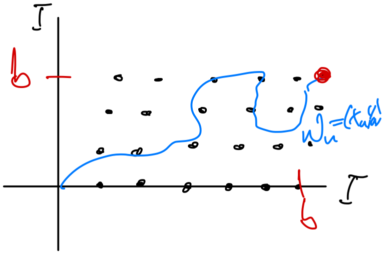
Pf: Consider two <u>indep.</u> copies of Markov chain $X_n \sim (\lambda, P)$
$Y_n \sim (\pi, P)$

Define Markov chain in 2 variables $W_n := (X_n, Y_n) \leftarrow$ MC on $I \times I$

Distrib. of W_n : $\mu_{(i,k)} = \lambda_i \pi_k$

Markov$(\mu, \tilde{P})$

$\tilde{P}_{(i,k)(j,l)} = P_{ij} P_{kl}$



Note: P aperiodic $\Rightarrow$ $\tilde{P}$ irreducible

$$\left(\begin{array}{l} P \text{ aperiodic } \Rightarrow \\ \exists n \quad \tilde{P}^{(n)}_{(i,k)(j,l)} \geq P^{(n)}_{ij} P^{(n)}_{kl} > 0 \end{array} \right)$$

$\tilde{P}$ has invariant distribution $\tilde{\pi}_{(i,k)} = \pi_i \pi_k \Rightarrow \tilde{P}$ POSITIVE RECURRENT

Fix $b \in I$ and set $T := \inf\{n \geq 1 : X_n = Y_n = b\}$.

$\uparrow$ Well defined (i.e.: $\mathbb{P}(T < \infty) = 1$)
b/c $\tilde{P}$ positive recurrent so it will hit <u>every</u> state w/ positive prob.

$$\mathbb{P}(T < \infty) = 1$$
$$\parallel$$
$$\mathbb{P}((X_n, Y_n) = (b, b) \text{ for some } n)$$

Define coupling:

$$Z_n := \begin{cases} X_n & \text{if } n < T \\ Y_n & \text{if } n \geq T \end{cases} \qquad Z'_n := \begin{cases} Y_n & \text{if } n < T \\ X_n & \text{if } n \geq T \end{cases}.$$

Note: $W_n' := (Z_n, Z_n')$ is Markov$(\mu, \tilde{P})$ b/c of Markov property i.e., once we get to b, the future only depends on that state so it doesn't matter who is X and who is Y.

$\Rightarrow$ Z_n is also Markov (by def.)

Z_n is Markov(λ, P) b/c it starts as λ i.e., $Z_n \sim X_n$

$$\mathbb{P}(Z_n = j) = \mathbb{P}(X_n = j \text{ and } n < T) + \mathbb{P}(Y_n = j \text{ and } n \geq T)$$

$X_n \sim Z_n$ and π_j is invariant (i.e., $(\pi P^n)_j = \pi_j \ \forall n$)

$$|\mathbb{P}(X_n = j) - \pi_j| = |\mathbb{P}(Z_n = j) - \mathbb{P}(Y_n = j)|$$

$$= |\mathbb{P}(X_n = j \text{ and } n < T) - \mathbb{P}(Y_n = j \text{ and } n < T)|$$

$$\leq \mathbb{P}(n < T) \xrightarrow{n \to \infty} 0$$

$\Rightarrow$ convergence to equilibrium (does not tell rate of converg.).

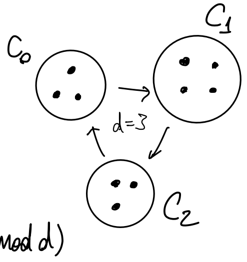
■

Thm: Let P be irreducible w/ period d.

Then $\exists$ partition $I = C_0 \cup C_1 \cup \cdots \cup C_{d-1}$

s.t. (1) $p_{ij}^{(n)} > 0$ only if $i \in C_r$ and $j \in C_{r+n \,(\text{mod } d)}$

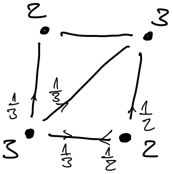
(2) $p_{ij}^{(nd)} > 0$ for all suff. large n and all $i, j \in C_r$.



Let λ be a distribution on C_0. Suppose $(X_n)_{n\geq 0}$ is Markov (λ, P). Then, for all $r = 0, \_, d-1$ and $j \in C_r$

$$\mathbb{P}(X_{nd+r} = j) \xrightarrow{n\to\infty} \frac{d}{m_j}$$

where $m_j = \mathbb{E}_j(\underset{\uparrow \text{1st passage time}}{T_j})$ is the expected return time to j.



(not directed)

<u>EXAMPLE</u>: G finite graph
Simple Random Walk on G

$$\begin{cases} v_i = \text{valency of } i \in I \\ P_{ij} = \dfrac{1}{v_i} \quad \text{if } (i,j) \in \text{Edge}(G) \end{cases}$$

<u>Q</u>: Invariant distribution ?

<u>Def</u>: Distribution λ is in <u>detailed</u> <u>balance</u> with P if

$$\lambda_i P_{ij} = \lambda_j P_{ji} \quad \text{for all } i,j \in I$$

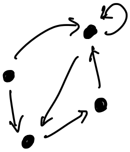
<u>Lemma</u>: λ detailed balance $\Longrightarrow$ λ invariant

$$(\lambda P)_i = \sum_j \lambda_j P_{ji} \overset{D.B.}{=} \sum_j \lambda_i P_{ij} = \lambda_i \sum_j P_{ij} = \lambda_i .$$

Take $\lambda_i = \dfrac{v_i}{\sum_{i\in G} v_i}$ distribution on G. Then $\lambda = (\lambda_i)$ is in detailed balance hence invariant.

LECTURE 5

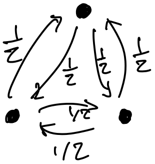
TIME REVERSAL

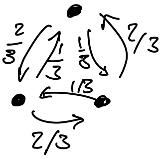


Given $(X_n)_{n\geq 0}$ Markov (λ, P) chain, consider, for N, $Y_n := X_{N-n}$ the backward process.

Q: Is it still a Markov chain?

Q: Can it have the same parameters as the forward one? (If so, called reversible)





Thm: Let P be irreducible with invariant distribution (π_i). Suppose $(X_n)_{n\geq 0}$ is Markov(π, P) and set $Y_n := X_{N-n}$. Then $(Y_n)_{n\geq 0}$ is Markov $(\pi, \hat{P})$ with

$$\left(\hat{P}_{ij} = \frac{\pi_j}{\pi_i} P_{ji}\right) \iff \pi_j \hat{P}_{ji} = \pi_i P_{ij}$$

→ i.e., same distrib. but the transition probs. are not necessarily equal

and $\hat{P}$ is also irreducible with distribution π.

Pf: $\hat{P}$ is stochastic: $\sum_i \hat{P}_{ji} = \frac{1}{\pi_j} \sum_i \pi_i P_{ij}$

π is invariant

$\hookrightarrow = \frac{\pi_j}{\pi_j} \sum_i P_{ij} = 1$ ✓

π is invariant for $\hat{P}$: $\sum_j \pi_j \hat{P}_{ji} = \sum_j \pi_i P_{ij} \overset{?✓}{=} \pi_i$

↑ π is P-invariant

Markov Property:

Need to check this

$$P(Y_0 = i_0, Y_1 = i_1, \ldots, Y_N = i_N) \overset{?}{=} \pi_{i_0} \hat{P}_{i_0 i_1} \cdots \hat{P}_{i_{N-1}, i_N}$$

$$\|$$

$$P(X_0 = i_N, X_1 = i_{N-1}, \ldots, X_N = i_0)$$

$$\|$$

$$\pi_{i_N} P_{i_N i_{N-1}} P_{i_{N-1} i_{N-2}} \cdots P_{i_1, i_0}$$

$=$ ✓

$$\pi_{i_3} P_{i_3 i_2} P_{i_2 i_1} P_{i_1 i_0} = \pi_{i_2} \hat{P}_{i_2 i_3} P_{i_2 i_1} P_{i_1 i_0} = \hat{P}_{i_2 i_3} \pi_{i_2} P_{i_2 i_1} P_{i_1 i_0}$$

$$= \hat{P}_{i_2 i_3} \hat{P}_{i_1 i_2} \pi_{i_1} P_{i_1 i_0} = \hat{P}_{i_2 i_3} \hat{P}_{i_1 i_2} \hat{P}_{i_0 i_1} \pi_{i_0}$$

$$= \pi_{i_0} \hat{P}_{i_0 i_1} \hat{P}_{i_1 i_2} \hat{P}_{i_2 i_3}$$

■

Def: The Markov chain $(X_n)_{n\geq 0} \sim (\lambda, P)$ is reversible if, for every N, $Y_n := X_{N-n}$, $(Y_n)_{n\geq 0}$ is also Markov (λ, P).

Def: λ and P are in detailed balance if

$$\lambda_i P_{ij} = \lambda_j P_{ji} \quad \forall i,j \in I.$$

Lemma: (λ, P) detailed balance $\Rightarrow$ λ is P-invariant.

Thm: Let P be a stochastic irreducible matrix and λ be any distribution. Suppose $(X_n)_{n\geq 0}$ is Markov (λ, P). TFAE

(i) $(X_n)_{n\geq 0}$ is reversible

(ii) (λ, P) are in detailed balance

Pf: Claim:

$$\begin{tikzcd} X_n \text{ reversible} \arrow[r, Rightarrow] \arrow[d] & \lambda \text{ invariant.} \\ (\lambda,P)\sim Y_n \sim X_n \arrow[r] & \begin{array}{l}\lambda = \text{initial distribution of } Y_0 \\ \text{but also } \underline{\text{time-1 distribution}} \\ \underline{\text{of } (X_n)} = \lambda P\end{array} \arrow[u, Rightarrow, "\lambda=\lambda P"'] \end{tikzcd}$$

Claim: (λ, P) detailed balance $\Rightarrow$ λ invariant

$\Downarrow$ previous theorem

Y_n is Markov$(\lambda, \hat{P})$

and $\lambda_i \hat{P}_{ij} = \lambda_j P_{ji}$

$\Downarrow$ detailed balance

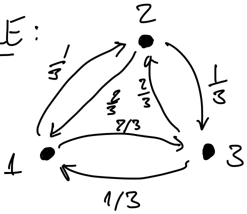
$P = \hat{P}$

$\Downarrow$

X_n is reversible

■

EXAMPLE:



$$P = \begin{pmatrix} 0 & 1/3 & 2/3 \\ 2/3 & 0 & 1/3 \\ 1/3 & 2/3 & 0 \end{pmatrix}$$

$\pi = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is s.t. $\pi P = \pi$

hence π is P-invariant.

$\pi_i \hat{P}_{ij} = \pi_j P_{ji} \rightsquigarrow \hat{P}_{ij} = P_{ji}$ i.e., $\hat{P} = P^T$.

◢

ERGODIC THEOREMS

(X_n) process

$f : I \to \mathbb{R}$ function

$$\frac{f(X_1)+\cdots+f(X_n)}{n} \quad \text{average} \quad \xrightarrow{?} \mu$$ with probability 1 ?
i.e., a.s. ?

e.g.: ?

LAW OF LARGE NUMBERS: let $(X_n)_{n\geq 0}$ be iid w/ $\mathbb{E}|X_1|<\infty$

then $$\frac{X_1+\cdots+X_n}{n} \longrightarrow \mathbb{E}X_1$$ a.s. (i.e., w/ prob. 1)

Rmk: if $X_n \geq 0$ $\forall n$, you don't need the L^1 assumption.
In fact $\mathbb{E}X_1=\infty$ is acceptable b/c we can define
$Y_n := \min\{X_n, N\}$ and apply LLN to Y_n.

$$\liminf_n \frac{X_1+\cdots+X_n}{n} \geq \liminf_n \frac{Y_1+\cdots+Y_n}{n} = \mathbb{E}[\min\{X_1, N\}]$$

Note $\mathbb{E}(\min\{X_1, N\}) \longrightarrow \infty$ as $N \to \infty$

So $$\liminf_n \frac{X_1+\cdots+X_n}{n} = \infty \quad \text{a.s.}.$$

Thm: (Ergodic Theorem) Let P be irreducible, λ any distribution, and $(X_n)_{n\geq 0} \sim \text{Markov}(\lambda, P)$. Then

$$V_i(n) := \sum_{k=0}^{n-1} \mathbb{1}_{\{X_k = i\}}$$

(# of visits to i)

satisfies

$$\frac{V_i(n)}{n} \xrightarrow{a.s.} \frac{1}{m_i} \quad \text{as } n \to \infty$$

where $m_i := \mathbb{E}_i(T_i)$ (expected return time)

In the positive recurrent case, for any $f: I \to \mathbb{R}$,

↳ does not work for null-recurrent b/c there's no invariant distribution there

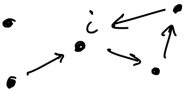
$$\frac{1}{n} \sum_{k=0}^{n-1} f(X_k) \xrightarrow{a.s.} \overline{f} := \sum_{i \in I} \pi_i f(i) \quad \text{as } n \to \infty$$

where (π_i) is an invariant distribution.

Pf: Case I: if P is transient, then $V_i(n)$ is finite a.s. so

$$\frac{V_i(n)}{n} \xrightarrow{a.s.} 0 .$$

Case II: if P is recurrent, fix $i \in I$. Can assume $\lambda = \delta_i$ because almost every sample path hits i, so the average # of visits does not depend



on the (finite) part of the path before you hit i.

Recall: $T_i^{(r)}$ rth hitting time of i

$S_i^{(r)} = T_i^{(r)} - T_i^{(r-1)}$ rth excursion time

Lemma!: The $(S_i^{(n)})_{n \geq 0}$ are iid. ← strong Markov property

Hence, by the LLN,

$$\frac{S_i^{(1)} + S_i^{(2)} + \dots + S_i^{(n)}}{n} \xrightarrow[n\to\infty]{a.s.} \mathbb{E}(S_i^{(1)}) = \mathbb{E}_i(T_i) =: m_i .$$

\# of visits before time n

$$\text{Note: } \underset{\displaystyle m_i}{\underset{\displaystyle\downarrow}{\frac{S_i^{(1)} + S_i^{(2)} + \dots + S_i^{(V_i(n)-1)}}{V_i(n)}}} \leq \frac{n}{V_i(n)} \leq \underset{\displaystyle m_i}{\underset{\displaystyle\downarrow}{\frac{S_i^{(1)} + \dots + S_i^{(V_i(n))}}{V_i(n)}}}$$

Thus: by Squeeze thm, $\frac{V_i(n)}{n} \xrightarrow{a.s.} \frac{1}{m_i}$ as $n \to \infty$

↳ obs: if null recurrent we just get zero.

Let $f: I \to \mathbb{R}$ be bounded and $|f| \leq 1$.

the X_k are some i
How many? $V_i(n)$.

$$\Big| \underbrace{\frac{1}{n} \sum_{k=0}^{n-1} f(X_k)}_{\text{Birkhoff average or ergodic average}} - \bar{f} \Big| = \Big| \sum_{i \in I} \left(\frac{V_i(n)}{n} - \pi_i \right) f(i) \Big|$$

Fix $J \subset I$

$$\hookrightarrow \leq \sum_{i \in J} \left| \frac{V_i(n)}{n} - \pi_i \right| + \sum_{i \notin J} \left(\frac{V_i(n)}{n} + \pi_i \right)$$

Note: $\displaystyle \sum_{i \in I} \frac{V_i(n)}{n} = 1 = \sum_{i \in I} \pi_i \Rightarrow \sum_{i \in I} \left(\pi_i - \frac{V_i(n)}{n} \right) = 0$

$$\Rightarrow 0 \leq \sum_{i \in J} \left| \pi_i - \frac{V_i(n)}{n} \right| + \sum_{i \notin J} \pi_i - \sum_{i \notin J} \frac{V_i(n)}{n}$$

$$\Rightarrow \sum_{i \notin J} \frac{V_i(n)}{n} \leq \sum_{i \in J} \left| \pi_i - \frac{V_i(n)}{n} \right| + \sum_{i \in J} \pi_i$$

$$\leq 2 \sum_{i \in J} \left| \frac{V_i(n)}{n} - \pi_i \right| + 2 \sum_{i \notin J} \pi_i < 2 \cdot \frac{\varepsilon}{4} + 2 \cdot \frac{\varepsilon}{4} = \varepsilon.$$

Fix $\varepsilon > 0$. Then $\exists J \subset I$ finite s.t. $\displaystyle \sum_{i \notin J} \pi_i < \frac{\varepsilon}{4}$.

here

Then $\exists N$ s.t. $\forall n \geq N$

$$\left| \frac{V_i(n)}{n} - \pi_i \right| < \frac{\varepsilon}{4} \quad \forall J \subset I$$

$\blacksquare$

CONTINUOUS-TIME MARKOV CHAINS

$\xcancel{(X_n)_{n \geq 0}} \rightsquigarrow (X_t)_{t \geq 0}$

~~transition probabilities~~ P_{ij} $\rightsquigarrow$ RATE of transition from i to j

Exponential variable w/ parameter λ

$$\mathbb{P}(t \in [a,b]) \sim \int_a^b e^{-\lambda s}\, ds$$

~~stochastic matrix~~ P $\rightsquigarrow$ Q-matrices and $P = e^{tQ}$.

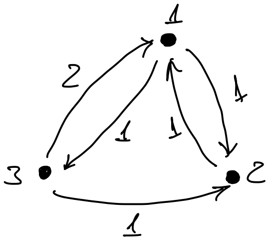
Def: Let I be a countable set. A Q-matrix is $Q = (q_{ij})$ s.t.

(i) $0 \leq -q_{ii} < \infty \quad \forall i$

(ii) $q_{ij} \geq 0 \quad \forall i \neq j$

(iii) $\sum_j q_{ij} = 0 \quad \forall i$

e.g.: $Q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix} \leftrightsquigarrow$



~~time-n state by powers of P~~ $\rightsquigarrow$ Exponential of Q-matrix

i.e., if Q is a Q-matrix then

e^{tQ} is the distribution after time $t \in \mathbb{R}$.

Def: $e^{Q} := \sum_{k=0}^{\infty} \frac{Q^k}{k!}$ (convergences in probabilistic cases... proof to come)

Thm: Let Q be a matrix on a finite I. Set $P(t) := e^{tQ}$. Then

(i) $P(s+t) = P(s)P(t) \quad \forall s,t$ (semi-group property)

(ii) $\frac{d}{dt} P(t) = P(t)Q, \quad P(0) = I$ (unique solution)

(iii) $\frac{d}{dt} P(t) = Q P(t), \quad P(0) = I$ (unique solution)

LECTURE 6

CONTINUOUS TIME MARKOV CHAINS

Q-matrix $\sum_j q_{ij} = 0$, $q_{ij} \geq 0$ $\forall i \neq j$, $q_{ii} \leq 0$.

Suppose I is finite, then

$$P(t) = e^{tQ} := \sum_{k=0}^{\infty} \frac{(tQ)^k}{k!}$$

← Well defined $\forall t \in \mathbb{C}$ because $\left\| \frac{Q^k}{k!} \right\| \leq \frac{\|Q\|^k}{k!} \leq \lambda^{-k}$ for every $\lambda > 1$. Any matrix norm works b/c they are all equivalent for finite matrices.

Note: given $s, t \in \mathbb{C}$

$$e^{tQ} \cdot e^{sQ} = e^{sQ} \cdot e^{tQ} = e^{(t+s)Q}$$

$\mathbb{C}[Q]$ commutative algebra or just take product term-by-term since both series converge absolutely:

$$\left(\sum_{k=0}^{\infty} \frac{t^k Q^k}{k!} \right) \left(\sum_{h=0}^{\infty} \frac{s^h Q^h}{h!} \right) = \sum_{k,h} \frac{t^k s^h Q^{h+k}}{k!\, h!}$$

$$\overset{n=h+k}{=} \sum_{n=0}^{\infty} Q^n \sum_{k=0}^{n} \frac{t^k s^{n-k}}{k!\,(n-k)!}$$

$$= \sum_{n=0}^{\infty} \frac{Q^n}{n!} \sum_{k=0}^{\infty} t^k s^{n-k} \binom{n}{k}$$

$$= \sum_{n=0}^{\infty} \frac{Q^n (s+t)^n}{n!} .$$

<u>Thm</u>: Let Q be a Q-matrix on a finite set I. Set $P(t) := e^{tQ}$. Then

(i) $P(t)P(s) = P(t+s)$ (semigroup property)

(ii) $P(t)$ is the unique solution to

$$\begin{cases} P(0) = I \\ \frac{d}{dt} P(t) = P(t) Q \end{cases}$$

(forward equation)

(iii) $P(t)$ is the unique solution to

$$\begin{cases} P(0) \\ \frac{d}{dt} P(t) = Q P(t) \end{cases}$$

(backward equation)

(iv) $$\left.\frac{d^k}{dt^k}\right|_{t=0} P(t) = Q^k \quad \text{for } k \geq 0.$$

Pf: (iii) $P'(t) = P(t)Q$, $P(t) = e^{tQ}$

Suppose $M(t)$ solution

$$\frac{d}{dt}\left(M(t)e^{-tQ}\right) = M'(t)e^{-tQ} - M(t)Qe^{-tQ}$$

$$\overset{M \text{ solution?}}{=} M(t)Qe^{-tQ} - M(t)Qe^{-tQ} = 0$$

$$\Rightarrow M(t)e^{-tQ} =: M_0$$

$$M(0) = M_0 = I.$$

$$\frac{d}{dt}\left(e^{tQ}\right) \underset{\uparrow \text{ power series.}}{=} Qe^{tQ}$$

(iv) $$P(t) = \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!} \Rightarrow \frac{d^k}{dt^k}P(t) = \frac{Q^k}{k!}k! + O(t) \checkmark$$

■

Thm: A matrix Q (on a finite I) is a Q-matrix iff e^{tQ} is stochastic for any $t \geq 0$.

Pf: $P(t) = I + tQ + O(t^2)$ (as $t \to 0$)

$P_{ij} \geq 0 \rightsquigarrow$ $\underline{i \neq j}$: $P_{ij}(t) = t q_{ij} + O(t^2)$.

So, $P_{ij} \geq 0 \iff q_{ij} \geq 0$ for $i \neq j$.

$\underline{i = j}$: $P_{ii}(t) = 1 + t q_{ii} + O(t^2)$

So $P_{ii} \geq 0$ independently of q_{ii}.

$$\sum_j P_{ij} = 1 \overset{?}{\iff} \sum_j q_{ij} = 0$$

$(\Leftarrow)$ If $\sum_j q_{ij} = 0$ then $\sum_j q_{ij}^{(n)} = 0 \ \forall n$ since by induction:

$$\sum_j q_{ij}^{(n+1)} = \sum_{j,k} q_{ik}^{(n)} q_{kj} = \sum_k q_{ik}^{(n)} \underbrace{\sum_j q_{kj}}_{=0} = 0 .$$

Now, $P_{ij}(t) = \delta_{ij} + \sum_{k=1}^{\infty} \frac{t^k}{k!} q_{ij}^{(k)}(t)$

$$\sum_j P_{ij}(t) = 1 + \sum_{k=1}^{\infty} \frac{t^k}{k!} \underbrace{\sum_j q_{ij}^{(k)}(t)}_{=0} = 1 .$$

($\Rightarrow$) If $\sum_j P_{ij}(t) = 1$ $\forall t$, then the way to go from P to Q is by taking derivatives; i.e.,

$$\sum_j P'_{ij}(t) = 0 \quad \forall t .$$

So,

$$0 = \sum_j P'_{ij}(0) = \sum_j q_{ij} .$$

■

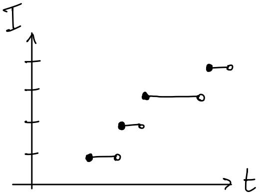
CONTINUOUS TIME PROCESSES

Def: Let I be countable. A <u>continuous-time random process</u> is $(X_t)_{t \geq 0}$ a family of random variables indexed by $t \in \mathbb{R}_+$, $X_t : \Omega \to I$.

Def: The process is <u>right-continuous</u> if $\forall \omega \in \Omega$, $t \geq 0$, there exists $\varepsilon > 0$ s.t.

$$X_s(\omega) = X_t(\omega) \text{ for } t \leq s \leq t + \varepsilon .$$

("locally" constant)



(The above are special cases of these fcts → CADLAG FUNCTION: $\lim_{s\to t^+} f(s) = f(t)$

$\lim_{s\to t^-} f(s)$ exists)

<u>REMARK</u>: A right-continuous process is determined by its <u>finite-dimensional</u> <u>distributions</u> i.e.,

$$\mathbb{P}(X_{t_0}=i_0, X_{t_1}=i_1, \dots, X_{t_n}=i_n)$$

(only need to know distrib. on finitely-many times.)

for $n\geq 0$, $t_0\leq t_1\leq \dots \leq t_n$, $(i_k)\subset I$.

<u>EXAMPLE</u>:

$$\mathbb{P}(X_t=i \text{ for some } t\in \overset{\substack{\text{uncountable}\\ \downarrow}}{[0,\infty)}) = \mathbb{P}(X_t=i \text{ for some } t\in \overset{\substack{\text{countable}\\ \downarrow}}{\mathbb{Q}_+})$$

$$= 1 - \lim_{n\to\infty} \sum_{\substack{(j_1,\dots,j_n)\\ j_k\neq i\ \forall k}} \mathbb{P}(X_{q_1}=j_1, \dots, X_{q_n}=j_n)$$

$$(q_i)_{i=1}^{\infty} = \mathbb{Q}$$

LECTURE 7

POISSON PROCESSES

Cont. time Markov chain

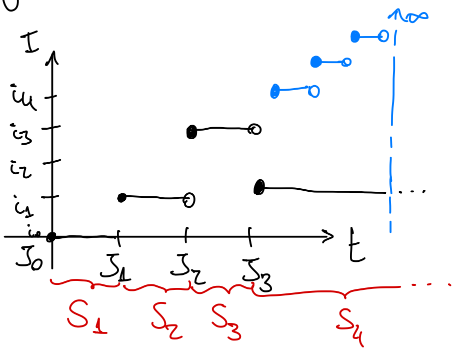
$I = \mathbb{N}$

Q - matrix $\quad Q = (q_{ij})$

$P(t) = e^{tQ}$ stochastic matrix $\forall t \geq 0$.

NOTE: $P(nt) = e^{ntQ} = (e^{tQ})^n = (P(t))^n$

Right - Continuous Process: (X_t) s.t. $\forall t \ \exists \varepsilon > 0$ s.t. $X_{t+\eta} = X_t$ $\forall \eta < \varepsilon$.



$\iff$

Def: JUMP TIMES $\quad J_0 = 0$

$$J_{n+1} = \inf\{t \geq J_n : X_t \neq X_{J_n}\}.$$

HOLDING TIMES $\quad S_n := \begin{cases} J_n - J_{n-1}, & \text{if } J_{n-1} < \infty \\ \infty, & \text{otherwise} \end{cases}$

EXPLOSION TIME: $\xi := \sup_n J_n = \sum_n S_n \in [0, \infty]$.

JUMP PROCESS: (Y_n), $Y_n := X_{J_n}$

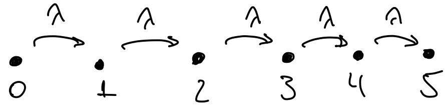
$\iff (Y_n) = (i_0, i_1, i_3, i_2, \dots)$.

Def: A process is MINIMAL iff $X_t = \infty$ if $t \geq \xi$
(extra state)

the state is not minimal
but the chain is active for a "minimal" time

Rmk: A minimal process can be reconstructed by knowing the holding times + jump process.

Let $\lambda > 0$



$$Q = \begin{pmatrix} -\lambda & \lambda & & 0 \\ & -\lambda & \lambda & \\ & & -\lambda & \lambda \\ 0 & & & \ddots \end{pmatrix}$$

$P(t) = e^{tQ}$

$P'(t) = P(t)\,Q \implies P'_{ii}(t) = -\lambda P_{ii}(t) \quad \forall i$.

$$\begin{pmatrix} P_{11} & P_{12} & P_{13} \\ \cancelto{0}{P_{21}} & P_{22} & P_{23} \\ \cancelto{0}{P_{31}} & \cancelto{0}{P_{32}} & P_{33} \end{pmatrix} \begin{pmatrix} -\lambda & \lambda & 0 \\ 0 & -\lambda & \lambda \\ 0 & 0 & -\lambda \end{pmatrix} = \begin{pmatrix} P_{11}' & P_{12}' & P_{13}' \\ \cancelto{0}{P_{21}'} & P_{22}' & P_{23}' \\ \cancelto{0}{P_{31}'} & \cancelto{0}{P_{32}'} & P_{33}' \end{pmatrix}$$

Because we can only go up the chain or stay put.

$$P_{12}'(t) = \lambda p_{11}(t) - \lambda p_{12}(t)$$

$$P_{ii}'(t) = -\lambda p_{ii}(t) \qquad P_{ii}(0) = 1$$

for $i<j$, $P_{ij}'(t) = -\lambda p_{ij}(t) + \lambda p_{i,j-1}(t)$ $\qquad P_{ij}(0) = 0 \quad i<j$

$$P_{ij}(t) = 0 \quad i>j \ \forall t$$

So, $\quad P_{ii}(t) = e^{-\lambda t}, \ \forall i$

$$\left(e^{\lambda t} P_{i,i+1}(t)\right)' = \lambda\left(e^{\lambda t} P_{ii}(t)\right) = \lambda$$

$$e^{\lambda t} P_{i,i+1}(t) = \lambda t$$

$$\left(e^{\lambda t} P_{i,i+2}(t)\right)' = \lambda(\lambda t) = \lambda^2 t$$

$$e^{\lambda t} P_{i,i+2}(t) = \frac{\lambda^2 t^2}{2}$$

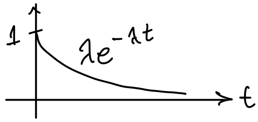
For any k: $\quad P_{i,i+k}(t) = e^{-\lambda t} \dfrac{\lambda^k t^k}{k!} \iff \text{Poisson}(\lambda t)$

Def: A right-continuous process (X_t) with values in $\mathbb{N} \cup \{0\}$ is a Poisson process of rate λ iff its holding times are independent exponentials of par. λ and its jump process is $Y_n = n$.

EXPONENTIAL RANDOM VARIABLES

Def: $T \sim \mathrm{Exp}(\lambda)$ iff $\mathbb{P}(T > t) = e^{-\lambda t}$ $\forall t \geq 0$.

Density: $f_T(t) = \lambda e^{-\lambda t} \mathbb{1}_{\{t \geq 0\}}$



Expectation: $\mathbb{E}(T) = \int_0^\infty \mathbb{P}(T > t)\, dt = \frac{1}{\lambda}$.

Thm: $T : \Omega \to [0, +\infty]$ has exponential distribution iff $\mathbb{P}(T > t+s \mid T > s) = \mathbb{P}(T > t)$ $\forall t, s \geq 0$.

Pf: $\mathbb{P}(T > t+s \mid T > s) = \dfrac{e^{-\lambda(t+s)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(T > t)$

$(\Leftarrow)$ $g(t) := \mathbb{P}(T > t)$

$g(t+s) = g(s)\, g(t)$ $\forall s, t \geq 0$ $\Rightarrow g(1) = g\left(\frac{1}{n}\right)^n$ $\forall n \in \mathbb{N}$

$g\left(\frac{p}{q}\right) = g\left(\frac{1}{q}\right)^p = g(1)^{p/q}$ $\forall p, q \in \mathbb{N}$

$\Rightarrow g(t) = e^{-\lambda t}$ by continuity.

∎

Thm: Let (S_n) be indep. exp. r.v.'s of pars (λ_n). Then:

(i) if $\sum_n \frac{1}{\lambda_n} < \infty$, then $\mathbb{P}\left(\sum_n S_n < \infty\right) = 1$

(ii) if $\sum_n \frac{1}{\lambda_n} = +\infty$, then $\mathbb{P}\left(\sum_n S_n = +\infty\right) = 1$

Pf: $\sum_n \frac{1}{\lambda_n} = \mathbb{E}\left(\sum_n S_n\right) < \infty \Rightarrow \mathbb{P}\left(\sum_n S_n < \infty\right) = 1$.

If $\sum_n \frac{1}{\lambda_n} = +\infty$, $\prod_n \left(1 + \frac{1}{\lambda_n}\right) \geq \sum \frac{1}{\lambda_n} \to \infty$

$$\mathbb{E}\left(e^{-\sum S_n}\right) = \mathbb{E}\left(\prod_n e^{-S_n}\right) = \prod_n \mathbb{E}\left(e^{-S_n}\right)$$

$$\mathbb{E}\left(e^{-S_n}\right) = \int_0^\infty e^{-t} \lambda e^{-\lambda t} dt = \lambda \int_0^\infty e^{-(1+\lambda)t} dt$$

$$= \frac{\lambda}{1+\lambda}$$

So $\mathbb{E}\left(e^{-\sum S_n}\right) = \prod_n \frac{\lambda_n}{1+\lambda_n} = \prod_n \frac{1}{1+1/\lambda_n} = \frac{1}{\prod_n (1+1/\lambda_n)}$

$$\longrightarrow 0$$

$$\Rightarrow e^{-\sum_n S_n} \longrightarrow 0 \overset{w/\ \mathbb{P}=1}{\vee} \Rightarrow \sum_n S_n \to +\infty \ w/\ \mathbb{P}=1 .$$

■

POISSON PROCESSES

(X_t) right-continuous
- holding times are indep. and $\sim \text{Exp}(\lambda)$
- jump process $Y_n = n$.

NOTE: $\mathbb{P}\left(\lim_n J_n = +\infty\right) = 1$, $\mathbb{P}\left(\lim_n (S_1 + \cdots + S_n) = +\infty\right) \overset{LLN}{=} 1$

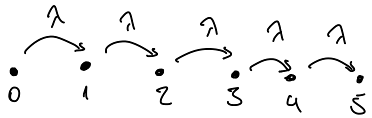
Thm: (Markov Property) Let (X_t) be Poisson (λ), let T be stopping time for (X_t) (i.e., $\{T \leq t\} \in \sigma((X_s)_{s \leq t})$). Then, conditional on $T < \infty$, $(X_{T+t} - X_T)_{t \geq 0}$ is also Poisson (λ) indep. of $(X_s : s \leq T)$.

Def: (X_t) has stationary increments if distribution of $X_{t+s} - X_s$ does not depend on s.

(X_t) has independent increments if for any $0 \leq t_1 \leq u_1 \leq t_2 \leq u_2 \leq \cdots \leq t_k \leq u_k$, $(X_{u_1} - X_{t_1}, X_{u_2} - X_{t_2}, \ldots, X_{u_k} - X_{t_k})$ are independent.

LECTURE 8

POISSON PROCESSES & BIRTH PROCESSES



rate = λ   can only go up with rate λ

Equivalent Characterizations of Poisson processes:

Thm: Let (X_t) be a right-continuous integer valued process and let $0<\lambda<\infty$. TFAE:

(i) Jump process is $Y_n = n$ ← only go up by one

Holding times are independent exponentials $\text{Exp}(\lambda)$.

(ii) infinitesimal (X_t) has independent increments and, unif. on t, as $h \to 0$

$$\mathbb{P}(X_{t+h} - X_t = 1) = \lambda h + o(h)$$ → prob of jumping

$$\mathbb{P}(X_{t+h} - X_t = 0) = 1 - \lambda h + o(h)$$ → prob of not jumping

(iii) (X_t) has stationary independent increments and, for each t, X_t is $\text{Poisson}(\lambda t)$.

$$\mathbb{P}(X_t = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

stationary increments := law of $X_{t+s} - X_t$ only depends on s

Thm: If (X_t) and (Y_t) are independent Poisson of rates λ, μ then (X_t+Y_t) is also Poisson of rate $\lambda+\mu$.

Pf: Use 2^{nd} characterization

$$\mathbb{P}(X_{t+h}-X_t=0)=1-\lambda h+o(h), \quad \mathbb{P}(X_{t+h}-X_t=1)=\lambda h+o(h)$$

$$\mathbb{P}(Y_{t+h}-Y_t=0)=1-\mu h+o(h), \quad \mathbb{P}(Y_{t+h}-Y_t=1)=\mu h+o(h)$$

Then

$$\mathbb{P}(X_{t+h}+Y_{t+h}-X_t-Y_t=0)\overset{\text{indep.}}{=}\mathbb{P}(X_{t+h}-X_t=0)\,\mathbb{P}(Y_{t+h}-Y_t=0)$$

$$=\Big(1-\lambda h+o(h)\Big)\Big(1-\mu h+o(h)\Big)$$

$$=1-(\lambda+\mu)h+o(h).$$

And

$$\mathbb{P}(X_{t+h}+Y_{t+h}-X_t-Y_t=0)=\overbrace{\mathbb{P}(X_{t+h}-X_t=1)}^{X\text{ jumps}}\,\overbrace{\mathbb{P}(Y_{t+h}-Y_t=0)}^{Y\text{ doesn't jump}}$$

$$+\underbrace{\mathbb{P}(X_{t+h}-X_t=0)}_{X\text{ doesn't jump}}\,\underbrace{\mathbb{P}(Y_{t+h}-Y_t=1)}_{Y\text{ jumps}}$$

$$=(\lambda h+o(h))\Big(1-\mu h+o(h)\Big)+(\mu h+o(h))(1-\lambda h+o(h))$$

$$=\lambda h+\mu h+o(h)=h(\lambda+\mu)+o(h).$$

Finally, since X and Y have independent increments and they are indep, X+Y has indep increments.

■

Thm: Let (X_t) be Poisson. Then, conditional on (X_t) having exactly one jump in $[s, s+t]$, the time of jump is <u>uniformly distributed</u> on $[s, s+t]$.

Know bus arrived between 9 and 9:15 ⇒ prob. of arriving is uniform in these 15 mins

Pf: WLOG $\underline{s = 0}$ by stationarity. Let $0 = s \leq u \leq t$.

$$P(J_1 \leq u \mid X_t = 1) = \frac{P(J_1 \leq u \text{ and } X_t = 1)}{P(X_t = 1)}$$

$T \sim \text{Poisson}(\lambda)$

$P(T=k)$

$= \dfrac{\lambda^k e^{-\lambda}}{k!}$

$$= \frac{\overset{\text{Poisson}(\lambda u)}{\underline{P(X_u \leq 1)}}\ \underline{P(X_t - X_u = 0)}^{= P(X_{t-u}=0)}}{\underline{P(X_t = 1)} = \text{Poisson}(\lambda t)}$$

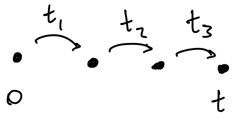
$$= \frac{\lambda u e^{-\lambda u} e^{-\lambda(t-u)}}{\lambda t e^{-\lambda t}} = \frac{u}{t}$$

⇒ Uniform distribution on $[0, t]$ ← total mass $= t$...

■

Thm: Let (X_t) be Poisson. Then, conditional on $\{X_t = n\}$ (i.e., n jumps), the jump times $J_1, -, J_n$ have joint density function

$$f(t_1, -, t_n) = \frac{n!}{t^n} \mathbb{1}\{0 \le t_1 \le t_2 \le \cdots \le t_n\}$$



Pf: The holding times have joint density function

$$\lambda^{n+1} e^{-\lambda(s_1 + \cdots + s_{n+1})} \mathbb{1}\{s_1, -, s_{n+1} \ge 0\}$$

The jump times (t_n) satisfy $t_n = s_1 + \cdots + s_n$

$$s_n = t_n - t_{n-1}$$

and have density

$$\lambda^{n+1} e^{-\lambda t_{n+1}} \mathbb{1}\{0 \le t_1 \le \cdots \le t_n\}.$$

Given $A \subset \mathbb{R}^n$,

$$\mathbb{P}\big((J_1, -, J_n) \in A \text{ and } X_t = n\big)$$

$$= \mathbb{P}\big((J_1, -, J_n) \in A \text{ and } J_n < t < J_{n+1}\big) \Big/ \underline{\mathbb{P}(X_t = n)}$$

$$= e^{-\lambda t} \lambda^n \int_{(t_1,\dots,t_n)\in A} \mathbb{1}_{\{0\le t_1\le \dots \le t_n\}} \, dt_1 \cdots dt_n$$

$$\underbrace{e^{-\lambda t} \frac{(\lambda t)^n}{n!}}_{X_t \text{ is Poisson}}$$

Upshot:

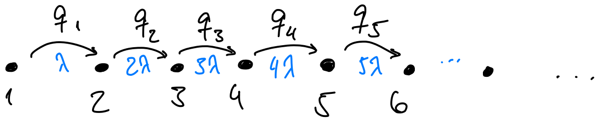
$$\mathbb{P}\big((J_1,\dots,J_n)\in A \mid X_t = n\big) = \frac{n!}{t^n} \int_{(t_1,\dots,t_n)\in A} \mathbb{1}_{\{0\le t_1\le \dots\le t_n\}} \, dt_1 \cdots dt_n .$$

■

Note: $P(x\in A \text{ and } x<t<y) = \lambda \int_{\substack{x\in A\\ x<t}} dx \; \lambda \int_{\substack{y\in A\\ y>t}} dy \; e^{-\lambda y}$

BIRTH PROCESSES

Jump rates depend on where you are:



EXAMPLE: population where each individual gives birth independently with parameter λ.

SIMPLE BIRTH PROCESS : $q_i = i\lambda$

$T =$ time of 1st birth $\qquad X_t :=$ size of pop. at time t

$X_0 = 1$

$$\mathbb{E}(X_t) = \mathbb{E}(X_t \mathbb{1}_{\{T \leq t\}}) + \mathbb{E}(X_t \mathbb{1}_{\{T > t\}})$$

$$= \int_0^t \lambda e^{-\lambda s} \underbrace{\mathbb{E}(X_t \mid T = s)}_{} ds + e^{-\lambda t}$$

$$\mu(t) := \mathbb{E}(X_t)$$

$$\mathbb{E}(X_t \mid \underbrace{T=s}_{\Downarrow}) = 2\mu(t-s)$$

birth happened
i.e., rate multiplies by 2

$$= \int_0^t \lambda e^{-\lambda s}\, 2\, \mu(t-s)\, ds + e^{-\lambda t}$$

Thus: $$\mu(t) = \int_0^t \lambda e^{-\lambda s}\, 2\, \mu(t-s)\, ds + e^{-\lambda t}$$

$t - s =: r$

$$\Longleftrightarrow \quad 2\lambda \int_0^t e^{-\lambda t} e^{\lambda r} \mu(r)\, dr + e^{-\lambda t} = \mu(t)$$

$$\Longleftrightarrow \quad \boxed{2\lambda \int_0^t e^{\lambda r} \mu(r)\, dr + 1 = e^{\lambda t} \mu(t)}$$

Take derivatives to solve:

$$\frac{d}{dt}: \quad 2\lambda e^{\lambda t}\mu(t) = \lambda e^{\lambda t}\mu(t) + e^{\lambda t}\mu'(t)$$

$$\boxed{\lambda \mu(t) = \mu'(t)}$$

i.e., $\boxed{\mu(t) = E(X_t) = e^{\lambda t}}$ b/c $X_0 = 1$

↳ Makes sense b/c nobody dies w/ birth rate $q_i = i\lambda$

Thm: Let (X_t) be a birth process with rates $\{q_j\}$.

(i) If $\sum_j \frac{1}{q_j} < \infty$ then $P(\xi < \infty) = 1$. ← EXPLOSION

(ii) If $\sum_j \frac{1}{q_j} = \infty$ then $P(\xi = \infty) = 1$.

$\xi :=$ explosion time $= \sum_{n=0}^{\infty} S_n$.

Pf: $S_n \sim \text{Exp}(q_n)$ indep.

Thm: (Markov Property) Let (X_t) be a birth process of rates (q_j). Then, conditional on $\{X_s = i\}$, $(X_{s+t})_{t \geq 0}$ is a birth process of rates (q_j) starting from i and independent of $(X_r)_{r \leq s}$.

CHARACTERIZATION OF BIRTH PROCESSES:

Thm: Let (X_t) be an increasing right continuous process with values on $\mathbb{N} \cup \{\infty\}$. Let $0 \leq q_j < \infty$.

TFAE

(i) Conditional on $X_0 = i$, the holding times are independent exponentials r.v.'s of parameters $q_i, q_{i+1}, q_{i+2}, \dots$ and the jump process is $Y_n = i + n$.

(ii) $\forall t, h \geq 0$, conditional on $X_t = i$, X_{t+h} is independent of $(X_s)_{s \leq t}$ and, uniformly in t, as $h \to 0$,

$$\mathbb{P}(X_{t+h} = i \mid X_t = i) = 1 - q_i h + o(h)$$

$$\mathbb{P}(X_{t+h} = i+1 \mid X_t = i) = q_i h + o(h)$$

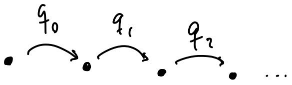
(iii) $\forall\, n \in \mathbb{N}$, times $0 \leq t_1 \leq \cdots \leq t_{n+1}$, states $i_0, \ldots, i_{n+1}$,

$$\mathbb{P}\left(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \ldots, X_{t_n} = i_n\right) = P_{i_n i_{n+1}}(t_{n+1} - t_n)$$

where $P_{ij}(t)$ is the unique solution to the forward equation

$$P(t) = e^{tQ} \qquad \boxed{\begin{aligned} P'(t) &= P(t)\,Q \\ P(0) &= \mathrm{Id} \end{aligned}}$$

Forward Eq.
(*)



In components:

$$P_{i0}'(t) = -q_0\, P_{i0}(t), \quad P_{i0}(0) = \delta_{i0}$$

and for $j \geq 1$: $P_{ij}'(t) = P_{i,j-1}(t)\, q_{j-1} - P_{ij}(t)\, q_j, \quad P_{ij}(0) = \delta_{ij}$.

$$Q = \begin{pmatrix} -q_0 & q_0 & & & 0 \\ & -q_1 & q_1 & & \\ 0 & & -q_2 & q_2 & \ddots \\ & & & \ddots & \end{pmatrix}$$

SOLVE: $P_{i0}'(t) = -q_0 P_{i0}(t)$

$$\rightsquigarrow \boxed{P_{i0}(t) = e^{-q_0 t}\, \delta_{i0} \quad \forall t.}$$

(*)
NOTE: For Poisson:

$$P_{ij}(t) = \begin{cases} e^{-\lambda t} \dfrac{(\lambda t)^{j-i}}{(j-i)!}, & \text{if } i \leq j \\ 0, & \text{if } i > j \end{cases}$$

$$\begin{aligned} P_{i1}'(t) &= P_{i0}(t)\, q_0 - P_{i1}(t)\, q_1 \\ &= q_0\, e^{-q_0 t} \delta_{i0} - P_{i1}(t)\, q_1 \end{aligned}$$

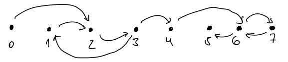
$$\Rightarrow \boxed{P_{i1}(t) = \delta_{i1}\, e^{-q_1 t} + \delta_{i0} \int_0^t q_0\, e^{-q_0 s} e^{-q_1 (t-s)}\, ds.}$$

Def: A minimal right continuous process (X_t) on I is called a *Markov chain* with *initial distribution* λ and *generator matrix* Q if its jump-process (Y_n) is discrete-Markov(λ, Π) and, conditional on $(Y_0, \dots, Y_{n-1})$, its holding times $S_1, \dots, S_n$ are indep. exponentials of parameters $q(Y_0), \dots, q(Y_{n-1})$.

We say (X_t) is Markov(λ, Q).

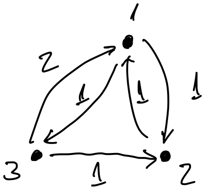
$$Q = Q\text{-matrix} \begin{cases} \bullet\ 0 \leq -q_{ii} < \infty \quad \leftarrow \boxed{q_i := q_{ii}} \\ \bullet\ q_{ij} \geq 0 \quad \forall i,j \\ \bullet\ \sum_{j} q_{ij} = 0 \quad \forall i \end{cases}$$

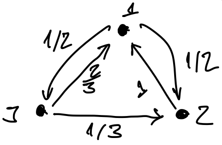
$\Pi :=$ *Jump Matrix*
$$\Pi_{ij} := \begin{cases} \frac{q_{ij}}{q_i} & \text{if } j \neq i,\ q_i \neq 0 \\ 0 & \text{if } j \neq i,\ q_i = 0 \end{cases}$$



$$\Pi_{ii} := \begin{cases} 0 & \text{if } q_i \neq 0 \\ 1 & \text{if } q_i = 0 \end{cases}$$

Ex: $$q = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -3 \end{pmatrix} \rightsquigarrow \text{Jump Matrix} \quad \Pi = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 \end{pmatrix}$$





LECTURE 9

CONTINUOUS TIME MARKOV CHAINS

I = countable set

Q = Q-matrix

$\| $

(q_{ij}) $\quad q_i := q(i) := -q_{ii}$

Π = jump matrix

$$\pi_{ij} = \frac{q_{ij}}{q_i} \quad \text{if } q_i \neq 0,\ i \neq j.$$

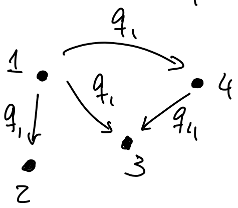
$$\sum_{ij} q_{ij} = 0 = \sum_{j \neq i} q_{ij} - q_i \longrightarrow \sum_{j \neq i} \frac{q_{ij}}{q_i} = 1$$

Def: A continuous-time Markov chain (λ, Q) is a process (X_t) where:

(1) The jump process (Y_n) is a discrete-time Markov (λ, π)

(2) The holding times are independent exponentials of rates $\underline{q(Y_n)}$.

$q(Y_n) = q_i$ if $Y_n = i$



REALIZATION: (1) • Let (Y_n) be discrete-time Markov (λ, Π).

• Let $T_1, \ldots, T_n$ be exponential indep. r.v.'s of parameter 1.

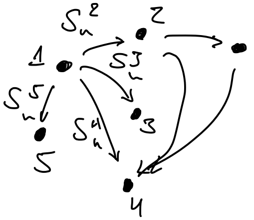
- Set $S_n := \dfrac{T_n}{q(Y_{n-1})}$, $J_n := S_1 + \dots + S_n$.

- Set $X_t = \begin{cases} Y_n & \text{if } J_n \le t < J_{n+1} \\ \infty & \text{otherwise} \end{cases}$.

(2) • Let $(T_n^j : j \in I, n \in \mathbb{N})$ indep. exponentials of par. 1.

Initial state $X_0 = Y_0$ has distrib. λ.

Then, if $Y_n = i$, we set



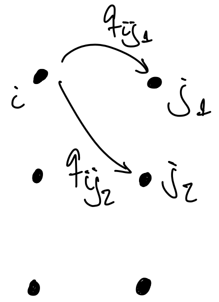
$$S_{n+1}^j := \frac{T_{n+1}^j}{q_{ij}}$$

$$S_{n+1} := \underbrace{\inf_{j \neq i}}_{} S_{n+1}^j \qquad Y_{n+1} = \begin{cases} j, & \text{if } S_{n+1} = S_{n+1}^j \\ i, & \text{if } S_{n+1} = \infty \end{cases}$$

can jump to possibly more than one state, but we choose to jump somewhere as soon as we can.

<u>NOTE</u>: S_{n+1} is <u>exponential</u> of rate $\sum_{j \neq i} q_{ij} = q_i$ and is <u>independent</u> of Y_{n+1}.

(3) (Using Poisson processes)



Let $\{(N_t^{ij})_{t\geq 0} : i\neq j\}$ be indep. Poisson processes of rate q_{ij}.

Start with $X_0 = Y_0$ and distrib. λ.

$$J_{n+1} := \inf\{t > J_n : N_t^{Y_n j} \neq N_{J_n}^{Y_n j} \text{ for some } j\neq Y_n\}$$

if the Poisson process jumps, we jump.

since Poisson processes always go up in the integers, this time t always increases too (in (2), it could go to zero).

$$\text{Set } Y_{n+1} = \begin{cases} j, & J_{n+1} < \infty \text{ and } N_{J_{n+1}}^{Y_n j} \neq N_{J_n}^{Y_n j} \\ i, & \text{if } J_{n+1} = \infty \end{cases}$$

EXPLOSION IN CONTINUOUS TIME

Explosion time $\xi := \sum_{n=0}^{\infty} S_n \in [0,\infty]$.

Explosion happens if $\xi < \infty$.

Lemma: Let (X_t) be Markov(λ, Q). Then (X_t) does not explode if one of the following holds:

(1) I is finite

(2) $\sup_i q_i < \infty$

(3) $X_0 = i$ and i recurrent for jump process

Pf: $S_n = \dfrac{T_n}{q(Y_n)}$ ← Exp(1) $\qquad \xi = \sum_{n=0}^{\infty} S_n$

← $= q_i$ if $Y_n = i$

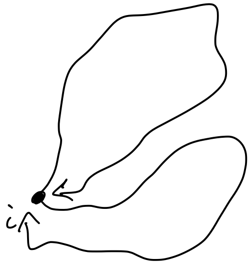
If $\sup_i q_i < \infty$, then

countably many indep exps. of same rate

$$\left(\sup_i q_i\right) \sum_{n=0}^{\infty} S_n \geq \sum_{n=0}^{\infty} T_n \longrightarrow +\infty \text{ w/ } \mathbb{P} = 1.$$

hence $\sum_{n=0}^{\infty} S_n \longrightarrow \infty$ with $\mathbb{P} = 1$ (2), (1) ✓

(3) If i is recurrent, $\exists (N_k)$ s.t. $Y_{N_k} = i$ with $\mathbb{P} = 1$.



Conditional on $\{Y_{N_k} = i\}$,

countable sum of indep exps of same rate

$$q_i \sum_n S_n \geq \sum_n T_n \geq \sum_k T_{N_k} \longrightarrow +\infty$$

a.s.

$$\Rightarrow \sum_n S_n \longrightarrow \infty \Rightarrow \text{no explosion}$$

■

Thm: (Criterion for Explosion) Let (X_t) be continuous-time Markov(λ, Q), let ξ = explosion time.

Let $\theta > 0$, $z_i := \mathbb{E}_i(e^{-\theta \xi})$. Then

(i) $|z_i| \leq 1$ for all i

(ii) $Qz = \theta z \qquad (z = (z_i)_{i \in I})$

Moreover: if $\tilde{z}$ satisfies (i) and (ii), then $\tilde{z}_i \leq z_i \ \forall i$

<u>Corollary</u>: For each $\theta > 0$, TFAE:

(i) Q is non-explosive

(ii) $Qz = \theta z$ and $|z_i| \leq 1 \ \forall i$ implies $z_i = 0 \ \forall i$.

<u>Upshot</u>: If Q has only the trivial eigenvector $\equiv 0$, then Q is non-explosive. But if Q has a nontrivial eigenvector, then it explodes.

Pf: (Corollary) (i) $\Rightarrow$ (ii):

Q non-explosive $\Leftrightarrow \mathbb{P}_i(\xi = \infty) = 1$

so $\mathbb{E}_i(e^{-\theta\xi}) = 0$.

$Qz = \theta z$ for z with $|z_i| \leq 1 \ \forall i$ $\Longrightarrow$ $z_i \leq \mathbb{E}_i(e^{-\theta\xi}) = 0$

By symmetry $z_i \mapsto -z_i$,

$z_i = 0 \quad \forall i$.

(ii) $\Rightarrow$ (i): $z_i = \mathbb{E}_i(e^{-\theta\xi})$ is solution to $Qz = \theta z$,

hence $z_i = 0 \ \forall i \Rightarrow \xi = +\infty$ a.s.

■

LECTURE 10

$(X_t)_{t\geq 0}$ Markov chain (λ, Q)

Initial distrib.

Q - matrix

$Q = (q_{ij})$

Π - matrix

$\pi_{ij} = \frac{q_{ij}}{q_i}$

$q(i) := q_i := -q_{ii}$

(1) Jump process is discrete Markov (λ, Π)

(2) Holding times are $\mathrm{Exp}(q_i)$

Thm: Let $(X_t)_{t\geq 0}$ be a minimal right-continuous process with values on a countable set I. Let Q be a Q-matrix. Then TFAE:

(i) Conditional on $\{X_0 = i\}$, the jump process is discrete Markov (δ_i, Π) and for each $n \geq 1$, conditional on $Y_0, \dots, Y_{n-1}$, the holding times $S_1, \dots, S_n$ are independent exponentials of parameters $q_{Y_0}, \dots, q_{Y_{n-1}}$.

(ii) $\forall\, n = 0, 1, \dots$, all times $0 \leq t_0 \leq \dots \leq t_{n+1}$, and all states $i_0, \dots, i_{n+1}$,

$$\mathbb{P}\left(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \dots, X_{t_n} = i_n\right) = p_{i_n i_{n+1}}(t_{n+1} - t_n)$$

where $P_{ij}(t)$ is the minimal nonnegative solution to the backward equation $\begin{cases} P'(t) = Q\,P(t) & \text{for } t \geq 0, \\ P(0) = Id \end{cases}$

- MINIMAL: If $\tilde{P}(t)$ satisfies $\tilde{P}'(t) = Q\,\tilde{P}(t)$ and $\tilde{P}(t) \geq 0$ $\forall t$, then $\tilde{P}(t) \geq P(t)$ $\forall t$.

Def: A process (X_t) is minimal if $X_t = \infty$ for $t \geq \zeta$ (explosion times)

- NOTE:

$$P(t) = e^{tQ}$$

(1) $P'(t) = P(t) \cdot Q$ ← Forward Equation

(2) $P'(t) = Q \cdot P(t)$ ← Backward Equation (easier)

! WARNING:

(1) $$P'_{ij}(t) = \sum_{k} P_{ik}(t)\, q_{kj}$$ ← conditioning on future.

(2) $$P'_{ij}(t) = \sum_{k} \underline{q_{ik}\, P_{kj}(t)}$$

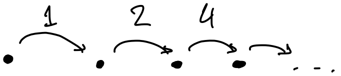
Conditioning on $X_{J_1} = k$

i.e., conditioning on past

↑ Easier to deal with

Examples of non-minimal processes:

1. Birth process (X_t) starting from 0 with rates $q_i = 2^i$



$$\sum_{i=0}^{\infty} q_i^{-1} = \sum_{i=0}^{\infty} \frac{1}{2^i} < +\infty$$

so X_t explodes a.s.

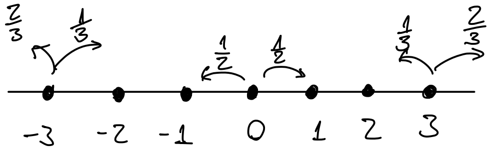
Set $X_\zeta := 0$ and restart the process. This violates minimality. It also violates uniqueness b/c can restart from anywhere (e.g. $X_\zeta = 5$) and we get process w/ same transition probabilities.

$$(P(t))_{ij} = p_{ij}(t) = \mathbb{P}_i(X_t = j)$$

If we allow restarting, then $\tilde{P}(t) \geq P(t)$ because

$$\tilde{P}(t) = \mathbb{P}_i(X_t = j \text{ (includes restarting)}) \geq \mathbb{P}_i(X_t = j \text{ (before explosion)})$$

2. Random Walk on $\mathbb{Z}$ with bias towards infinity:



Jump process

$$\pi_{0,1} = \pi_{0,-1} = \frac{1}{2}$$

$i > 0 : \pi_{i,i+1} = 2/3, \quad \pi_{i,i-1} = 1/3$

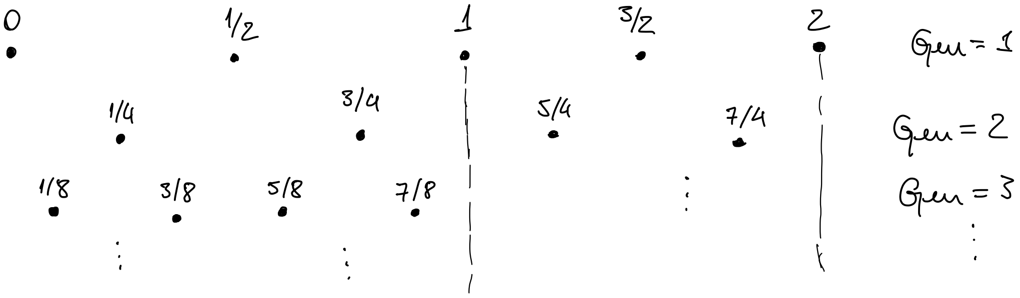
$i < 0 : \pi_{i,i+1} = 1/3, \quad \pi_{i,i-1} = 2/3$

Rates: $q_i = 2^{|i|}$

Thus, a.s., $\xi < +\infty$ and restart as follows

$$X_\xi = \begin{cases} 0 & \text{if } \lim\limits_{t\to\xi} X_t = +\infty \\ +1 \quad P = 1/2 & \\ -1 \quad P = 1/2 & \text{if } \lim\limits_{t\to\xi} X_t = -\infty \end{cases}$$

3. Non-minimal, non-right-continuous process



$D_n = \left\{ \frac{k}{2^n} : k \in \mathbb{Z} \right\}$ (diatic rationals) $\qquad I := \bigcup_n D_n$ (countable)

$\# D_n \cap [0,1) \approx 2^n$

For $i \in D_n \setminus D_{n-1}$, associate indep. r.v. S_i of par. $(2^n)^2$.

$$\mathbb{E}\Big(\sum_{j\le i} S_j\Big) \le \underbrace{(i+1)}_{\text{Number of blocks of } [0,1)} \sum_{n=0}^{\infty} \underbrace{2^{n-1}}_{\#(D_n \setminus D_{n-1})} \underbrace{(2^{-2n})}_{\text{rates}} < +\infty \;.$$

$$\mathbb{P}\Big(\sum_{j\le i} S_j \longrightarrow \infty \text{ as } i \to \infty\Big) = 1 .$$

Define the process

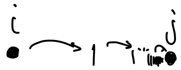
$$X_t = \begin{cases} i, & \text{if } \sum_{j<i} S_j \le t < \sum_{j\le i} S_j \text{ for some } i \in I \\ \text{NAN}, & \text{otherwise} \end{cases}$$

X_t lives in $i \in D_n \setminus D_{n-1}$ for exp. time w/ par. 1

Between any distinct states $i<j$ it makes ∞-many visits to NAN

$\mathbb{P}(X_t = \text{NAN}) = 0 \quad \forall t \geq 0.$

↑ accumulation pt.



So, the process increases towards the right but makes ∞-many jumps in finite time. It's not right-cont. b/c its' not even continuous since we don't spend time on NAN.

◢

In theorem, if I is finite, we have a third equivalence:

(iii) $\forall t, h > 0$, conditional on $X_t = i$, X_{t+h} is indep. of $(X_s : s \leq t)$ and, as $h \searrow 0$, uniformly in t, $\forall i \in I$,

$$\mathbb{P}(X_{t+h} = j \mid X_t = i) = \delta_{ij} + q_{ij} h + o(h).$$

Thm: Let Q be a Q-matrix. Then the backward eq.

$$P'(t) = QP(t), \quad P(0) = Id$$

has a minimal, non-negative solution. The solution satisfies

$$P(t+s) = P(t)P(s) \qquad \forall s,t \geq 0.$$

Pf: (Equivalences) $\quad p_{ij}(t) = \mathbb{P}_i(X_t = j)$ where (X_t) satisfies (i)

((i) $\Rightarrow$ (ii))

Step 1: $P(t)$ satisfies backw. eq.

Conditional on $X_0 = i$, $\quad J_1 \sim Exp(q_i)$

$$X_{J_1} \sim (\pi_{ik} : k \in I)$$

$$\mathbb{P}_i(X_t = j,\ t < J_1) = \delta_{ij} e^{-q_i t}$$

easier to condition on past

$$\mathbb{P}_i(J_1 \leq t, X_{J_1} = k, X_t = j) \overset{J_1 = s}{=} \int_0^t q_i e^{-q_i s} \pi_{ik} p_{kj}(t-s)\, ds.$$

$$\boxed{p_{ij}(t) = \delta_{ij} e^{-q_i t} + \sum_{k \neq i} \int_0^t q_i e^{-q_i s} \pi_{ik} p_{kj}(t-s)\, ds}$$

INTEGRAL FORM

set $u := t - s$

$$e^{q_i t} p_{ij}(t) = \delta_{ij} + \int_0^t \sum_{k \neq i} q_i e^{q_i u} \pi_{ik} \, p(u) \, du$$

$\Rightarrow$ $p_{ij}(t)$ is continuous in t

$\Rightarrow$ $p_{ij}(t)$ is differentiable in t

$$q_i \cancel{e^{q_i t}} p_{ij}(t) + \cancel{e^{q_i t}} p_{ij}'(t) = \sum_{k \neq i} q_i \cancel{e^{q_i t}} \pi_{ik} \, p_{kj}(t)$$

$q_{ii} = -q_i$

$q_i \pi_{ik} = q_{ik}$

$$p_{ij}'(t) = \sum_{k} q_{ik} \, p_{kj}(t)$$

DIFFERENTIAL FORM

i.e., $P(t) = Q P(t)$ ✓

Step 2 : If $\tilde{P}(t)$ is another nonnegative solution of backw. eq.
then $\tilde{P}(t) \geq P(t)$ $\forall t$.

$$\mathbb{P}_i(X_t = j, \, t < J_{n+1}) = \underbrace{e^{-q_i t} \delta_{ij}}_{\text{"stay at } i\text{"}}$$

$\mathbb{P}$ jumping to k

$\mathbb{P}$ jumping to j

$$+ \sum_{k \neq i} \int_0^t q_i e^{-q_i s} \pi_{ik} \, \mathbb{P}_k(X_{t-s} = j, \, t-s < J_n) \, ds$$

If $\tilde{P}(t)$ is solution then:

$$\tilde{P}_{ij}(t) = \delta_{ij} e^{-q_i t} + \sum_{k \neq i} \int_0^t q_i e^{-q_i s} \pi_{ik} \tilde{P}_{kj}(t-s)\, ds .$$

<u>By induction</u>: $(n=0)$

since $\tilde{P}(t) \geq 0$, $\mathbb{P}_i(X_t = j, t < J_0) = 0 \leq \tilde{P}_{ij}(t)$ ✓

Assume $\mathbb{P}_i(X_t = j, t < J_n) \leq \tilde{P}_{ij}(t)$. Then, comparing integral eqs., we have it for $n+1$. Hence

$$\underset{\mathbb{P}_i(X_t = j)}{\underset{\Vert}{\lim_{n \to \infty} \mathbb{P}_i(X_t = j, t < J_n)}} \leq \tilde{P}_{ij}(t)$$

<u>Step 3</u>: $P(t)$ is semi-group.

$$P_{ij}(t+s) = \sum_k \underbrace{\mathbb{P}_i(X_{s+t} = j \mid X_s = k)}_{\downarrow} \mathbb{P}_i(X_s = k)$$

Markov property

$$= \sum_k \mathbb{P}_k(X_t = j)\, \mathbb{P}_i(X_s = k)$$

$$= \sum_k P_{kj}(t) P_{ik}(s) = \big(P(t)P(s)\big)_{ij} .$$

Step 4: By Markov property,

$$\mathbb{P}\left(X_{t_{n+1}} = i_{n+1} \mid X_{t_0} = i_0, \ldots, X_{t_n} = i_n\right)$$

$$\overset{\text{Markov property}}{=} \mathbb{P}_{i_n}\left(X_{t_{n+1}-t_n} = i_{n+1}\right) \overset{\text{def}}{=} P_{i_n, i_{n+1}}(t_{n+1} - t_n).$$

((b) ⇒ (a)) For a right-continuous process, the finite dim. distributions determine the law of the process (cf. Poisson).

■

Also proved that backw. eq. has the unique minimal nonnegative solution that is a semi-group.

FORWARD EQUATION

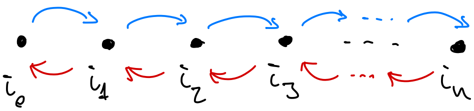
Thm: The minimal nonnegative sol. $P(t)$ of backw. eq. is also the minimal nonnegative sol. of forw. eq.:

$$P'(t) = P(t)\, Q, \quad P(0) = \mathrm{Id}.$$

Time Reversal Lemma:

$$q_{i_n} \mathbb{P}\left(J_n \leq t < J_{n+1} \mid Y_0 = i_0, \ldots, Y_n = i_n\right) \quad \text{LHS}$$

$$= q_{i_0} \mathbb{P}\left(J_n \leq t < J_{n+1} \mid Y_0 = i_n, \ldots, Y_n = i_0\right). \quad \text{RHS}$$



Pf: (Lemma) Conditional on $Y_0 = i_0, \ldots, Y_n = i_n$,

$$S_k \sim \mathrm{Exp}\left(q_{i_{k-1}}\right)$$

$$\text{LHS} = \int_{\Delta(t)} q_{i_n} e^{-q_{i_n}(t - s_1 - \cdots - s_n)} \prod_{k=1}^{n} q_{i_{k-1}} e^{-q_{i_{k-1}} s_k} \, ds_k$$

$$\Delta(t) := \{(s_1, \ldots, s_n) : s_1 + \cdots + s_n = t,\ s_i \geq 0\}$$

$$\begin{cases} u_1 := t - s_1 - \cdots - s_n \\ u_2 := S_{n-k+2} \end{cases} \qquad S_n = t - u_1 - \cdots - u_n$$

$$= \int_{\Delta(t)} q_{i_0} e^{-q_{i_0}(t - u_1 - \cdots - u_n)} \prod_{k=1}^{n} q_{i_{n-k+1}} e^{-q_{n-k+1} u_k} \, du_k$$

$$= \text{RHS}.$$

$\blacksquare$

Apply Lemma and integral eq. to prove the theorem.

LECTURE 11

Chapter 3 of Norris

$Q = (q_{ij})$ Q-matrix, I countable

$\Pi = (\pi_{ij})$, $\pi_{ij} = \frac{q_{ij}}{q_i}$, $q_i := -q_{ii}$

$P(t)$ Substochastic Matrix $(P(t))_{ij} = P_{ij}(t) = \mathbb{P}_i(X_t = j)$

satisfies $P'(t) \overset{\text{backward}}{=} QP(t)$, $P(0) = Id$

forward $= P(t)Q$

Substochastic: $\forall i \ \sum_j P_{ij}(t) \leq 1$ ← can be < 1 b/c of explosion (then the chain is not defined for $t > \zeta$)

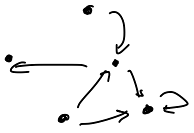
explosion time

Assume X_t is minimal; i.e., $X_t = NAN$ for $t > \zeta$

CLASS STRUCTURE FOR CONTINUOUS TIME MARKOV CHAIN

Def: i leads to j ($i \to j$) if

$$\mathbb{P}_i(X_t = j \text{ for some } t \geq 0) > 0$$



Communicating class of $i := \{ j : i \to j \text{ and } j \to i \}$

Thm: For distinct states $i \neq j$.

TFAE

(i) $i \to j$

(ii) $i \to j$ for the jump chain

(iii) $\exists\ i = i_0, i_1, \dots, i_n = j$ s.t. $q_{i_0 i_1} q_{i_1 i_2} \cdots q_{i_{n-1}, i_n} > 0$

(iv) $p_{ij}(t) > 0 \quad \forall t > 0$

(v) $p_{ij}(t) > 0$ for some $t > 0$.

Pf: (i) $\Rightarrow$ (ii) Trivial

(iv) $\Rightarrow$ (v) Trivial

(v) $\Rightarrow$ (i) Trivial

(ii) $\Rightarrow$ (iii) $\quad \frac{q_{ij}}{q_i} \overset{\text{def}}{=} \pi_{ij}$ $\quad$ By using the jump chain:

$\exists\ i_0, \dots, i_n$ s.t. $\underbrace{\pi_{i_0 i_1} \cdots \pi_{i_{n-1}, i_n}}_{*} > 0$

$$\Rightarrow * = \frac{q_{i_0 i_1}}{q_{i_0}} \frac{q_{i_1 i_2}}{q_{i_1}} \cdots \frac{q_{i_{n-1}, i_n}}{q_{i_{n-1}}} > 0$$

(iii) $\Rightarrow$ (iv)

$$P_{ij}(t) \geq \mathbb{P}_i\Big(\boxed{\underbrace{J_1}_{\sim \text{Exp}(q_i)} \leq t}, \overset{\pi_{ij}}{\boxed{Y_1 = j}}, \overset{\sim \text{Exp}(q_j) \text{ b/c we're in } j \text{ now}}{\boxed{S_2 > t}}\Big)$$

$$= \boxed{\left(1 - e^{-q_i t}\right)} \cdot \boxed{\frac{q_{ij}}{q_i}} \cdot \boxed{e^{-q_j t}}$$

$$> 0$$

By (iii), $\exists\, i_0 = i, i_1, \ldots, i_k = j$ s.t. $q_{i_k i_{k+1}} > 0 \;\forall k$

$$P_{ij}(t) \geq \underset{\substack{\vee \\ 0}}{P_{i_0 i_1}\left(\frac{t}{n}\right)} \; \underset{\substack{\vee \\ 0}}{P_{i_1 i_2}\left(\frac{t}{n}\right)} \cdots \underset{\substack{\vee \\ 0}}{P_{i_{n-1}, i_n}\left(\frac{t}{n}\right)}$$

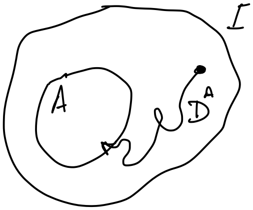
■

HITTING TIMES & ABSORPTION PROBABILITIES

$A \subset I$

Def: The hitting time of A is

$$D^A := \inf\{t \geq 0 : X_t \in A\}$$



Def: the absorption probability of A is

$$h_i^A := \mathbb{P}_i(D^A < \infty).$$

Thm: The vector $h^A = (h_i^A)_{i\in I}$ is the minimal, nonnegative solution to

$$\begin{cases} h_i^A = 1 & i \in A \\ \sum_j q_{ij} h_j^A = 0 & i \notin A \end{cases}$$

cf. Discrete Case: had

$$\begin{cases} h_i^A = \sum_j P_{ij} h_j^A & (i \notin A) \\ h_i^A = 1 & (i \in A) \end{cases}$$

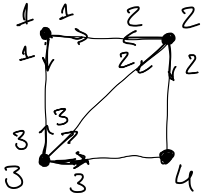
$\Longrightarrow$ the system in the continuous case is like the "derivative" of the discrete system.

$\pi_{ij} = \frac{q_{ij}}{q_i}$ $(i \neq j)$

Def: The average hitting time is

$$k_i^A := \mathbb{E}_i(D^A)$$

Example:



$k_i^A = \mathbb{E}_i(D^A)$

$$Q = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 2 & -6 & 2 & 2 \\ 3 & 3 & -9 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$q_1 = 2$
$q_2 = 6$
$q_3 = 9$
$q_4 = 1$

$$S_1 \sim \mathrm{Exp}(\lambda) \rightsquigarrow \mathbb{E}(S_1) = \lambda^{-1}$$

$$\mathbb{E}(\text{holding at } 1) = \mathbb{E}(S_1) = \frac{1}{2}$$

Expectation of holding at that state

$$k_1^{\{4\}} = \frac{1}{2} + \frac{1}{2} k_2^{\{4\}} + \frac{1}{2} k_3^{\{4\}}$$

$$k_2^{\{4\}} = \frac{1}{6} + \frac{1}{3} k_1^{\{4\}} + \frac{1}{3} k_3^{\{4\}}$$

$$k_3^{\{4\}} = \frac{1}{9} + \frac{1}{3} k_1^{\{4\}} + \frac{1}{3} k_4^{\{4\}}$$

$$k_4^{\{4\}} = 0$$

Thm: Assume $q_i > 0$ for $i \notin A$. Then $k^A := (k_i^A)_{i \in I}$ is the minimal nonnegative solution to

$$\begin{cases} k_i^A = 0 & i \in A \\ -\sum_j q_{ij} k_j^A = 1 & i \notin A \end{cases}$$

TRANSIENCE & RECURRENCE

Def: State $i \in I$ is recurrent if

$$\mathbb{P}_i(\{t : X_t = i\} \text{ is unbounded}) = 1.$$

Def: State $i \in I$ is transient if

$$\mathbb{P}_i(\{t : X_t = i\} \text{ is unbounded}) = 0.$$

Thm:

(i) If i is recurrent for the jump chain, then i is recurrent for continuous process.

(ii) If i is transient for the jump chain, then i is transient for continuous process.

(iii) Every state is either transient or recurrent

(iv) Recurrence & transience are class properties.

Pf: (i) i recurrent for (Y_n) jump chain $\Rightarrow$ rec. for (X_t).

Obs: Have a lemma that if jump chain is recurrent then (X_t) does not explode $\overset{a.s}{}$ b/c of the exponential rates.

(ii) i is transient for (Y_n). $\Rightarrow$ return times are finitely many, hence bounded

$\Rightarrow$ (X_t) transient.

LECTURE 12

Ch. 3 of Norris

CONTINUOUS TIME MARKOV CHAINS

I countable, $Q = (q_{ij})$, $\Pi = (\pi_{ij})$ Jump chain

$$(i\neq j)\quad \pi_{ij} = \begin{cases} \frac{q_{ij}}{q_i}, & q_i \neq 0 \\ 0, & \text{else} \end{cases}$$

$$(i=j)\quad \pi_{ii} = \begin{cases} 0, & q_i \neq 0 \\ 1, & q_i = 0 \end{cases}.$$

Thm: This dichotomy holds:

(i) if $q_i = 0$ or $\mathbb{P}_i(T_i<\infty) = 1$ then i is recurrent and

$$\int_0^\infty p_{ii}(t)\,dt = +\infty$$

(ii) if $q_i > 0$ and $\mathbb{P}_i(T_i<\infty) < 1$ then i is transient and

$$\int_0^\infty p_{ii}(t)\,dt < +\infty.$$

Pf: $\Pi = (\pi_{ij})$ jump chain.

WTS: $$\int_0^\infty P_{ii}(t)\,dt = \frac{1}{q_i}\sum_{n=0}^{\infty} \pi_{ii}^{(n)}$$

so that we can use the discrete dichotomy here

Fubini: $$\int_0^\infty P_{ii}(t)\,dt = \int_0^\infty \mathbb{E}_i\left(\mathbb{1}_{\{X_t = i\}}\right) dt$$

$$= \mathbb{E}_i\left(\int_0^\infty \mathbb{1}_{\{X_t = i\}}\,dt\right)$$

$$= \mathbb{E}_i\left(\sum_{n=0}^{\infty} S_{n+1}\,\mathbb{1}_{\{Y_n = i\}}\right)$$

$$= \sum_{n=0}^{\infty} \underbrace{\mathbb{E}_i\left(S_{n+1} \mid Y_n = i\right)}_{\mathrm{Exp}(q_i) \Rightarrow \mathbb{E}_i(\cdot) = \frac{1}{q_i}} \mathbb{P}_i(Y_n = i)$$

$$= \frac{1}{q_i}\sum_{n=0}^{\infty} \pi_{ii}^{(n)}.$$

■

Thm: Let $h > 0$ and $Z_n = X_{nh}$. Then,

(i) if i is recurrent for $(X_t)_{t\geq 0}$, then i is recurrent for $(Z_n)_{n\geq 0}$.

(ii) if i is transient for $(X_t)_{t\geq 0}$ then i is transient for $(Z_n)_{n\geq 0}$

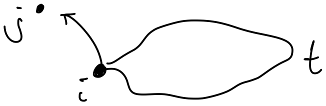
Pf: (ii) Trivial, it never goes back.

(i)

$$\overset{\overset{i \text{ is recurrent}}{\downarrow}}{+\infty} = \int_0^\infty P_{ii}(t)\,dt \overset{?}{\leq} h e^{q_i h} \sum_{n=1}^{\infty} P_{ii}(nh)$$

Note: for $nh \leq t \leq (n+1)h$,

$$P_{ii}((n+1)h) \geq \underbrace{e^{-q_i h} P_{ii}(t)}_{\mathbb{P}\text{ don't jump between } t \text{ and } (n+1)h}$$



Sum over n on both sides and we're done. ∎

INVARIANT CONTINUOUS DISTRIBUTIONS

Def: For a Q-matrix Q, λ is *invariant* if $\lambda Q = 0$.

Thm: TFAE

(i) λ is invariant for Q

(ii) $\mu \underset{\substack{\uparrow \\ \text{jump} \\ \text{chain}}}{\Pi} = \mu$ for $\mu_i = \lambda_i q_i$

Pf: $q_i(\pi_{ij} - \delta_{ij}) = q_{ij}$ by def. of $\Pi = (\pi_{ij})$.

$\mu_i = \lambda_i q_i$

$$\big(\mu(\Pi - \mathrm{Id})\big)_j = \sum_i \mu_i(\pi_{ij} - \delta_{ij}) = \sum_i \mu_i \frac{q_{ij}}{q_i}$$

$$= \sum_i \lambda_i q_{ij}$$

$$= (\lambda Q)_j$$

Upshot: $\lambda Q = 0 \iff \mu(\Pi - \mathrm{Id}) = 0$

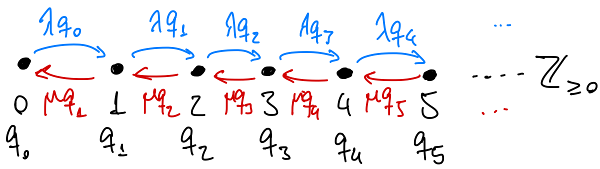
$\iff \mu \Pi = \mu$. ∎

EXISTENCE OF INVARIANT DISTRIBUTIONS

DISCRETE CASE: $m_i = \frac{1}{\pi_i}$
but I countable can
make this not well-defined

Expected Return Time: $m_i := \mathbb{E}_i(T_i)$.

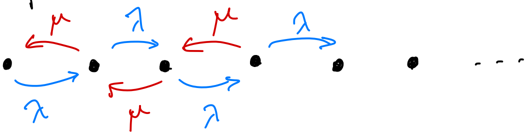
EXAMPLE: Existence of invariant distrib. in continuous case is not sufficient for positive recurrence!



Fix $0 < \lambda = 1 - \mu < 1$

$(q_i) > 0$

Jump Chain Π:



Transient: $\lambda > \mu$

Recurrent: $\lambda \leq \mu$

Invariant Distribution:

detailed balance

$$v_i q_{ij} = v_j q_{ji}$$

$$v_i \lambda q_i = v_{i+1} \mu q_{i+1} \quad (j = i+1) \quad \rightsquigarrow \quad \frac{v_{i+1}}{v_i} = \frac{\lambda}{\mu} \frac{q_i}{q_{i+1}}$$

$$\rightsquigarrow \quad v_i = \frac{1}{q_i} \left(\frac{\lambda}{\mu} \right)^i$$

Can make v_i finite by choosing large enough q_i.

e.g.: $q_i = 2^i$ and $\underbrace{1 < \frac{\lambda}{\mu} < 2}_{\Rightarrow \text{ transient}} \Rightarrow \nu_i = \frac{1}{q_i}\left(\frac{\lambda}{\mu}\right)^i$ finite

& invariant.

Upshot: Found an invariant distrib. that is transient

(PROBLEM = Q here is explosive)

Thm: Let Q be an irreducible Q-matrix.

associated jump chain is irreducible

TFAE

(i) Some state is positive recurrent

(ii) Every state is positive recurrent

(iii) Q is non-explosive and has a unique invariant distribution λ and $m_i = \frac{1}{\lambda_i q_i}$

LECTURE 13

Ch. 1 Le Gall

GAUSSIAN PROCESSES & BROWNIAN MOTION

GAUSSIAN RANDOM VARIABLES

Def: A r.v. $X : \Omega \to \mathbb{R}$ is a standard Gaussian $N(0,1)$ if its law is

$$P_X(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} .$$

- The Laplace Transform! of X is:

$$\mathbb{E}(e^{zX}) = e^{z^2/2} \qquad \forall z \in \mathbb{C}$$

Pf: $\lambda \in \mathbb{R}$.

complete the square

$$e^{\lambda x - \frac{x^2}{2}} = e^{-\left(\frac{x}{\sqrt{2}} - \frac{\lambda}{\sqrt{2}}\right)^2} e^{\lambda^2/2}$$

$$\mathbb{E}(e^{\lambda X}) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\lambda x} e^{-x^2/2} \, dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{\lambda^2/2} e^{-\frac{(x-\lambda)^2}{2}} \, dx$$

$$= e^{\lambda^2/2} .$$

Both sides are holomorphic in z. ☐

- The Fourier Transform of X is

Laplace with $z = i\xi$.

$$\mathbb{E}(e^{i\xi X}) = e^{-\xi^2/2} \quad \text{for } \xi \in \mathbb{R}.$$

Note:

$$\mathbb{E}(e^{i\xi X}) = 1 + i\xi\,\mathbb{E}(X) + \frac{(i\xi)^2}{2}\mathbb{E}(X^2) + \cdots$$

$$\Rightarrow \mathbb{E}(X) = 0 \quad \text{and} \quad \mathbb{E}(X^2) = 1.$$

n-th Moments:

$$\mathbb{E}(X^{2n}) = \frac{(2n)!}{2^n n!} \qquad \mathbb{E}(X^{2n+1}) = 0.$$

Def: If $m \in \mathbb{R}$ and $\sigma > 0$, we say Y is Gaussian of mean m and standard deviation σ, write $Y \sim N(m, \sigma^2)$ if

(i) $Y = \sigma X + m$, $X \sim N(0,1)$ or equiv.

(ii) $p_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(y-m)^2/2\sigma^2}$ or equiv. (Density)

(iii) $\mathbb{E}(e^{i\xi Y}) = e^{\left[im\xi - \frac{\sigma^2}{2}\xi^2\right]}$. (characteristic Function)

NOTE: For $Y \sim N(m, \sigma^2)$,

$$\mathbb{E}(Y) = m \quad , \quad Var(Y) = \sigma^2 .$$

SUM OF INDEP. GAUSSIAN R.V.S

Claim: If $Y_1 \sim N(m_1, \sigma_1^2)$ and $Y_2 \sim N(m_2, \sigma_2^2)$ then $Y_1 + Y_2 \sim N(m_1 + m_2, \sigma_1^2 + \sigma_2^2)$.

Pf:

$$\mathbb{E}\left(e^{i\xi(Y_1+Y_2)}\right) \overset{\text{indep}}{=} \mathbb{E}\left(e^{i\xi Y_1}\right) \mathbb{E}\left(e^{i\xi Y_2}\right)$$

$$= e^{\left[im_1\xi + \frac{\sigma_1^2}{2}\xi^2\right]} \quad e^{\left[im_2\xi + \frac{\sigma_2^2}{2}\xi^2\right]}$$

$$= e^{i(m_1+m_2)\xi + \frac{(\sigma_1^2+\sigma_2^2)}{2}\xi^2} .$$

$\square$

CONVERGENCE OF GAUSSIAN VARIABLES

Thm: Let $(X_n)_{n\geq 0}$ be a sequence of (real) Gaussian r.v.s $X_n \sim N(m_n, \sigma_n^2)$. Suppose $X_n \xrightarrow{L^2} X$.

Then

(i) $X \sim (m, \sigma^2)$ with $m = \lim_n m_n$ and $\sigma = \lim_n \sigma_n^2$.

(ii) The convergence also holds in L^p, $1 \leq p < \infty$.

Pf:

(i) $X_n \xrightarrow{L^2} X \iff \mathbb{E}(X_n) \longrightarrow \mathbb{E}(X)$ $(m_n \to m)$

$\mathrm{Var}(X_n) \longrightarrow \mathrm{Var}(X)$ $(\sigma_n \to \sigma)$

Char. Fct:

$$\mathbb{E}(e^{i\xi X}) = \lim_n \mathbb{E}(e^{i\xi X_n})$$

$$= \lim_n e^{i m_n \xi - \frac{\sigma_n^2}{2}\xi^2}$$

$$= e^{i m \xi - \frac{\sigma^2}{2}\xi^2}$$

i.e., $X \sim N(m, \sigma^2)$.

(ii) Note $X_n \sim \sigma_n N + m_n$, $(m_n), (\sigma_n)$ bounded

$N \sim N(0,1)$.

$\Longrightarrow \forall p > 1, \sup_n \mathbb{E}[|X_n|^p] < +\infty$.

Also $\sup_n \mathbb{E}[|X_n - X|^p] < +\infty$.

Since $X_n \xrightarrow{L^2} X$ then $X_n \xrightarrow{\text{prob}} X$

so $|X_n - X|^p \xrightarrow{\text{prob}} 0$

so $|X_n - X|^p$ is uniformly integrable b/c it's bounded in L^2.

So $|X_n - X|^p \xrightarrow{L^1} 0$. ✓

$\square$

Def: Sequence $(X_n) \subset L^1$ is uniformly integrable if

(i) $\sup_n E[|X_n|] < +\infty$ or equiv.

(ii) $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t. if $\mu(E) < \delta$, then

$$\int_E |X_n| \, d\mu < \varepsilon \quad \forall n.$$

GAUSSIAN VECTORS

Let $E = \mathbb{R}^d$, $d \in \mathbb{N}$.

Def: A r.v. $X : \Omega \to \mathbb{R}^d$ is a Gaussian vector if, $\forall u \in \mathbb{R}^d$, $\langle u, X \rangle$ is a Gaussian real 1-dim r.v..

↳ Easy to generalize to Hilbert spaces

Note: There exists $m_X \in \mathbb{R}^d$ (mean) s.t.

$$\mathbb{E}(\underset{\substack{\uparrow \\ \text{linear functional} \\ \text{on } \mathbb{R}^d}}{\langle u, X\rangle}) = \langle u, m_X\rangle \text{ . Say } m_X := \mathbb{E}(X) \text{ .}$$

Note: There exists a quadratic form $q_X : \mathbb{R}^d \longrightarrow \mathbb{R}$

$$q_X(u) = \mathrm{Var}(\langle u, X\rangle) \quad \forall u \in \mathbb{R}^d .$$

Pf: Let $(e_1, \ldots, e_d)$ be an orthonormal basis for $\mathbb{R}^d$

Write

$$X = \sum_{j=1}^{d} e_j X_j \quad \text{s.t.} \quad X_j = \langle e_j, X\rangle \text{ is Gaussian}$$

Then

$$m_X = \sum_{j=1}^{d} e_j \, \mathbb{E}(X_j) =: \mathbb{E}(X)$$

If $\underset{\mathbb{R}^d}{\underset{\rotatebox{90}{$\in$}}{u}} = \sum_{j=1}^{d} e_j u_j \qquad u_j \in \mathbb{R}$

$$q_X(u) = \mathrm{Var}\left(\left\langle \sum_{j=1}^{n} e_j u_j , \sum_{j=1}^{d} e_j X_j \right\rangle\right)$$

$$= \sum_{j,k} u_j u_k \, \mathrm{Cov}(X_j, X_k)$$

Characteristic Function

$$\mathbb{E}\left(e^{i\langle u, X\rangle}\right) = e^{i\langle u, m_X\rangle - \frac{1}{2} q_X(u)} \qquad \forall u \in \mathbb{R}^d.$$

Thm: The r.v.s $X_1, \ldots, X_d$ are independent $\left(\begin{array}{c} X = \sum e_j X_j \\ \text{Gaussian} \end{array} \right)$ if and only if $\mathrm{Cov}(X_j, X_k)$ = Identity Matrix $\forall j, k$.

Pf: $(\Rightarrow)$ If (X_j) are indep. then $\underset{j,k}{\mathrm{Cov}} (X_j, X_k) = \delta_{jk}$.

$(\Leftarrow)$ If $\mathrm{Cov} = \mathrm{Id}$, write $q_X(u) = \sum_{j=1}^{d} \lambda_j u_j^2$, $\lambda_j \in \mathbb{R}$.

So

$$\mathbb{E}\left(e^{i \sum_j u_j X_j}\right) = e^{i \sum_j u_j \mathbb{E}(X_j) - \frac{1}{2} \sum_j \lambda_j u_j^2}$$

$$= \prod_j \left(e^{i u_j \mathbb{E}(X_j) - \frac{1}{2} \lambda_j u_j^2} \right)$$

$$= \prod_{j} \mathbb{E}\left(e^{iu_j X_j}\right) .$$

∎

<u>Thm</u>: Given q_X, there is a unique symmetric endomorphism $\gamma_X : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ s.t., $\forall u \in \mathbb{R}^d$,

$$q_X(u) = \langle u, \gamma_X(u) \rangle \geq 0 .$$

(γ_X has matrix $= \underset{j,k}{\mathrm{Cov}}(X_j, X_k)$ which is symmetric)

<u>Thm</u>:

(i) Let γ be a symmetric non-negative endomorphism of $\mathbb{R}^d$. Then, there exists a Gaussian vector X s.t. $\gamma_X = \gamma$.

$E(X) = 0$ by a simple change of coords.

(ii) Let X be a centered Gaussian vector. Let $(\varepsilon_1, —, \varepsilon_d)$ be a basis for $\mathbb{R}^d$ for which γ_X is diagonal: $\gamma_X \varepsilon_j = \lambda_j \varepsilon_j$, $1 \leq j \leq d$

$$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0 = \lambda_{r+1} = \dots = \lambda_d$$

$r := \mathrm{rank}(\gamma_X)$

Then $X = \sum_{j=1}^{d} Y_j \varepsilon_j$, where (Y_j) are indep. real 1d Gaussian r.v.'s with $\mathrm{Var}(Y_j) = \lambda_j$.

Then the topological support of P_X (distrib. of X) is $\mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_d\} \subset \mathbb{R}^d$. So, P_X is absolutely continuous w.r.t. Lebesgue on $\mathbb{R}^d$ iff $r = d$, and if so

$$P_X(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det \gamma_X}} e^{-\frac{1}{2}\langle x, \gamma_X^{-1}(x)\rangle} \qquad \forall x \in \mathbb{R}^d .$$

GAUSSIAN SPACES

$L^2(\Omega, \mathcal{F}, \mathbb{P})$ Prob. Space

$X : \Omega \to \mathbb{R}^d$ Gaussian

$L^2(\Omega) \ni X_1, \ldots, X_d : \Omega \to \mathbb{R}$ Gaussian

Def: A centered Gaussian space is a closed linear subspace of $L^2(\Omega, \mathcal{F}, \mathbb{P})$ such that all elements are real centered Gaussian r.v.s.

They do not to be indep (In fact, in this space indep. $\longleftrightarrow$ orthogonality)

Def: A real valued random process $(X_t)_{t \in T}$ is a Gaussian process if any finite linear combination of X_t's is Gaussian.

Thm: The closed linear subspace generated (spanned + closure) by X_t is a Gaussian space.

Pf: The L^2 limit of Gauss. is Gauss.. $\square$

Def: A collection H of r.v.'s on $(\Omega, \mathcal{F}, \mathbb{P})$, the σ-algebra $\sigma(H)$ is the smallest σ-algebra that contains all $h^{-1}(B)$ $\forall h \in H$, $B \in \mathcal{B}(\mathbb{R})$.

Thm: Let H be a centered Gaussian space and let $(H_i)_{i \in I}$ be a collection of linear subspaces of H. Then the (H_i) are pairwise orthogonal iff $\sigma(H_i)$ are independent.

WARNING: It's important that all H_i's are subspaces of the same Gaussian space.

If not, can have things like

$X \sim N(0,1)$

$$\begin{tikzcd} & +1 \quad \mathbb{P}=1/2 \\ \varepsilon \arrow[ur] \arrow[dr] & \\ & -1 \quad \mathbb{P}=1/2 \end{tikzcd}$$

(indep of X)

$X_1 = X \sim N(0,1)$

$\updownarrow$ Not indep $|X_1| = |X_2|$

$X_2 = \varepsilon X \sim N(0,1)$

However: $\mathbb{E}(X_1 X_2) = \mathbb{E}(\varepsilon X^2) = \underbrace{\mathbb{E}(\varepsilon)}_{=0} \mathbb{E}(X^2) = 0$

$\Rightarrow$ X_1, X_2 uncorrelated "orthogonal". (X_1, X_2 don't live in same space b/c of ε...)

"L^2 inner prod."

Pf: $(\Leftarrow)$ $\sigma(H_i)$ indep $\Rightarrow$ $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y) = 0$
$\forall X \in H_1, Y \in H_2$.

$(\Rightarrow)$ Conversely, suppose $H_1 \perp H_2$

WTS: $X_1^{(1)}, \dots, X_{n_1}^{(1)} \in H_1$ vectors $(X_1^{(1)}, \dots, X_{n_1}^{(1)}), (X_1^{(2)}, \dots, X_{n_2}^{(2)})$
$X_1^{(2)}, \dots, X_{n_2}^{(2)} \in H_2$ are indep.

Find orthonormal bases $\tilde{X}_1^{(1)}, \dots, \tilde{X}_{n_1}^{(1)}$ of $\operatorname{span}\{X_1^{(1)}, \dots, X_{n_1}^{(1)}\} \subset H_1$

$\tilde{X}_1^{(2)}, \dots, \tilde{X}_{n_2}^{(2)}$ of span $\{X_1^{(2)}, \dots, X_{n_2}^{(2)}\} \subset H_2$

Consider $(\tilde{X}_1^{(1)}, \dots, \tilde{X}_{n_1}^{(1)}, \tilde{X}_1^{(2)}, \dots, \tilde{X}_{n_2}^{(2)})$ uncorrelated Gaussian variables

$H_1 \perp H_2$ by assumption

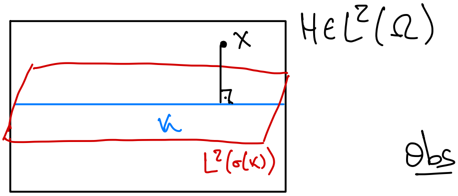
$\Downarrow$

They're indep.

$\blacksquare$

Corollary: (Conditional Expectation) Let $X \in H$. then

$$\underbrace{K}_{\text{closed, lin. subsp.}} \subset \underbrace{H}_{\text{Gaussian}} \subset L^2(\Omega) \qquad \mathbb{E}(X \mid \sigma(K)) = \underset{\substack{\uparrow \\ \text{orthog. proj. onto } K}}{\text{proj}_K}(X)$$



Obs: $K \subsetneq L^2(\sigma(K))$.

LECTURE 14

GAUSSIAN WHITE NOISE

Prob. Space - $(\Omega, \mathcal{F}, \mathbb{P})$

Gaussian Space: closed subspace $H \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$ s.t. every element is centered Gaussian.

Def: Given a σ-finite measure space $(E, \mathcal{E}, \mu)$.
A Gaussian white noise with intensity μ is an isometry

$$G : L^2(E, \mathcal{E}, \mu) \longrightarrow H \subset L^2(\Omega, \mathcal{F}, \mathbb{P})$$

into a Gaussian space H.

Note: If $f \in L^2(E, \mathcal{E}, \mu)$ then $G(f)$ is Gaussian with variance:

G isometry ↔ preserves inn. prod.

$$\mathbb{E}\left(G(f)^2\right) \overset{\downarrow}{=} \int_E f^2 \, d\mu \, .$$

If $f, g \in L^2(E, \mathcal{E}, \mu)$

G isometry

$$\mathbb{E}\left(G(f)\, G(g)\right) \overset{\downarrow}{=} \int_E f g \, d\mu \, .$$

If $f = \mathbb{1}_A$, then $G(f)$ has variance

$$\mathbb{E}\left(G(f)^2\right) = \int_E f^2 \, d\mu = \mu(A).$$

Define: $G(A) := G(\mathbb{1}_A)$.

If (A_i) are disjoint, $\mu(A_i) < \infty$, then $\left(G(A_1), \_\_, G(A_n)\right)$ is Gaussian vector with diagonal covariance matrix.

$$\mathbb{E}\left[G(A_i)G(A_j)\right] = \int \mathbb{1}_{A_i} \mathbb{1}_{A_j} \, d\mu = \overset{\swarrow\; i \neq j}{0}$$

$$\Downarrow$$

$G(A_i)$ are independent

Thm: Let $(E, \mathcal{E})$ be a measurable space and μ σ-finite on $\mathcal{E}$. Then, there exist $(\Omega, \mathcal{F}, \mathbb{P})$ and a Gaussian white noise with intensity μ.

Pf: $(f_i)_{i \in I}$ orthonormal basis for (Hilbert space) $L^2(E, \mathcal{E}, \mu)$.
For $f \in L^2(E, \mathcal{E}, \mu)$, let $\alpha_i := \langle f, f_i \rangle$ so $f = \sum_{i \in I} \alpha_i f_i$.

Let $(X_i)_{i\in I}$ be a collection of indep. RVs with law $N(0,1)$ on some space $(\Omega, \mathcal{F})$. Define $G(f) = \sum_{i\in I} \alpha_i X_i$

$\square$

Ex: $E=\mathbb{R}$, $\mathcal{E} = \mathcal{B}(\mathbb{R})$

$(\xi_n)_{n\in\mathbb{N}}$ seq. of iid $N(0,1)$

$(\varphi_n)_{n\in\mathbb{N}}$ orthonormal basis of $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mu)$

$$G(f) = \sum_{n\in\mathbb{N}} \langle f, \varphi_n \rangle \xi_n .$$

Thm: Let G be a Gaussian white noise on $(E, \mathcal{E})$ with intensity μ. Let $A \subset \mathcal{E}$ with $\mu(A) < \infty$. Assume there is a sequence of partitions

$$A = A_1^{(n)} \sqcup \cdots \sqcup A_k^{(n)} , \quad n\in\mathbb{N}$$

s.t. $\lim\limits_{n\to\infty} \left(\sup\limits_j \mu(A_j^{(n)}) \right) = 0$.

Then,

$$\lim_{n\to\infty} \sum_{j=1}^{n_k} G(A_j^{(n)})^2 = \mu(A) \quad \text{in } L^2 .$$

Note: $\mathbb{E}\left(\sum_j G(A_j^{(n)})^2 \right) = \sum_j \mu(A_j^{(n)}) = \mu(A)$.

Pf: $$\mathbb{E}\left[\left(\sum_j G(A_j^{(n)})^2 - \mu(A)\right)^2\right] = \mathrm{Var}\left[\sum_j G(A_j^{(n)})^2\right]$$

$X \sim N(0,\sigma^2)$

$\mathrm{Var}(X^2) = \mathbb{E}(X^4) - \sigma^4$

$= \dots$

$= 3\sigma^4 - \sigma^4$

$= 2\sigma^4$

$$= \sum_j \mathrm{Var}\left(G(A_j^{(n)})^2\right)$$

$$= 2 \sum_j \left(\mathrm{Var}\left(G(A_j^{(n)})\right)\right)^2$$

$$= 2 \sum_j \mu(A_j^{(n)})^2$$

$$\leq 2 \sup_j \left(\mu(A_j^{(n)})\right)\mu(A) \xrightarrow[n\to\infty]{} 0 .$$

$\square$

PRE-BROWNIAN MOTION

Def: Let G be a Gaussian white noise on $\mathbb{R}_+$ with intensity $\mu =$ Lebesgue. Define the pre-Brownian motion as

$$B_t := G(\mathbb{1}_{[0,t]}) \quad \text{for } t \in \mathbb{R}_+ .$$

Thm: Pre-Brownian motion is a centered Gaussian process and its covariance $\mathbb{E}[B_s B_t] = \min\{s,t\} =: s \wedge t$, $s,t \in \mathbb{R}_+$.

Pf:

$$\mathbb{E}[B_s B_t] \overset{\text{def}}{=} \mathbb{E}\left[G(\mathbb{1}_{[0,s]})\,G(\mathbb{1}_{[0,t]})\right]$$

$$= \int_{\mathbb{R}_+} \mathbb{1}_{[0,s]}(x)\ \mathbb{1}_{[0,t]}(x)\ dx$$

$$= \min\{s,t\}.$$

$\square$

Thm: Let $(X_t)_{t\geq 0}$ be a random process.

TFAE

(i) X_t is a pre-Brownian motion

(ii) X_t is centered Gaussian with covariance $\mathbb{E}(X_s X_t) = s \wedge t$

(iii) $X_0 = 0$ a.s. and for any $0 \leq s \leq t$, the increment $X_t - X_s$ is independent of $\sigma(X_r : r \leq s)$ and $X_t - X_s \sim N(0, \underbrace{t-s}_{\text{variance grows linearly}})$.

(iv) $X_0 = 0$ a.s. and for any choices of $0 = t_0 < \dots < t_p$ the increments $X_{t_{i+1}} - X_{t_i}$ are indep. and of law $N(0, t_{i+1} - t_i)$

Pf: (iii) $\Rightarrow$ (iv) Induction

(i) $\Rightarrow$ (ii) Done above.

(ii) $\Rightarrow$ (iii) $r \leq s \leq t$, $t = s+u$

$$\mathbb{E}[X_r(X_{s+u} - X_s)] = \min\{r, s+u\} - \min\{r, s\}$$

$$= r - r = 0 \Rightarrow X_r \text{ and } X_{s+u} - X_s$$

are uncorrelated.

Since X_r, $X_{s+u} - X_s$ are uncorrelated Gaussians, they are independent. Finally,

$$\mathrm{Var}(X_{s+u} - X_s) = \mathbb{E}[(X_{s+u} - X_s)^2]$$

$$= \mathbb{E}[X_{s+u}^2] - 2\mathbb{E}[X_{s+u}X_s] + \mathbb{E}[X_s^2]$$

$$= s+u-2s+s = u.$$

$$\Rightarrow X_t - X_s \sim N(0, t-s).$$

(iv) $\Rightarrow$ (i) $0 = t_0 < t_1 < \dots < t_p$

$$f = \sum_i \lambda_i \mathbb{1}_{(t_{i-1}, t_i]} \quad \text{set} \quad G(f) := \sum_i \lambda_i \overbrace{(X_{t_i} - X_{t_{i-1}})}^{\text{process satisfying (iv)}}$$

WTS: G is an L^2-isometry.

$\Rightarrow$ Notation: $G(f) = \int_0^t f(s)\, dB_s \approx \sum_i f(t_i)(B_{t_i} - B_{t_{i-1}})$ $\square$

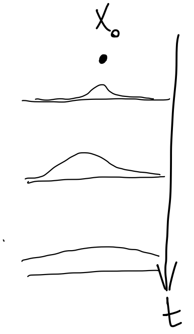
LECTURE 15

Pre-Brownian Motion: (X_t) is centered Gaussian

$X_t - X_s \sim N(0, t-s)$ for $s < t$

Independent increments: for $0 = t_0 < t_1 < \dots$,

$$(X_{t_{i+1}} - X_{t_i})_i \text{ are independent.}$$



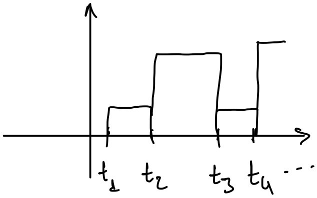
White Noise: Isometric embedding

$$G : L^2(E, \mathcal{E}) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$$

Typical ex. of white noise:

$$G : L^2(\mathbb{R}_+, \mathcal{B}(\mathbb{R}_+), dt) \longrightarrow L^2(\Omega, \mathcal{F}, \mathbb{P})$$

$$f := \sum_i \lambda_i \overset{\cup}{\mathbb{1}}_{(t_i, t_{i-1}]} \longmapsto G(f) = \sum_i \lambda_i \underbrace{(X_{t_i} - X_{t_{i-1}})}_{\text{random increments}}$$



(Wiener Integral) $G(f) = \int_0^\infty f(t)\, dB_t$ ← NOTATION

TERMINOLOGY: Brownian motion = Wiener process

NOTE: If $g : \mathbb{R}_+ \to \mathbb{R}$ is of bounded variation, then can define the Stieltjes integral

$$\int_0^\infty f(t)\, dg(t) = \lim \sum_i f(t_i)[g(t_i) - g(t_{i-1})]$$

WARNING: Brownian motion (B_t) does NOT have bounded variation a.s. ?!

ISSUE: With this definition, we don't know if $t \mapsto X_t$ is continuous a.s.

Def: The functions $t \mapsto X_t(\omega)$ are called SAMPLE PATHS i.e., for each $\omega \in \Omega$, this is a fct. $\mathbb{R}_+ \to \mathbb{R}$.

$(X, \mu) \xrightarrow{\Phi} Y \qquad \Phi_* \mu(A) = \mu(\Phi^{-1}(A))$

NOTE:

$$\Phi : \Omega \longrightarrow C^0(\mathbb{R}_+, \mathbb{R})$$
$$\omega \longmapsto (t \mapsto X_t(\omega))$$

The pushforward $\Phi_* \mathbb{P}$ is called the WIENER MEASURE W on $C^0(\mathbb{R}_+, \mathbb{R})$.

Cylinder set

$$A := \{ w \in C^0(\mathbb{R}_+, \mathbb{R}) : w(t_0) \in A_0, w(t_1) \in A_1, \_\_, w(t_n) \in A_n \}$$

for $t_0 < t_1 < \dots < t_n$ and measurable $A_0, A_1, \_\_, A_n \subset \mathbb{R}$.

Then

$$W\left(\{ w : w(t_0) \in A_0, \_\_, w(t_n) \in A_n \}\right)$$

$$= \mathbb{1}_{A_0}(0) \int_{A_1 \times \dots \times A_n} \frac{dx_1 \cdots dx_n}{(2\pi)^{n/2} \sqrt{t_1 (t_2 - t_1) \cdots (t_n - t_{n-1})}} \exp\left(- \sum_{i=1}^{n} \frac{(X_{t_i} - X_{t_{i-1}})^2}{2(t_i - t_{i-1})} \right)$$

(WIENER MEASURE)

HOW TO GO FROM PRE-BROWNIAN TO BROWNIAN ?

Def: Given processes (X_t) and $(\tilde{X}_t)$, we say $\tilde{X}_t$ is a modification of X_t if $\forall t \in \mathbb{R}_+$ $\mathbb{P}(X_t = \tilde{X}_t) = 1$.

Note: X_t continuous $\not\Longrightarrow$ $\tilde{X}_t$ continuous.

Much stronger

Def: $\tilde{X}_t$ is indistinguishable from X_t if there exists $N \subset \Omega$ s.t. $\mathbb{P}(N) = 0$ and, $\forall \omega \in \Omega \setminus N$, $\tilde{X}_t(\omega) = X_t(\omega)$ $\forall t$.

Note: If X is a modification of pre-Brownian then X is also pre-Brownian.

REMARK: If $X, \tilde{X}$ are continuous (except on a prob-0 set) then X is a modification of $\tilde{X}$ iff X is indistinguishable from $\tilde{X}$.

Pf: X is mod. of $\tilde{X}$ $\Rightarrow$ $\forall t \in \mathbb{Q}_+$, $X_t(\omega) = \tilde{X}_t(\omega)$
on a set Ω_t, $\mathbb{P}(\Omega_t) = 1$.

So on $\bigcap_{t \in \mathbb{Q}_+} \Omega_t$, $X_t(\omega) = \tilde{X}_t(\omega)$ $\forall t \in \mathbb{Q}_+$, hence by continuity,

$X_t(\omega) = \tilde{X}_t(\omega)$ $\forall t \in \mathbb{R}_+$ $\Rightarrow$ indistinguishable.

□

Thm: (Kolmogorov's Lemma) Let $(X_t)_{t\in I}$ be a random process with I being a bounded interval, taking values on a complete metric space (E, d). Suppose $\exists\, q, \varepsilon, C > 0$ s.t. $\forall s, t \in I$

$$\mathbb{E}\left[d(X_s, X_t)^q\right] \leq C\,|t-s|^{\varepsilon+1}.$$

Then, there is a modification $\tilde{X}$ of X whose sample paths are Hölder continuous with exponent α for any $\alpha \in (0, \varepsilon/q)$; i.e., $\forall \omega \in \Omega$ and $\forall \alpha \in (0, \varepsilon/q)$ $\exists\, C_\alpha(\omega)$ s.t.

$$d(\tilde{X}_s(\omega), \tilde{X}_t(\omega)) \leq C_\alpha(\omega)\,|t-s|^\alpha \quad \forall s, t \in I.$$

Corollary: A pre-Brownian motion (B_t) has a modification whose sample paths are locally Hölder conts. with exponent $\frac{1}{2} - \delta$, $\forall \delta \in (0, \frac{1}{2})$.

Pf (Cor): $B_s - B_t \sim N(0, t-s)$

$$\mathbb{E}(|B_s - B_t|^q) \overset{?}{\leq}$$

$$\Rightarrow U := \frac{B_s - B_t}{\sqrt{t-s}} \sim N(0,1)$$

INCREMENTS OF THE MODIFIED pre-Brownian motion!

$\forall q > 0$:

$$\mathbb{E}\left(|B_s - B_t|^q\right) \leq \mathbb{E}\left[(t-s)^{q/2} |U|^q\right] = (t-s)^{q/2} \underbrace{\mathbb{E}\left[|U|^q\right]}_{=:C}$$

$1 + \varepsilon = \frac{q}{2}$ and by the lemma $\alpha < \frac{\varepsilon}{q}$

i.e., $\alpha < \frac{\varepsilon}{q} = \frac{q-2}{2q} = \frac{1}{2} - \frac{1}{q}$.

i.e., $\forall \delta > 0 \ \exists q$ s.t. $\frac{1}{2} - \frac{1}{q} = \frac{1}{2} - \delta$.

$\square$

Def: A process $(B_t)_{t \geq 0}$ is a Brownian motion if it is pre-Brownian and continuous.

Note: Well-defined b/c modifications of continuous process are indistinguishable.

WIENER MEASURE

$\omega \longmapsto (t \mapsto B_t(\omega)) \in C^0(\mathbb{R}_+, \mathbb{R})$ defines a prob. measure on $C^0(\mathbb{R}_+, \mathbb{R})$ and is uniquely determined by its value on cylinder sets.

If equip $(C^0(\mathbb{R}_+, \mathbb{R}), W)$ then the CANONICAL PROCESS is $X_t(w) = w(t)$ is a Brownian motion.

$\uparrow$ randomly pick a function and look at time t

GEOMETRIC PROPERTIES OF BROWNIAN MOTION

$(B_t)_{t \geq 0}$ Brownian motion

$$\mathcal{F}_t := \sigma(B_s : s \leq t)$$

$$\mathcal{F}_{0^+} := \bigcap_{s>0} \mathcal{F}_s$$

Thm: (Blumenthal's 0-1 law) The σ-algebra $\mathcal{F}_{0^+}$ is trivial i.e., $\forall A \in \mathcal{F}_{0^+}$, $\mathbb{P}(A) \in \{0, 1\}$.

Pf: $0 < t_1 < t_2 < \cdots < t_k$, $g : \mathbb{R}^k \to \mathbb{R}$ continuous and bdd

$A \in \mathcal{F}_{0^+}$.

$$\mathbb{E}\left[\mathbb{1}_A \, g(B_{t_1}, \ldots, B_{t_k})\right] = \lim_{\varepsilon \to 0} \mathbb{E}\left[\mathbb{1}_A \, g(B_{t_1} - B_\varepsilon, \ldots, B_{t_k} - B_\varepsilon)\right]$$

by continuity of sample paths.

$B_{t_1} - B_\varepsilon, —, B_{t_k} - B_\varepsilon$ are indep. of B_ε $(\varepsilon < t_1)$, hence indep. of $\mathcal{F}_{0^+}$.

Thus:

$$= \mathbb{E}(\mathbb{1}_A) \lim_{\varepsilon \to 0} \mathbb{E}\left[g\left(B_{t_1} - B_\varepsilon, —, B_{t_k} - B_\varepsilon\right)\right]$$

$$= \mathbb{P}(A) \; \mathbb{E}\left[g\left(B_{t_1}, —, B_{t_k}\right)\right]$$

$\Rightarrow$ $\mathcal{F}_{0^+}$ is indep. of $\sigma(B_{t_1}, —, B_{t_k})$.

$\Rightarrow$ $\mathcal{F}_{0^+}$ is indep of $\sigma(B_t : t > 0) = \sigma(B_t : t \geq 0)$.

Note: $\mathcal{F}_{0^+} \subset \sigma(B_t : t \geq 0)$ $\Rightarrow$ $\mathcal{F}_{0^+}$ is indep. of itself hence trivial.

$\mathbb{P}(X \in A) = \mathbb{P}(X \in A, X \in A) = \mathbb{P}(X \in A)^2$. $\square$

Proposition: (i) For every $\varepsilon > 0$

$$\sup_{0 \leq s \leq \varepsilon} B_s > 0 \text{ a.s. and } \inf_{0 \leq s \leq \varepsilon} B_s < 0 \text{ a.s.}$$

(ii) For any $a \in \mathbb{R}$, let $T_a := \inf\{t \geq 0 : B_t = a\}$.

Then, $\forall a \in \mathbb{R}$, $T_a < \infty$ a.s. and

$$\limsup_{t \to \infty} B_t = +\infty \text{ a.s. and } \liminf_{t \to \infty} B_t = -\infty \text{ a.s.}$$

Pf: (i) Let $\varepsilon_p \to 0$ and set

$$A := \bigcap_p \left\{ \sup_{0 \le s \le \varepsilon_p} B_s > 0 \right\} \in \mathcal{F}_{0^+}$$

$$\mathbb{P}(A) = \lim_{p \to \infty} \underbrace{\mathbb{P}\left(\sup_{0 \le s \le \varepsilon} B_s > 0 \right)}_{\geq}$$

$$\mathbb{P}(B_{\varepsilon_p} > 0) = \frac{1}{2} \text{ (centered Gaussian)}$$

$\Longrightarrow \mathbb{P}(A) = 1$ by Blumenthal's 0-1.

i.e., $\forall \varepsilon > 0 \quad \mathbb{P}\left(\sup_{0 \le s \le \varepsilon} B_s > 0 \right) = 1$

$\square$

LECTURE 16

Brownian Motion: $(B_t)_{t \ge 0}$

$B_0 = 0$

$B_t - B_s \sim N(0, t-s)$

Indep. increments

$t \mapsto B_t(\omega)$ continuous for a.e. ω.

(i) $\forall \varepsilon > 0 \quad \sup_{0 \le s \le \varepsilon} B_s > 0, \quad \inf_{0 \le s \le \varepsilon} B_s < 0$ a.s.

(ii) $\forall a \in \mathbb{R}$, $T_a := \inf\{t \ge 0 : B_t = a\}$, $T_a < +\infty$ a.s., $\limsup_{t \to \infty} B_t = +\infty$

$\liminf_{t\to\infty} B_t = -\infty$

Proposition: Let $(B_t)_{t\geq 0}$ be Brownian motion. Then

(i) $(-B_t)_{t\geq 0}$ is also Brownian motion (HW04)

(ii) $\forall \lambda > 0$, $B_t^{\lambda} := \frac{1}{\lambda} B_{\lambda^2 t}$ is also BM

(iii) $\forall s \geq 0$, $B_t^{(s)} := B_{s+t} - B_t$ is also BM

Pf: (ii) $\frac{1}{\lambda} B_{\lambda^2 t}$; $B_t \sim N(0, t)$

$$\frac{1}{\lambda} B_t \sim N(0, t/\lambda^2)$$

$$\frac{1}{\lambda} B_{\lambda^2 t} \sim N(0, t).$$

□

Claim: $\forall a \in \mathbb{R}$, $T_a := \inf\{t \geq 0 : B_t = a\}$. Then $\boxed{T_a < +\infty \text{ a.s.}}$ and $\limsup_{t\to\infty} B_t = +\infty$ and $\liminf_{t\to\infty} B_t = -\infty$

implies the limits by taking $a \in \mathbb{R}$ bigger and bigger or smaller and smaller.

Pf: $1 = \mathbb{P}\left(\sup_{0\leq s\leq 1} B_s > 0\right)$

Fix $a > 0$

$$= \lim_{\delta \downarrow 0} \mathbb{P}\left(\sup_{0\leq s\leq 1} B_s > a\delta\right)$$

$B_s \sim \delta B_{\frac{s}{\delta^2}}$

$s = u\delta^2$

$$= \lim_{\delta \searrow 0} \mathbb{P}\left(\sup_{0 \le u \le \frac{1}{\delta^2}} \delta B_u > a\delta \right)$$

$$= \lim_{\delta \searrow 0} \mathbb{P}\left(\sup_{0 \le u \le \frac{1}{\delta^2}} B_u > a \right)$$

$$= \mathbb{P}\left(\sup_{u > 0} B_u > a \right)$$

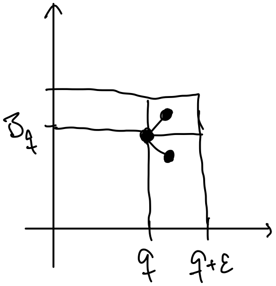
For $a < 0$, use inf instead of sup.

$\square$

<u>Corollary</u>: A.s., the function $t \mapsto B_t$ is <u>not</u> monotone on any interval.

<u>Pf</u>: $\forall q \in \mathbb{Q}$, $q > 0$ and $\varepsilon > 0$

$$\sup_{q < t < q+\varepsilon} B_t > B_q \qquad (*)$$



Consider $\tilde{B}_t := B_{t+q} - B_q$ which is a BM.

Then apply the prop. above to get $(*)$.

Also, $\inf_{q \le t < q+\varepsilon} B_t < B_q$

$\Longrightarrow B_t$ is <u>not</u> monotone on $[q, q+\varepsilon]$ a.s.

Take any $q \in \mathbb{Q}^+$ and $\varepsilon \in \mathbb{Q}^+$. Then

$\mathbb{P}(B_t$ is not monotone in <u>any</u> interval$) = 1$.

$\square$

Proposition: (B_t) has finite quadratic variation.

That is, let $0 = t_0^n < t_1^n < \dots < t_{p_n}^n = t$ be a seq. of subdivisions with

$$\sup_{1 \le i \le p_n} \left(t_i^n - t_{i-1}^n \right) \xrightarrow{n \to \infty} 0 \quad \text{a.s.}$$

(e.g. equispaced mesh)

Then,

$$\lim_{n \to \infty} \sum_{i=1}^{p_n} \left(B_{t_i^n} - B_{t_{i-1}^n} \right)^2 = t \quad \text{in } L^2 .$$

Corollary: A.s., the function $t \mapsto B_t$ has infinite variation on any interval.

$\Rightarrow$ Very non-differentiable!

$$\mathrm{Var}(f) := \sup_{\{\text{partitions}\}} \sum_{i=1}^{p_n} \left| f(t_i) - f(t_{i-1}) \right|$$

Note: B_t is a.s. locally $\left(\frac{1}{2} - \varepsilon\right)$-Hölder but not of bdd variation?!

Pf: (Cor) Taking a subsequence, $\exists$ a seq. of partitions s.t.

$$t \xleftarrow[=]{\text{a.s.}} \sum_{i=1}^{p_n} \left(B_{t_i^n} - B_{t_{i-1}^n} \right)^2$$

$$\le \underbrace{\sup_i \left| B_{t_i^n} - B_{t_{i-1}^n} \right|}_{\longrightarrow 0 \text{ as } n \to \infty \text{ a.s.}} \cdot \underline{\sum_{i=1}^{p_n} \left| B_{t_i^n} - B_{t_{i-1}^n} \right|}$$

since B_t is continuous $\Rightarrow$ $\nearrow$ must go to $+\infty$ as $n \to \infty$

$\square$

MARKOV PROPERTY (in Brownian Motion)

$$\mathcal{F}_\infty := \sigma(B_s : s \geq 0)$$

$$\mathcal{F}_t := \sigma(B_s : 0 \leq s \leq t)$$

Def: $T : \Omega \to [0,\infty]$ is a stopping time iff $\forall t$, $\{T \leq t\} \in \mathcal{F}_t$.

Note: $\{T < t\} = \bigcup_{q \in [0,t)\cap\mathbb{Q}} \{T \leq q\} \in \mathcal{F}_t$

Def: The σ-algebra of the past before T is

$$\mathcal{F}_T := \{A \in \mathcal{F}_\infty : \forall t \geq 0,\ A \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

Thm: (Strong Markov Property) Let T be a stopping time with $\mathbb{P}(T<\infty) > 0$. Set

$$B_t^{(T)} := \mathbb{1}_{\{T<\infty\}}(B_{T+t} - B_T).$$

Then, conditional on $\{T<\infty\}$, the process $(B_t^{(T)})_{t\geq 0}$ is BM independent of $\mathcal{F}_T$.

LECTURE 17

PROPERTIES OF BM

Strong Markov Property: Let T be a stopping time s.t. $\mathbb{P}(T<\infty)>0$ and set

$$B_t^{(T)} := \mathbb{1}_{\{T<\infty\}}\left(B_{T+t}-B_T\right).$$

Then, conditional on $\{T<\infty\}$, the process $(B_t^{(T)})_{t\geq 0}$ is BM and independent of $\mathcal{F}_T$. ← σ-algebra of the past

Pf: Given $A\in\mathcal{F}_T$, $t_1<\cdots<t_p$, $F:\mathbb{R}^p\to\mathbb{R}$ bdd & conts.,

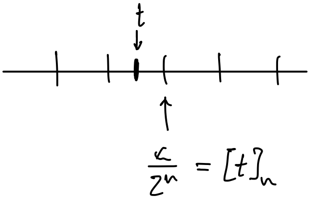
WTS:

$$\mathbb{E}\left[\mathbb{1}_A F\left(B_{t_1}^{(T)},\_\_,B_{t_p}^{(T)}\right)\right]=\mathbb{P}(A)\,\mathbb{E}\left[F\left(B_{t_1},\_\_,B_{t_p}\right)\right]$$

This implies that $B_t^{(T)}$ is independent of $\mathcal{F}_T$.

Setting $A=\Omega$, $B_{t_1}^{(T)}\sim B_{t_1}$ so it's also BM.

Define: $[t]_n := \inf\left\{\frac{k}{2^n} : \frac{k}{2^n} > t\right\}$



Note: $F\left(B_{t_1}^{(T)}, \dots, B_{t_p}^{(T)}\right) = \lim_n F\left(B_{t_1}^{[T]_n}, \dots, B_{t_p}^{[T]_n}\right)$ a.s.

So,

$$\mathbb{E}\left[\mathbb{1}_A F\left(B_{t_1}^{(T)}, \dots, B_{t_p}^{(T)}\right)\right] = \lim_n \mathbb{E}\left[\mathbb{1}_A F\left(B_{t_1}^{[T]_n}, \dots, B_{t_p}^{[T]_n}\right)\right]$$

$$= \lim_n \sum_{k=0}^{\infty} \mathbb{E}\left[\mathbb{1}_A \mathbb{1}_{\left\{\frac{k-1}{2^n} < T \leq \frac{k}{2^n}\right\}} F\left(B_{t_1+\frac{k}{2^n}} - B_{\frac{k}{2^n}}, \dots, B_{t_p+\frac{k}{2^n}} - B_{\frac{k}{2^n}}\right)\right]$$

Note: if $A \in \mathcal{F}_T$, then

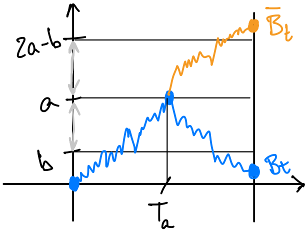
$$A \cap \left\{\frac{k-1}{2^n} < T \leq \frac{k}{2^n}\right\} = \left(A \cap \left(T \leq \frac{k}{2^n}\right)\right) \cap \left(T \leq \frac{k-1}{2^n}\right)^c \in \mathcal{F}_{\frac{k}{2^n}} .$$

$$= \lim_n \sum_k \mathbb{P}\left(A \cap \left\{\frac{k-1}{2^n} < T \leq \frac{k}{2^n}\right\}\right) \mathbb{E}\left(F\left(B_{t_1}, \dots, B_{t_p}\right)\right)$$

$$= \mathbb{P}(A) \cdot \mathbb{E}\left(F\left(B_{t_1}, \dots, B_{t_p}\right)\right).$$

$\square$

REFLECTION PRINCIPLE: If $a \geq 0$, $b \in (-\infty, a]$, set $S_t := \sup_{0 \leq s \leq t} B_s$.



Then,

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(B_t \geq 2a-1)$$

Moreover, S_t has the same distribution of $|B_t|$.

Pf: $T_a := \inf\{t \geq 0 : B_t = a\} < +\infty$ a.s..

$$\mathbb{P}(S_t \geq a, B_t \leq b) = \mathbb{P}(T_a \leq t, B_t \leq b)$$

$$= \mathbb{P}(T_a \leq t, \underbrace{B^{(T_a)}_{t-T_a}}_{\text{progress between } T_a \text{ and } t} \leq b-a)$$

$\tilde{B} := B^{(T_a)}$ is BM indep. of $\mathcal{F}_{T_a}$.

$(T_a, \tilde{B}) \sim (T_a, -\tilde{B})$

$H := \{(s,w) \in \mathbb{R}_+ \times C^0(\mathbb{R}_+, \mathbb{R}) : s \leq t, w(t-s) \leq b-a\}$.

$$= \mathbb{P}((T_a, \tilde{B}) \in H) = \mathbb{P}((T_a, -\tilde{B}_t) \in H) = \mathbb{P}(T_a \leq t, -B^{[T_a]}_{t-T_a} \leq b-a)$$

$$= \mathbb{P}(T_a \leq t, B_t \geq a + a - b = 2a-b) = \mathbb{P}(B_t \geq 2a-b).$$

$2a-b \geq a$ since $a \geq b$

Compute distrib. of S_t:

$$\begin{aligned}
\mathbb{P}(S_t \geq a) &= \mathbb{P}(S_t \geq a, B_t \leq b) + \mathbb{P}(S_t \geq a, B_t \geq b) \\
&= \mathbb{P}(S_t \geq a, B_t \leq a) + \mathbb{P}(S_t \geq a, B_t \geq a) \\
&= \mathbb{P}(B_t \geq a) + \mathbb{P}(B_t \geq a) \\
&= 2\,\mathbb{P}(B_t \geq a) \\
&\overset{\text{symmetry around zero}}{=} \mathbb{P}(|B_t| \geq a).
\end{aligned}$$

$\square$

<u>Corollary</u>: For every $a > 0$, $T_a \sim \frac{a^2}{B_1^2}$ and the density of T_a is

$$f(t) = \frac{a}{\sqrt{2\pi t^3}} \exp\left(-\frac{a^2}{2t}\right) \mathbb{1}_{\{t>0\}} .$$

Moreover $\mathbb{E}(T_a) = \int f(t)\,dt = +\infty$.

Pf: $\mathbb{P}(T_a \leq t) = \mathbb{P}(S_t \geq a) = \mathbb{P}(|B_t| \geq a) = \mathbb{P}(B_t^2 \geq a^2)$

$$= \mathbb{P}(t B_1^2 \geq a^2) = \mathbb{P}\left(\frac{a^2}{B_1^2} \leq t\right).$$

$\square$

<u>Def</u>: If Z is a random variable, a process $(X_t)_{t\geq 0}$ is a <u>BM started at Z</u> if $X_t = B_t + Z$ with (B_t) standard BM.

<u>Def</u>: A random process $B_t = (B_t^1, \ldots, B_t^d)$ with values in $\mathbb{R}^d$ is a <u>d-dim. BM</u> if its components are independent $\mathbb{R}$-valued BMs.

FILTRATIONS & MARTINGALES (Ch. 3 of Le Gall)

<u>Def</u>: A <u>FILTRATION</u> $(\mathcal{F}_t)_{0\leq t\leq\infty}$ is a collection of σ-algebras on $(\Omega, \mathcal{F}, \mathbb{P})$ s.t. $\mathcal{F}_s \subset \mathcal{F}_t$ for $s \leq t$.

<u>Canonical</u> <u>Filtration</u>: $\mathcal{F}_t := \sigma(B_s : s \leq t)$.

$$\mathcal{F}_{t^+} := \bigcap_{s>t} \mathcal{F}_s .$$

<u>Def</u>: The filtration is <u>RIGHT-CONTINUOUS</u> if $\mathcal{F}_{t^+} = \mathcal{F}_t \ \forall t > 0$.

Def: A process $(X_t)_{t\geq 0}$ is measurable if

$$(\Omega \times \mathbb{R}_+, \mathcal{F} \otimes \mathcal{B}(\mathbb{R}_+)) \longrightarrow (E, \mathcal{E}) \text{ metric space}$$

$$(\omega, t) \longmapsto X_t(\omega)$$

is measurable.

Note: This is stronger than saying $\omega \mapsto X_t(\omega)$ is measurable for each t because above we check measurability for both coords.

Def: A random process is ADAPTED to a filtration $(\mathcal{F}_t)$ if for every $t \geq 0$, X_t is $\mathcal{F}_t$ - measurable.

This process is PROGRESSIVE if for every $t \geq 0$,

$$(\omega, s) \longmapsto X_s(\omega)$$

$$\Omega \times [0,t] \longrightarrow E$$

is measurable for $\mathcal{F}_t \otimes \mathcal{B}([0,t])$.

Note: $\boxed{\text{Progressive}} \Longrightarrow \boxed{\text{adapted \& measurable}}$

(stronger b/c we check both coords.)

Proposition: Let (X_t) be a random process. If X_t is adapted and the sample paths are right-continuous, then X is progressive.

Def: A random variable $T : \Omega \to [0,\infty]$ is a stopping time for $(\mathcal{F}_t)$ if $\{T \leq t\} \in \mathcal{F}_t$ for every $t \geq 0$.

The σ-algebra of the past of T is

$$\mathcal{F}_T := \{ A \in \mathcal{F}_\infty : \forall t \geq 0 , \ A \cap \{T \leq t\} \in \mathcal{F}_t \}.$$

Def: An adapted real-valued process (X_t) s.t. $X_t \in L^1$ for every t is called a

- MARTINGALE if for $0 \leq s < t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$
- SUPERMARTINGALE if for $0 \leq s < t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] \leq X_s$
- SUBMARTINGALE if for $0 \leq s < t$, $\mathbb{E}[X_t \mid \mathcal{F}_s] \geq X_s$

Example: A process (Z_t) has *independent increments* w.r.t. $(\mathcal{F}_t)$ if it is adapted and, for $0 \leq s < t$, $Z_t - Z_s$ is independent of $\mathcal{F}_s$.

Thm: if Z_t is indep. increments,

(i) if $Z_t \in L^1$ $\forall t \geq 0$, then $\tilde{Z}_t := Z_t - \mathbb{E}[Z_t]$ is a martingale.

$$\mathbb{E}[\tilde{Z}_t \mid \mathcal{F}_s] = \mathbb{E}\big[Z_t - \mathbb{E}[Z_t] \mid \mathcal{F}_s\big]$$

$$= \mathbb{E}\Big[\boxed{Z_t - Z_s} \underbrace{+ Z_s - \mathbb{E}[Z_t - Z_s] - \mathbb{E}[Z_s] \mid}_{\text{indep.}} \boxed{\mathcal{F}_s}\Big]$$

$$= \cancel{\mathbb{E}[Z_t - Z_s]} + Z_s - \cancel{\mathbb{E}[Z_t - Z_s]} - \mathbb{E}[Z_s]$$

$$= \tilde{Z}_s .$$

(ii) if $Z_t \in L^2$ $\forall t \geq 0$, then $Y_t := \tilde{Z}_t^2 - \mathbb{E}[\tilde{Z}_t^2]$ is a martingale.

$$\mathbb{E}[Y_t \mid \mathcal{F}_s] = \mathbb{E}[Y_t - Y_s + Y_s \mid \mathcal{F}_s]$$

$$Y_t = (Z_t - Z_s + Z_s)^2 - \mathbb{E}\big[(Z_t - Z_s + Z_s)^2\big]$$

$$= (Z_t - Z_s)^2 - 2(Z_t - Z_s)Z_s + Z_s^2 - \mathbb{E}\big[(Z_t - Z_s)^2\big]$$

$$+ 2\,\mathbb{E}\big[(Z_t - Z_s)Z_s\big] + \mathbb{E}[Z_s^2]$$

$$\Rightarrow \mathbb{E}\left[\tilde{Z}_t^2 \mid \mathcal{F}_s\right] = \tilde{Z}_s^2 + 2\,\mathbb{E}\left[\tilde{Z}_t - \tilde{Z}_s \mid \mathcal{F}_s\right]^{=0} + \mathbb{E}\left[(\tilde{Z}_t - \tilde{Z}_s)^2 \mid \mathcal{F}_s\right]$$

$$= \tilde{Z}_s^2 + \mathbb{E}\left[(\tilde{Z}_t - \tilde{Z}_s)^2\right]$$

$$= \tilde{Z}_s^2 + \mathbb{E}\left[\tilde{Z}_t^2\right] - \mathbb{E}\left[\tilde{Z}_s^2\right]$$

(iii) if for some $\theta \in \mathbb{R}$, $\mathbb{E}[e^{\theta Z_t}] < +\infty$ $\forall t \geq 0$, then

$$X_t := \frac{e^{\theta Z_t}}{\mathbb{E}[e^{\theta Z_t}]}$$ is a martingale.

$$\mathbb{E}\left[\frac{e^{\theta Z_t}}{\mathbb{E}[e^{\theta Z_t}]} \,\middle|\, \mathcal{F}_s\right] = \mathbb{E}\left[\frac{e^{\theta(Z_t - Z_s)} \cdot e^{\theta Z_s}}{\mathbb{E}[e^{\theta(Z_t - Z_s)}\, e^{\theta Z_s}]} \,\middle|\, \mathcal{F}_s\right]$$

$$\overset{\text{indep. increments}}{=} \underbrace{\mathbb{E}\left[\frac{e^{\theta(Z_t - Z_s)}}{\mathbb{E}[e^{\theta(Z_t - Z_s)}]}\right]}_{=1} \cdot \mathbb{E}\left[\frac{e^{\theta Z_s}}{\mathbb{E}[e^{\theta Z_s}]} \,\middle|\, \mathcal{F}_s\right]$$

$$= \frac{e^{\theta Z_s}}{\mathbb{E}[e^{\theta Z_s}]}.$$

<u>Corollary</u>: If $(B_t)_{t \geq 0}$ is BM, then

B_t, $B_t^2 - t$, $e^{\theta B_t - \theta^2 t/2}$ are martingales.

<u>NOTE</u>: For any $f \in L^2(\mathbb{R}_+)$, let

$$Z_t = \int_0^t f(s)\, dB_s \,. \quad \text{“=”} \sum_i f(t_i)\left(B_{t_{i+1}} - B_{t_i}\right)$$

Then Z_t has independent increments w.r.t. canonical filtration of (B_t).

So, $\displaystyle\int_0^t f(s)\,dB_s \,, \ \left(\int_0^t f(s)\,dB_s\right)^2 - \int_0^t f(s)^2\,ds \,,$

$$\text{and} \quad \exp\left(\theta \int_0^t f(s)\,dB_s - \frac{\theta^2}{2}\int_0^t f(s)^2\,ds\right)$$

are all martingales.

<u>Proposition</u>: Let (X_t) be an adapted process and let $f : \mathbb{R} \to \mathbb{R}_+$ be a convex function s.t. $\mathbb{E}[f(X_t)] < \infty \ \forall t \geq 0$

(i) If (X_t) is martingale then $f(X_t)$ is submartingale.

(ii) If (X_t) is submart. and f is nondecreasing, then $(f(X_t))_{t\geq 0}$ is submartingale.

Pf: $\mathbb{E}[f(X_t) \mid \mathcal{F}_s] \overset{\text{Jensen}}{\geq} f\left(\mathbb{E}[X_t \mid \mathcal{F}_s]\right) \overset{\substack{f \text{ nondec.} \\ \& \ X_t \text{ submart.}}}{\geq} X_s \,.$

$\square$

<u>Thm</u>: (Martingale Convergence) Let (X_t) be a supermartingale with right-continuous sample paths. Assume $(X_t)_{t\geq 0}$ is bounded (uniformly) in L^1.

Then, $\exists!$ random variable $X_\infty \in L^1$ s.t.

$$\lim_{t\to\infty} X_t = X_\infty .$$

<u>Def</u>: A martingale is <u>CLOSED</u> if $\exists Z \in L^1$ s.t.

$$X_t = \mathbb{E}[Z \mid \mathcal{F}_t] \quad \forall t \geq 0 .$$

<u>Thm</u>: Let (X_t) be a martingale with right-continuous sample paths. TFAE:

(i) X_t is closed

(ii) $(X_t)_{t\geq 0}$ is uniformly integrable

(iii) X_t converges a.s. and in L^1 as $t \to \infty$.

Moreover, if these hold, $X_t = \mathbb{E}[X_\infty \mid \mathcal{F}_t]$ $\forall t \geq 0$ with $X_\infty = \lim_{t\to\infty} X_t$ a.s..

LECTURE 18

MARTINGALES

$(\mathcal{F}_t)_{t\geq 0}$ increasing filtration of Ω

Def: An adapted process $(X_t)_{t\geq 0}$ s.t.

- $X_t \in L^1 \ \forall t \geq 0$ and
- $\mathbb{E}[X_t | \mathcal{F}_s] = X_s$ for any $0 \leq s < t$ is a martingale.
- $\mathbb{E}[X_t | \mathcal{F}_s] \leq X_s$ " " " " " supermartingale
- $\mathbb{E}[X_t | \mathcal{F}_s] \geq X_s$ " " " " " submartingale

Examples: B_t, $B_t^2 - t$, $e^{\theta B_t - \frac{\theta^2 t}{2}}$ are all martingales

Thm: Let $(M_t)_{t\geq 0}$ be a square-integrable martingale (i.e., $M_t \in L^2 \ \forall t$).
Let $0 \leq s < t$, $s = t_0 < t_1 < \dots < t_p = t$. Then,

$$\mathbb{E}\left[\sum_{i=1}^{p} (M_{t_i} - M_{t_{i-1}})^2 \,\Big|\, \mathcal{F}_s\right] = \mathbb{E}\left[M_t^2 - M_s^2 \,|\, \mathcal{F}_s\right]$$

$$= \mathbb{E}\left[(M_t - M_s)^2 \,|\, \mathcal{F}_s\right].$$

Pf:

$$\mathbb{E}\left[\sum_{i=1}^{p}(M_{t_i}-M_{t_{i-1}})^2 \,\middle|\, \mathcal{F}_s\right] = \sum_{i=1}^{p}\mathbb{E}\left[(M_{t_i}-M_{t_{i-1}})^2 \mid \mathcal{F}_s\right]$$

$$= \sum_{i=1}^{p}\mathbb{E}\left[\mathbb{E}\left[(M_{t_i}-M_{t_{i-1}})^2 \mid \mathcal{F}_{t_{i-1}}\right] \,\middle|\, \mathcal{F}_s\right]$$

$M_{t_{i-1}}$ is measurable w.r.t. $\mathcal{F}_{t_{i-1}}$

$$= \sum_{i=1}^{p}\mathbb{E}\left[\mathbb{E}\left[M_{t_i}^2 \mid \mathcal{F}_{t_{i-1}}\right] - 2\,\underline{M_{t_{i-1}}}\,\underbrace{\mathbb{E}\left[M_{t_i} \mid \mathcal{F}_{t_{i-1}}\right]}_{=M_{t_{i-1}} \text{ by martingale}} + \underline{M_{t_{i-1}}^2} \,\middle|\, \mathcal{F}_s\right]$$

$$= \sum_{i=1}^{p}\mathbb{E}\left[\mathbb{E}\left[M_{t_i}^2 \mid \mathcal{F}_{t_{i-1}}\right] - M_{t_{i-1}}^2 \,\middle|\, \mathcal{F}_s\right]$$

$\sigma(\mathcal{F}_{t_{i-1}}) \supset \sigma(\mathcal{F}_s)$

$$= \sum_{i=1}^{p}\mathbb{E}\left[M_{t_i}^2 - M_{t_{i-1}}^2 \mid \mathcal{F}_s\right]$$

Telescoping sum once inside $\mathbb{E}$

$$= \mathbb{E}\left[M_t^2 - M_s^2 \mid \mathcal{F}_s\right].$$

■

Thm: (i) (MAXIMAL INEQUALITY) Let $(X_t)_{t\geq 0}$ be a supermartingale with right-continuous sample paths. Then, for any $t>0$ and every $\lambda > 0$,

$$\mathbb{P}\left(\sup_{0\leq s\leq t} |X_s| > \lambda\right) \leq \frac{1}{\lambda}\mathbb{E}[|X_0|] + \frac{2}{\lambda}\mathbb{E}[|X_t|]$$

(ii) (DOOB'S INEQUALITY) Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths. Then, $\forall t>0$ and $\forall p>1$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t} |X_s|^p\right] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}[|X_t|^p] .$$

Thm: (OPTIONAL STOPPING) Let $(X_t)_{t\geq 0}$ be a uniformly integrable martingale with right-continuous sample paths. Let S, T be stopping times with $S\leq T$ a.s.. Then X_S and X_T are in L^1 and

$$\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S .$$

$(X_t)_{t\geq 0}$ uniformly integrable if

$$\lim_{\kappa\to\infty} \sup_t \mathbb{E}\left(|X_t| \mathbb{1}_{\{|X_t|\geq \kappa\}}\right) = 0$$

without integrability condition, and bounded stopping times

Corollary 1: (from OST) Let $(X_t)_{t\geq 0}$ be a martingale with right-continuous sample paths and let $S \leq T$ be bounded stopping times. Then, $X_S, X_T \in L^1$ and

$$\mathbb{E}[X_T \mid \mathcal{F}_S] = X_S$$

Pf: Say $S \leq T \leq a \in \mathbb{R}$. Then $\tilde{X}_t := X_{t\wedge a}$. WTS: $\tilde{X}_t$ is unif. integrable.

Note: $\tilde{X}_t$ is closed by X_a b/c

$$\tilde{X}_t \overset{\text{def}}{=} X_{t\wedge a} = \mathbb{E}[X_a \mid \mathcal{F}_t] \quad \forall\, t \leq a$$

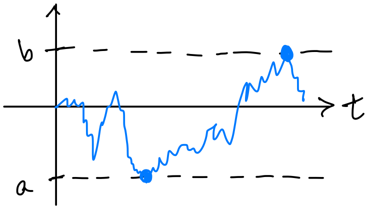
$$\overset{\text{Thm from}}{\underset{\text{last time}}{\Longleftrightarrow}} \quad \tilde{X}_t \text{ unif. integrable.}$$

□

Corollary 2: If T is stopping time and (M_t) martingale then $M_{t\wedge T}$ is martingale.

APPLICATIONS TO BROWNIAN MOTION !

- Set $T_a := \inf\{t \geq 0 : B_t = a\}$.



! **Thm:** For $a < 0 < b$,

$$\mathbb{P}(T_a < T_b) = \frac{b}{b-a}.$$

Pf: Set! $\boxed{T := T_a \wedge T_b}$. Then $M_t := B_{t \wedge T}$ is a martingale.

Moreover, $|M_t| \le \max\{|a|, |b|\}$ $\forall t$.

Set $S = 0$. So, by the Optional Stopping Thm,

$$\mathbb{E}[M_T] = \mathbb{E}[M_0] = 0 .$$

$S = 0$ so we have no conditioning...

$B_0 = 0$

$$\begin{aligned} &\| \\ &a\,\mathbb{P}(T_a < T_b) + b\,\mathbb{P}(T_a > T_b) \\ &\| \\ &a\,\mathbb{P}(T_a < T_b) + b\left(1 - \mathbb{P}(T_a < T_b)\right) \end{aligned}$$

$$\Rightarrow \quad \mathbb{P}(T_a < T_b) = \frac{b}{b-a} .$$

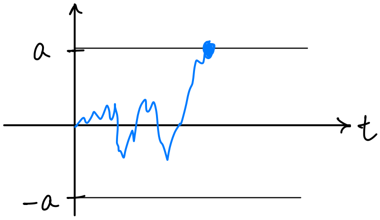
∎

!Thm: For $a > 0$, let $U_a := \inf\{t \ge 0 : |B_t| = a\}$.

Then,

$$\boxed{\mathbb{E}[U_a] = a^2}$$

BIZARRE b/c $\mathbb{E}[T_a] = +\infty$



$U_a << T_a$

Pf: $M_t := B_t^2 - t$ and take stopping times ! $\left\{\begin{array}{l} T := U_a \\ \qquad\ \rotatebox[origin=c]{-90}{$\geq$} \\ S := 0 \end{array}\right.$

Then

$$\underset{\displaystyle \mathbb{E}\left[(B_{t\wedge U_a})^2\right] - \mathbb{E}\left[(t\wedge U_a)\right]}{\underset{\Vert}{\mathbb{E}\left[M_{t\wedge U_a}\right]}} \overset{\substack{\text{Cor 2 above since } M_t \text{ is martingale} \\ \downarrow}}{=} \mathbb{E}\left[M_0\right] = 0 .$$

i.e., $\mathbb{E}\left[(B_{t\wedge U_a})^2\right] = \mathbb{E}\left[(t\wedge U_a)\right]$

Now,

$$\begin{aligned} \mathbb{E}\left[U_a\right] &\overset{\substack{\text{Monotone Convergence} \\ \swarrow}}{=} \lim_{t\to\infty} \mathbb{E}\left[t\wedge U_a\right] \\ &= \lim_{t\to\infty} \mathbb{E}\Big[\underbrace{(B_{t\wedge U_a})^2}_{\text{is integrable b/c } (B_{t\wedge U_a})^2 \leq a^2}\Big] \\ &\overset{\text{and DCT}}{=} \mathbb{E}\left[B_{U_a}^2\right] \\ &= a^2 . \end{aligned}$$

∎

! <u>Thm</u>: (LAPLACE TRANSFORM OF HITTING TIMES) For $\lambda > 0$, and $a > 0$,

$$\boxed{\mathbb{E}\left[e^{-\lambda T_a}\right] = e^{-a\sqrt{2\lambda}} .}$$

Pf: Consider the martingale $\boxed{N_t^\lambda := \exp\left(\lambda B_t - \frac{\lambda^2}{2} t\right)}$

$L^1 \ni \rightarrow$

for appropriate λ

and stopping times $T := T_a$, $S := 0$!

Then, $N^\lambda_{t \wedge T_a}$ stopped martingale is bounded (b/c $B_{t\wedge T_a} \le a$)

$$\Downarrow$$

$$N^\lambda_{t\wedge T_a} \le e^{\lambda a}.$$

So,

$$\mathbb{E}\left[N^\lambda_{T_a}\right] \overset{\substack{\text{Optional stopping Thm} \\ \downarrow}}{=} \mathbb{E}\left[N^\lambda_0\right] = 1$$

$$\|$$

$$\mathbb{E}\left[\exp\left(\lambda a - \frac{\lambda^2}{2} T_a\right)\right] \Rightarrow \mathbb{E}\left[e^{-\frac{\lambda^2}{2} T_a}\right] = e^{-\lambda a}$$

Set $\mu := \frac{\lambda^2}{2}$ $\longrightarrow$ $\mathbb{E}\left[e^{-\mu T_a}\right] = e^{-a\sqrt{2\mu}}$.

$\Rightarrow \lambda = \sqrt{2\mu}$

∎

<u>NOTE</u>: (i) If (X_t) is unif. integrable, then $(X_{t\wedge T})$ is unif. integ. martingale: $\mathbb{E}[X_T | \mathcal{F}_t] = X_{t\wedge T}$.

(ii) If (X_t) is martingale, then $(X_{t\wedge T})$ is martingale.

<u>Cl</u>: (i) $\Rightarrow$ (ii)

CONTINUOUS SEMI-MARTINGALES

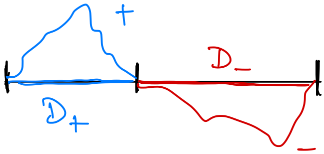
(Finite Variation Process + local martingales)

FINITE VARIATION PROCESSES

Def: Let $T \geq 0$. A continuous function $a : [0,T] \to \mathbb{R}$ s.t. $a(0) = 0$ has finite variation if $\exists$ signed measure μ on $[0,T]$ s.t. $a(t) = \mu([0,t]) \ \forall t \in [0,T]$.

Cl: Every signed measure μ can be written as $\mu = \mu_+ - \mu_-$ of two positive finite measures μ_+, μ_- supported in disjoint Borel sets. Define the total variation measure as

$$|\mu| := \mu_+ + \mu_-$$



Obs: $|\mu|$ is a positive measure

- NOTE: $$\frac{d\mu}{d|\mu|} = \mathbb{1}_{D_+} - \mathbb{1}_{D_-} .$$

- If $f : [0,T] \to \mathbb{R}$ is measurable and $\int_{[0,T]} |f(s)| \, |\mu|(ds) < +\infty$, let $$\int_0^T f(s) \, da(s) = \int_{[0,T]} f(s) \, \mu(ds)$$

"Stjeltjes" Integral $\int f(s) da(s) \approx \sum_i f(s_i) \cdot (a(s_i) - a(s$

Proposition: $\forall t \in [0,T]$,

$$\int_0^t |da|(s) = \sup_{\{\text{partitions}\}} \left\{ \sum_{i=1}^{p} |a(t_i) - a(t_{i-1})| \right\} =: \operatorname{Var}_{[0,T]}(a).$$

Pf: $\lim_{n\to\infty} \sum_{i=1}^{p_n} \left| a(t_i^{(n)}) - a(t_{i-1}^{(n)}) \right| = \int_0^T |da|(s)$

where $0 =: t_0 < \cdots < t_{p_n} = T$ partition w/ $\sup_i |t_i^{(n)} - t_{i-1}^{(n)}| \xrightarrow{n\to\infty} 0$

$(\leq)$ $|a(t_i) - a(t_{i-1})| = |\mu([t_{i-1}, t_i))| \leq |\mu|([t_{i-1}, t_i))$.

$(\geq)$ $\begin{cases} \Omega := [0,T] \\ \mathcal{B}([0,T]) \\ \mathbb{P} = |\mu| \end{cases}$ $\mathcal{B}_n := \sigma\left([t_{i-1}^{(n)}, t_i^{(n)}] ; i \right)$

$X(s) := \dfrac{d\mu}{d|\mu|}(s)$

Claim: $X_n = \mathbb{E}[X \mid \mathcal{B}_n]$ is a martingale.

Now, X_n is constant on $[t_{i-1}^{(n)}, t_i^{(n)})$ b/c here it's just an actual "average" over some real intervals. Moreover,

$$X_n = \frac{a(t_i^{(n)}) - a(t_{i-1}^{(n)})}{|\mu|([t_{i-1}^{(n)}, t_i^{(n)}))}.$$

Also, (X_n) is $\overset{a}{\vee}$ closed martingale closed by X.

So, by Martingale Convergence Thm,

$$X_n \xrightarrow{L^1} X \qquad \text{i.e.,} \qquad \lim_n \mathbb{E}[|X_n|] = \mathbb{E}[|X|] = 1$$

$$\|$$

$$\lim_n \sum_i \frac{|a(t_i^{(n)}) - a(t_{i-1}^{(n)})|}{|\mu|([0,T])} .$$

i.e.,

$$\lim_n \sum_i |a(t_i^{(n)}) - a(t_{i-1}^{(n)})| \le |\mu|([0,T]) .$$

■

LECTURE 19

Obs: Let X_t be martingale and T stopping time.

X_T ← not a martingale...

$(X^T)_t := X_{t\wedge T}$ ← is a martingale (stopped process)

Proposition: (i) if (X_t) is unif. integrable martingale, then $X_{t\wedge T}$ is a unif. integ. mart.

(ii) if (X_t) is mart. (even if not unif. integ.), then $X_{t\wedge T}$ is mart.

NOTE: $\mathbb{E}[X_T] \neq \mathbb{E}[X_0]$ but $\mathbb{E}[X_{t\wedge T}] = \mathbb{E}[X_0]$.

Claim: (i) $\Rightarrow$ (ii)

Pf: WTS that $\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s] \overset{?}{=} X_{s\wedge T}$ $(s<t)$

Take $a > \max\{s,t\}$ and set $\tilde{X}_t := X_{t\wedge a}$. Then $\tilde{X}_t$ is unif. integ. b/c it's closed by X_a.

So, by (i), $\mathbb{E}[\tilde{X}_{t\wedge T} \mid \mathcal{F}_s] \overset{(i)}{=} \tilde{X}_{s\wedge T} \overset{a>s}{=} X_{s\wedge T}$

$a>t \rightarrow \parallel$

$\mathbb{E}[X_{t\wedge T} \mid \mathcal{F}_s]$ $=$ as desired.

$\square$

//

FINITE VARIATION PROCESS (Let $(\mathcal{F}_t)_{t\geq 0}$ be a filtration)

Def: An adapted process (A_t) is a *finite variation process* if all of its sample paths are finite variation continuous functions on every $[0,T]$; i.e.,

$t \mapsto A_t(\omega)$ is continuous and has finite variation $\forall \omega \in \Omega$

NOTE: If A_t has finite variation, then

$$V_t := \int_0^t |dA_s| \text{ is an increasing process in } t.$$

NOTE: $a(t)$ is finite variation $\iff$ $a(t) = \mu([0,t))$ for some signed (non-atomic) measure μ

$$\int_0^t |da(s)| = |\mu|([0,t)).$$

$$\Updownarrow$$

$$\sup_{\left\{\substack{\text{partitions} \\ \text{of } [0,T]}\right\}} \sum_{i=1}^{P_n} |a(t_i) - a(t_{i-1})| < +\infty \quad \forall T.$$

CONTINUOUS LOCAL MARTINGALES

Def: A continuous adapted process (M_t) is a *continuous local martingale* if there is a non-decreasing sequence of stopping times $(T_n)_{n\geq 0}$ such that $T_n(\omega) \to \infty$ $\forall \omega \in \Omega$ and

$$M^{T_n} - M_0 = M_{t \wedge T_n} - M_0$$

can be anything

is a uniformly integrable martingale.

We say (T_n) REDUCES M.

NOTE: There is no L^1 assumption on M_t. ← (Good old martingales used to be in L^1, unlike here)

PROPERTIES: (i) M martingale with continuous sample paths then $T_n = n$ reduces M.

(ii) If M is a continuous local martingale, then for any stopping time T, $M^T = M_{t \wedge T}$ is also a continuous local martingale.

(iii) If $(T_n)_{n \geq 0}$ reduces M and $S_n \nearrow \infty$ is another sequence of stopping times, then $(T_n \wedge S_n)_{n \geq 0}$ also reduces M.

$$\left[(M^{T_n})^{S_n} \overset{\text{def}}{=} M^{T_n \wedge S_n} \right]$$

(iv) The space of continuous local martingales is a vector space.

- Multiplication by scalar ✓ clear
- M, M' are cont. loc. mart. $\overset{?}{\Longrightarrow}$ $M + M'$ cont. loc. mart.

Pf: (T_n) reduces M and (S_n) reduces M', then $(T_n \wedge S_n)_n$ reduces $M + M'$. So, $M + M'$ is cont. loc. mart.

Proposition: (i) A nonnegative cont. loc. mart. M s.t. $M_0 \in L^1$ is a supermartingale.

(ii) A cont. loc. mart. M s.t. there exists $Z \in L^1$ with $|M_t| \leq Z$ $\forall t \geq 0$ is a uniformly integrable martingale.

(iii) If M is a cont. loc. mart. with $M_0 \in L^1$, the sequence of stopping times

$$T_n := \inf\{t \geq 0 : |M_t| \geq n\}$$

reduces M.

Pf: (i) $N_t := M_t - M_0$. By def. of cont. loc. mart., $\exists$ seq. of stopping times $(T_n) \nearrow \infty$ s.t. $N^{T_n} = N_{t\wedge T_n}$ is a unif. integ. mart.; i.e.,

$$\mathbb{E}[N_{t\wedge T_n} | \mathcal{F}_s] = N_{s\wedge T_n}$$

$$\Big\Updownarrow \; M_0 = 0$$

$$\begin{tikzcd} \mathbb{E}[M_{t\wedge T_n} | \mathcal{F}_s] \arrow[r, phantom, "="] \arrow[d, "n\to\infty"] & M_{s\wedge T_n} \arrow[d, "n\to\infty"] \\ \mathbb{E}[M_t | \mathcal{F}_s] \arrow[r, phantom, "\leq"] & M_s \end{tikzcd}$$

Definition of supermartingale

FATOU b/c M is nonnegative

Fatou's lemma: $(f_n) \geq 0 \Rightarrow \varliminf \int f_n \geq \int \varliminf f_n$

Set $s=0$, then

$$\mathbb{E}[M_t \mid \mathcal{F}_0] = \mathbb{E}[M_t] \leq \mathbb{E}[M_0] < +\infty$$

(ii) $M_{s\wedge T_n} = \mathbb{E}[M_{t\wedge T_n} \mid \mathcal{F}_s]$ and $M_{s\wedge T_n}$ is unif. integr. by the def. of cont. loc. mart..

Now, $|M_{t\wedge T_n}| \leq Z \in L^1$. $\Rightarrow$ Use DCT to get

Of course $M_{t\wedge T_n} \to M_t$ as $n\to\infty$. So, by DCT

$$M_s = \mathbb{E}[M_t \mid \mathcal{F}_s].$$

(iii) $M^{T_n} - M_0$ is cont. loc. mart. unif. bdd by $n+|M_0|$ so it is unif. integ.

■

LECTURE 20

Def: (X_t) is a <u>continuous local martingale</u> if the sample paths are continuous and there is a seq. $(T_n)_{n\geq 0} \to \infty$ of stopping times s.t.

$$(X^{T_n})_t - X_0 \overset{\text{def}}{=} X_{t\wedge T_n} - X_0$$

is a uniformly integrable martingale $\forall n$.

E.g.: If (X_t) is martingale, then $T_n = n$ <u>reduces</u> (X_t)

i.e., $X^n := X_{t\wedge n}$ is UI martingale.

Thm: Let M be a cont. loc. martingale with $M_0 = 0$. If M is also a finite variation process over finite intervals (i.e., its sample paths have finite variation), <u>then</u> $M_t \equiv 0 \;\; \forall t \geq 0$ a.s.

Pf: $$\tau_n := \inf\Big\{ t\geq 0 : \underbrace{\int_0^t |dM_s|}_{\text{total variation of process between 0 and } t} \geq n \Big\}$$

Set $N_t = M^{\tau_n} = M_{t\wedge\tau_n}$

$$|N_t| = |M_{t\wedge\tau_n}| \leq \int_0^{t\wedge\tau_n} |dM_s| \overset{(*)}{\leq} n \Rightarrow N_t \text{ is bdd martingale.}$$

$$\mathbb{E}\left[|N_t|^2\right] = \sum_{i=1}^{p} \mathbb{E}\left[|N_{t_i} - N_{t_{i-1}}|^2\right]$$

Thm about martingales from before

$$\leq \mathbb{E}\left[\sup_i |N_{t_i} - N_{t_{i-1}}| \underbrace{\sum_i |N_{t_i} - N_{t_{i-1}}|}_{\leq n \text{ by } (*)} \right]$$

$$\leq n \, \mathbb{E}\left[\sup_i |N_{t_i} - N_{t_{i-1}}| \right] \xrightarrow{?} 0$$

$\downarrow$ $p \to \infty$ by DCT b/c sample paths of N_t are cont.

0

$\Downarrow$

finer partitions make the $\sup_i(\ldots) \to 0$.

Thus $N_t = 0$ a.s. $\forall t$ $\Rightarrow$ $M_t = 0$ a.s. $\forall t$.

■

Thm: (Quadratic Variation) Let (M_t) be a cont. loc. mart.. Then there exists a unique (a.s.) increasing process $\langle M, M \rangle_t$ s.t.

$$M_t^2 - \langle M, M \rangle_t$$

is a cont. loc. martingale.

Moreover,

$$\langle M, M \rangle_t = \lim_{n \to \infty} \sum_{i=1}^{p_n} \left(M_{t_i^n} - M_{t_{i-1}^n} \right)^2$$

in probability, for any sequence of partitions $0 = t_0^n < \cdots < t_{p_n}^n = t$ with $\sup_i |t_i^n - t_{i-1}^n| \to 0$

Def: If (M_t) and (N_t) are cont. loc. martingales, then the BRACKET of M, N is a finite variation process

$$\langle M, N\rangle_t := \frac{1}{2}\left(\langle M+N, M+N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t\right).$$

Note: $\langle M, N\rangle_t$ has finite variation since it's the difference of increasing processes.

Alternative definition:

$$\langle M, N\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{P_n} \left(M_{t_i^n} - M_{t_{i-1}^n}\right)\left(N_{t_i^n} - N_{t_{i-1}^n}\right)$$

in probability.

EXAMPLE: (B_t) Brownian motion. Then

$$\langle B, B\rangle_t \overset{\text{HW04}}{=} t \quad \forall\, t > 0$$

$B_t^2 - t$ is cont. loc. martingale.

If (B_t) and (B_t') are independent BMs, then

$$\langle B, B'\rangle_t = ?$$

Note: $X_t = \frac{1}{\sqrt{2}}\left(B_t + B_t'\right)$ is martingale b/c sum of martingales.

In fact, X_t is BM. <u>Note</u>: (B_t, B_t') is BM in $\mathbb{R}^2$.

$$\Rightarrow \langle X, X\rangle_t = t = \left\langle \frac{B+B'}{\sqrt{2}}, \frac{B+B'}{\sqrt{2}} \right\rangle_t$$

$$= \frac{1}{2}\langle B, B\rangle_t + \frac{1}{2}\langle B', B'\rangle_t + \langle B, B'\rangle_t$$

$$= \frac{t}{2} + \frac{t}{2} + \langle B, B'\rangle_t$$

$$\Rightarrow \boxed{\langle B, B'\rangle_t = 0}.$$

◢

<u>Pf</u>: (Uniqueness)

$$M_t^2 = \overset{\text{increasing}}{I_t} + \overset{\text{CLM}}{C_t} = \tilde{I}_t + \tilde{C}_t$$

$$\underbrace{I_t - \tilde{I}_t}_{\text{fin. var.}} = \underbrace{\tilde{C}_t - C_t}_{\text{CLM}}$$

$\Rightarrow$ $I_t - \tilde{I}_t$ is CLM of fin. var.

hence $I_t - \tilde{I}_t = 0 = \tilde{C}_t - C_t$.

$$\Rightarrow \quad I_t = \tilde{I}_t$$

$$C_t = \tilde{C}_t .$$

■

Proposition: If M is CLM and T is stopping time. Then

$$\langle M^T, M^T\rangle_t = \langle M, M\rangle_{t\wedge T}.$$

Pf:

$$(M^T)^2_t - \langle M, M\rangle_{t\wedge T} = \underbrace{M^2_{t\wedge T} - \langle M, M\rangle_{t\wedge T}}_{\text{is CLM}}$$

b/c we know
$M_t^2 - \langle M, M\rangle_t$ is CLM
then stopping at T gives
another CLM.

$\square$

Proposition: Let M be CLM with $M_0 = 0$. Then

$$\langle M, M\rangle_t = 0 \quad \forall t \iff M = 0.$$

Pf: $(\Rightarrow)$ If $\langle M, M\rangle = 0$, M_t^2 is CLM and is nonneg..
But this means M_t^2 is supermart. $\Rightarrow \mathbb{E}[M_t^2] \leq \mathbb{E}[M_0] = 0$.
$\Rightarrow M_t = 0 \ \forall t$ a.s.

$(\Leftarrow)$ ...

$\square$

__NOTE__: $\langle M,M\rangle_t$ is increasing so $\langle M,M\rangle_\infty = \underset{\substack{\cap \\ [0,\infty]}}{\lim_{t\to\infty}\langle M,M\rangle_t}$.

__Thm__: Let M be a CLM, $M_0 \in L^2$. Then:

(i) TFAE:

(a) M is a martingale bounded in L^2

(b) $\mathbb{E}\left[\langle M,M\rangle_\infty\right] < +\infty$

If these hold, then $M_t^2 - \langle M,M\rangle_t$ is a UI mart.

(ii) TFAE

(a) M is a martingale with $M_t \in L^2$

(b) $\mathbb{E}\left[\langle M,M\rangle_t\right] < +\infty \quad \forall t \geq 0$

If these hold, then $M_t^2 - \langle M,M\rangle_t$ is a martingale.

__Def__: A __continuous semimartingale__ is a process

$$X_t = \underset{\substack{\nearrow \\ \text{finite} \\ \text{variation} \\ \text{process}}}{\underline{A_t}} + \underset{\substack{\nwarrow \\ \text{CLM}}}{\underline{M_t}}$$

LECTURE 21

STOCHASTIC INTEGRATION (Ch. 5.1 - 5.2 Le Gall)

- If M, N are CLM then $\langle M, N\rangle_t$ is finite variation
- $M_t N_t - \langle M, N\rangle_t$ is CLM
- Can define

$$\langle M, N\rangle_t = \lim_{n\to\infty} \sum_{i=1}^{P_n} \left(M_{t_i^n} - M_{t_{i-1}^n}\right)\left(N_{t_i^n} - N_{t_{i-1}^n}\right)$$

in probability for an increasing seq. of partitions of $[0, t]$.

<u>Proposition</u>: If M, N are bounded in L^2, then

$$M_t N_t - \langle M, N\rangle_t$$

is a uniform integrable martingale. So

$$X_\infty := \lim_{t\to\infty} \langle M, N\rangle_t \quad \text{a.s.}$$

is well-defined. Moreover, $\mathbb{E}\left[|\langle M, N\rangle_\infty|\right] < +\infty$.

$$\langle M, N\rangle_t = \frac{1}{2}\left(\langle M+N, M+N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t\right)$$

$\Rightarrow$ L^2 martingales form a Hilbert space.

Hilbert Space of L^2 Martingales

$$\mathbb{H}^2 := \{ \text{continuous, } L^2\text{-bdd martingales } M \text{ s.t. } M_0 = 0 \}.$$

(i) M L^2-bdd $\Rightarrow$ M L^1-bdd $\Rightarrow$ $M_\infty := \lim_t M_t$ exists.

$M_t = \mathbb{E}[M_\infty | \mathcal{F}_t]$, M is uniformly integrable.

Def: $M, N \in \mathbb{H}^2$,

$$(M,N)_{\mathbb{H}^2} := \mathbb{E}\left[\langle M, N\rangle_\infty\right] = \mathbb{E}\left[M_\infty N_\infty\right].$$

Remark: $(M,M)_{\mathbb{H}^2} = 0 \iff M = 0$

$\Downarrow \qquad\qquad\qquad\qquad \nwarrow$

$\mathbb{E}[M_\infty^2] = 0 \Rightarrow M_\infty = 0 \Rightarrow M_t = \mathbb{E}[M_\infty | \mathcal{F}_t] = 0$

Proposition: $(\mathbb{H}^2, (\cdot,\cdot)_{\mathbb{H}^2})$ is a Hilbert space.

Pf: NTS: Completeness $(M^n)_{n\geq 0}$ Cauchy for $(\cdot,\cdot)_{\mathbb{H}^2}$,

$$\lim_{m,n\to\infty} \mathbb{E}\left[(M^n_\infty - M^m_\infty)^2\right] \overset{\text{def}}{=} \lim_{m,n} \left(M^n - M^m, M^n - M^m\right)_{\mathbb{H}^2} = 0$$

So $(M^n_\infty)_{n\geq 0}$ converges in L^2 to $Z \in L^2$.

Prove Z continuous a.s.. Set $M^\infty_t := \mathbb{E}[Z | \mathcal{F}_t]$.

WIENER INTEGRAL : $f \in L^2(\mathbb{R}_+)$

$$\int_0^t f(s)\, dB_s \underset{\uparrow}{\approx} \sum_i f(t_i)\left[B(t_i) - B(t_{i-1})\right] .$$

more complicated b/c $B(t_i) - B(t_{i-1})$ are never finite variation

STOCHASTIC INTEGRAL : Want to integrate martingales now.

$$\int_0^t H_s\, dM_s \quad \text{with } H, M \in \mathbb{H}^2$$

$$\wr\wr$$

$$\sum_{i=1}^{P_n} H(\underset{\sim}{t_{i-1}})\left[M(t_i) - M(t_{i-1})\right]$$

Important to have t_{i-1} here

(using t_i gives a different integral)

NOTE : Let $M \in \mathbb{H}^2$. Then $t \mapsto \langle M, M \rangle_t$ is finite variation.

So, $d\langle M, M \rangle_s$ is a signed measure on $\mathbb{R}$.

Def :

$$L^2(\overset{\in \mathbb{H}^2}{M}) := \left\{ H \text{ progressive process s.t. } \mathbb{E}\left[\int_0^\infty H_s^2\, \underline{d\langle M,M\rangle_s}\right] < +\infty \right\}$$

signed measure $\Rightarrow \int \cdots d\langle M,M\rangle_s$ is finite variation

H progressive : $\forall t \geq 0$,

$\Omega \times [0,t] \ni (\omega, s) \longmapsto H_s(\omega)$ is $\mathcal{F}_t \otimes \mathcal{B}([0,t])$ - measurable

Proposition: $L^2(M)$ is a Hilbert space with inner product

$$(H,K)_{L^2(M)} := \mathbb{E}\left[\int_0^\infty H_s K_s \, d\langle M,M\rangle_s\right]$$

Def: An elementary process is a progressive process of the form

$$H(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega)\, \mathbb{1}_{(t_i,\, t_{i+1}]}(s) \quad ,$$

open ! closed

where $0 = t_0 < t_1 < \dots < t_p$ and $H_{(i)}$ is an $\underline{\underline{\mathcal{F}_{t_i}}}$ - measurable r.v.

has to be indep. of $\mathbb{1}_{(t_i, t_{i+1}]}$

Lemma: The space $\mathcal{E}$ of elementary processes is a dense linear subspace of $L^2(M)$.

(In Hilbert spaces, showing there's no orthogonal is the same as density)

Pf: Assume $K \in L^2(M)$ is orthogonal to $\mathcal{E}$.

WTS: $K = 0$.

Set

$$X_t := \int_0^t K_u \, d\langle M,M\rangle_u \, .$$

Note: By Cauchy-Schwarz

$$\mathbb{E}\Big[\underbrace{\int_0^t |K_u| \, d\langle M,M\rangle_u}_{\text{finite a.s.}}\Big] \leq \Big(\underbrace{\mathbb{E}\Big[\int_0^t |K_u|^2 d\langle M,M\rangle_u\Big]}_{< \infty \text{ b/c } K \in L^2(M)}\Big)^{1/2} \underbrace{\Big(\mathbb{E}\big[\langle M,M\rangle_\infty\big]\Big)^{1/2}}_{\substack{< \infty \\ \text{b/c } M \in L^2}}$$

Let F be bdd and $\mathcal{F}_s$-measurable, and set

$$M_r(\omega) = F(\omega) \, \mathbb{1}_{(s,t]}(r) \in \mathcal{E} .$$

$$(K,M)_{L^2(M)} = 0 = \mathbb{E}\Big[F \int_s^t K_u \, d\langle M,M\rangle_u\Big]$$

$$= \mathbb{E}\Big[F(X_t - X_s)\Big] \Rightarrow X_t - X_s \text{ is indep. of } F$$

$\Rightarrow$ increments are indep. of past

$\Rightarrow$ X_t martingale.

Note: X_t also has finite variation (since K is continuous)

$X = 0$

by a previous result

$\Downarrow$

$K_u = 0$ $d\langle M,M\rangle_s$ - a.e., a.s.

$\Rightarrow K = 0$ in $L^2(M)$. ∎

! **Thm**: Let $M \in \mathbb{H}^2$. For every $H \in \mathcal{E}$ of the form

$$H_s(\omega) = \sum_{i=0}^{p-1} H_{(i)}(\omega)\, \mathbb{1}_{(t_i, t_{i+1}]}(s),$$

define

$$(H \bullet M)_t := \sum_{i=0}^{p-1} H_{(i)} \left(M_{t_{i+1} \wedge t} - M_{t_i \wedge t} \right).$$

Then, $H \bullet M \in \mathbb{H}^2$ and $H \longmapsto H \bullet M$ extends to an isometry from $L^2(M)$ into $\mathbb{H}^2$.

Moreover, $H \bullet M$ is the unique element in $\mathbb{H}^2$ s.t.

$$\langle H \bullet M, N \rangle_{\mathbb{H}^2} = H \bullet \langle M, N \rangle_{\mathbb{H}^2} \quad \forall N \in \mathbb{H}^2.$$

If T is a stopping time,

$$\left(\mathbb{1}_{[0,T]} H \right) \bullet M = (H \bullet M)^T = H \bullet M^T.$$

Denote, more commonly,

$$(H \bullet M)_t =: \int_0^t H_s \, dM_s.$$

Pf: (Isometry) $M^i_t = H_{(i)} \left(M_{t_{i+1} \wedge t} - M_{t_i \wedge t} \right)$ is a continuous martingale b/c $M_{\cdot \wedge t}$ is mart. and $H_{(i)}$ only depends on past. So, $(H \bullet M)_t$ is a martingale, at least for all $H \in \mathcal{E}$.

Now,

$$\langle M^i_t, M^i_t \rangle_t = H_{(i)}^2 \left(\langle M, M \rangle_{t_{i+1} \wedge t} - \langle M, M \rangle_{t_i \wedge t} \right).$$

Summing over i, note $\langle M^i_t, M^j_t \rangle_t = 0 \quad \forall i \neq j$.

Thus,

$$\langle H \cdot M, H \cdot M \rangle_t = \sum_{i=0}^{p-1} H_{(i)}^2 \left(\langle M, M \rangle_{t_{i+1} \wedge t} - \langle M, M \rangle_{t_i \wedge t} \right)$$

$$= \int_0^t H_s^2 \, d\langle M, M \rangle_s$$

Take $t \to +\infty$, then

$$\| H \cdot M \|^2_{H^2} = \int_0^\infty H_s^2 \, d\langle M, M \rangle_s = \| H \|^2_{L^2(M)}$$

hence $\int_0^t H_s \, dM_s$ can be def. by completion $\forall H \in L^2(M)$.

//

$$(H \cdot M)_t = \sum_i M^i_t$$

$$\langle H \cdot M, N \rangle = \sum_i \langle M^i, N \rangle = \sum_i H_{(i)} \left(\langle M, N \rangle_{t_{i+1} \wedge t} - \langle M, N \rangle_{t_i \wedge t} \right).$$

$$\langle \boxed{H_{(i)}} M_{t_{i+1} \wedge t}, N_t \rangle = H_{(i)} (M_{t_{i+1} \wedge t}, N)_t$$

independent of $(M_{t_{i+1}} - M_{t_i})$

$$= \int_0^t H_s \, d\langle M, N \rangle_s$$

$$\overset{\text{def}}{=} H \cdot \langle M, N \rangle$$

■

STOCHASTIC CALCULUS

$dX_t = \mu_t\,dt + \sigma_t\,dB_t$, μ_t and σ_t are functions of t.

Want: Process (X_t) s.t.

$$X_t - X_0 = \int_0^t \mu_s\,ds + \int_0^t \sigma_s\,dB_s$$

Obs: Deterministic case $\rightsquigarrow$ $\sigma_t \equiv 0$ i.e., $dX_t = \mu_t\,dt$

$$\frac{dX}{dt} = \mu_t .$$

Example: (Asset / Stock prices) S_t = price at time t

$$dS_t = \mu S_t\,dt + \sigma S_t\,dB_t \iff S_t - S_0 = \int_0^t \mu S_s\,ds + \int_0^t \sigma S_s\,dB_s, \quad \underline{\underline{\forall t}}$$

$$\underbrace{\frac{dS_t}{S_t}}_{\parallel} = \underbrace{\mu\,dt}_{\text{w/out risk, increments}} + \overset{\text{volatility of stock price}}{\downarrow\atop\sigma}\,dB_t$$

Geometric Brownian Motion

(Model the $\overset{\text{percent}}{\vee}$ increment of price increments)

SEMIMARTINGALES

Def: A continuous semimartingale (CSM) is the sum

$$X_t = M_t + A_t ,$$

where M_t is a CLM and A_t is finite variation.

Note: Decomposition is unique...

Def: Say $X_t = M_t + A_t$ and $Y_t = N_t + B_t$ are CSM, then

$$\langle X, Y\rangle_t = \lim_{t\to\infty} \sum_i (X_{t_{i+1}} - X_{t_i})(Y_{t_{i+1}} - Y_{t_i}) \overset{\uparrow}{=} \langle M, N\rangle_t$$

Obs: bracketing w/ finite variation gives zero at the end

Note: $$\langle N, A\rangle_t = \lim_{t\to\infty} \sum_i (N_{t_{i+1}} - N_{t_i})\underbrace{(A_{t_{i+1}} - A_{t_i})}_{\text{finite variation}}$$

$$= \lim_t \Big(\underbrace{\sup_i |N_{t_{i+1}} - N_{t_i}|}_{\to\, 0} \underbrace{\sum_i |A_{t_{i+1}} - A_{t_i}|}_{\leq \int_0^t |dA_s| < \infty} \Big)$$

b/c $t \mapsto N_t$ is cont.

Thm: (Itô's Formula) Let $X^1, \text{—}, X^p$ be p continuous semimartingales, and let $F \in C^2(\mathbb{R}^p, \mathbb{R})$. Then, $\forall t \geq 0$,

$$F(X^1_t, \text{—}, X^p_t) = F(X^1_0, \text{—}, X^p_0)$$

CLM
since it's a stochastic integral

$$+ \sum_{i=1}^{p} \int_0^t \frac{\partial F}{\partial x^i}(X^1_s, \text{—}, X^p_s)\, dX^i_s$$

Finite Variation Process
b/c bracket has finite variation

$$+ \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial F}{\partial x^i \partial x^j}(X^1_s, \text{—}, X^p_s)\, d\langle X^i, X^j \rangle_s$$

EXAMPLE: INTEGRATION BY PARTS

Take $F(x,y) = xy$ and say X, Y are CSM. Then $\forall t \geq 0$,

$$X_t Y_t = X_0 Y_0 + \int_0^t Y_s\, dX_s + \int_0^t X_s\, dY_s + \int_0^t d\langle X, Y \rangle_s$$

$$= X_0 Y_0 + \int_0^t (X_s\, dY_s + Y_s\, dX_s) + \langle X, Y \rangle_t .$$

Obs: if $X = Y$, then

$$X_t^2 = X_0^2 + 2 \int_0^t X_s\, dX_s + \langle X, X \rangle_t .$$

Remark: Let X_t be martingale.

since it extends ...

$$\int_0^t X_s \, dX_s \overset{(1)}{=} \lim_{n \to \infty} \sum_{i=1}^{P_n} \underline{X_{t_i^n}} \left(X_{t_{i+1}^n} - X_{t_i^n} \right) \quad \text{in } \downarrow \text{probability} .$$

Note:

$$\sum_{i=1}^{P_n} \underline{X_{t_{i+1}^n}} \left(X_{t_{i+1}^n} - X_{t_i^n} \right) \overset{(2)}{=} \sum_{i=1}^{P_n} X_{t_i^n} \left(X_{t_{i+1}^n} - X_{t_i^n} \right) + \sum_{i=0}^{P_n} \left(X_{t_{i+1}^n} - X_{t_i^n} \right)^2$$

$$\Big\downarrow \, n \to \infty$$

$$\int_0^t X_s \, dX_s + \langle X, X \rangle_t$$

Adding (1) and (2): get telescoping series

$$X_t^2 - X_0^2 = 2 \int_0^t X_s \, dX_s + \langle X, X \rangle_t \quad .$$

LECTURE 22

ITÔ'S FORMULA & APPLICATIONS

Itô's Formula: Let $X^1, \_, X^p$ be continuous martingale, and let $F: \mathbb{R}^p \to \mathbb{R}$ be of class C^2. Then

$$F(X_t^1, \_, X_t^p) = F(X_0^1, \_, X_0^p) + \sum_{i=0}^{p} \underbrace{\int_0^t \frac{\partial F}{\partial x^i} \, dX_s^i}_{\text{CLM (b/c stochastic integral)}}$$

This term is not necessarily true if $X^1, \_, X^p$ are CSM

$$+ \frac{1}{2} \sum_{i,j} \underbrace{\int_0^t \frac{\partial F}{\partial x^i \partial x^j} \, d\langle X^i, X^j \rangle_s}_{\text{Finite Variation Process}}$$

(b/c $d\langle X^i, X^j \rangle_s$ is a signed measure b/c the bracket of CSM is finite variation)

In particular $F(X_t^1, \_, X_t^p)$ is CSM.

EXAMPLE: $f(B_t)$, $f \in C^2$, $\langle B, B \rangle_t = t$

$$f(B_t) = \underset{B_0 = 0}{f(0)} + \underbrace{\int_0^t f'(B_s) \, dB_s}_{\text{Martingale}} + \frac{1}{2} \underbrace{\int_0^t f''(B_s) \, ds}_{\text{Finite Variation}}$$

$F(t,B_t)$, $F \in C^2$,

Not a martingale

$$F(t,B_t) = F(0,0) + \int_0^t \frac{\partial F}{\partial t}\,ds + \underbrace{\int_0^t \frac{\partial F}{\partial x}\,dB_s}_{\text{Martingale}}$$

$$\left\{\begin{array}{l} \langle B_t, t\rangle = 0 \\ \langle t, t\rangle = 0 \end{array}\right. \qquad + \frac{1}{2}\int_0^t \frac{\partial^2 F}{\partial x^2}\,ds$$

$$= F(0,0) + \int_0^t \frac{\partial F}{\partial x}\,dB_s + \int_0^t \left(\frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2}\right) ds$$

Upshot: $F(t,B_t)$ is CLM $\iff \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial x^2} \equiv 0$

◢

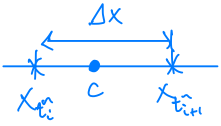
Pf: (Itô's Formula) $p = 1$, $F : \mathbb{R} \to \mathbb{R}$, $F \in C^2$.

$$F(X_t) = F(X_0) + \sum_{i=0}^{P_n-1}\left[F(X_{t_{i+1}^n}) - F(X_{t_i^n})\right]$$

Taylor expand near $X_{t_i^n}$

$$= F(X_0) + \sum_{i=0}^{P_n-1}\Big[F'(X_{t_i^n})(X_{t_{i+1}^n} - X_{t_i^n})$$

$$+ \frac{1}{2}\underbrace{f_{n,i}}\,(X_{t_{i+1}^n} - X_{t_i^n})^2\Big]$$



$f_{n,i} := F''(X_{t_i^n} + c(X_{t_{i+1}^n} - X_{t_i^n}))$, for some $c \in [0,1]$.

Now,

$$\sum_{i=0}^{p_n-1}\left[F'(X_{t_i^n})(X_{t_{i+1}^n}-X_{t_i^n})\right]\xrightarrow[n\to\infty]{\text{probability}}\int_0^t F'(X_s)\,dX_s \qquad \checkmark_{1^{st}\text{ term}}$$

And,

$$\sup_{0\le i\le p_n}\left|f_{n,i}-F''(X_{t_i^n})\right|\le\sup_{0\le i\le p_n}\ \sup_{x\in[X_{t_i^n},X_{t_{i+1}^n}]}\left|F''(x)-F''(X_{t_i^n})\right|$$

$$\xrightarrow[\substack{\text{b/c sample paths}\\\text{are cont! b/c } F\in C^2}]{a.s.}0$$

So,

$$\left|\sum_i f_{n,i}(X_{t_{i+1}^n}-X_{t_i^n})^2-\sum_i F''(X_{t_i^n})(X_{t_{i+1}^n}-X_{t_i^n})^2\right|\xrightarrow{\text{prob.}}0$$

Thus:

$$\sum_i (X_{t_{i+1}^n}-X_{t_i^n})^2\xrightarrow{\text{prob.}}\langle X,X\rangle_t$$

<u>WTS</u>: $\displaystyle\sum_{i=1}^{p_n}F''(X_{t_i^n})(X_{t_{i+1}^n}-X_{t_i^n})^2\xrightarrow{\text{prob.}}\int_0^t F''(X_s)\,d\langle X,X\rangle_s$

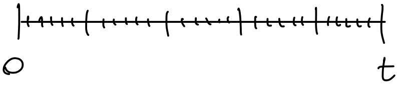
Consider seq. of measures

$$\mu_n := \sum_{i=0}^{p_n-1} \left(X_{t^n_{i+1}} - X_{t^n_i}\right)^2 \delta_{t^n_i} \text{ on } [0,t] .$$

Then

$$\int_0^t F''(X_s)\, \mu_n(ds) \xrightarrow{?} \int_0^t F''(X_s)\, d\langle X,X\rangle_s$$

$$D := \{ t^n_i : 0 \le i \le p_n ,\ n \ge 0 \}$$



But, by def. of quad. variation, we know that for $r \in D$,

$$\mu_n([0,r]) \xrightarrow{\text{prob.}} \langle X,X\rangle_r$$

i.e.,

$$\sum_{t_i \le r} \left(X_{t^n_{i+1}} - X_{t^n_i}\right)^2 \xrightarrow{\text{prob.}} \langle X,X\rangle_r .$$

By diagonal extraction, $\exists$ a subseq. (n_k) of n s.t. $\forall r \in D$

$$\mu_{n_k}([0,r]) \xrightarrow{n_k \to \infty} \langle X,X\rangle_r \quad \underline{\text{a.s.}} .$$

Thus, $\mu_{n_k} \xrightarrow{\text{weak-}*} d\langle X,X\rangle$. So, since F'' is continuous,

$$\int_0^t F''(X_s)\, d\mu_n(s) \longrightarrow \int_0^t F''(X_s)\, d\langle X,X\rangle_s .$$

■

ITÔ PROCESSES : Process X_t governed by SDE

$$X_t = X_0 + \underbrace{\int_0^t \mu_s(X_s)\,ds}_{\text{Finite Variation}} + \underbrace{\int_0^t \sigma_s(X_s)\,dB_s}_{\text{Stochastic Integral}} .$$

$$\Longleftrightarrow \quad dX_t = \mu_t\,dt + \sigma_t\,dB_t$$

GEOMETRIC BROWNIAN MOTION (with drift)

$$dX_t = r X_t\,dt + \sigma X_t\,dB_t \quad , \quad r, \sigma \in \mathbb{R} .$$

↳ Model asset prices b/c can find the proportional change day-by-day :

$$\frac{dX_t}{X_t} = r\,dt + \sigma\,dB_t$$

(Derivative)
Option price : $V(t, X_t)$

$$V(t,X_t) - V(0,X_0) \overset{\text{Itô}}{=} \int_0^t \frac{\partial V}{\partial t}\,ds + \int_0^t \frac{\partial V}{\partial x}\,\underbrace{dX_s}_{\text{Use eq.}} + \frac{1}{2}\int_0^t \frac{\partial^2 V}{\partial x^2}\,\overset{\sigma^2 X_s^2\,ds}{\overset{\shortparallel}{\boxed{d\langle X,X\rangle_s}}}$$

$$= \int_0^t \frac{\partial V}{\partial t}\, ds + \frac{\partial V}{\partial x} r X_s\, ds + \frac{\partial V}{\partial x} \sigma X_s\, dB_s + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} \sigma^2 X_s^2\, ds$$

$$= \int_0^t \left(\frac{\partial V}{\partial t} + r X_s \frac{\partial V}{\partial x} + \frac{1}{2} \frac{\partial^2 V}{\partial x^2} X_s^2 \right) ds + \int_0^t \frac{\partial V}{\partial x} \sigma X_s\, dB_s$$

$$\Rightarrow \quad \frac{\partial V}{\partial t} + r X \frac{\partial V}{\partial x} + \frac{1}{2} \sigma^2 X^2 \frac{\partial^2 V}{\partial x^2} = r V \qquad \text{(Black - Scholes)}$$