LECTURE 1

INTRODUCTION

11/09/2023

- <u>Office hours</u>: PG 201 B   Come to office hours!

  Mon 12 – 1 pm
  Wed 11 – 12 pm

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* **MOTIVATIONS**: New measure theory developed after this

BANACH – TARSKI THM (1924): $\forall\, U, V$ open sets in $\mathbb{R}^{n \geq 3}$, $\exists\, k \in \mathbb{N}$ $E_1, \_\_, E_k \subset U$, $F_1, \_\_, F_k \subset V$,

s.t.
- $E_i \cap E_j = \emptyset$ and $F_i \cap F_j = \emptyset$ $\forall i, j$,
- $\bigsqcup_i E_i = U$ and $\bigsqcup_i F_i = V$,
- $E_i \sim F_i$ ← very nasty sets.

congruent (i.e., can rotate & translate each piece and form the original other open sets).

$\Rightarrow$ There are some nasty sets / fcts. out there...

This encouraged the development of new analysis (which we will see in this course). This was based on working w/ "good" sets / fcts. Start from the beginning.

Def: (σ-ALGEBRA) Let $X \neq \emptyset$ and consider $\mathcal{P}(X) = \{Y \subset X\}$ (power set of X). Then $\mathcal{M} \subset \mathcal{P}(X)$ is an <u>algebra</u> if

- $\forall E \in \mathcal{M}$, $E^c \in \mathcal{M}$,
- $\forall E, F \in \mathcal{M}$, $E \cup F \in \mathcal{M}$.

Now, $\mathcal{M}$ is a <u>σ-algebra</u> if it is an algebra and if

$$E_i \in \mathcal{M},\ i \in \mathbb{N} \implies \bigcup_i E_i \in \mathcal{M}.$$

<u>PROPERTIES</u>: • $\underbrace{\{\emptyset, X\}}_{\text{"Trivial } \sigma\text{-algebra"}} \in \mathcal{M}$ on X

⋮

⋮

- $\mathcal{P}(X)$ is a σ-algebra.

How to check something is a σ-algebra?

NOTE: • It is enough to consider only disjoint sets (this is b/c if E_i are arbitrary,

$$E_i = E_i \setminus \bigcup_{j<i} E_j \;)$$

• It is enough to check an increasing sequence

$$E_i \subset E_{i+1} \subset \dots$$

QUESTION: ARE OPEN SETS AN ALGEBRA? No!

Not closed under complements!

So... how to define σ-algebras?

FACT: Let M_α be a collection of σ-algebras on X. Then $\bigcap_\alpha M_\alpha$ is a σ-algebra.

Notation: $\alpha \rightsquigarrow$ arbitrarily many elements.

Def: Let $\mathcal{C} \subset P(X)$ be a collection of sets. Define <u>the</u> σ-algebra generated by $\mathcal{C}$, denoted $\langle \mathcal{C} \rangle$, as

possibly uncountable

$$\langle \mathcal{C} \rangle := \bigcap_{\alpha} \mathcal{M}_{\alpha},$$

where $\mathcal{M}_{\alpha}$ are σ-algebras containing $\mathcal{C}$.

ASIDE: if $\mathcal{M} \subset \mathcal{M}'$ then $\mathcal{M}$ is coarser than $\mathcal{M}'$ and $\mathcal{M}'$ is finer than $\mathcal{M}$.

$\Rightarrow$ We can define <u>the</u> σ-algebra generated by open sets!

Def: (BOREL σ-ALGEBRA) Let $X \neq \emptyset$ and $\mathcal{T}$ be a topology on X. Then $\langle \mathcal{T} \rangle$ is called <u>the</u> Borel σ-algebra of $(X, \mathcal{T})$.

$\mathcal{T}$ is just a collection of subsets of X that we declare to be open.

Prop: Let τ be the usual topology on $\mathbb{R}$.
Let $\mathcal{B} := \mathcal{B}_\tau$, then

(i) $\mathcal{B} = \langle \{(a,b) : a<b\} \rangle$

(ii) $\mathcal{B} = \langle \{(a,b] : a<b\} \rangle$ ← or $[a,b)$

(iii) $\mathcal{B} = \langle \{[a,b] : a<b\} \rangle$

(iv) $\mathcal{B} = \langle \{(a,\infty) : \forall a \in \mathbb{R}\} \rangle$ ← or $(-\infty, a)$

(v) $\mathcal{B} = \langle \{[a,\infty) : \forall a \in \mathbb{R}\} \rangle$ ← or $(-\infty, a]$

* **EXAMPLE**: Take $A \subset X$. What is $\langle \{A\} \rangle$?

$$\langle \{A\} \rangle = \{\emptyset, A, A^c, X\}.$$

Take $A, B \subset X$. What is $\langle \{A, B\} \rangle$?

LEMMA: Let $\mathcal{E}, \mathcal{F} \subset \mathcal{P}(X)$. If $\mathcal{E} \subset \langle \mathcal{F} \rangle$, then $\langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$.

Pf: $\langle \mathcal{E} \rangle = \bigcap_\alpha \mathcal{M}_\alpha$, $\mathcal{M}_\alpha$ σ-algebra and $\mathcal{E} \subset \mathcal{M}_\alpha$ $\forall \alpha$. Now, note that $\langle \mathcal{F} \rangle$ is a σ-algebra s.t.

$\langle \mathcal{F} \rangle \supset \mathcal{E}$. Thus, $\langle \mathcal{E} \rangle \subset \langle \mathcal{F} \rangle$. $\square$

Pf of Proposition: (i) Note that

$$\mathcal{B} \supset \{(a,b) : a < b\}$$

by definition (b/c $\mathcal{B}$ is generated by open sets, so it need to contain them). So, by the Lemma above, $\langle \{(\cdot,\cdot)\} \rangle \subset \mathcal{B}$.

any open set can be written as unions of open intervals.

Obs: U open set $\Rightarrow U = \bigcup_i (a_i, b_i)$. So,

$$\mathcal{T} \subset \langle \{(a,b)\} \rangle$$

$$\Downarrow$$

$$\mathcal{B} \subset \langle \{(a,b)\} \rangle$$

$\square$

Obs. Borel sets (sets in Borel σ-algebra) can be open, closed, we have ∞ intersec. of open (G_δ) and ∞ union of closed (F_σ), intersections of G_δ & F_δ, etc. etc. Can get complicated w/ transfinite induction. But this is enough philosophy.

Def: (MEASURE) Let $X \neq \emptyset$ and $\mathcal{M}$ be a σ-algebra. We say that $(X, \mathcal{M})$ is a <u>measurable space</u> (because we can attach a measure to it). We say that $E_i \in \mathcal{M}$ are <u>measurable sets</u> in $(X, \mathcal{M})$.

Now, a function $\mu : \mathcal{M} \longrightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$ is called a <u>measure</u> <u>on</u> $(X, \mathcal{M})$ if

- $\mu(\emptyset) = 0$

Additivity is NOT enough: $\mu(E) + \mu(F) = \mu(E \cup F)$ is not enough, $E \cap F = \emptyset$.

- (σ-additivity) let $E_i \in \mathcal{M}$ s.t. $E_i \cap E_j = \emptyset$ then

$$\sum_{i \in \mathbb{N}} \mu(E_i) = \mu\left(\bigcup_{i \in \mathbb{N}} E_i\right).$$

EXAMPLES: • $\mu(E) = 0 \ \forall E$ ← Trivial ex. that's good for counterex.

• let $x \in X$ and define $\mu_x(E) = \begin{cases} 1, & \text{if } x \in E \\ 0, & \text{else} \end{cases}$

DIRAC MASS

- if $X = \mathbb{R}$, $\mu(E) = \#(E \cap \mathbb{Z})$
- if $X = \mathbb{Z}$, $\mu(E) = \# E$ (Counting measure on $\mathcal{P}(\mathbb{Z})$)

Def: (FINITE MEASURE) μ is <u>finite</u> if $\mu(X) < +\infty$ $\forall X \in \mathcal{M}$.

useful in probability b/c we can normalize $\mu(X) \overset{!}{=} 1$.

Def: (σ – FINITE) μ is <u>σ – finite</u> if

$$\exists \{E_i\}_{i \in \mathbb{N}}, \quad \mu(E_i) < \infty, \quad X = \bigcup_i E_i$$

Def: (SEMI FINITE) μ is semi-finite if

$$\forall E \in \mathcal{M} \text{ s.t. } \mu(E) = \infty \quad \exists \underset{\neq \emptyset}{F} \subset E \text{ s.t. } 0 < \mu(F) < \infty.$$

Def: (MEASURE SPACE) $(X, \mathcal{M}, \mu)$

Set

σ-algebra on X

Measure

Lecture 2

Pushforwards & Pullbacks and Product σ-Algebras

13/09/2023

Little name hint: "σ-" $\rightsquigarrow$ "sum"

Recap:
- **σ-algebra**: collection of sets in a power set closed under complements & countable unions. (notation: $\mathcal{M}, \mathcal{N}, \dots$)
- **measures**: $\mu : \mathcal{M} \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ s.t. $\mu(\emptyset) = 0$ and $\sum_{i \in \mathbb{N}} \mu(E_i) = \mu\left(\bigcup_{i \in \mathbb{N}} E_i\right)$ for $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{M}$ s.t. $E_i \cap E_j = \emptyset$ $\forall i, j$.
- **generated σ-algebra**: For $\mathcal{E} \subset P(X)$, the σ-algebra generated by $\mathcal{E}$ is
$$\langle \mathcal{E} \rangle = \text{smallest } \sigma\text{-algebra containing } \mathcal{E}.$$
- **Unassuming Lemma**: (TVUL) If $\mathcal{F} \subset \mathcal{M}$ then $\langle \mathcal{F} \rangle \subset \mathcal{M}$ (follows directly by definition)

* **FUNCTIONS**: that are compatible w/ measure structures.

Def: (MEASURABLE FUNCTION) Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be measurable spaces. A function $f : X \to Y$ is called $(\mathcal{M}, \mathcal{N})$-measurable if $\forall F \in \mathcal{N}$, $f^{-1}(F) \in \mathcal{M}$.

RECALL:
- $\forall F \subset Y$, $f^{-1}(F^c) = [f^{-1}(F)]^c$
- $\forall F_\alpha \subset Y$, $f^{-1}\left(\bigcup_\alpha F_\alpha\right) = \bigcup_\alpha f^{-1}(F_\alpha)$.

Might even be an uncountable collection of sets.

Def: (PULLBACK) Given $f : X \to Y$, take $(Y, \mathcal{N})$ then the pullback of f is defined as

$$f^* \mathcal{N} := \{ f^{-1}(F) : F \in \mathcal{N} \}.$$

Note that $f^* \mathcal{N}$ is a σ-algebra on X (it is the coarsest σ-algebra that makes f measurable*.

Def: (PUSHFORWARD) Let $(X, \mathcal{M})$ and $f: X \to Y$. The pushforward of f is

$$f_* \mathcal{M} = \{ F \subset Y : f^{-1}(F) \in \mathcal{M} \}.$$

Note that this is a σ-algebra on Y. In particular, it is the finest σ-algebra that makes f measurable**.

NOTE: * if f is $(\mathcal{M}', \mathcal{N})$-measurable, then $f^* \mathcal{N} \subset \mathcal{M}'$.

** if f is $(\mathcal{M}, \mathcal{N}')$-measurable, then $\mathcal{N}' \subset f_* \mathcal{M}$.

! **EXAMPLE:** Let $f: X \to Y$ be a constant fct. (i.e., $\exists y \in Y$ s.t. $\forall x \in X$, $f(x) = y$). Then

$$\forall \mathcal{N} \subset P(Y), \quad f^* \mathcal{N} = \{ \emptyset, X \}$$

$$\forall \mathcal{M} \subset P(X), \quad f_* \mathcal{M} = P(Y)$$

Lemma: Let $\mathcal{N} = \langle \mathcal{F} \rangle$, then
f is $(\mathcal{M}, \mathcal{N})$-measurable $\iff f^*\mathcal{F} \subset \mathcal{M}$.

Pf: $(\Rightarrow)$ Tautology.

$(\Leftarrow)$ Note $\mathcal{F} \subset f_* \mathcal{M}$, so, $\langle \mathcal{F} \rangle \subset f_* \mathcal{M}$.

TVUL

$\square$

IMPORTANT

Borel σ-algebras

Corollary: Let $(X, \mathcal{B}_X)$ and $(Y, \mathcal{B}_Y)$ be measurable spaces. Then, any continuous functions $f: X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$-measurable

continuity & measurability are not always very friendly to each other.

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* **PRODUCT OF SPACES**: Let $(Y_\alpha, \mathcal{N}_\alpha)$ be a (possibly uncountable) family of measurable spaces. Let $f_\alpha : X \to (Y_\alpha, \mathcal{N}_\alpha)$. Then

$$\Big\langle \underbrace{\bigcup_\alpha f_\alpha^* \mathcal{N}}_{\text{Possibly uncountable union of sets on } X, \text{ so it might not be a } \sigma\text{-algebra on } X} \Big\rangle = \text{coarsest } \sigma\text{-algebra s.t. } f_\alpha \text{ is measurable } \forall \alpha.$$

RECALL: $Y := \prod_{\alpha \in I} Y_\alpha$ is the set of all maps $\phi : I \longrightarrow \bigcup_\alpha Y_\alpha$ such that $\phi(\alpha) \in Y_\alpha$.

Very natural to define the **CANONICAL PROJECTION**

$$\pi_\alpha(\underset{\substack{\cap \\ Y}}{y}) = \phi_y(\alpha) \in Y_\alpha$$

Def: (PRODUCT σ-ALGEBRA) The product σ-algebra on $Y = \prod_\alpha Y_\alpha$ is defined as

$$\bigotimes_\alpha \mathcal{N}_\alpha := \Big\langle \bigcup_\alpha \pi_\alpha^* \mathcal{N}_\alpha \Big\rangle$$

$$= \Big\langle \{ \pi_\alpha^{-1}(F_\alpha) : F_\alpha \in \mathcal{N}_\alpha,\ \alpha \in I \} \Big\rangle$$

Lemma: Let

$$\prod_\alpha \mathcal{N}_\alpha := \left\langle \left\{ \prod_\alpha F_\alpha : F_\alpha \in \mathcal{M}_\alpha,\ \alpha \in I \right\} \right\rangle .$$

"Box σ-algebra"

Then,

$$\bigotimes_\alpha \mathcal{N}_\alpha \subseteq \prod_\alpha \mathcal{N}_\alpha$$

Equality $\Leftrightarrow$ I is countable

Pf: Note that

$$\pi_\alpha^{-1}(F_\alpha) \in \left\{ \prod_\alpha F_\alpha : F_\alpha \in \mathcal{N}_\alpha,\ \alpha \in I \right\}$$

So, by TVUL,

$$\bigotimes_\alpha \mathcal{N}_\alpha \stackrel{\text{def}}{=} \left\langle \left\{ \pi_\alpha^{-1}(F_\alpha) \right\} \right\rangle \subset \prod_\alpha \mathcal{N}_\alpha .$$

Conversely,

$$E \in \left\{ \prod_\alpha F_\alpha : F_\alpha \in \mathcal{N}_\alpha,\ \alpha \in I \right\} = \bigcap_\alpha \pi_\alpha^{-1}(F_\alpha)$$

$$\Rightarrow \prod_\alpha \mathcal{N}_\alpha \subset \bigotimes_\alpha \mathcal{N}_\alpha .$$

$\square$

LECTURE 3

PROPERTIES OF MEASURES

18/09/2023

Recall: We say $(X, \mathcal{M})$ is a measurable space.

EXAMPLE: PRODUCT SPACES

$$(Y_\alpha, \mathcal{N}_\alpha) \rightsquigarrow \left(\prod_\alpha Y_\alpha, \bigotimes_\alpha \mathcal{N}_\alpha \right)$$

where the product σ-algebra is generated by

$$\bigotimes_\alpha \mathcal{N}_\alpha = \left\langle \bigcup_\alpha \pi_\alpha^* \mathcal{N}_\alpha \right\rangle$$

Lemma: Let $f : X \to \prod_\alpha Y_\alpha$ and $(X, \mathcal{M})$ be a measurable space. Then

$$f \text{ is } (\mathcal{M}, \otimes_\alpha \mathcal{N}_\alpha)\text{-measurable}$$

$$\Updownarrow$$

$$\pi_\alpha \circ f \text{ is measurable } \forall \alpha.$$

Pf: ($\Downarrow$) Trivial. Because all the projections are measurable, and the composition of measurable maps is measurable.

($\Uparrow$) Suffices to check on a generating collection (and then use TVOL to extend it to the whole σ-algebra) such as: $\bigcup_\alpha \pi_\alpha^* \mathcal{N}_\alpha$.

But, this is

$$f^{-1}\left(\pi_\alpha^{-1}(F)\right) \text{ for some } F \in \mathcal{N}_\alpha$$

$$\parallel$$

$$(\pi_\alpha \circ f)^{-1}(F) \in \mathcal{M}$$

since $(\pi_\alpha \circ f)^{-1}$ is assumed to be measurable.

$\square$

Recall: A measure is $\mu : \mathcal{M} \to [0, +\infty]$ s.t. $\mu(\emptyset) = 0$ and $\mu\left(\bigcup_{j \in \mathbb{N}} E_j\right) = \sum_{j \in \mathbb{N}} \mu(E_j)$ for all $\{E_j\}_{j\in\mathbb{N}} \subset \mathcal{M}$ pairwise disjoint.

Prop: (PROPERTIES OF MEASURES) Let $(X, \mathcal{M}, \mu)$ be a measure space.

(i) If $E, F \in \mathcal{M}$ s.t. $E \subset F$, then

(MONOTONICITY)

$$\mu(E) \leq \mu(F)$$

Pf: $\mu(F) = \mu(E \sqcup (F\setminus E)) = \mu(E) + \mu(F\setminus E) \geq \mu(E)$.

(ii) For <u>any</u> countable collection of measurable sets $\{E_j\}_{j\in\mathbb{N}} \subset \mathcal{M}$,

(σ-SUBADDITIVITY)

$$\mu\left(\bigcup_{j\in\mathbb{N}} E_j\right) \leq \sum_{j\in\mathbb{N}} \mu(E_j).$$

(iii) For all $\{E_j\}_{j\in\mathbb{N}} \subset \mathcal{M}$ such that $E_j \subset E_{j+1}$ $\forall j \in \mathbb{N}$,

(UPWARD MONOTONE CONVERGENCE)

$$\mu\left(\bigcup_{j\in\mathbb{N}} E_j\right) = \lim_{j\to\infty} \mu(E_j).$$

Pf: Let $F_1 := E_1$

$F_2 := E_2 \setminus E_1$ all disjoint

$\vdots$

$F_n := E_n \setminus E_{n-1}$

$\vdots$

Thus,

$$\mu\left(\bigcup_i E_i\right) = \mu\left(\bigsqcup_i F_i\right) \overset{\sigma\text{-additivity}}{=} \sum_{i=1}^{\infty} \mu(F_i)$$

$$\overset{\text{def}}{=} \lim_{N\to\infty} \sum_{i=1}^{N} \mu(F_i)$$

$$\overset{\text{finite additivity}}{=} \lim_{N\to\infty} \mu(E_N).$$

(iv) For all $\{E_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$ such that $E_i \supset E_{i+1}$ $\forall i\in\mathbb{N}$ and $\mu(E_1) < +\infty$, then

DOWNWARD MONOTONE CONVERGENCE

$$\mu\left(\bigcap_{i\in\mathbb{N}} E_i\right) = \lim_{i\to\infty} \mu(E_i).$$

Pf: Let $E_i' := E_1 \setminus E_i$. Then $\{E_i'\}$ is increasing seq. (by construction). Apply upward monotone convergence:

$$\mu(E_1) = \mu(E_i) + \mu(E_i') \quad \forall i$$

$$\lim_{i\to\infty} \mu(E_1 \setminus E_i') = \mu\left(\bigcap_i E_i\right) + \mu\left(\bigcup_i E_i'\right)$$

$$\longrightarrow \lim_{i\to\infty} \mu(E_i')$$

Now, since $\mu(E_1) < +\infty$, subtract from both sides.

Def: (NULL SET) Let $(X, \mathcal{M}, \mu)$ be a measure space. Then, a set $E \in \mathcal{M}$ is called μ-null if $\mu(E) = 0$.

Philosophy: null sets are negligible for the measure. So much so that we say a property holds μ-almost-everywhere (a.e.) if the property holds for all sets except for measure zero sets.

PATHOLOGY: Sometimes, pathological things happen on subsets of measure zero sets... So, we care about "non-pathological" measures:

Def: (COMPLETE MEASURE) $\mathcal{M}$ is called complete with μ if every subset of all μ-null sets are measurable. More specifically, $\forall N$ μ-null set, $\mathcal{P}(N) \subset \mathcal{M}$.

E.g., Lebesgue measure is <u>not</u> complete on the σ-algebra of Borel sets.

<u>BUT</u>, good news, easy to extend σ-algebras so that the measure is complete:

* <u>COMPLETION SCHEME</u>: Let $(X, \mathcal{M}, \mu)$ be a measure space. Let

$$\mathcal{N} := \{ N \in \mathcal{M} : \mu(N) = 0 \}$$

collection of null sets in $\mathcal{M}$

Define

COMPLETION OF σ-ALGEBRA $\mathcal{M}$

$$\overline{\mathcal{M}} := \{ E \cup F : E \in \mathcal{M}, \exists N \in \mathcal{N} \text{ s.t. } F \subset N \}$$

Extended σ-algebra on which μ is complete

Prop: $\overline{\mathcal{M}}$ is a σ-algebra and there exists a completion $\overline{\mu}$ on $\overline{\mathcal{M}}$ s.t. $\forall E \in \mathcal{M}$, $\overline{\mu}(E) = \mu(E)$ and $\overline{\mathcal{M}}$ is complete w/ $\overline{\mu}$.

$\overline{\mathcal{M}}$ is the coarsest (smallest) σ-algebra that is complete w.r.t. μ and that contains $\mathcal{M}$.

Pf: $\overline{\mathcal{M}}$ is closed under countable unions b/c $\mathcal{M}$ is (trivially) and so is $\mathcal{N}$ (b/c by σ-additivity the measure of countable unions of null sets is again zero).

Now, let $A = E \cup F \in \overline{\mathcal{M}}$. WTS: $A^c \in \overline{\mathcal{M}}$.

Then, we can choose E s.t. $E \cap N = \phi$.

$$(E \cup F)^c = \left[(E \cup N) \cap (F \cup N^c)\right]^c$$

$$= \underbrace{(E \cup N)^c}_{\in \mathcal{M}} \cup \underbrace{(F \cup N^c)^c}_{\subset N \in \mathcal{N}}$$

$\Rightarrow \overline{\mathcal{M}}$ closed under complements.

Now, define $\boxed{\bar{\mu}(E \cup F) =: \mu(E).}$

Suppose

$E_1 \cup F_1 = E_2 \cup F_2 \Rightarrow E_1 \subset E_2 \cup N_2.$

By monotonicity: $\mu(E_1) \leq \mu(E_2) + \cancel{\mu(N_2)}^{=0}$

Conversely, $E_2 \subset E_1 \cup N_1.$

By monotonicity: $\mu(E_2) \leq \mu(E_1) + \cancel{\mu(N_1)}^{=0}$

Thus, $\mu(E_2) = \mu(E_1)$, so it is well-defined.

$\square$

Now, we'll see a tool to relax the properties and still have good/useful features.

OUTER MEASURE

$$\mu^*: \mathcal{P}(X) \longrightarrow [0,+\infty]$$

Def: We say that $E \in \mathcal{P}(X)$ is μ^*-Carathéodory measurable if for all $A \in \mathcal{P}(X)$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

We can nicely split any set in $\mathcal{P}(X)$ using E!

NOTE: By subadditivity of μ^*, $\leq$ is guaranteed. So, we only need to check $\geq$ for finite sets (nothing will be larger than $+\infty$).

Equivalently,

$$E \text{ is } \mu^*\text{-measurable} \iff \forall A \in \mathcal{P}(X),\ \mu^*(A) < \infty$$
$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Now, how to construct these useful measure theories:

Thm: (CARATHÉODORY EXTENSION) Let μ^* be an outer measure and let

$$\mathcal{M} := \{ E \in P(X) : E \text{ is } \mu^*\text{-measurable} \}.$$

Define $\mu := \mu^*|_{\mathcal{M}}$. Then $(X, \mathcal{M}, \mu)$ is a complete measure space.

Pf: Need to check:

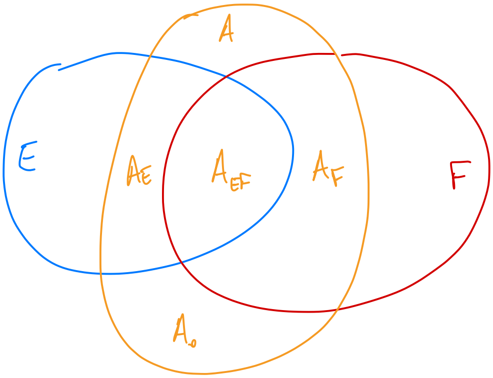
(1) $\mathcal{M}$ is a σ-algebra

(2) μ is a measure

(3) $\mathcal{M}$ is complete w.r.t. μ.

Proof of (1): $\mathcal{M}$ is closed under complements is is direct from the "complement symmetry" of Carathéodory's criterion.

Show that $\mathcal{M}$ is closed under disjoint countable unions. Take finite disjoint unions first



Now,

$$\mu^*(A) \overset{E\in\mathcal{M}}{=} \mu^*(A_E \cup A_{EF}) + \mu^*(A_0 \cup A_F)$$

$$\overset{F\in\mathcal{M}}{=} \mu^*(A_E) + \mu(A_{EF}) + \mu^*(A_0) + \mu^*(A_F)$$

$$= \cdots = \mu^*(A_E \cup A_F \cup A_{EF}) + \mu^*(A_0).$$

So, $\mathcal{M}$ is an algebra. So, we can now only consider disjoint countable unions. For that, let $\{E_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$ s.t. $E_i \cap E_j = \emptyset$, $i \neq j$.

Let

$$\mathcal{M} \ni F_N := \bigcup_{n=1}^{N} E_n \qquad F_\infty = \bigcup^{\infty} E_n$$

$$\Rightarrow F_N \nearrow F_\infty. \quad \leftarrow F_\infty \supset F_N \Rightarrow F_\infty^c \subset F_N^c \; (*)$$

Now, $\forall A \in P(X)$, since F_N is μ^*-measurable,

$$\mu^*(A) = \mu^*(A \cap F_N) + \mu^*(A \cap F_N^c)$$

Monotonicity of μ^* $\Downarrow$ (‡)

$$\mu^*(A) \geq \mu^*(A \cap F_N) + \mu^*(A \cap F_\infty^c)$$

Now, cut with F_{N-1}:

$$\mu^*(A) \geq \mu^*(A \cap F_{N-1}) + \mu^*(A \cap E_N) + \mu^*(A \cap F_\infty^c)$$

Cut with F_{N-2}:

F_{N-3}

$\vdots$

F_i: (always using measurability of F_i)

$$\mu^*(A) \geq \sum_{i=1}^{N} \mu^*(A \cap E_i) + \mu^*(A \cap F_\infty^c)$$

Take limit as $N \nearrow \infty$:

$$\mu^*(A) \geq \sum_{i=1}^{\infty} \mu^*(A \cap E_i) + \mu^*(A \cap F_\infty^c)$$

Subadditivity of μ^* $\rightarrow \geq \mu^*(A \cap F_\infty) + \mu^*(A \cap F_\infty^c)$.

This concludes the proof that $\mathcal{M}$ is a σ-algebra. $\square$

Proof of (2): ($\mu \overset{\text{def}}{=} \mu^*|_{\mathcal{M}}$ is a measure)

Note that

$$\mu = \mu^*|_{\mathcal{M}} \implies \mu(\emptyset) = \mu^*(\emptyset) = 0.$$

So, only need to check σ-additivity: if $\{E_i\}_{i\in\mathbb{N}} \subset \mathcal{M}$, $E_i \cap E_j = \emptyset$ $i \neq j$, then

$$\sum_{i\in\mathbb{N}} \mu(E_i) = \mu\Big(\underbrace{\bigcup_{i\in\mathbb{N}} E_i}_{=: F_\infty}\Big)$$

So,

$$\mu^*(F_\infty) \geq \sum_{i=1}^{\infty} \mu^*(\underbrace{E_i}_{\substack{\text{Measurable,} \\ \text{so the star drops}}}) + \cancel{\mu^*(\emptyset)} \rightarrow = 0$$

and this gives σ-additivity. $\square$

Proof of (3): ($\mathcal{M}$ is complete w.r.t. μ) Note that $\forall F \subset N$ s.t. $\mu^*(N) = 0$, we have that $0 = \mu^*(F) \leq \mu^*(N) = 0$.

So, if N is a null-set and $F \subset N$,

$$\mu^*(F) = \mu^*(N) = \mu(N) = 0.$$

So, only need to show that null sets are Carathéodory measurable (trivial): $\forall A \in \mathcal{P}(X)$ let F be s.t. $\mu^*(F) = 0$, then by monotonicity,

$$\mu^*(A) \leq \mu^*(A \cap F) + \mu^*(A \cap F^c) \leq \mu^*(A)$$

$\square$

$\blacksquare$

Lecture 4

Hahn - Kolmogorov

20/09/2023

<u>Recall</u>: By Carathéodory's criterion, a set $E \in \mathcal{M}$ (σ-algebra on X) is μ^*-measurable (outer measure) iff $\forall A \subset X$,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

Then, by Carathéodory Extension Thm, for any X $(X, \mathcal{M}, \mu = \mu^*|_{\mathcal{M}})$ is a complete measure space.

Now, construct the Lebesgue measure using Hahn-Kolmogorov.

<u>Def</u>: (**Premeasure**) Let $\mathcal{A}$ be an <u>algebra</u> $\mathcal{A} \subset P(X)$. Then, $\mu_0 : \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a <u>premeasure</u> if

- $\mu_0(\emptyset) = 0$
- (conditional σ-additivity) $\{E_i\}_{i \in \mathbb{N}} \subset \mathcal{A}$

s.t. $\boxed{\bigcup_{i\in\mathbb{N}} E_i \in \mathcal{A}}$ and $E_i \cap E_j \neq \emptyset \quad \forall i \neq j$,

Need to assume this since $\mathcal{A}$ is an algebra

Then

$$\mu_0\left(\bigcup_{i\in\mathbb{N}} E_i\right) = \sum_{i\in\mathbb{N}} \mu_0(E_i).$$

This statement is a bit sketchy. See pg. 38 for the correct statement

<u>Thm</u>: (HAHN-KOLMOGOROV 1933) Let $\mathcal{A}$ be an <u>algebra</u> on X and μ_0 a <u>premeasure</u> on $\mathcal{A}$.

Then, μ_0 can be extended to a complete measure μ on $\langle \mathcal{A} \rangle$ (s.t. $\mu|_{\mathcal{A}} = \mu_0$).

Not necessarily complete

Moreover, if ν is another extension to $\mathcal{A}$, then $\forall E \in \langle \mathcal{A} \rangle$, $\nu(E) \leq \mu(E)$.

"=" holds if $\mu(E) < \infty$ or $\forall E \in \langle \mathcal{A} \rangle$ if X is σ-finite.

Pf: <u>OUTLINE</u> • <u>Step 1</u>: Construct an outer measure

μ^* on X from μ_0.

- Step 2: Invoke Carathéodory Extension Thm to get $(X, \mathcal{M}, \mu)$ and show $\mathcal{M} \supset \mathcal{A}$.

- Step 3: Show that $\mu\big|_{\mathcal{A}} = \mu_0$.

- Step 4: Deal w/ uniqueness.

Pf: Step 1: For all $A \subset X$, define the outer measure from a covering of A:

$$\mu^*(A) := \inf\left\{ \sum_i \mu_0(E_i) : E_i \in \mathcal{A} \text{ s.t. } \bigcup_i E_i \supset A \right\}.$$

Lemma (*): μ^* is an outer measure.

Pf:

This gives the desired outer measure on X.

Step 2: Now, use Caratheodory Extension Thm to construct $(X, \mathcal{M}, \mu)$ complete measure space.

WTS: $\forall E \in \mathcal{A}$, $\mu^*(A) < \infty$

$$\mu^*(A) \geq \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

For that, fix $\varepsilon > 0$. By def. of μ^*, there exists a covering $\{E_i\}$, $E_i \in \mathcal{A}$, s.t.

$$\sum_i \mu_0(E_i) \leq \mu^*(E) + \varepsilon.$$

Take $E_i \cap E \in \mathcal{A}$ (finite intersection of elements in $\mathcal{A}$). Note that $E_i \cap E$ cover $A \cap E$ (since the E_i's cover A by assumption). Thus,

$$\mu^*(A \cap E) \leq \sum_i \mu_0(E_i \cap E)$$

since μ^* is the inf over coverings.

Can do the same w/ $E_i \cap E^c \in \mathcal{A}$ that cover $A \cap E^c \Rightarrow \mu^*(A \cap E^c) \leq \sum_i \mu_0(E_i \cap E^c)$.

Thus,

$$\mu^*(A \cap E) + \mu^*(A \cap E^c)$$

$$\leq \sum_i \mu_0(E_i \cap E) + \sum_i \mu_0(E_i \cap E^c)$$

$$= \sum_i \mu_0(E_i) \leq \mu^*(A) + \varepsilon$$

Since $\varepsilon > 0$ is arbitrary, we are done w/ Step 2.

Step 3: WTS: $\forall E \in \mathcal{A}$ s.t. $\mu(E) = \mu_0(E)$.

Indeed, as E covers itself, $\mu^*(E) \leq \mu_0(E)$.

Conversely, let $\{E_i\}$ cover E. Then

$$\sum_i \mu_0(E_i) \geq \mu_0(E)$$

(WLOG: Assume the E_i's are disjoint)

Let $\tilde{E}_i := E_i \cap E$ so that $\bigcup_i \tilde{E}_i = E$.

Thus, since μ^* is an outer measure,

$$\sum_i \mu_0(E_i) \geq \sum_i \mu_0(\tilde{E}_i)$$

$$= \mu_0(E).$$

Thus, $\mu|_{\mathcal{A}} = \mu_0$.

<u>Step 4</u>: Assume $\nu : \langle \mathcal{A} \rangle \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ is s.t. $\nu(E) = \mu_0(E) \;\; \forall E \in \langle \mathcal{A} \rangle$.

Let $E \in \langle \mathcal{A} \rangle$ and let $\{E_i\} \subset \mathcal{A}$ be a covering of E. By subadditivity,

$$\nu(E) \leq \sum_i \nu(E_i) = \sum_i \mu_0(E_i)$$

Take the inf over all possible coverings

$$\nu(E) \leq \mu^*(E) = \mu(E).$$

NOTE: $\nu\left(\bigcup_{i=1}^{\infty} E_i\right) = \lim_{N\to\infty} \nu\Big(\underbrace{\bigcup_{i=1}^{N} E_i}_{\in \mathcal{A}}\Big)$

upward monotone convergence

$$= \lim_{N\to\infty} \mu_0\left(\bigcup_{i=1}^{N} E_i\right)$$

$$= \mu\left(\bigcup_{i=1}^{\infty} E_i\right).$$

So, fix $\varepsilon > 0$. If $\mu(E) < \infty$, then for some covering $\{E_i\}$ of E, we have that

$$\cancel{\mu(E)} + \varepsilon \geq \sum_{i\in\mathbb{N}} \mu(E_i)$$

subadditivity of μ

$$\geq \mu\left(\bigcup_{i\in\mathbb{N}} E_i\right)$$

$$\geq \cancel{\mu(E)} + \mu\left(\left(\bigcup_{i\in\mathbb{N}} E_i\right) \setminus E\right)$$

That is why we NEED to assume $\mu(E) < \infty$.

(otherwise couldn't do this)

Then,

$$\mu\left(\left(\bigcup_{i\in\mathbb{N}} E_i\right) \setminus E\right) \leq \varepsilon .$$

Thus,

$$\begin{aligned}
\mu(E) &\leq \mu\left(\bigcup_{i\in\mathbb{N}} E_i\right) \\
&= \nu\left(\bigcup_{i\in\mathbb{N}} E_i\right) \\
&= \nu(E) + \nu\left(\left(\bigcup_{i\in\mathbb{N}} E_i\right) \setminus E\right) \\
&\leq \nu(E) + \mu\left(\left(\bigcup_{i\in\mathbb{N}} E_i\right) \setminus E\right) \\
&\leq \nu(E) + \varepsilon \leftarrow \text{arbitrary.}
\end{aligned}$$

Therefore, if $\mu(E) < \infty$, $\nu(E) = \mu(E)$.

Lastly, if $X = \bigcup_{i\in\mathbb{N}} X_i$, X_i all disjoint, then

$$\nu(E) = \nu\left(\bigcup_i (E\cap X_i)\right) = \sum_i \nu(E\cap X_i) = \mu(E). \quad \blacksquare$$

Lecture 5

Lebesgue Measure

25/09/2023

Recall: !!! IMPORTANT & POWERFUL

Thm: (Hahn-Kolmogorov) Let $\mathcal{A}$ be an algebra and μ_0 a premeasure on $\mathcal{A}$. Then, there exists a measure μ on $\langle \mathcal{A} \rangle$ so that $\mu|_{\mathcal{A}} = \mu_0$.

Any other measure ν extending μ_0 on $\langle \mathcal{A} \rangle$ is such that $\nu(E) \leq \mu(E)$ $\forall E \in \langle \mathcal{A} \rangle$ with $\nu(E) = \mu(E)$ if $\mu(E) < \infty$. If $(X, \langle \mathcal{A} \rangle, \mu)$ is σ-finite, then $\nu(E) = \mu(E)$ $\forall E \in \langle \mathcal{A} \rangle$ (i.e., the extension is <u>unique</u> if the space is σ-finite).

No guarantee of completeness!

---

* **CONSTRUCTION OF LEBESGUE MEASURE**: Strategy is to use Hahn-Kolmogorov to turn a premeasure on an algebra into a measure on the σ-algebra generated by that algebra.

Recall: From Prop Check 3, the following collection of intervals on $\mathbb{R}$ form an elementary family:

$$\mathcal{E} := \{ \underset{a<b}{(a,b]}, \underset{a\in\mathbb{R}}{(a,\infty)}, \underset{b\in\mathbb{R}}{(-\infty,b]}, \emptyset \}$$

ELEMENTARY FAMILY

$$\Rightarrow \mathcal{A} := \left\{ \begin{array}{c} \text{finite disjoint unions of} \\ \text{elements of } \mathcal{E} \end{array} \right\}$$

is an algebra. Moreover, $\langle \mathcal{A} \rangle = \mathcal{B}_{\mathbb{R}}$.

GOAL: find a premeasure on $\mathcal{A}$ to extend it to a measure on $\langle \mathcal{A} \rangle = \mathcal{B}_{\mathbb{R}}$.

Def: $\forall E \in \mathcal{A}$, define $\mu_0 : \mathcal{A} \to \mathbb{R}_{\geq 0} \cup \{\infty\}$ as

$$\mu_0(E) := \begin{cases} 0, & \text{if } E = \emptyset \\ \infty, & \text{if } E \text{ contains } (a,\infty) \text{ or } (-\infty, b] \\ \sum_{i=1}^{N} (b_i - a_i), & \text{if } E = \bigcup_{i=1}^{N} (a_i, b_i] \end{cases}$$

Claim: μ_0 is a premeasure on $\mathcal{A}$.

NOTE: Applying Hahn-Kolmogorov to $\mathcal{A}$ and μ_0 gives a Borel measure (i.e. on $\mathcal{B}_{\mathbb{R}}$) which is the Lebesgue measure. ← This measure will not be complete (will have to extend the σ-algebra)

NOTE: $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ is σ-finite since we can always write

$$\mathbb{R} = \bigsqcup_{k \in \mathbb{Z}} (k, k+1]$$

Countable union of finite measure sets

with $\mu((k,k+1]) = \mu_0((k,k+1]) = 1$.

Pf: (WTS μ_0 is a premeasure) First, need to check μ_0 is well-defined. We have

$$E = \bigcup_{i=1}^{N} (a_i, b_i]$$

If $b_i = a_j$ for some i,j, we can "join" them, i.e.,

$$(a_i, b_i] \cup (a_j, b_j] \rightsquigarrow (a_i', b_i']$$

where $a_i' = a_i$ and $b_i' = b_j$. In doing so,

$$\begin{aligned}
\mu_0((a_i,b_i]\cup(a_j,b_j]) &= \cancel{b_i} - a_i + b_j - \cancel{a_j} \\
&= b_j - a_i \\
&= b_i' - a_i' \\
&= \mu_0((a_i', b_i']),
\end{aligned}$$

so the value of μ_0 is the same, as it ought to be. Thus, μ_0 is well-defined and finitely additive.

Now, need "conditional σ-additivity"; i.e.,
WTS: if $\{E_j\}_{j\in\mathbb{N}} \subset \mathcal{A}$ s.t. $\bigcup_{j\in\mathbb{N}} E_j \in \mathcal{A}$, then

$$\mu_0\left(\bigcup_{j\in\mathbb{N}} E_j\right) = \sum_{j\in\mathbb{N}} \mu_0(E_j).$$

For this, WLOG assume that the E_j's are disjoint. Note that we can assume that E_j is in fact a half-interval $(a_i, b_i]$. Moreover, assume no E_j is an ∞-interval. Lastly, we can assume that $\bigcup_{j\in\mathbb{N}} E_j$ is a finite union of half-intervals.

Case 1: finite half-intervals.

Case 2: infinite half-intervals.

So, it suffices to show the claim in the situation where $\bigcup_{j\in\mathbb{N}} E_j$ is <u>one</u> <u>finite</u> <u>half-interval</u>.

$\Longrightarrow$ **<u>WTS</u>**: if $\bigcup_{j\in\mathbb{N}} E_j = (a,b]$ then $\sum_{j\in\mathbb{N}} \mu_0(E_j) = b-a$.

Indeed, note that

$$E := (a,b]$$

$$\begin{aligned}
b-a &= \mu_0\Big(\bigcup_{j\in\mathbb{N}} E_j\Big) \\
&= \mu_0\Big(\underbrace{\bigcup_{j=1}^{N} E_j}_{\in \mathcal{A}} \cup \big(E \setminus \underbrace{\bigcup_{j=1}^{N} E_j}_{\in\mathcal{A}}\big)\Big) \\
&= \mu_0\Big(\bigcup_{j=1}^{N} E_j\Big) + \underbrace{\mu_0\Big(E \setminus \bigcup_{j=1}^{N} E_j\Big)}_{\geq 0} \\
&\geq \mu_0\Big(\bigcup_{j=1}^{N} E_j\Big) \quad \forall N \\
&= \sum_{j=1}^{N} \mu_0(E_j) \quad \forall N.
\end{aligned}$$

Taking the limit $N \nearrow \infty$, we find that

$$b - a \geq \sum_{j=1}^{\infty} \mu_0(E_j)$$

So, now need to show the reverse inequality:

Want to extract a finite subcollection ($\rightsquigarrow$ compact).

For that, $\forall \varepsilon > 0$, note that

compact $\to$ $E' := [a+\varepsilon, b] \subset E =: (a, b]$

let $E_j' := (a_j, b_j + \underbrace{\varepsilon 2^{-j}}) \supset E_j = (a_j, b_j]$

open $\nearrow$

$b_j' := b_j + \varepsilon 2^{-j}$

Common strategy to enlarge a countable collection by a "little bit".

Let E_j'' be a finite cover of E' w/ elements in $\{E_j'\}_{j\in\mathbb{N}} \rightsquigarrow E_j'' := (a_j'', b_j'')$.

NOTE: When we enlarge, there might be overlaps.

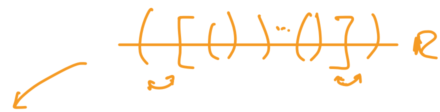
But we can assume that $a_j'' < a_{j+1}''$ and that

$b_j'' > a_{j+1}$.

Now,

$$\mu_0(E) = b - a$$

$$= b - (a+\varepsilon) + \varepsilon$$



$$\le b_N'' - a_1'' + \varepsilon$$

$$\le b_N'' - a_N'' + \sum_{j=1}^{N-1} a_{j+1}'' - a_j'' + \varepsilon$$

$$\le b_N'' - a_N'' + \sum_{j=1}^{N-1} b_j'' - a_j'' + \varepsilon$$

$$\le \sum_{j=1}^{N} b_j'' - a_j'' + \varepsilon$$

$$\le \sum_{j=1}^{N} \mu_0(E_j) + \sum_{j=1}^{N} \varepsilon 2^{-j} + \varepsilon$$

$$\leq \sum_{j=1}^{\infty} \mu_0(E_j) + \underbrace{2\varepsilon}$$

arbitrary, so we are done. ∎

Hahn-Kolmogorov $\Longrightarrow$

b/c σ-finite

$$\boxed{\exists!\ \mu \text{ on } \mathcal{B}_{\mathbb{R}} \text{ s.t. } \mu((a,b]) = b-a.}$$

Next we show this behaves well w/ open intervals.

---

* **REGULARITY OF THE LEBESGUE MEASURE**

Thm: If $E \in \mathcal{B}_{\mathbb{R}}$, then

$$\mu(E) = \inf\{\mu(U) : U^{\text{open}} \supset E\}$$

$$= \sup\{\mu(K) : K^{\text{compact}} \subset E\}$$

i.e., can approximate E from the outside using open sets and from the inside using compact sets.

Now, a technical lemma.

Recall that

$$\mu(E) \overset{\text{def}}{=} \inf\left\{ \sum_{i\in\mathbb{N}} \mu_0(E_i) : E_i \in \mathcal{A} \text{ and } \bigcup_{i\in\mathbb{N}} E_i \supset E \right\}$$

$$= \inf\left\{ \sum_{i\in\mathbb{N}} \mu_0\big((a_i, b_i]\big) : \bigcup_{i\in\mathbb{N}} (a_i, b_i] \supset E \right\}.$$

Lemma: $\forall E \in \mathcal{B}_{\mathbb{R}}$,

$$\mu(E) = \inf\left\{ \sum_{i\in\mathbb{N}} \mu\big((a_i, b_i)\big) : \bigcup_{i\in\mathbb{N}} (a_i, b_i) \supset E \right\}$$

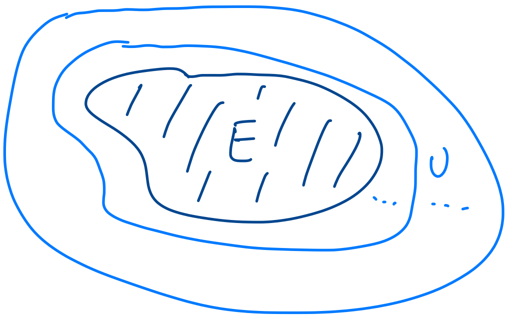
Same measure if we consider open intervals.

Pf of Lemma: Page 35-36 of Folland. $\square$

Pf of Regularity Theorem:

Step 1: $\forall E \in \mathcal{B}_{\mathbb{R}}$,

$$\mu(E) = \inf \{ \mu(U) : U^{\text{open}} \supset E \}.$$



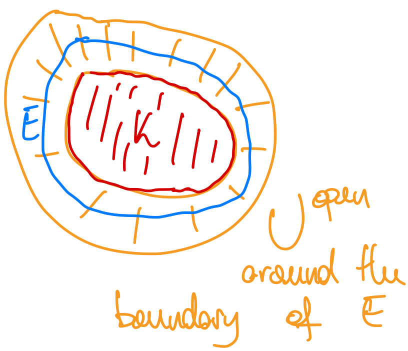
By the lemma and the fact that every $U^{\text{open}} \subset \mathbb{R}$ is a countable union of open intervals $\bigcup_{i \in \mathbb{N}} (a_i, b_i)$, we conclude that the claim is true.

Step 2: Approximation by compact sets from the inside.

(i) <u>Suppose $E \in \mathcal{B}_{\mathbb{R}}$ is bounded</u>. <u>If E is closed</u>, then it is compact by Heine-Borel. So, by monotonicity, $\mu(E) \geq \mu(K)$ $\forall K^{\text{compact}} \subset E$, and, clearly, the sup is realized by E.

Now, <u>if E is not closed</u>, then $\forall \varepsilon > 0$

$\exists\, U^{\text{open}} \supset \bar{E} \setminus E$ s.t.



$$\mu(U) \le \mu(\bar{E} \setminus E) + \varepsilon.$$

Now, let

$$K^{\text{compact}} := \bar{E} \setminus U.$$

Then,

$$\mu(K) \overset{\text{additivity}}{=} \mu(E) - \mu(E \cap U)$$

$$= \mu(E) - \left[\mu(U) - \mu(U \setminus E)\right]$$

$$= \mu(E) - \mu(U) + \mu(U \setminus E)$$

Monotonicity: $U \supset \bar{E} \setminus E$

$$\ge \mu(E) - \underbrace{\mu(U) + \mu(\bar{E} \setminus E)}_{\ge \varepsilon}$$

$$\ge \mu(E) + \varepsilon.$$

Arbitrary.

(ii) Suppose $E \in \mathcal{B}_{\mathbb{R}}$ is unbounded. Then,

let
$$E = \bigsqcup_{j \in \mathbb{Z}} \underbrace{E \cap [j, j+1]}_{=: E_j}.$$

So, $\forall j \in \mathbb{Z}$ $\exists K_j^{\text{compact}} \subset E_j$ s.t.
$$\mu(K_j) \geq \mu(E_j) - \varepsilon 2^{-|j|}.$$

Let
$$H_N := \bigcup_{j=-N}^{N} K_j$$
← compact since finite union of compact.

Then,
$$\mu(H_N) \geq \mu\left(\bigcup_{j=-N}^{N} E_j\right) - \sum_{j=-N}^{N} \varepsilon 2^{-|j|}$$
$$\geq \mu\left(\bigcup_{j=-N}^{N} E_j\right) - 2\varepsilon$$
← arbitrary

$N \to \infty$ gives the desired result. ■

❗ **IMPORTANT PROOF TECHNIQUE**: Start proving something for bounded sets then approximate unbounded sets w/ bounded sets and hope the claim holds. :)

LECTURE 6

LEBESGUE MEASURE (ctd.)

27/09/2023

Recall: Last time we took the algebra $\mathcal{A}$ of finite unions of half-open intervals on $\mathbb{R}$ and a premeasure μ_0 on $\mathcal{A}$. Then, we noted that $\langle \mathcal{A} \rangle = \mathcal{B}_{\mathbb{R}}$.

Then, we constructed a measure μ on $\mathcal{B}_{\mathbb{R}}$ s.t. $\mu|_{\mathcal{A}} = \mu_0$ using Hahn-Kolmogorov. In particular

premeasure μ_0 on $\mathcal{A}$ (disjoint unions of half-intervals), s.t. $\langle \mathcal{A} \rangle = \mathcal{B}_{\mathbb{R}}$	$\rightarrow$	Carathéodory extension to get μ^* outer measure on $(\mathbb{R}, \mathcal{M}, \mu)$ (complete measure space	$\rightarrow$	μ measure on $\langle \mathcal{A} \rangle \subset \mathcal{M}$ s.t. $\mu|_{\mathcal{A}} = \mu_0$

where

$$\langle \mathcal{A} \rangle = \mathcal{B}_{\mathbb{R}} \subset \mathcal{M} := \overline{\mathcal{B}_{\mathbb{R}}}$$

σ-algebra of Lebesgue measurable sets

Lastly:

$$\mu_0(E) = \begin{cases} \infty, & \text{if } E \text{ contains an } \infty\text{-h-interval} \\ 0, & \text{if } E = \emptyset \\ \sum_i b_i - a_i, & \text{if } E = \bigsqcup_i (a_i, b_i] \end{cases}$$

NOTE: Recall that a function $f : (X, \mathcal{M}) \to (Y, \mathcal{N})$ is measurable if $f^* \mathcal{N} \subset \mathcal{M}$.

Def: f is Lebesgue measurable if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-measurable; i.e., measurable as a function

$$f : (\mathbb{R}, \mathcal{M}) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}}).$$

$\mathcal{M} = \overline{\mathcal{B}_{\mathbb{R}}}$ = Lebesgue measurable sets

Note: this is not very well behaved with fct. compositions (e.g., $g \circ f$ is not necessarily Lebesgue measurable even if both g and f are).

But we can "forget" this by redefining "fct."
Moreover, usually we only deal with Borel measurable fcts. (e.g., continuous fcts).

NOTATION: Always denote the Lebesgue measure on $\mathbb{R}$ (and later on $\mathbb{R}^d$) as m.

Arbitrary measures = μ.

Thm: (REGULARITY OF LEBESGUE MEASURE) $\forall E \in \mathcal{B}_{\mathbb{R}}$,

$$m(E) = \inf\{m(U) : U^{\text{open}} \supset E\}$$ (Inner Regularity of m)

$$= \sup\{m(K) : K^{\text{compact}} \subset E\}$$ (Outer regularity of m)

Thm: The following are equivalent

(This is special for the Lebesgue measure. It uncomplicates our life a lot)

(i) $E \in \mathcal{M}$

(ii) $\exists$ G_δ-set V and a m-null set N_1 s.t.

$$E = V \setminus N_1.$$

(iii) $\exists$ F_σ-set W and a m-null set N_2 s.t.

$$E = W \cup N_2$$

Pf: (ii) $\Rightarrow$ (i) Automatic.

(iii) $\Rightarrow$ (i) Automatic.

(i) $\Rightarrow$ (ii) & (iii) Let $E \in \mathcal{M}$ and assume first that $\mu(E) < \infty$. Since $\mathcal{M}$ is the completion of $\mathcal{B}_{\mathbb{R}}$, we can write

$$E = \underset{\substack{\cap \\ \mathcal{B}_{\mathbb{R}}}}{E'} \cup \underset{\substack{\cap \\ N}}{F}$$

$N \rightsquigarrow$ m-null Borel set

$$\Rightarrow \quad \bar{E} = E \cup N.$$

So, there exists U_n open such that

$$m(U_n \setminus \bar{E}) < 2^{-n}, \quad U_n \supset E.$$

Now, $V := \bigcap_n U_n$ is G_δ by definition. Thus,

$$m(V \setminus E) < 2^{-n} \quad \forall n$$

and $E = V \setminus (V \setminus E)$.

If $m(E) = \infty$, just write E as a countable union of finite measure sets and do the argument above for each of the finite sets. $\blacksquare$

REMARK: The definition of μ (i.e., as the Hahn-Kolmogorov extension of μ_0) implies that

- $$m(E) = m(\{x+c : x \in E\}) \quad \forall c \in \mathbb{R}$$

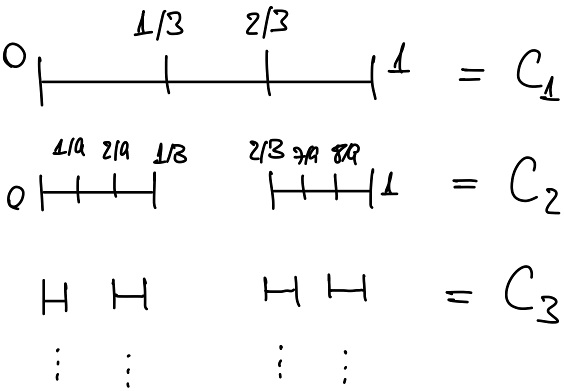
  (m is invariant under translations)

- $$|\lambda|\, m(E) = m(\{\lambda x : x \in E\}) \quad \forall \lambda \in \mathbb{R}.$$

  (m is invariant under dilations)

---

* **CANTOR SET**: Very weird $\Rightarrow$ good for counterexamples:



$$C = \bigcap_{n=1}^{\infty} C_n$$

Standard "middle thirds" Cantor set

Prop: (Properties of C)

(i) C is compact, totally disconnected, nowhere dense, perfect.

(ii) $m(C) = 0$

(iii) C is uncountable

Pf: (i) Topology.

(ii)

$$m(C) = m\left(\bigcap_n C_n\right) = \lim_{n\to\infty} m(C_n)$$

But

$$m(C_1) = 1$$

$$m(C_2) = 1 - 1/3$$

$$m(C_3) = 1 - 1/3 - 2/9$$

$$m(C_4) = 1 - 1/3 - 2/9 - 4/27$$

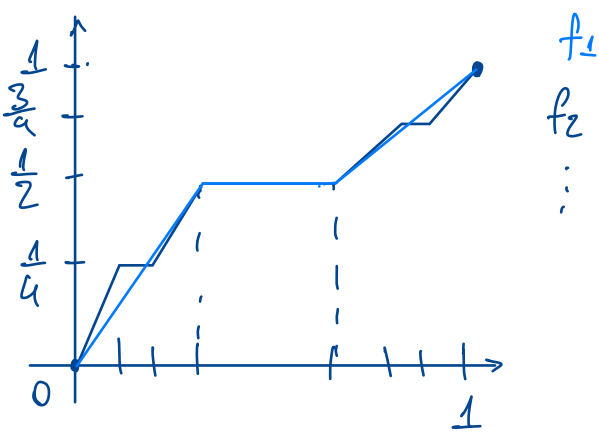
So,

$$m(C) = \lim_{n\to\infty} m(C_{n+1})$$

$$= \lim_{n\to\infty} \left(1 - \sum_{k=0}^{n-1} \frac{2^k}{3^{k+1}} \right)$$

$$= 1 - \frac{1}{3}\left(\frac{1}{1-2/3}\right) = 0 .$$

(iii) Consider the "Devil's Staircase" that is a surjection from C to $[0,1]$.



$$f := \lim_{n\to\infty} f_n$$

■

LECTURE 7

02/10/2023

INTEGRATION (Ch. 2 Folland)

Recall: The Borel σ-algebra $\mathcal{B}_{\mathbb{R}^n}$ on $\mathbb{R}^n$ is the σ-algebra generated by open sets in $\mathbb{R}^n$.

Equivalently, $\mathcal{B}_{\mathbb{R}^n}$ can be seen as the product σ-algebra of $\mathbb{R}$:

$$\mathcal{B}_{\mathbb{R}^n} = \left(U^{\text{open}} \subset \mathbb{R}^n \right) = \underbrace{\mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}}_{n \text{ times}}$$

Recall that $f : X \to \prod_\alpha Y_\alpha$ is <u>measurable</u> if and only if $\pi_\alpha \circ f$ is measurable $\forall \alpha$.

Since $\mathbb{C} \simeq \mathbb{R}^2$, the above notions extend to the complex plane quite naturally:

$$\boxed{f : X \to \mathbb{C} \text{ is measurable}} \iff \boxed{\operatorname{Re}(f), \operatorname{Im}(f) \text{ are measurable}}$$

NOTE: if $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are measurable, then

- $f+g$ is measurable
- fg is measurable

Since $\cdot, + : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous, hence Borel measurable.

Def: Extended reals $\boxed{\overline{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}}$.

Then

$$\overline{M} = \{ E \subset \overline{\mathbb{R}} : E \cap \mathbb{R} \in M \}$$

$$\overline{\mu}(E) = \mu(E \cap \mathbb{R})$$

Obs: $\overline{\mathcal{B}}_{\mathbb{R}} = \langle [a, \infty] : a \in \mathbb{R} \rangle$.

Prop: Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of measurable functions $f_n : X \to \overline{\mathbb{R}}$. Then, the supremum and infimum, etc, are measurable; i.e,

- $x \longmapsto \sup_n f_n(x)$ is measurable

- $x \mapsto \inf_n f_n(x)$ is measurable
- $x \mapsto \limsup_{n\to\infty} f_n(x)$ is measurable
- $x \mapsto \liminf_{n\to\infty} f_n(x)$ is measurable

$\Rightarrow$ if $\exists \lim_{n\to\infty} f_n(x) = f(x)$, then f is measurable.

Pf: ($\sup_n f_n(x)$ measurable) Let

$$E_n(a) = f_n^{-1}([a,\infty]).$$

Note that $E_n(a)$ is measurable $\forall n\in\mathbb{N}$. Then the set

$$\{x : \sup_n f_n(x) \geq a\} = \bigcap_{m\in\mathbb{N}} \underbrace{\bigcup_{n\in\mathbb{N}} E_n\left(a-\tfrac{1}{m}\right)}_{\text{Since } \sigma\text{-algebra, this is measurable}}$$

is measurable, and we are done.

(inf measurable) Use that $\inf_n(f_n) = -\sup_n(-f_n)$.

($\limsup_{n\to\infty}$ measurable) Also easy since

$$\limsup_{n\to\infty} f_n = \inf_m \sup_{n>m} f_n$$

($\liminf_{n\to\infty}$ measurable) Similar ⤴

Only used that $\mathcal{B}_{\mathbb{R}}$ is a σ-algebra → $\blacksquare$

Def: Given $f: X \to \overline{\mathbb{R}}$ measurable, define

$$f^+ := \max\{f, 0\}$$

$$f^- := \max\{-f, 0\}$$

Note that $f = f^+ + f^-$ and f^+, f^- are both nonnegative and measurable. ← by previous proposition

If $f: X \to \mathbb{C}$ & measurable, $f = (\operatorname{sgn} f)|f|$, where ← Nonnegative

$$\operatorname{sgn} z := \begin{cases} z/|z|, & \text{if } z \neq 0 \\ 0, & \text{if } z = 0 \end{cases}.$$

↑ $\operatorname{sgn} f$ and $|f|$ are measurable if f is

Def: (SIMPLE FUNCTION) A function f is *simple* if it is measurable and $f(X)$ is finite. ($\subset \mathbb{R}, \mathbb{C}$ not ∞)

Equivalently, f is simple if it is a finite linear combination of characteristic fcts. of measurable sets.

STANDARD REPRESENTATION OF SIMPLE FUNCTIONS:

$$f(x) = \sum_{y \in im(f)} y \cdot \chi_{f^{-1}(y)}(x)$$

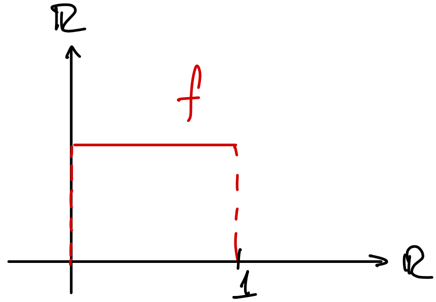
$\leftarrow$ image of f

$$= \sum_{i=1}^{N} a_i \chi_{E_i}(x), \qquad a_i \neq a_j \ \forall i,j, \quad E_i \cap E_j = \emptyset, \quad X = \bigsqcup_i E_i$$

- Example of simple function:

$$f(x) = \chi_{[0,1]} + 0 \cdot \chi_{(-\infty,0) \cup (1,\infty)}$$

$$= \chi_{[0,1/2]} + \chi_{(1/2,1]} + \dots$$



NOTATION: $\phi, \psi, \_ \rightsquigarrow$ simple fcts.

! IMPORTANT

Thm: (APPROX. W/ SIMPLE FUNCTION) Let $f: X \to \overline{\mathbb{R}}_{\geq 0}$ (nonnegative) be measurable. Then, there exists an increasing sequence of nonnegative simple functions

$$0 \leq \phi_1 \leq \phi_2 \leq \cdots \leq \phi_n \leq \cdots \leq f$$

such that $\phi_n \to f$ pointwise and uniformly on any set where f is bounded.

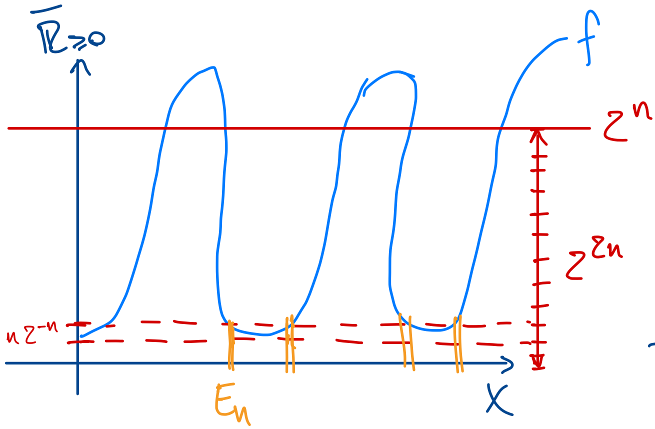
- If $f: X \to \mathbb{C}$ is measurable, then there exists a sequence $\{\phi_n\}$ such that

$$0 \leq |\phi_1| \leq |\phi_2| \leq \cdots \leq |\phi_n| \leq \cdots \leq |f|$$

so that $\phi_n \to f$ pointwise and uniformly on any set where f is bounded.

Pf: Let $n \in \mathbb{N}$, $0 \leq l \leq 2^{2n} - 1$ and consider

$$E_n^l := f^{-1}\left(\left[l \cdot 2^{-n}, (l+1)2^{-n}\right)\right)$$



Define

$$R_n := f^{-1}([2^n, \infty])$$

Now, define the simple functions as:

$$\phi_n(x) := \sum_{l=0}^{2^{2n}-1} l \cdot 2^{-n} \chi_{E_n^l}(x) + 2^n \chi_{R_n}(x)$$

- Of course, $\phi_n \leq \phi_{n+1} \quad \forall n$.
- We have that

$$\sup_{x \in R_n^c} |f(x) - \phi_n(x)| \leq 2^{-n}.$$

* For the complex version of the statement, simply split the fct. f as

$$f = (\mathrm{Re} f)^+ - (\mathrm{Re} f)^- + i\left[(\mathrm{Im} f)^+ - (\mathrm{Im} f)^-\right].$$

$\blacksquare$

IMPORTANT: ALLOWS US TO BE SLEEZY

Thm: Let $(X, \mathcal{M}, \mu)$ be a measure space and $(\bar{X}, \bar{\mathcal{M}}, \bar{\mu})$ its completion. If f is $\bar{\mathcal{M}}$-measurable, $\exists\, g$ $\mathcal{M}$-measurable s.t. $f = g$ $\bar{\mu}$-a.e.

E.g.: Can forget the distinction between Lebesgue-measurable fcts. and Borel-measurable.

Pf: If f is $\bar{\mathcal{M}}$-measurable, $\exists$ seq. of $\bar{\mathcal{M}}$-simple fcts. ϕ_n s.t. $\phi_n \to f$ ptwise and given by

$$\phi_n(x) = \sum_{l=0}^{L-1} a_l \, \chi_{\underbrace{E_n^l}_{\bar{\mu}\text{-measurable}}}$$

$E_n^l \in \bar{\mathcal{M}}$

$\mathcal{M} \ni E_n'^l \cup F_n^l \subset N_n^l$

$$\phi_n'(x) = \sum_{l=0}^{L-1} a_l \, \chi_{E_n'^l}$$

simple function w.r.t. $\mathcal{M}$

$$\{\phi_n \neq \phi_n'\} \subset \bigcup_l N_n^l = N_n \quad (\text{"null"})$$

Note that $\phi_n \to f$ ptwise. So,

$$\phi_n' = \phi_n \text{ on } \Big(\bigcup_n N_n\Big)^c =: N^c \text{ (null)}$$

Thus: $\phi_n' \to f$ on N^c.

Take $\phi_n'' := \chi_{N^c}\,\phi_n'$. So $\phi_n'' \to g$ $\mathcal{M}$-measurable and $g - f = 0$ on N^c.

$\blacksquare$

INTEGRALS: First, define a class of functions:

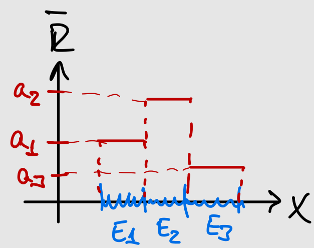
$$L^+(X, \mathcal{M}, \mu) := \{ f : X \to \overline{\mathbb{R}}_{\geq 0} : f \text{ measurable} \}$$

Def: If $\phi \in L^+$ is simple, then we can write

$$\phi = \sum_{i=0}^{N-1} a_i \chi_{E_i}.$$

Define

$$\int_X \phi \, d\mu := \sum_{i=0}^{N-1} a_i \, \mu(E_i).$$



For any $A \in \mathcal{M}$, $\phi \chi_A$ is again simple. So, we define

$$\int_A \phi \, d\mu := \sum_{i=0}^{N-1} a_i \, \mu(E_i \cap A).$$

Prop: $\forall \phi, \psi \in L^+$,

(a) if $c \geq 0$, then $\int c\phi = c \int \phi$

(b) $\int \phi + \psi = \int \phi + \int \psi$

(c) if $\phi \leq \psi$, then $\int \phi \leq \int \psi$

(d) fix ϕ and consider the set fct.

$$\nu : \mathcal{M} \ni A \longmapsto \int_A \phi \, d\mu.$$

Then ν is a measure.

Pf: $\nu(\emptyset) = 0$

Take $A = \bigsqcup_n A_n$, $\nu(A) = \int_A \phi \, d\mu = \sum_{j=0}^{L-1} a_j \, \mu(A \cap E_j)$

σ-additivity of μ $= \sum_{n,j} a_j \, \mu(A_n \cap E_j)$

$$= \sum_n \int_{A_n} \phi \, d\mu = \sum_n \nu(A_n).$$

Finally ...

!!! VERY IMPORTANT

Def: (INTEGRAL IN L^+) Let $f \in L^+(X, \mathcal{M}, \mu)$

$$\int_X f \, d\mu := \sup \left\{ \int_X \phi \, d\mu : \phi \le f \right\}$$

LEBESGUE INTEGRAL FOR $\mu = m$

$\Rightarrow$ Immediate to check that

(a) $f \le g$ a.e. $\Rightarrow \int f \le \int g$. (monotonicity)

(b) $\forall c \ge 0 \Rightarrow \int cf = c \int f$. (scaling)

❗ EXTREMELY IMPORTANT ❗

Thm: (MONOTONE CONVERGENCE THEOREM, a.k.a. MCT)

Let $\{f_n\}_{n\in\mathbb{N}} \subset L^+$ be monotone (i.e., $f_n \le f_{n+1}$).
Let $f := \sup_n f_n = \lim_{n\to\infty} f_n$. Then

$$\int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu .$$

LECTURE 8

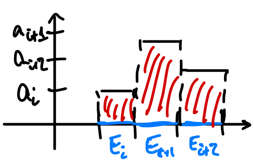
MCT & FATOU

04/10/2023

Recall: If $\phi \in L^+(X, \mathcal{M}, \mu)$ is simple, then

$$\int \phi \, d\mu = \sum_{\mathrm{im}(\phi) \ni y} y \cdot \mu(\phi^{-1}(y)) = \underbrace{\sum_{i=1}^{N} a_i \, \mu(E_i)}_{\text{Obs: } \infty \cdot 0 = 0}$$

E_i disjoint & partition X



Now, fix ϕ, then $A \longmapsto \int_A \phi \, d\mu$ is a measure.

For a general function $f \in L^+$, we define

$$\int f \, d\mu := \sup \left\{ \int \phi \, d\mu : 0 \leq \phi \leq f \right\}.$$

Thm: (MCT) If $\{f_\ell\}_{\ell \in \mathbb{N}} \subset L^+$ is monotonically increasing (i.e, $f_\ell \leq f_{\ell+1}$ $\forall \ell \in \mathbb{N}$) and converges to $f := \lim_{\ell \to \infty} f_\ell = \sup_{\ell \in \mathbb{N}} \{f_\ell\}$, then

since increasing

$$\int \lim_{\ell \to \infty} f_\ell \, d\mu = \lim_{\ell \to \infty} \int f_\ell \, d\mu.$$

Pf: By monotonicity of the integral, $\left\{ \int f_\ell \, d\mu \right\}_{\ell \in \mathbb{N}}$ also form a monotonically increasing sequence. So, $\forall \ell$,

$$\int f_\ell \leq \int f \Rightarrow \lim_{\ell \to \infty} \int f_\ell \leq \int f.$$

Now, need to show the other inequality (w/out Fatou's lemma ῒ). Fix $\alpha \in (0,1)$ and let $0 \leq \phi \leq f$. For all $n \in \mathbb{N}$, define

simple

$$E_n := \{ x \in X : f_n(x) \geq \alpha \phi(x) \}.$$

So, $E_n \nearrow X$, and

$$\int_X f_n \, d\mu \geq \int_{E_n} f_n \, d\mu \geq \alpha \int_{E_n} \phi \, d\mu$$

Now, note that

$$E_n \xmapsto{\nu_\phi} \int_{E_n} \phi \, d\mu$$

is a measure! So, since $E_n \nearrow X$, we have

$$\nu_\phi(X) = \int_X \phi \, d\mu = \lim_{n\to\infty} \int_{E_n} \phi \, d\mu$$

Monotone convergence from below of the measure

$$= \lim_{n\to\infty} \nu_\phi(E_n)$$

Thus,

$$\lim_{n\to\infty}\int_X f_n \geq \int_{E_n} f_n \geq \alpha \int_{E_n} \phi = \int_X f.$$

∎

Corollary: $\forall f, g \in L^+$,

$$\int (f+g) = \int f + \int g$$

Pf: $f, g \in L^+ \Rightarrow \exists\, \phi_n, \psi_n$ simple s.t.
$\phi_n \nearrow f$ and $\psi_n \nearrow g$.

Thus $(\phi_n + \psi_n) \nearrow (f+g)$

$$\begin{array}{ccc} \displaystyle\int f+g & & \displaystyle\int f + \int g. \\ \text{MCT} \longrightarrow \parallel & & \parallel \longleftarrow \text{MCT} \\ \displaystyle\lim_{n\to\infty}\int \phi_n+\psi_n & = & \displaystyle\lim_{n\to\infty}\int \phi_n + \lim_{n\to\infty}\int \psi_n \end{array}$$

∎

Claim: Can do better:

$$\int \sum_{i=1}^{\infty} f_i = \sum_{i=1}^{\infty} \int f_i$$

Pf: Let $F_N := \sum_{i=1}^{N} f_i$, then $\int F_N = \sum_{i=1}^{N} \int f_i$,

and

$$F_N \nearrow F := \sum_{i=1}^{\infty} f_i .$$

So, by MCT,

$$\int F = \lim_{N\to\infty} \int F_N = \sum_{i=1}^{\infty} \int f_i$$

∎

* NON-EXAMPLES OF MCT

(1) Let $f_n = \chi_{[n, n+1]}$. So, $f_n \to 0$ ptwise.

Not monotonic!

Thus

$$0 = \int \lim_{n\to\infty} f_n \neq \lim_{n\to\infty} \int f_n = 1 .$$

(2) Let $f_n = n\chi_{[0,1/n]}$. ← Not monotonic

Then,

$$0 = \int \lim_{n\to\infty} f_n \neq \lim_{n\to\infty} \int f_n = 1.$$

IMPORTANT

FATOU'S LEMMA: If $\{f_n\}_{n\in\mathbb{N}} \subset L^+$ (i.e., non-negative μ-measurable functions), then

$$\boxed{\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n.}$$

↖ Very useful since we basically have no assumptions...

Recall: $\liminf_{n\to\infty} a_n = \lim_{\ell\to\infty} \inf_{n\geq \ell} a_n$

Pf: (FATOU'S LEMMA) Note that $\forall \ell \geq 1$, by the def. of inf,

$$\inf_{n\geq \ell} f_n \leq f_j \quad \forall j \geq \ell.$$

Monotonicity of $\int$ $\Longrightarrow$ $\displaystyle\int \inf_{n\geq l} f_n \leq \int f_j \quad \forall j \geq l$

Arbitraging trick

$$\Longrightarrow \int \inf_{n\geq l} f_n \overset{(*)}{\leq} \inf_{j\geq l} \int f_j$$

MCT

$$\Longrightarrow \int \liminf_{n\to\infty} f_n = \lim_{l\to\infty} \int \inf_{n\geq l} f_n$$

$$\overset{(*)}{\leq} \lim_{l\to\infty} \inf_{n\geq l} \int f_j$$

$$= \liminf_{n\to\infty} \int f_n .$$

$\blacksquare$

LECTURE 9

11/10/2023

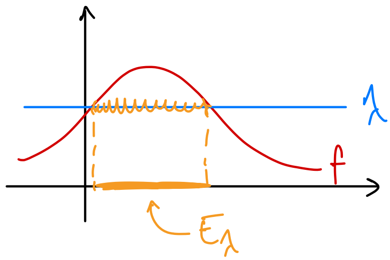
LEBESGUE DOMINATED CONVERGENCE THEOREM

Sometimes called Chebyshev

MARKOV INEQUALITY: Let $f \in L^+$ (nonnegative measurable fct.). For all $\lambda \in (0,\infty)$, define $E_\lambda := \{x : f(x) \geq \lambda\}$. Then

$$\mu(E_\lambda) \leq \frac{1}{\lambda} \int f \, d\mu .$$

VERY USEFUL!



Pf: Assume $\int f < \infty$. Let $\phi_\lambda = \lambda \cdot \chi_{E_\lambda}$ (clearly a simple function). Then,

$$\phi_\lambda \leq f \implies \int \phi_\lambda \, d\mu \leq \int f \, d\mu$$

$$\int \phi_\lambda \, d\mu = \lambda \mu(E_\lambda)$$

$\square$

Corollary 1: If $f \in L^+$ is such that $\int f < \infty$, then f is a.e. finite.

Pf:

$$\{x : f(x) = \infty\} = \bigcap_n E_n$$

$$\mu\{x : f(x) = \infty\} = \mu\left(\bigcap_n E_n\right) \overset{\text{Markov}}{\leq} \frac{1}{n} \int_{E_n} f \, d\mu \leq \frac{M}{n}$$

arbitrary

$\square$

Corollary 2: If $f \in L^+$ and $\int f = 0$, then $f = 0$ a.e.

Pf: Let

$$Z = \{x : f(x) > 0\} = \bigcup_n E_{1/n}$$

By contradiction, if $\mu(Z) > 0$, then $\exists n$ s.t.

$$\mu(E_{1/n}) > 0 \Rightarrow 0 < \mu(E_{1/n}) \leq n \int f \, d\mu$$

Markov

$\square$

Upgrade MCT to...

Prop: Assume $f_n \nearrow f$ $\underline{\mu\text{-a.e.}}$, then

$$\int \lim_{n\to\infty} f_n \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu.$$

Pf: $\exists N$ μ-null s.t. $\underbrace{f_n \chi_{N^c}}_{\tilde{f}_n :=} \nearrow \underbrace{f \chi_{N^c}}_{\tilde{f}}$

Thus, by MCT,

$$\int \lim_{n\to\infty} \tilde{f}_n \, d\mu \overset{\text{MCT}}{=} \lim_{n\to\infty} \int \tilde{f}_n \, d\mu$$

$$= \lim_{n\to\infty} \int f_n \, d\mu .$$

Corollary 2

$\square$

Variation of Fatou's Lemma:

Prop: If $\{f_n\} \subset L^+$ and $f_n \to f \in L^+$ μ-a.e., then

$$\int f \leq \liminf_{n\to\infty} \int f_n .$$

Def: (INTEGRABLE) Let f be a measurable $\mathbb{R}$-valued function. We say f is *integrable* on $E \in \mathcal{M}$ iff

$$\int_E |f| < \infty .$$

<u>REMARK:</u> The set of integrable functions form a linear vector space (linearity of $\int$ with triangle inequality).

If f is integrable, we define

$$\int f := \int f^+ - \int f^- \quad \text{(if } f \text{ is } \mathbb{R}\text{-valued)}$$

$$\int f := \int \operatorname{Re} f + i \int \operatorname{Im} f \quad \text{(if } f \text{ is } \mathbb{C}\text{-valued)}$$

Prop: If f is integrable on $(X, \mathcal{M}, \mu)$, then

$$\left| \int f \right| \leq \int |f| .$$

Pf: Assume f is $\mathbb{R}$-valued. Then

$$\left| \int f \right| \overset{\text{def}}{=} \left| \int f^+ - \int f^- \right| \leq \int f^+ + \int f^-$$

$$\leq \int f^+ + f^-$$

$$= \int |f| .$$

If f is $\mathbb{C}$-valued, then

- if $\int f = 0$, there's nothing to prove.
- if $\int f \neq 0$,

$$\left| \int \operatorname{Re} f \right| \overset{\text{from before}}{\leq} \int |\operatorname{Re} f| \leq \int |f|$$

So, for any $\alpha \in \mathbb{C}$, $|\alpha| = 1$, we have

$$\left| \int \operatorname{Re}(\alpha f) \right| \leq \int |\operatorname{Re}(\alpha f)| \leq \int |\alpha f| = \int |f|$$

The "optimal" inequality is attained for $\alpha = \overline{\operatorname{sgn} \int f}$.

Thus,

$$\int \operatorname{Re}(\alpha f) = \operatorname{Re}\left(\alpha \int f\right) = \int |f| . \qquad \square$$

Prop: If f is integrable, then

(i) the set $\{x : f(x) \neq 0\}$ is σ-finite

(ii) if f and g are integrable, then $\forall E \in \mathcal{M}$,

$$\boxed{\int_E f = \int_E g} \overset{\Rightarrow}{\underset{\Leftarrow}{\Longleftrightarrow}} \boxed{f = g \text{ a.e.}} \overset{\Leftrightarrow}{\Longleftrightarrow} \boxed{\int_E |f-g| = 0}$$

Also obvious ($\Leftarrow$)

obvious from Cor 2 ($\Leftrightarrow$)

Pf: ($\Rightarrow$) Take $E' = \{f \neq g\}$. Suppose it's not null. So, at least one of $\{f^{\pm} \neq g^{\pm}\}$ is not null. Thus,

$$\int_{E^+} |f^+ - g^+| \neq 0 \quad \hookrightarrow\!\!/\!\!/\!\!\hookleftarrow$$

$\square$

Upshot: For the purpose of integration, it doesn't matter what happens on null sets ($\int$ can't detect it anyway...).

Def: $f \sim g$ if $f = g$ μ-a.e.. Let

$f = [f] \in f \text{ integrable}/\sim$.

For the purpose of integration, $\int_E [f]\,d\mu$ is well-defined.

Def: (**L^1 SPACE**) Define

$$L^1(X, \mathcal{M}, \mu) := \{ [f] : \int |[f]| \, d\mu < \infty \}$$

Equivalence classes of functions that agree μ-a.e..

Can define a notion of distance $\rightsquigarrow d(f,g) = \int |f-g|$

This is only a distance since we define this equivalence class.

$\Rightarrow$ $L^1(X, \mathcal{M}, \mu)$ with d is a BANACH SPACE

! EXTREMELY IMPORTANT !

Thm: (**LEBESGUE DOMINATED CONVERGENCE**) Let $\{f_n\}$ be a sequence in L^1 such that $f_n \to f$. If there exists $g \in L^1 \cap L^+$ such that $|f_n| \le g$, then $f \in L^1$ and

don't need "a.e." b/c of equivalence class

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f.$$

LECTURE 10

16/10/2023

LDCT, APPROXIMATIONS, MODES OF CONVERGENCE

LDCT: Let $\{f_n\}_{n\in\mathbb{N}} \subset L^1$ such that $f_n \to f$ a.e..
If there exists $g \in L^1 \cap L^+$ such that $|f_n| \leq g$ a.e., ← $0 \leq |f_n| \leq g$
then

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n = \int f.$$

Corollary: Under the same assumptions of LDCT, we can conclude that $f_n \longrightarrow f$ in L^1. That is, f_n converges to f w.r.t. the distance in L^1:

$$d(g,h) = \int |g-h| \, d\mu.$$

L^1 convergence:

$$f_n \xrightarrow{L^1} f \iff \int |f_n - f| \, d\mu \xrightarrow{n\nearrow\infty} 0$$

$|f_n - f| \to 0$ a.e. and $|f_n - f| \leq 2g$
so, this follows by LDCT

Pf (LDCT): Assume the f_n's are real (otherwise, just apply the same argument for the real & imaginary parts). Of course, f is measurable and integrable.

Note that $\boxed{\begin{array}{l} g + f_n \\ g - f_n \end{array}}$ $\begin{array}{l} \geq 0 \\ \geq 0 \end{array}$

both are sequences of nonnegative measurable functions. So, apply Fatou:

$$\overset{\text{Fatou}}{\Longrightarrow} \quad \int \liminf (g + f_n) \leq \int g + \liminf \int f_n$$

$$\int \liminf (g+f_n) = \int g + \int f$$

$$\int \liminf (g - f_n) \leq \int g - \limsup \int f_n$$

$$\int \liminf (g-f_n) = \int g - \int f$$

can do this b/c g integrable

Thus, subtract $\int g$ from both sides and get the result. ∎

Thm: Let $\{f_n\}_{n\in\mathbb{N}} \subset L^1$ be such that

$$\sum_{j=1}^{\infty} \int |f_j| < \infty .$$

Then

$$\sum_{j=1}^{\infty} f_j \xrightarrow{\text{converges to some } f\in L^1} f \in L^1 \quad \text{and} \quad \int \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \int f_j$$

Pf: By MCT,

$$\int \sum_{j=1}^{\infty} |f_j| \overset{MCT}{=} \sum_{j=1}^{\infty} \int |f_j| < \infty .$$

So,

$$g(x) := \sum_{j=1}^{\infty} |f_j(x)| \underbrace{\in L^1}_{\Rightarrow g \text{ is finite a.e.}}$$

Let

$$h_n(x) := \sum_{j=1}^{n} f_j(x), \quad \text{then } |h_n| \le g \text{ a.e.}$$

Thus, by LDCT, $h_n \xrightarrow{L^1} h$. ∎

Upshot: Every absolutely convergent series converges in L^1. This is equivalent to L^1 being complete (i.e., every Cauchy sequence converges in L^1)

VERY IMPORTANT !! VERY USEFUL !!

Thm: (DENSITY OF SIMPLE FUNCTIONS IN L^1) Let $f \in L^1$.

(i) Then, $\forall \varepsilon > 0$, there exists a simple function $\phi \in L^1$ such that

$$\int |f - \phi| \, d\mu < \varepsilon.$$

(ii) If $f \in L^1(\mathbb{R}, \mathcal{M}, \underset{\text{Lebesgue measure}}{m})$, then the ϕ above can be "very simple" (i.e., finite linear combination of characteristic functions of open intervals).

(iii) In fact, we can find g continuous such that

$$\int |f - g| \, dm < \varepsilon.$$

Pf: Recall that we can always find a sequence $\{\phi_n\}$ of simple functions such that

$$0 \leq |\phi_1| \leq \cdots \leq |\phi_n| \leq \cdots \leq |f|$$

with $\phi_n \to f$ a.e.. By LDCT, we can upgrade this convergence to convergence in L^1. This shows (i).

Now, take $L^1(\mathbb{R}, \mathcal{M}, \underset{\text{Lebesgue}}{m})$. Then, given

$$\phi(x) = \sum_{i=1}^{N} a_i \chi_{E_i}(x),$$

note that, if $a_i \neq 0$, then

$$m(E_i) \leq |a_i^{-1}| \int_{E_i} |\phi| \leq |a_i^{-1}| \int_{\mathbb{R}} |\phi| < \infty.$$

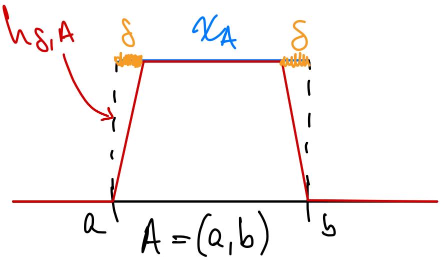
But, as shown before, we can approximate m-measurable sets as well as we want by the union of open intervals:

$$\underset{\text{open intervals}}{\tilde{E}_i} \supset E_i \quad \text{with} \quad m(\tilde{E}_i \setminus E_i) \leq |a_i^{-1}| \frac{\varepsilon}{N}.$$

This shows (ii).

Finally, let $\tilde{\phi}(x) := \sum_{i=1}^{N} a_i \chi_{\tilde{E}_i}(x)$. Then

$$\int |\phi - \tilde{\phi}| \, d\mu < \varepsilon.$$



In L^1, can approximate χ_A by a continuous function $h_{\delta,A}$ s.t.

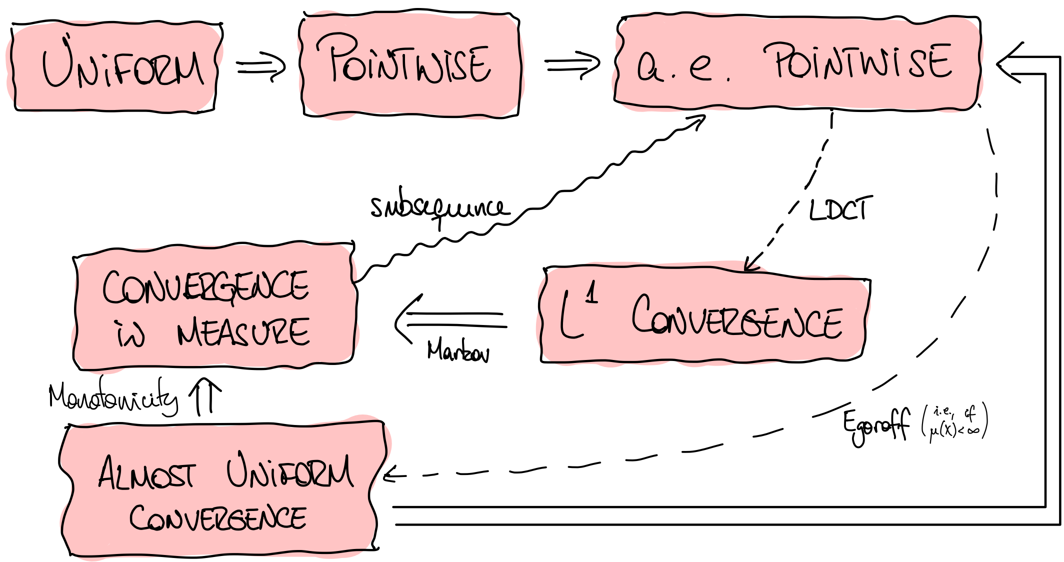
$$\int |\chi_A - h_{A,\delta}| = \int_a^{a+\delta} |\chi_A - h_{A,\delta}| + \int_{b-\delta}^{\delta} |\chi_A - h_{A,\delta}|$$

$$\leq 2\delta + 2\delta \longleftarrow \text{Arbitrarily small as } \delta \to 0$$

■

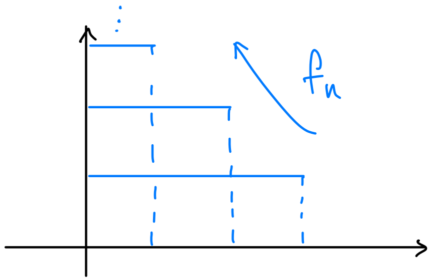
---

* **MODES OF CONVERGENCE** ❗IMPORTANT



EXAMPLES TO KEEP IN MIND: !!

1) SQUEEZE UP: $\boxed{f_n = n\,\chi_{[0,1/n]}}$.

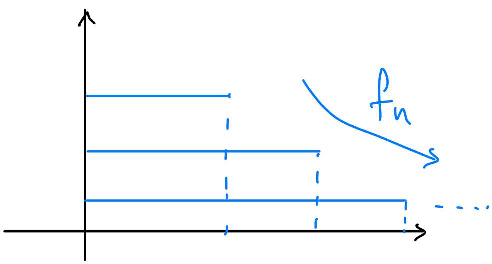


$$\int f_n = 1 \Rightarrow f_n \not\xrightarrow{L^1} 0$$

$$f_n \xrightarrow{\text{unif.}} 0 \text{ and } f_n \xrightarrow{\text{measure}} 0$$

2) SQUEEZE RIGHT: $\boxed{f_n = \frac{1}{n}\chi_{[0,n]}}$. $\int f_n = 1$

$$f_n \to 0 \text{ uniformly}$$



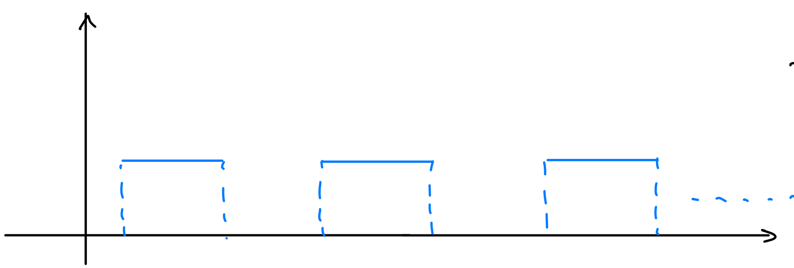
$$f_n \not\xrightarrow{L^1} 0$$

$$\lim_{n\to\infty}\int|0-f_n| = \lim_{n\to\infty}\int f_n = \lim_{n\to\infty}\left(\frac{1}{n}m([0,n])\right) = 1 \neq 0$$

$$f_n \xrightarrow{\text{measure}} 0$$

3) WALKING CAMEL: $\boxed{f_n = \chi_{[n,n+1]}}$

Not Cauchy in measure



f_n does <u>not</u> converge in measure but $f_n \xrightarrow{\text{ptwise}} 0$.

(measure $\not\Leftarrow$ ptwise)

4) ZENO'S PIANO:

$k \in \mathbb{Z}$ s.t. $0 \leq j < 2^{k+1}$

$$f_n = \chi_{\left[\frac{j}{2^k}, \frac{j+1}{2^k}\right]}, \quad n = j + 2^k.$$

$f_n \xrightarrow{\text{measure}} 0$ but does <u>not</u> convergence a.e.

$f_n \xrightarrow{L^1} 0$ but does <u>not</u> converge a.e.

Def: A sequence $\{f_n\}_{n\in\mathbb{N}} \subset L^1$ is <u>Cauchy in measure</u> if $\forall \varepsilon > 0$, $i \geq j \geq n$,

$$\lim_{n\to\infty} \mu(\{x : |f_i(x) - f_j(x)| \geq \varepsilon\}) = 0.$$

Def: (CONVERGENCE IN MEASURE) $f_n \xrightarrow{\text{measure}} f$ if for all $\varepsilon > 0$,

$$\lim_{n\to\infty} \mu(\{x : |f_n(x) - f(x)| \geq \varepsilon\}) = 0.$$

Lemma 1: If $f_n \xrightarrow{L^1} f$, then $f_n \xrightarrow{\text{measure}} f$.

Pf: $0 \leq \mu(\{x : |f_n - f| > \varepsilon\}) \underset{\text{Markov}}{<} \frac{1}{\varepsilon}\int |f_n - f| \longrightarrow 0.$ $\square$

Converse is not true by squeeze right.

Lemma 2: If f_n is Cauchy in measure, that is $f_{n_j} \xrightarrow{\text{measure}} f$, then $f_n \xrightarrow{\text{measure}} f$.

Lemma 3: If $f_n \xrightarrow{\text{measure}} f$ and $f = g$ a.e., then $f_n \xrightarrow{\text{measure}} g$.

Pf: (Lemma 3) Fix $\varepsilon > 0$, then

$$\underset{\overset{\text{\rotatebox{90}{$\leq$}}}{0}}{\mu\left(\{x : |f-g| > \varepsilon\}\right)} \leq \mu\left(\cancel{\{x : |f-f_n| > \tfrac{\varepsilon}{2}\}} \cup \cancel{\{x : |f_n-g| > \tfrac{\varepsilon}{2}\}}\right)$$

null

null

But, for all $n \in \mathbb{N}$,

$$|f-g| = |f - f_n + f_n - g|$$

Triangle ineq. in $\mathbb{R}$ $\rightarrow$ $$\leq |f-f_n| + |f_n - g|$$

$$< \varepsilon.$$

$\blacksquare$

Pf: (Lemma 2)

$$\mu(\{x : |f - f_n| > \varepsilon\}) \leq \mu\left(\{x : |f - f_m| > \tfrac{\varepsilon}{2}\} \cup \{x : |f_n - f_m| > \tfrac{\varepsilon}{2}\}\right)$$

$0 \Leftarrow$

$\forall m, j,\ m = n_j$

$$= \cancel{\mu\left(|f - f_{n_j}| > \tfrac{\varepsilon}{2}\right)} + \cancel{\mu\left(|f_n - f_{n_j}| > \tfrac{\varepsilon}{2}\right)}$$

$0 =$ $\qquad$ $0 =$

∎

Thm: If $\{f_n\}_{n \in \mathbb{N}} \subset L^1$ is Cauchy in measure, then there exists $f_n \xrightarrow{\text{measure}} f$ and there exists n_j such that $f_{n_j} \xrightarrow{\text{a.e. ptwise}} f$.

Pf: Let g_j be the subsequence f_{n_j} such that

$$\mu\Big(\underbrace{\{|g_j - g_{j+1}| > 2^{-j}\}}_{=:\, E_j}\Big) \leq 2^{-j}.$$

Let $F_\ell := \bigcup_{j=\ell}^{\infty} E_j$. Then $\mu(F_\ell) < 2 \cdot 2^{-\ell}$

Moreover, if $r \geq s \geq \ell$, then, $\forall x \notin F_\ell$,

$$|g_r - g_s| \leq \sum_{j=s}^{r-1} |g_{j+1} - g_j| \leq 2 \cdot 2^{-s}.$$

This, $\{g_j\}$ is Cauchy (in the $\mathbb{R}$-sense) as long as $x \notin F_\ell$.

continuity from above

Set $F = \bigcap_\ell F_\ell$, then $\mu(F) = 0$.

Let

$$f(x) = \underbrace{\lim_{j\to\infty} g_j(x)}_{\text{a.e. ptwise limit}} \quad \text{for } x \in F.$$

Now, we are left w/ showing that g_j converges to f in measure. Indeed,

$$\mu(\{|f-g_s| > 2 \cdot 2^{-s}\}) \leq \mu(F_s) \longrightarrow 0.$$

$$\Rightarrow g_s \xrightarrow{\text{measure}} f. \qquad \blacksquare$$

LECTURE 11

18/10/2023

EGOROFF'S THEOREM

Def: A sequence $\{f_n\}$ of measurable functions converge **almost uniformly** to f if $\forall \varepsilon > 0$ $\exists E \in \mathcal{M}$ with $\mu(E) < \varepsilon$ such that $f_n \longrightarrow f$ uniformly on E^c

IMPORTANT

Crucial assumption

Thm: (Egoroff) If $\boxed{\mu(X) < \infty}$, any a.e. converging sequence of measurable functions $\{f_n\}$ converges almost uniformly.

Pf: WLOG, assume $f_n \to f$ $\forall x \in X$.

Fix $\ell, n \in \mathbb{N}$ and consider

$$E_n(\ell) := \bigcup_{m \geq n} \left\{ x : |f_m(x) - f(x)| \geq \frac{1}{\ell} \right\}$$

For fixed ℓ, $E_n(\ell) \searrow \emptyset$ as $n \nearrow \infty$. That is:

$$\bigcap E_n(\ell) = \emptyset$$

Continuity from above (using that $\mu(X) < \infty$), we have that $\mu(E_n(\ell)) \searrow 0$. Fix $\varepsilon > 0$, then $\exists n_\ell$ s.t.

$$\mu(E_{n_\ell}(\ell)) < \varepsilon 2^{-\ell}.$$

Just let $E = \bigcup_\ell E_{n_\ell}$ so that $\mu(E) < \varepsilon$.

Now, if $x \in E^c$, then $\forall \ell$ and $\forall n \geq n_\ell$, we have

$$|f_n(x) - f(x)| < \frac{1}{\ell}.$$

$\Rightarrow$ Uniform convergence on E^c. ■

Philosophy: In measure theory, show that something is true on a "big set" and perhaps false somewhere (just make sure this "somewhere" is small enough).

* **PRODUCT MEASURES** (towards Fubini)

MONOTONE CLASS LEMMA: A collection $\mathcal{C} \subset P(X)$ is a monotone class if

- $\forall E_n \subset E_{n+1}$, $\bigcup E_n \in C$ (closed under countable increasing unions)
- $\forall F_n \supset F_{n+1}$, $\bigcap F_n \in C$ (closed under countable decreasing intersections)

Just as for σ-algebras, for any $\mathcal{E} \subset P(X)$, denote by $\langle \mathcal{E} \rangle_c$ the monotone class generated by $\mathcal{E}$.

<u>**Lemma**</u>: (Monotone Class) Let $\mathcal{A}$ be an algebra on X, then $\langle \mathcal{A} \rangle_c = \langle \mathcal{A} \rangle_\sigma$ ← σ-algebra .

Pf: Since σ-algebras are monotone classes, trivial to see that $\langle \mathcal{A} \rangle_c \subset \langle \mathcal{A} \rangle_\sigma$.

Conversely, need to show that $\langle \mathcal{A} \rangle_c$ <u>is</u> a σ-algebra.

Let $\langle \mathcal{A} \rangle_c =: \mathcal{C}$ for short, for $B \in \mathcal{C}$, define

$$\mathcal{C}(B) = \{ B' \in \mathcal{C} : B' \setminus B,\ B \setminus B',\ B \cap B' \in \mathcal{C} \}.$$

Note

- $\emptyset \in \mathcal{C}(B)$.

- $B' \in \mathcal{C}(B) \iff B \in \mathcal{C}(B')$

- $\mathcal{C}(B)$ is a monotone class.

- $\forall E \in \mathcal{A}$, $\mathcal{A} \subset \mathcal{C}(E)$. This implies that

$$\langle \mathcal{A} \rangle_{\mathcal{C}} = \mathcal{C} \subset \mathcal{C}(E).$$

smallest monotone class containing $\mathcal{A}$

- $\forall B \in \mathcal{C}$, $\mathcal{A} \subset \mathcal{C}(B)$ (by the previous bullet). So, if $E, F \in \mathcal{C}$, then

$$E \setminus F \text{ and } E \cap F \text{ are also in } \mathcal{C}.$$

$\Rightarrow \mathcal{C}$ is an <u>algebra</u>.

Finally, since $\mathcal{C}$ is a monotone class <u>and</u> an algebra by the above, it follows directly that $\mathcal{C}$ is a σ-algebra.

∎

LECTURE 12

FUBINI - TONELLI

23/10/2023

GOAL: Prove Fubini - Tonelli. For that, need to define product measures and a theory of integration on this product space.

Recall: For two measure spaces $(X, \mathcal{M}, \mu)$
$(Y, \mathcal{N}, \nu)$
we define the product σ-algebra:

$$\mathcal{M} \otimes \mathcal{N} = \langle \{ \underset{\mathcal{M}}{E \in} \times \underset{\mathcal{N}}{F \in} \} \rangle .$$

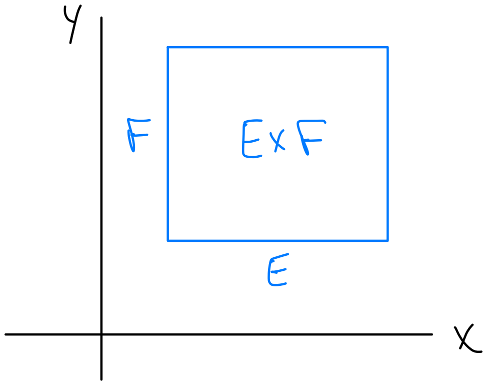
So, we have a measurable product space

$$(X \times Y, \mathcal{M} \otimes \mathcal{N}, \ ?)$$

Monotone Class Lemma: $\mathcal{A}$ algebra, then

$$\langle \mathcal{A} \rangle_\sigma = \langle \mathcal{A} \rangle_c$$

← smallest monotone class containing $\mathcal{A}$.



$$(\mu\times\nu)(E\times F) \overset{?}{=} \mu(E)\nu(E)$$

1) Rectangles form an elementary family.

$$\Downarrow$$

Disjoint unions of rectangles form an algebra $\mathcal{A}$

$$\mathcal{A} = \left\{ \bigsqcup_{i=1}^{N} E_i \times F_i \ : \ E_i \in \mathcal{M},\ F_i \in \mathcal{N} \right\}.$$

Define for $A \in \mathcal{A}$

$$(\mu\times\nu)(A) := \sum_{i=1}^{N} \mu(E_i)\nu(E_i).$$

<u>Claim 1</u>: $(\mu\times\nu)$ is a premeasure on $\mathcal{A}$.

So that we can apply Hahn-Kolmogorov to get a measure λ on $\langle\mathcal{A}\rangle$ s.t. $\lambda|_{\mathcal{A}} = (\mu\times\nu)$

Pf: Clearly $(\mu\times\nu)(\emptyset)=0$.

(WTS: conditional σ-additivity) Let

$$E\times F=\bigsqcup_{i=1}^{\infty} E_i\times F_i \in \mathcal{A}.$$

Then, $\chi_{E\times F}(x,y)=\chi_E(x)\,\chi_F(y)$ all disjoint

All disjoint $$=\sum_{i=1}^{\infty}\chi_{E_i}(x)\,\chi_{F_i}(y)$$

Now, integrate both sides w.r.t. μ

$$\text{LHS}=\int \chi_{E\times F}(x,y)\,d\mu=\int \chi_E(x)\,\chi_F(y)\,d\mu$$

$$=\mu(E)\,\chi_F(y)$$

On the other hand,

$$\text{RHS}=\int\sum_{i=1}^{\infty}\chi_{E_i}(x)\,\chi_{F_i}(y)\,d\mu$$

Monotone Conv. Thm $$=\sum_{i=1}^{\infty}\int\chi_{E_i}(x)\,d\mu\;\chi_{F_i}(y)$$

$$= \sum_{i=1}^{\infty} \mu(E_i)\, \chi_{F_i}(y)$$

Integrate w.r.t. to ν now:

$$\text{LHS} = \int \chi_F(y)\, \mu(E)\, d\nu = \mu(E)\, \nu(F)$$

$$\text{RHS} = \int \sum_{i=1}^{\infty} \mu(E_i)\, \chi_{F_i}(y)\, d\nu$$

Monotone Convergence Thm $= \displaystyle\sum_{i=1}^{\infty} \mu(E_i)\, \nu(F_i).$

$\blacksquare$

Thus, apply Hahn-Kolmogorov to get a full product measure space

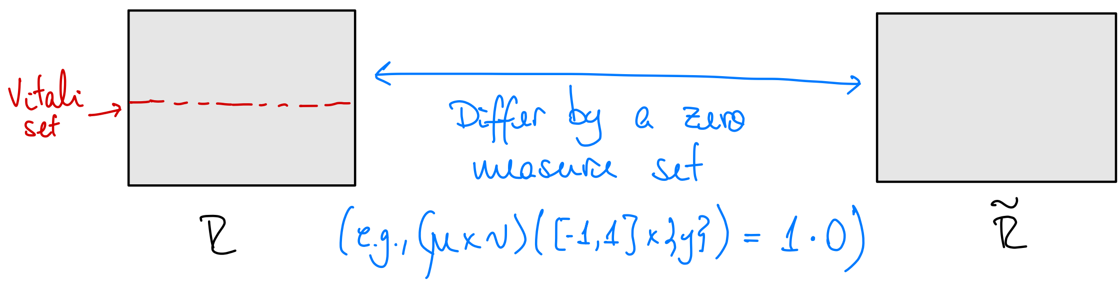
Hahn-Kolmogorov

$$(X \times Y,\ \mathcal{M} \otimes \mathcal{N},\ \mu \times \nu)$$

NOTE: $\mu \times \nu$ is uniquely defined if $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite.

Remarks about $\mu \times \nu$:

(1) $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ is not necessarily complete. Embed a Vitali set in a rectangle



Very unlikely that the product measure is complete by the following proposition:

Prop: (i) If $E \in \mathcal{M} \otimes \mathcal{N}$, $\boxed{\begin{array}{ll} \forall x \in X & E_x \in \mathcal{N} \\ \forall y \in Y & E^y \in \mathcal{M} \end{array}}$ $(*)$

(ii) If f $(\mathcal{M} \otimes \mathcal{N})$-measurable, then

$f(x,y) \leftarrow y : f_x$ is $\mathcal{N}$-measurable $\forall x \in X$

$f(x,y) \leftarrow x : f^y$ is $\mathcal{M}$-measurable $\forall y \in Y$

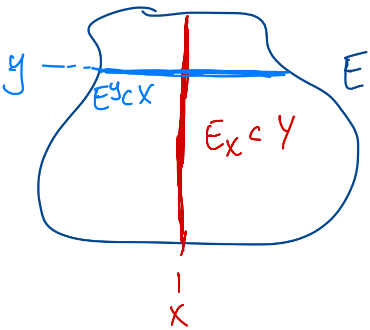
Pf: (i) Let $\mathcal{R}$ be the collection of subsets $E \subset X \times Y$ s.t. $(*)$ holds.

$\underline{\text{WTS}}$: $\left.\begin{array}{l}\text{(a) } \mathcal{R} \text{ contains rectangles} \\ \text{(b) } \mathcal{R} \text{ is a } \sigma\text{-algebra}\end{array}\right\} \Rightarrow \mathcal{M}\otimes\mathcal{N} \subset \mathcal{R}$

Now, (a) holds trivially:

slice

$$E = A \times B \rightsquigarrow E_x = \begin{cases} B, & \text{if } x \in A \\ \emptyset, & \text{otherwise} \end{cases}$$



$$E^y = \begin{cases} A, & \text{if } y \in B \\ \emptyset, & \text{otherwise} \end{cases}.$$

(b): $\mathcal{R}$ is a σ-algebra if $E \in \mathcal{R} \Rightarrow E^c \in \mathcal{R}$

$$\forall x \in X, \quad (E^c)_x = \{ y \in Y : \underbrace{(x,y) \in E^c}_{(x,y)\notin E \Rightarrow y \notin E_x \Rightarrow y \in (E_x)^c} \}$$

$$\subset$$

$$(E_x)^c$$

etc etc and so on.

(ii) f_x is $\mathcal{N}$-measurable $\Rightarrow f_x^{-1}(A) \in \mathcal{N} \;\; \forall A \in \mathcal{M}\otimes\mathcal{N}$.

Observe that $(f_x)^{-1}(A) = \underbrace{[f^{-1}(A)]_x}_{} \in \mathcal{N}$

$\in \mathcal{M} \otimes \mathcal{N}$

(2) This construction of the product measure can be done for finite products:

$$\left(X_1 \times X_2 \times \cdots \times X_n, \right.$$

$$\mathcal{M}_1 \otimes \mathcal{M}_2 \otimes \cdots \otimes \mathcal{M}_n,$$

$$\left. \mu_1 \times \mu_2 \times \cdots \times \mu_n \right)$$

Ex: Take all X_i above to be $(\mathbb{R}, \mathcal{L}, m)$

Not the Lebesgue measure on $\mathbb{R}^n$, need to complete it to get the Lebesgue measure on $\mathbb{R}^n$. $\longrightarrow (\mathbb{R}^n, \mathcal{L}^{(n)}, m^n)$

!!

Thm: (FUBINI – TONELLI) Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be σ-finite measure spaces, crucial, otherwise theorem is false.

(1) (Tonelli) If $f \in L^+(X \times Y)$, then

$$g(x) := \int f_x \, d\nu \quad \text{and} \quad h(y) := \int f^y \, d\mu$$

are in $L^+(X)$ and $L^+(Y)$, respectively. Moreover,

$$\underbrace{\int f \, d(\mu \times \nu)}_{(*)} = \int \left(\int f(x,y) \, d\nu \right) d\mu = \int \left(\int f(x,y) \, d\mu \right) d\nu$$

Note that these can be infinite

(2) (Fubini) If $f \in L^1(\mu \times \nu)$, then for a.e. x $f_x \in L^1(\nu)$ and for a.e. y $f^y \in L^1(\mu)$, and the equality $(*)$ above holds.

Thm: (Baby Tonelli) Tonelli above but for indicator fcts. Suppose $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ are σ-finite. Let $E \in \mathcal{M} \otimes \mathcal{N}$, then

$$x \xmapsto{g} \nu(E_x) \quad \text{and} \quad y \xmapsto{h} \mu(E^y)$$

are measurable and

$$(\mu\times\nu)(E) = \int \underbrace{\mu(E^y)}_{h(y)}\, d\nu = \int \underbrace{\nu(E_x)}_{g(x)}\, d\mu .$$

Pf: (Tonelli)

! STRATEGY: To show things about σ-finite spaces, first prove result for finite measure spaces and then approximate the whole space by finite sets (usually need MCT for this last step).

Suppose first that X, Y are finite measure spaces.

Let $\mathcal{C} \subset \mathcal{M}\otimes\mathcal{N}$ be the collection of E s.t. the statement holds.

WTS: $\mathcal{C}$ is a monotone class and contains disjoint unions of rectangles.

$\mathcal{M}\otimes\mathcal{N} \subset \mathcal{C}$ by MCL $\Leftarrow$

Claim: $E := \underset{\substack{\cap\\ \mathcal{M}}}{A} \times \underset{\substack{\cap\\ \mathcal{N}}}{B} \in \mathcal{C}$ and $\nu(E_x) = \nu(B)\chi_A(x)$

$\mu(E^y) = \mu(A)\chi_B(y)$

$\rightsquigarrow \mu(A)\nu(B) = (\mu\times\nu)(E) = \int \mu(A)\chi_B(y)\, d\nu$

$$= \int \nu(B) \chi_A(x) \, d\mu$$

$\Rightarrow$ $\mathcal{C}$ contains rectangles.

<u>Claim</u>: $\mathcal{C}$ is a monotone class

<u>Pf</u>: Let $\{E_n\} \subset \mathcal{C}$ be an increasing sequence of sets (i.e., $E_n \subset E_{n+1}$) and set $E := \bigcup_{n \in \mathbb{N}} E_n$. Let

$$h_n(y) := \mu((E_n)^y), \quad g_n(x) := \nu((E_n)_x).$$

Note that h_n is increasing & measurable. Say $h_n \xrightarrow{\text{ptwise}} h$. So, h is measurable. By MCT,

$$\int h \, d\nu \overset{\text{MCT}}{=} \lim \int h_n \, d\nu$$

$$= \lim \int \mu((E_n)^y) \, d\nu$$

$$= \lim (\mu \times \nu)(E_n)$$

Continuity from below

$$= (\mu \times \nu)(E).$$

Do the same for g_n. So, $\mathcal{C}$ is closed under increasing unions.

For decreasing intersections, we use LDCT with the bound function given by the measure of the whole space (since it's finite): $g(x) \leq \nu(Y) < \infty$

$$\Rightarrow g(x) \text{ is integrable b/c } \mu(X) < \infty$$

$\leadsto$ Use a similar argument as above w/ LDCT and measure continuity from above.

$\square$

PROOF OF TONELLI: By the above,

Tonellino $\Rightarrow$ Tonelli for indicator fcts. $f = \chi_E$.

For general fcts., approximate by simple fcts.: let $\phi_n \nearrow f$. Set

simple

$$g_n(x) := \int (\phi_n)_x \, d\nu, \quad h_n(y) := \int (\phi_n)^y \, d\mu.$$

Then, by MCT, $g_n \nearrow g$ and $h_n \nearrow h$ ptwise; hence as the ptwise limit of measurable is measurable, we have that h and g are measurable and the equalities of switching integrals holds.

Tonelli → $\square$

LECTURE 13

25/10/2023

L^p SPACES

Recall: $L^1(\mu) \ni f \longmapsto \int |f| \, d\mu$ is a norm.

LINEAR SPACE: X on K (either $\mathbb{R}$ or $\mathbb{C}$)

$$X \ni v, w \rightsquigarrow v + w \in X$$

$$K \ni \lambda \rightsquigarrow \lambda v \in X$$

Def: (SEMI-NORM) Function $p : X \to [0, \infty]$ s.t.

- homogeneous ($p(x) \geq 0 \ \forall x$)
- triangle inequality ($p(x+y) \leq p(x) + p(y)$)

If $p(x) = 0 \iff x = 0$, then p is a norm. V.I

Def: $(X, \|\cdot\|)$ is a normed space is called **BANACH** if it is complete w.r.t. the metric induced by $\|\cdot\|$.

$d(x,y) = \|x-y\|$

Prop:

$(X, \|\cdot\|)$ is Banach $\iff$ Every absolutely convergent series converges.

Recall that we showed that in L^1 every abs. convergent series converges $\Rightarrow (L^1, \|\cdot\|_1)$ is Banach!

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach, then $(X \times Y, \|\cdot\|_{X\times Y})$ is Banach.

e.g., $\|(x,y)\|_{X\times Y} = \max(\|x\|_X, \|y\|_Y)$

LINEAR OPERATOR: $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$

Denote the space of linear operators $T: X \to Y$ as

$$\{T: X \to Y\} =: \mathcal{L}(X, Y).$$

Def: (BOUNDED OPERATOR) $T \in \mathcal{L}(X,Y)$ is <u>bounded</u> if $\exists\, C \geq 0$ such that

$$\|Tx\|_Y \leq C \|x\|_X \quad \forall x \in X.$$

Equivalently, T is bounded iff

$$\sup_{\|x\| \leq 1} \|Tx\|_Y < \infty$$

$$\iff \sup_{\|x\| = 1} \|Tx\|_Y < \infty$$

$$\iff \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X} < \infty$$

Prop: TFAE

(i) T ← linear is continuous w.r.t. $d_X(x,x') := \|x - x'\|_X$ and $d_Y(y,y') = \|y - y'\|_Y$.

(ii) T ← linear is continuous at 0.

(iii) T ← linear is bounded.

Def: The graph of $T: X \to Y$ is

$$X \times Y \supset \Gamma_T := \{(x,y) \in X \times Y : y = Tx\}$$

NOTE: $\mathcal{L}(X,Y)$ is itself a linear space.

Def: (OPERATOR NORM) On the space $\mathcal{L}(X,Y)$ can define the operator norm $\|\cdot\|_{\mathcal{L}(X,Y)}$ as follows

$$\|T\| = \sup_{x \in X} \frac{\|Tx\|_Y}{\|x\|_X} .$$

$\Leftrightarrow \sup\limits_{\|x\| \leq 1} \|Tx\|_Y$.

or $\|x\| = 1$

Lemma: On the space of bounded operators $\mathcal{B}(X,Y) \subset \mathcal{L}(X,Y)$, $\|\cdot\|_{op}$ is a norm.

Prop: If $(Y, \|\cdot\|_Y)$ is Banach, then the space of bounded operators w/ op. norm $(\mathcal{B}(X,Y), \|\cdot\|)$ is Banach.

Pf: If T_n is Cauchy, then $T_n \to T \in \mathcal{B}(X,Y)$

$\exists\, T_{n_j}$ s.t. $\| T_{n_{j+1}} - T_{n_j} \| < 2^{-j}$

$$\forall x \in X \qquad T_{n_0} \overset{(*)}{+} \sum_{j=0}^{N-1} \underbrace{(T_{n_{j+1}} - T_{n_j})}_{= T_{n_j}} x \quad \text{is absolutely}$$

convergent in Y. $\Longrightarrow$ We can define $Tx := \lim\limits_{N\to\infty}$ (*)

$$\rightsquigarrow \quad \|T\| = \lim_{n\to\infty} \|T_n\| .$$

∎

Corollary: Given X, consider $\mathcal{B}(X, \mathbb{K})$ (space of bdd op. onto the field; i.e., bdd linear functionals). Then, $\int \in \mathcal{B}(L^1, \mathbb{K})$.

either $\mathbb{R}$ or $\mathbb{C}$

Pf: For $f \in L^1$, $\quad f \mapsto Tf := \int f \, d\mu$ ← clearly linear

$$\|Tf\| \le \int |f| \, d\mu = \|f\|_1 < \infty$$

$f \in L^1$

$$\Rightarrow \|T\| = 1 \quad \rightsquigarrow \int \text{ is bounded op. on } L^1$$

□

Always BANACH

Def: Given X, $B(X, \mathbb{K}) = X^*$ is the space of <u>bounded</u> <u>linear</u> <u>functionals</u> and it is called the <u>dual</u> <u>space</u> of X.

Q: But is X^* non trivial ?

A: Hahn – Banach ! VERY IMPORTANT

Def: (SUBLINEAR FUNCTIONAL) Let X be a vec. space, then $p: X \to \mathbb{R}$ is a <u>sublinear</u> <u>functional</u> on X if

sublinear

- $p(x + x') \leq p(x) + p(x') \quad \forall x, x' \in X$
- $p(\lambda x) = \lambda p(x) \quad \forall x \in X \ \forall \lambda \in \mathbb{R}$.

VERY IMPORTANT !!

VERY IMPORTANT AND HYPERPOWERFUL!

Thm: (HAHN – BANACH) Let X be a $\mathbb{R}$-vector space and let p be a sublinear functional on X. If $M \subset X$ is a subspace and f is a linear functional on M s.t. $f(x) \le p(x)$ $\forall x \in M$, then $\exists F: X \to \mathbb{R}$ linear functional s.t.

$$F(x) \le p(x) \quad \forall x \in X \quad \text{and} \quad F|_M = f.$$

LECTURE 14

HAHN – BANACH

01/11/2023

Thm: (HAHN – BANACH) Let X be a $\mathbb{R}$-vector space and $p: X \to \mathbb{R}$ a sublinear functional on X.

$$\left.\begin{array}{l} p(x+y) \le p(x) + p(y) \quad \forall x, y \in X \\ p(\lambda x) = \lambda p(x) \quad \forall x \in X, \ \lambda \ge 0 \end{array}\right\} \text{e.g., seminorm}$$

Let $\ell: M \subset X \to \mathbb{R}$ be a linear functional s.t.

$$\ell(x) \le p(x) \quad \forall x \in X.$$

Then, $\exists \tilde{\ell}: X \to \mathbb{R}$ linear functional on X s.t.

$$\tilde{\ell}(x) \le p(x) \quad \forall x \in X, \quad \tilde{\ell}|_M = \ell. \qquad (*)$$

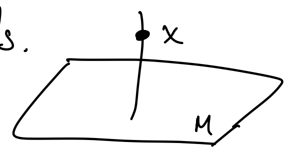
Idea of proof: extend ℓ to some strictly bigger subspace $M' \supset M$ s.t. (*) holds.

ZORN'S LEMMA: Let S be a partially ordered set s.t. every totally ordered subset has an upper bound (i.e., if $S' \subset S$ totally ordered, then $\exists y \in S$ s.t. $y \geq x \ \forall x \in S'$). Then, there exists a maximal element in S (i.e., $\exists \tilde{y} \in S$ s.t. if $y \geq \tilde{y} \Rightarrow y = \tilde{y}$).

NEED: Use Zorn's Lemma to show Hahn-Banach.

Pf: (Hahn-Banach)

Claim 1: Let $M \subsetneq X$ be a subspace of X. Let $x \in X \setminus M$. Then, there exists an extension ℓ' of M to $M \oplus \mathbb{R}x =: M'$ s.t. (*) holds.



Assume this claim to be true.

Let

$$S = \left\{ \begin{array}{l} \text{family of all functionals defined} \\ \text{on some subspace } N \supset M \text{ which} \\ \text{satisfy (*)} \end{array} \right\}$$

Recall: given $l \in S$, we can define the graph of l:

$$\Gamma_l = \{ (x, l(x)) : x \in N \} \subset N \times \mathbb{R} \subset X \times \mathbb{R}.$$

Define the partial order $\leq$:

$$l_1 \leq l_2 \overset{\text{def}}{\iff} \Gamma_{l_1} \subset \Gamma_{l_2}$$

<u>WTS</u>: there is an upper bound in S.

But this only corresponds to an increasing seq. of subspaces of X: $\bigcup_\alpha \Gamma_\alpha$ ← May be uncountable

→ upper bound of S

$\Rightarrow$ Can apply Zorn's lemma.

<u>Pf</u>: (Claim 1) Let $x \in X \setminus M$. Want to define $\tilde{l}$ on $M \oplus \mathbb{R}x$.

Note: if $y_1, y_2 \in M$,

$$\begin{aligned} l(y_1) + l(y_2) = l(y_1 + y_2) &\leq p(y_1 + y_2) \\ &\leq p(y_1 + x - x + y_2) \\ &\leq p(y_1 - x) + p(x + y_2) \end{aligned}$$

$$\Rightarrow l(y_1) - p(y_1 - x) \leq p(x + y_2) - l(y_2) \quad \forall x \in X, \ \forall y_1, y_2 \in M$$

Arbitrage

$\Longrightarrow \forall x \in X, \ \sup_{y \in M} \left(\ell(y) - p(y-x) \right) \leq \alpha \leq \inf_{y \in M} \left(p(x+y) - \ell(y) \right)$

Let $\alpha \in \mathbb{R}$ be s.t.

Set $\tilde{\ell}(\overset{\in M \oplus \mathbb{R}x}{z = y + \lambda x}) := \ell(y) + \lambda \alpha$

- linear functional ✓
- $\tilde{\ell}|_M = \ell$ ✓
- $\tilde{\ell}(x) \leq p(x) \quad \forall x \in X$.
  - if $\lambda = 0$, clearly holds.
  - if $\lambda > 0$, then

$$\tilde{\ell}(y+\lambda x) = \lambda \left(\frac{\ell(y)}{\lambda} + \alpha \right)$$

$$\leq \lambda \left[\cancel{\ell\left(\frac{y}{\lambda}\right)} + p\left(x + \frac{y}{\lambda}\right) - \cancel{\ell\left(\frac{y}{\lambda}\right)} \right]$$

$$\leq \lambda \, p\left(x + \frac{y}{\lambda}\right) = p(y + \lambda x)$$

  - if $\lambda < 0$, perform same computations to get

$\tilde{\ell}(y+\lambda x) = \dots \leq p(y+\lambda x)$.

■

Has stronger assumptions

Thm: ($\mathbb{C}$-Hahn-Banach) X is $\mathbb{C}$ a vector space and p is a seminorm, then the same conclusions of the $\mathbb{R}$-version hold.... ($|\ell(x)| \leq p(x)$)

! HAHN-BANACH $\Rightarrow$ Non-triviality of the dual of any normed space.

Prop: Let $(X, \|\cdot\|)$ be a normed vector space. Then

(i) if $M \subset X$ is a closed subspace, $x \in X \setminus M$, $\exists\, \ell \in X^*$ s.t. $\ell|_M = 0$ and $\ell(x) \neq 0$ (in fact, let $\delta := \inf_{y \in M} \|x - y\|$, then $\exists\, \ell$, $\|\ell\| = 1$, $\ell(x) = \delta$)

(ii) if $x \neq 0$, $\exists\, \ell \in X^*$ s.t. $\|\ell\| = 1$, $\ell(x) = \|x\|$

(iii) $\forall x, y \in X$ $\exists\, \ell \in X^*$ s.t. $\ell(x) \neq \ell(y)$

(iv) if $x \in X$, let $\hat{x} : X^* \to \mathbb{C}$, $\hat{x}(\ell) := \ell(x)$, then $x \mapsto \hat{x}$ is a linear isometry from $X \to X^{**}$

$$\|\hat{x} - \hat{y}\|_{X^{**}} = \|x - y\|_X$$

↳ Hahn-Banach $\Rightarrow$ $\exists$ enough functionals to separate pts. of X. So, we can embed X in X^{**} (obs: X^{**} can be bigger than X).

Always Banach (every dual space is complete)

Let $\hat{X} = \{ \hat{x} : x \in X \} \subset X^{**}$ (usually these are not the same). Let $\overline{\hat{X}} \subset X^{**}$. Then $\overline{\hat{X}}$ is called the COMPLETION of X.

- If X is Banach, then $\overline{\hat{X}} = \hat{X}$, but still $\overline{\hat{X}} \subset X^{**}$ (equality holds when X finitely-dim. but not necessarily for ∞-dim. spaces, e.g., L^1).

→ If X Banach is s.t. $\hat{X} = X^{**}$, then X is called REFLEXIVE (e.g., L^2).

Lecture 15 — Banach Spaces

13/11/2023

Topology of Metric Spaces:

Def: Let X be a metric space.

- $S \subset X$ is <u>nowhere dense</u> if $\bar{S}$ does not contain any ball in X
- $S \subset X$ is <u>meager</u> if it is the countable union of nowhere dense sets.

meager sets were called 1^{st}-category...

<u>Baire Category Theorem</u>: If X is a <u>complete</u> metric space and $S \subset X$ is meager, then S does not contain any balls.

<u>Obs</u>: Meager sets are the topological equivalents of null sets. But note that there are no relations between these notions.

<u>Thm</u>: (Uniform Boundedness Principle) Let X, Y be vector spaces and $A \subset B(X,Y)$. ← bounded linear operators

(i) if $\exists S \subset X$ non-meager and s.t.

$$\sup_{T\in A} \|Tx\| < \infty \quad \forall x \in S,$$

then $\sup_{T\in A} \|T\| < \infty$.

(ii) if X is Banach and $\sup_{T\in A} \|Tx\| < \infty \quad \forall x \in X$, then

$$\sup_{T\in A} \|T\| < \infty .$$

Pf: (i) $\xRightarrow{\text{Baire}}$ (ii). So, prove (i)

<u>Proof of (i)</u>: Consider

$$E_n = \{ x \in X : \sup_{T\in A} \|Tx\| \le n \}$$

$$= \bigcap_{T\in A} \{ x \in X : \underbrace{\|Tx\| \le n}_{= T^{-1}\left(\overline{B_0(n)}\right)} \}$$

So the E_n's are closed. Thus, $\bigcup_n E_n \supset \underbrace{S}_{\text{non-meager}}$.

Thus, $\exists n$ s.t. E_n is $\underbrace{\text{somewhere dense}}_{\text{(not nowhere dense)}}$. So, this set contains some ball $\overline{B_{r_0}(x_0)} \subset E_n$.

Thus, by assumption, $\forall T \in A$,

$$T\left(\overline{B_{r_0}(x_0)}\right) \subset \overline{B_0(n)} = \{y \in Y : \|y\| \leq n\}.$$

WTS: $\sup_{T \in A} \|T\| < \infty$.

Claim: $\overline{B_0(r_0)} \subset E_{2n}$.
$$\left(\begin{array}{l} \Rightarrow T\left(\overline{B_0(r_0)}\right) \subset \overline{B(0, 2n)} \\ \Rightarrow T\left(\overline{B_0(1)}\right) \subset \overline{B(0, 2n/r_0)} \end{array}\right)$$

Indeed, let $\|x\| < r_0$ and $\forall T \in A$

$$\|Tx\| = \|T(x + x_0 - x_0)\| \leq \|\underbrace{T(x+x_0)}_{\in B_{x_0}(r_0) \subset E_n}\| + \|T(x_0)\|$$

$$\leq n + n \leq 2n.$$

$\square$

WHY DO WE LIKE BANACH? For example, consider the space

$$c_{00} = \{(a_n) \subset \mathbb{C} : \text{only finitely many elements are nonzero}\}$$

with $\|a_n\| := \sup_{n \in \mathbb{N}} |a_n|$. Then $(c_{00}, \|\cdot\|)$ is NOT COMPLETE.

Consider a sequence (of seqs.)

$$\left.\begin{aligned} a_n^1 &= (1, 0, \text{---}) \\ a_n^2 &= (1, \tfrac{1}{2}, 0, \text{---}) \\ a_n^3 &= (1, 1/2, 1/4, 0, \text{---}) \end{aligned}\right\} \begin{aligned} &\text{Cauchy in } (c_{00}, \|\cdot\|) \text{ but} \\ &a_n^\infty = (1, \tfrac{1}{2}, \tfrac{1}{4}, \text{---}) \notin c_{00}. \end{aligned}$$

$$\Downarrow$$

c_{00} is <u>not</u> Banach

Consider the operator $T: c_{00} \to c_{00}$ such that

$$\big(T(a_n)\big)_n := \left(\frac{1}{n} a_n\right)_n$$

so that $T(a_n^1) = a_n^1$

$$T(a_n^2) = (1, 1/4, 0, \text{---})$$

$$\vdots$$

Thus, $\ker T = \{0\}$ and $\underbrace{\operatorname{im} T = c_{00}}$.

$$\big(T^{-1}(a_n)\big)_n = (n a_n)_n$$

Moreover,

$$\|Ta_n\| = \sup \left|\frac{1}{n} a_n\right| \leq \sup |a_n| = \|a_n\| < \infty.$$

So, T is bounded. HOWEVER, T^{-1} is <u>UNBOUNDED</u>!

Problem: c_{00} not Banach.

OPEN MAPPING THEOREM: Let X, Y be Banach spaces. If $T \in B(X,Y)$ is surjective, then T is open

(i.e., images of open are open)
(continuous $\rightarrow$ preimages of open are open)

Corollary: If $T \in B(X,Y)$ is bijective, then T is open; i.e., T^{-1} is continuous, i.e., T^{-1} is bounded.

REMARK: T is open if (and only if) $T(B_1(0))$ contains a ball centred at zero.

Pf: (Open Mapping theorem) Write

$$X = \bigcup_{n=1}^{\infty} B_n(0).$$

Since T is surjective, $Y = \bigcup_{n=1}^{\infty} T(B_n(0))$. But, Y is Banach (and, so, it cannot be meager); i.e., by Baire's Category Thm, $\exists n$ s.t. $T(B_n(0))$ is somewhere dense.

By linearity, $T(B_n(0)) = n\,T(B_1(0))$.
So, $T(B_1(0))$ is somewhere dense. So, $\exists y_0 \in Y$ s.t.

$\overline{T(B_1(0))} \supset B_{r_0}(y_0)$, $\quad r_0 > 0$.

First, recover a ball centered at zero. Let

$$y_1 = Tx_1 \quad \text{s.t.} \quad \|y_1 - y_0\| < r_0/2.$$

Then, by Δ-ineq., $B_{r_0/2}(y_1) \subset \overline{T(B_1(0))}$. Thus, if $\|y\| < r_0/2$, then

$$\|y\| \le \|\underbrace{y + y_0}_{\in \overline{T(B_1(0))}}\| + \|\underbrace{y_0}_{\in \overline{T(B_1(0))}}\| \Longrightarrow y \in \overline{T(B_2(0))}$$

So, $\overline{T(B_2(0))} \supset B_{r_0/2}(0) \Rightarrow \overline{T(B_1(0))} \supset B_{r_0/4}(0)$.

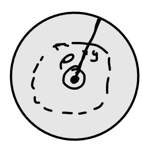
Now, need to show that the <u>actual</u> image of the ball contains a ball; i.e., $T(B_1(0)) \supset B_{r'}(0)$ for some r'!

Let $\|y\| < \frac{\rho}{2}$ where $\rho = \frac{r_0}{4}$ so that

$$\overline{T(B_{2^{-n}}(0))} \supset B_{\rho 2^{-n}}.$$

<u>WTS</u>: $\exists x \in B_1(0) \subset X$ s.t. $Tx = y$, $\|y\| < \frac{\rho}{2}$.

(then we'd have $T(B_1(0)) \supset B_{\rho/2}(0)$).



But, by construction,

$\exists x_1 \in B_{1/2}(0)$ s.t. $\|y - Tx_1\| < \rho/4$

iterate this

$\exists x_2 \in B_{1/4}(0)$ s.t. $\|(y - Tx_1) - Tx_2\| < \rho/8$

$\vdots$

$\exists x_m \in B_{2^{-m}}(0)$ s.t. $\left\| y - \sum_{j=1}^{m} Tx_j \right\| < \rho\, 2^{-m-1}$

Then, $\|x_m\| < 2^{-m}$. So, since X is Banach,

$$\sum_{j=1}^{N} x_j \quad \text{converges to } x \text{ in } B_1(0), \text{ s.t.}$$

$$\|x\| \leq \sum_j \|x_j\| < \sum 2^{-j} < 1 \Rightarrow Tx = y.$$

$\square$

<u>Recall</u>: Given $T \in \mathcal{L}(X,Y)$, $\Gamma_T = \{(x,y) \in X \times Y : y = Tx\}$.

<u>Def</u>: T is closed iff Γ_T is closed in $X \times Y$.

not necessarily bounded

<u>CLOSED GRAPH THEOREM</u>: If X, Y are Banach and $T \in \mathcal{L}(X,Y)$, then

$$T \text{ closed} \Rightarrow T \text{ bounded}.$$

(very useful for showing ops. are bounded...)

Pf: Let π_X, π_Y be the projections onto X, Y, respectively.
Note

$$\pi_X \in \mathcal{B}(X\times Y, X), \quad \pi_Y \in \mathcal{B}(X\times Y, Y)$$ Both π_X, π_Y are linear & bounded

$\| \|_{X\times Y} = \max(\| \|_X, \| \|_Y)$

Since both X, Y are Banach, so is $X\times Y$. Since Γ_T is a closed subspace of Banach, it is also Banach.

Note that $\pi_X : \Gamma_T \to X$ is bijective, hence bounded.

By the Open Mapping Theorem, π_X^{-1} is bounded. Thus,

$$T = \pi_Y \circ \pi_X^{-1} \in \mathcal{B}(X, Y).$$

$\square$

Pf: (Baire Category Theorem) Note that if E is nowhere dense, then $(\overline{E})^c$ is dense and open. By assumption, $S = \bigcup_n E_n$. Assume, by contradiction, that there exists

nowhere dense

a ball $B_{r_0}(x_0) \subset S$. But, E_1 is nowhere dense, hence $(\overline{E_1})^c$ is dense, hence $\exists x_1 \in B_{r_0/10}(x_0)$ and $\exists r_1 > 0$ s.t. $B_{r_1}(x_1) \subset (\overline{E_1})^c$. Assume further that $B_{r_1}(x_1) \subset B_{r_0/10}(x_0)$.

Note $B_{r_1}(x_1)$ is disjoint from E_1.

Iterate: $\exists x_2, r_2$ s.t. $B_{r_2}(x_2) \subset B_{r_1/10}(x_1)$ disjoint from E_2

$\vdots$

Consider a seq. x_m, r_m s.t. $r_m \leq \frac{r_{m-1}}{10}$ and

$$d(x_m, x_{m+\ell}) \leq 2 \cdot \frac{r_{m-1}}{10}.$$

Then (x_m) is Cauchy, hence converges b/c X complete; say $x_m \to x \in X$. Then, $x \in B_{r_0}(x_0)$, so $B_{r_0}(x_0)$ is disjoint from every E_n.

$\square$

Recall: X^* is the dual of X (i.e., space of bdd linear functionals on X). Also, X^{**} double dual of X and we have a natural (isometric) inclusion $X \subset X^{**}$.

Def: $x_n \xrightarrow{\text{weakly}} x$ iff $\forall \ell \in X^* \ \ell(x_n) \longrightarrow \ell(x)$.

weaker

Def: $x_n \xrightarrow{S\text{-weakly}} x$ iff $\forall \ell \in \underset{\uparrow \text{subspace}}{S} \subset X^* \ \ell(x_n) \longrightarrow \ell(x)$.

Def: $\ell_n \xrightarrow{\text{weakly}} \ell$ iff $\forall \phi \in X^{**} \ \phi(\ell_n) \longrightarrow \phi(\ell)$.

Def: $\ell_n \xrightarrow{\text{weak-}*} \ell$ iff $\forall x \in X$, $\underbrace{x(\ell_n)}_{} \longrightarrow \ell(x)$

$X \subset X^{**}$ by setting $\forall x \in X$ and $\ell \in X^*$, $x(\ell) := \ell(x)$

weak-* topology is the weakest

Banach–Alaoglu: If X is normed, then the unit ball of X^* is precompact w.r.t. weak-$*$ convergence.

Lecture 16

15/11/2023

Signed Measures (Ch. 3)

Recall: $\ell \in X^*$, $\ell : X \to \mathbb{R}$ or $\mathbb{C}$, e.g. $\ell(f) = \int f \, d\mu$. $f \in L^1(\mu)$

Moreover, if $f \geq 0$, then $\ell(f) \geq 0$.

Cone of positive fcts. C

Cone $\subset X$ s.t. $x, y \in$ Cone, $x+y \in$ Cone, $\lambda x \in$ Cone $\forall \lambda \geq 0$

Def: A functional is positive semi-definite iff $\forall x \in C$ we have $\ell(x) \geq 0$.

any cone on some Banach space

—— // ——

Def: (Signed Measure) A signed measure $\nu : \mathcal{M} \to \mathbb{R} \cup \{\pm\infty\}$ such that

- $\nu(\emptyset) = 0$
- ν may assume either $+\infty$ or $-\infty$ (never both) !
- if $\{E_j\}_{j \in \mathbb{N}} \subset \mathcal{M}$ and disjoint, then

$$\nu\left(\bigcup_{j\in\mathbb{N}} E_j\right) = \sum_{j\in\mathbb{N}} \nu(E_j),$$ and if LHS is finite then RHS converges absolutely. !

cannot have rearrangement issues

Ex 1: Let μ_1, μ_2 be finite measures on (X, M), then $\mu_1 - \mu_2 =: \nu$ is a signed measure.

Ex 2: Let μ be a measure and $f \in L^1(\mu)$, then

$$E \longmapsto \int_E f \, d\mu \quad \text{is a signed measure.}$$

Obs: In fact, every signed measure can be written in either one of these ways.

FACTS: • Continuity from below: Let ν be a signed measure and $E_j \nearrow E$, then $\nu(\lim_j E_j) = \nu(\bigcup_j E_j) = \lim_j \nu(E_j)$.

• Continuity from above: if $E_j \searrow E$ and $|\nu(E_1)| < \infty$, then $\nu(\lim_j E_j) = \nu\left(\bigcap_j E_j\right) = \lim_j \nu(E_j)$.

Pf: Verbatim of the one for positive measures.

Def: • A set $E \in \mathcal{M}$ is **positive w.r.t. ν** if

$$\forall \underset{\mathcal{M}}{\underset{\cap}{F}} \subset E, \quad \nu(F) \geq 0$$

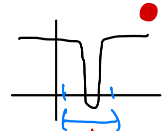
• A set $E \in \mathcal{M}$ is **negative w.r.t. ν** if

$$\forall \underset{\mathcal{M}}{\underset{\cap}{F}} \subset E, \quad \nu(F) \leq 0$$

• A set $E \in \mathcal{M}$ is **null w.r.t. ν** if it is both positive and negative.

Ex: For the signed measure $\nu(E) = \int_E f \, d\mu$, let

$$X^+ := \{f > 0\} \qquad\qquad X^+ \text{ is positive w.r.t. } \nu$$
$$X^- := \{f < 0\} \quad \Rightarrow \quad X^- \text{ is negative w.r.t. } \nu$$
$$X^0 := \{f = 0\} \qquad\qquad X^+ \cup X^0 \text{ is positive w.r.t. } \nu$$



Not a positive set! (but has positive measure as a whole)

FACT: Countable unions of positive sets is positive. Same for negative.

Thm: (HAHN DECOMPOSITION) If ν is a signed measure, there exists $P, N \subset X$ s.t. $P \cap N = \emptyset$, $P \sqcup N = X$, and P is a positive set and N is a negative set. If P', N' satisfy the same properties of P, N, then $P' \Delta P$ and $N' \Delta N$ are null sets.

Pf: (Greedy → Wall Street 1987) Assume ν does not attain $+\infty$. Let

$$m := \sup_{P \text{ positive}} \nu(P) < \infty$$

i.e., $\exists P_n$ positive s.t. $\nu(P_n) \longrightarrow m$.

Set $P := \bigcup_n P_n$. By continuity from below, $\nu(P) = m$.

Now, P countable union of positive $\Rightarrow$ P positive.

Let $N := X \setminus P$. WTS: N negative.

Fact 1: N does not contain any non-null positive set.

(otherwise, let $A \subset N$ be a non-null positive set, then $A \cap P = \emptyset$ and $\nu(A) > 0$, but $\nu(A \cup P) = \nu(A) + \nu(P) > m$ $\rightarrow\!\leftarrow$)

Fact 2: if $A \subset N$ w/ $\nu(A) > 0$ $\exists B \subset A$ s.t. $\nu(B) > \nu(A)$

$\left(\exists A' \subset A \text{ with } \nu(A') < 0, \text{ then } \nu(A \setminus A') = \nu(A) - \underbrace{\nu(A')}_{<0} > \nu(A) \right)$

Construct a sequence $n_j^{\in \mathbb{N}}, A_j^{\in N}$ as follows:

- let n_0 be the smallest integer s.t. A_0 is s.t. $\nu(A_0) > \frac{1}{n_0} > 0$
- let n_1 be the smallest integer s.t. $\exists A_1 \subset A_0$ s.t. $\nu(A_1) > \nu(A_0) + \frac{1}{n_1}$

$\vdots$

- n_ℓ " " " " " $\exists A_\ell \subset A_{\ell-1}$ s.t. $\nu(A_\ell) > \nu(A_{\ell-1}) + \frac{1}{n_\ell}$

$\vdots$

Let $A := \bigcap_j A_j$. Then

$$\infty > \nu(A) \overset{\text{cont. from above}}{=} \lim_j \nu(A_j) > \sum_{\ell=0}^{\infty} \frac{1}{n_\ell},$$

hence the series converges $\Rightarrow n_\ell \to \infty$.

But then $\nu(A) > 0$ $\exists B \subset A$ s.t. $\nu(B) > \nu(A) + \frac{1}{n}$ for some n. But since $n_\ell \nearrow \infty$, $\exists \ell$ s.t. $n_\ell > n$. This means that $B \subset A \subset A_{\ell-1}$, i.e.,

$$\nu(B) > \nu(A) + \frac{1}{n} \underset{\text{construction}}{>} \nu(A_{\ell-1}) + \frac{1}{n} \Rightarrow n < n_\ell \quad \lightning$$

Thus, N is negative. TBC $\rightarrow$ $\blacksquare$

LECTURE 17

SIGNED MEASURES (ctd)

20/11/2023

Hahn Decomposition Theorem: Let ν be a signed measure on $(X, \mathcal{M})$. Then, $\exists P, N$ s.t. $X = P \sqcup N$ w/ P positive and N negative. Moreover, any P', N' as above are so that $P \triangle P' = N \triangle N'$ are null sets.

Pf: .... Continuing from last time:

$P \setminus P' \subset N' \subset P$ is null and likewise for $P' \setminus P$. ■

positive sets that are also negative must be null

Recall: $d\nu = f\,d\mu \longrightarrow$

$$P = \{f > 0\}$$
$$N = \{f < 0\}$$
$$Z = \{f = 0\}$$

IMPORTANT!

JORDAN DECOMPOSITION THEOREM: Let ν be a signed measure. Then, there exist unique positive measures ν^+ and ν^- such that $\nu^+ \perp \nu^-$ and $\nu = \nu^+ - \nu^-$.

Mutually singular: $\exists E, F \in \mathcal{M}$ s.t. $X = E \cup F$ and $\nu^+(E) = \nu^-(F) = 0$.

Pf: Let $X = P \sqcup N$ using Hahn Decomposition. Set for any $A \in \mathcal{M}$

$$\nu^+(A) := \nu(A \cap P)$$

$$\nu^-(A) := -\nu(A \cap N)$$

Then, $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$ (b/c P is ν^--null and N is ν^+-null).

<u>Uniqueness</u>: Let $\mu^\pm$ be another pair of positive measures s.t. $\mu^+ \perp \mu^-$ and $\nu = \mu^+ - \mu^-$. Then

$$\mu^+ \perp \mu^- \iff \exists E^+, E^- \in \mathcal{M} \text{ s.t. } E^+ \sqcup E^- = X \text{ and } \mu^+(E^-) = \mu^-(E^+) = 0$$

Since $\nu = \mu^+ - \mu^-$, if $A \subset E^+$, then

$$\nu(A) = \mu^+(A) - \underbrace{\cancel{\mu^-(A)}}_{=0} \geq 0$$

If $B \subset E^-$, then

$$\nu(B) = \underbrace{\cancel{\mu^+(B)}}_{=0} - \mu^-(B) \leq 0$$

So, E^+, E^- form a Hahn decomposition of $(X, \mathcal{M}, \nu)$, which is unique (up to null sets) $\Rightarrow \nu^\pm = \mu^\pm$. ∎

EXAMPLE: if μ is positive, $f \in L^1(\mu)$

$\int_X |f|\,d\mu < \infty$

$$E \longmapsto \int_E f\,d\mu$$

$\int_X f^+\,d\mu + \int_X f^-\,d\mu$

Def: (EXTENDED INTEGRABLE FUNCTIONS)

$$L^1_{ext}(\mu) = \left\{ \begin{array}{l} \text{measurable fcts } f \text{ s.t. at least} \\ \text{one of } \int_X f^+ d\mu \text{ or } \int_X f^- d\mu \text{ is } < \infty \end{array} \right\}$$

Not a vector space !!

ABSOLUTE CONTINUITY: if ν is a signed measure and μ is a positive measure, then $\nu \ll \mu$ if every μ-null set is also a ν-null set.

Def: If $\nu(E) = \int_E f\,d\mu$, then $|\nu|(E) = \int_E |f|\,d\mu$.

If we cannot do this (e.g., $f \in L^1_{ext}(\mu)$), then define

Total variation measure $|\nu| := \nu^+ + \nu^-$ from Jordan dec.

Note: $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu \Leftrightarrow \begin{cases} \nu^+ \ll \mu \\ \nu^- \ll \mu \end{cases}$.

Def: Let μ, ν be signed measures, then $\mu \precsim \nu$ iff X is a positive measure set for $\nu - \mu$. ← partial order

$\Updownarrow$ $E \in \mathcal{M}$

$\mu \precsim \nu$ on $E \iff E$ is a positive set for $\nu - \mu$.

Upshot: Since, given a signed measure, we can always decompose it into 2 positive measures (by Jordan dec.), we can recover all of the integration theory from before:

ν signed measure, $L^1(\nu) :=$ "$L^1(\nu^+) \cap L^1(\nu^-)$"

$$L^1(\nu) \ni f \rightsquigarrow \int f \, d\nu := \int f \, d\nu^+ - \int f \, d\nu^- \in \mathbb{R}.$$

FACT: Let ν be signed and μ be positive. Suppose $\nu \ll \mu$ and $\nu \perp \mu$. Then $\nu = 0$.

$$\left(\nu \perp \mu \Rightarrow \exists E, F \text{ s.t. } E \sqcup F = X \ \& \ \underbrace{\nu(E) = 0 = \mu(F)}_{\nu \ll \mu \Rightarrow \nu(F) = 0} \right)$$

CONTINUITY FOR ABS. CONT. FUNCTIONS: Let ν be finite and signed and μ be a positive measure.

$\nu \ll \mu \iff \forall \varepsilon > 0\ \exists \delta > 0$ s.t. $\mu(E) < \delta \Rightarrow |\nu(E)| < \varepsilon$.

Pf: Note $\forall E$

$$|\nu(E)| = |\nu^+(E) - \nu^-(E)| \le \nu^+(E) + \nu^-(E) = |\nu|(E)$$

So, can drop the "signed" assumption (i.e., just assume ν, μ are both positive).

($\Leftarrow$) Immediate.

($\Rightarrow$) Assume $\exists \varepsilon > 0$ s.t. $\forall n \in \mathbb{N}$ $\exists E_n$ w/ $\mu(E_n) \le 2^{-n}$ and $\nu(E_n) \ge \varepsilon$. Let $F_\ell = \bigcup_{i=\ell}^{\infty} E_i$ Then $\mu(F_\ell) \le 2^{1-\ell}$.

$\nu(F_\ell) \ge \varepsilon$

Set $F := \bigcap_\ell F_\ell$. Then by continuity, (from above) $\mu(F) = 0$. Now ν finite $\Rightarrow$ can use continuity from above to get $\nu(F) \ge \varepsilon$. ■

EXTREMELY IMPORTANT!

LEBESGUE-RADON-NIKODYM THEOREM: Let ν be a σ-finite signed measure and μ be a σ-finite positive measure. Then, there exists unique σ-finite signed measures λ, ρ s.t.

$$\lambda \perp \mu, \quad \rho \ll \mu, \quad \nu = \rho + \lambda. \qquad \text{(Lebesgue)}$$

Moreover, there exists$\overset{\text{unique}}{\vee}$ $f \in L^1_{ext}(\mu)$ s.t. $\rho(E) = \int_E f\, d\mu$ and we denote

$$\frac{d\rho}{d\mu} := f \quad \text{Radon-Nikodym derivative.}$$

(Radon-Nikodym)

Pf: <u>Uniqueness</u>: if $\nu = \lambda + \rho$ and also $\nu = \lambda' + \rho'$.

Then, $\rho - \rho' = \lambda' - \lambda$. But $(\rho - \rho') \ll \mu$ and $(\lambda' - \lambda) \perp \mu$.

So, $\rho - \rho' = \lambda' - \lambda = 0$.

$\lambda \perp \mu \Leftrightarrow E \sqcup F$ s.t. E λ-null, F μ-null

$\lambda' \perp \mu' \Leftrightarrow E' \sqcup F'$ s.t. E' λ'-null, F' μ'-null

$\lambda + \lambda' = \lambda''$ then $E'' := E \cap E'$ is λ''-null, $F'' := F \cap F'$ is μ-null

<u>Existence</u>: Since we have σ-finite measures, always first assume they are finite.

<u>Step 1</u>: Assume ν, μ are finite. Prove separately for $\nu^{\pm}$ (i.e., we can assume ν is in fact positive).

positive from Jordan dec.

Consider

$$\mathcal{F} = \left\{ \underset{\text{measurable}}{f : X \to [0, \infty]} : \int_E f \, d\mu \leq \nu(E) \right\}$$

Note: $0 \in \mathcal{F}$ (so $\mathcal{F} \neq \emptyset$).

Note: $f, g \in \mathcal{F} \Rightarrow \max(f, g) \in \mathcal{F}$. In fact, let $h := \max(f, g)$ and $A := \{x \in X : f > g\}$. Then

$$\int_E h = \int_{E \cap A} h + \int_{E \setminus A} h = \int_{E \cap A} f + \int_{E \setminus A} g$$

$$\leq \nu(E \cap A) + \nu(E \setminus A)$$

$$= \nu(E).$$

↙ assumed ν finite

Let $a := \sup_{f \in \mathcal{F}} \int_X f \, d\mu$. <u>Note</u>: $a \leq \nu(X) < \infty$

Take $\{f_n\} \subset \mathcal{F}$ s.t. $\int_X f_n \, d\mu \longrightarrow a$.

Define $g_n := \max(f_1, \ldots, f_n)$. Then $g_n \nearrow g \in \mathcal{F}$.

Let $f := \sup_n f_n$. Then $g_n \nearrow f$ ptwise. Moreover

$$a \geq \int g_n \geq \int f_n \longrightarrow a$$

Thus, by MCT, $\int f \, d\mu = \lim \int g_n \, d\mu = a$. Thus, this f is the candidate density to be the Radon-Nikodym derivative: $\rho := f \, d\mu$. → i.e., $\rho(E) = \int_E f \, d\mu$

WTS: $(\nu - \rho) \perp \mu$.

<u>Lemma</u>: Let ν, μ be finite positive measures. Then either $\nu \perp \mu$ or $\exists \, \varepsilon > 0$ and $E \in \mathcal{M}$ w/ $\mu(E) > 0$ and $\nu \geq \varepsilon \mu$ on E.

But by this lemma, if $(\nu - \rho) \not\perp \mu$, then $\exists \, E$ with $\mu(E) > 0$ s.t. $(\nu - \rho) \geq \varepsilon \mu$ on E. So,

$$\varepsilon \chi_E \, d\mu \leq d(\nu - \rho) = d\nu - d\rho = d\nu - f \, d\mu$$

i.e., $f + \varepsilon \chi_E \, d\mu \leq d\nu$ ↪ // ⟵ contradicts maximality of f.

∎

LECTURE 18

LRN & HARDY-LITTLEWOOD

22/11/2023

Thm: (Lebesgue – Radon – Nikodym) $\forall \nu$ signed σ-finite measure on $(X, \mathcal{M})$ and $\forall \mu$ positive σ-finite measure on $(X, \mathcal{M})$, $\exists!\ \lambda, \rho$ w/ $\lambda \perp \mu$ and $\rho \ll \mu$ s.t. $\mu = \rho + \lambda$. Moreover, $\exists!\ f \in L^1_{ext}(\mu)$ s.t. $\rho(A) = \int_A f\, d\mu$. We denote this f by $f =: \frac{d\rho}{d\mu}$ (Radon-Nikodym derivative).

used this to prove LRN-Thm

Lemma: Let ν, μ be finite positive measures. Then either $\nu \perp \mu$ or $\exists\, \varepsilon > 0$ and $E \in \mathcal{M}$ w/ $\mu(E) > 0$ and $\nu \geq \varepsilon \mu$ on E.

Pf: Let $P_n \sqcup N_n$ be a seq. of Hahn dec. for $(X, \nu - \frac{1}{n}\mu)$.

Let

$$P := \bigcup_{n=1}^{\infty} P_n \quad \text{and} \quad N := \bigcap_{n=1}^{\infty} N_n$$

obs: $X = P \sqcup N$

N is a negative set for every $\nu - \frac{1}{n}\mu$.

In particular,

ν positive

$$0 \leq \nu(N') \leq \frac{1}{n}\mu(N') \quad \forall N' \subset N$$

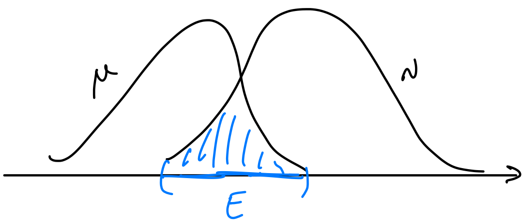
$$\Rightarrow \nu(N') = 0 \quad \forall N' \subset N$$

$$\Rightarrow N \text{ is } \nu\text{-null}.$$

by def. of $\perp$ (since $X = P \sqcup N$)

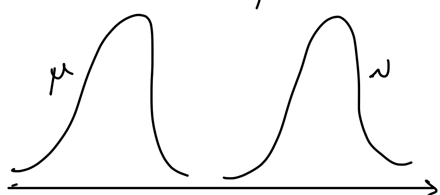
Now, if $\mu(P) = 0$, then $\mu \perp \nu$. Otherwise, if $\mu(P) > 0$, then $\exists n$ s.t. $\mu(P_n) > 0$. So, P_n is a positive set for $\nu - \frac{1}{n}\mu \Rightarrow \nu \geq \frac{1}{n}\mu$ on P_n ($=: E$ on the lemma). ∎

<u>Remark:</u> $\exists \varepsilon > 0$ s.t. $\nu \geq \varepsilon\mu$ on E



or

$\nu \perp \mu$



"Disjoint supports"

Lemma: If ν is σ-finite signed on $(X, \mathcal{M})$ and μ, λ are positive σ-finite measures on $(X, \mathcal{M})$ s.t. $\nu \ll \mu$ and $\mu \ll \lambda$ ($\Rightarrow \nu \ll \lambda$). Then

(i) if $g \in L^1(\nu)$, then $g \frac{d\nu}{d\mu} \in L^1(\mu)$ and

$$\int g \, d\mu = \int g \frac{d\nu}{d\mu} \, d\mu.$$

(ii) $$\frac{d\nu}{d\lambda} = \frac{d\nu}{d\mu} \frac{d\mu}{d\lambda}, \quad \lambda\text{-a.e.}.$$

SIGNED MEASURES & DENSITY: Consider the Lebesgue measure m on $\mathbb{R}^n$ and take another measure $\nu \ll m$. Then, $\forall E \in \mathcal{L}$

$$\text{Density} = \frac{\text{mass}(E)}{\text{Leb}(E)} \rightsquigarrow \text{Density} = \frac{\nu(E)}{m(E)}.$$

Replace E w/ $B_r(x) = \{y \in \mathbb{R}^n : \|x-y\| < r\} \in \mathcal{L}$. Consider

$$\lim_{r \to 0} \frac{\nu(B_r(x))}{m(B_r(x))} \stackrel{?}{=} \frac{d\nu}{dm}.$$

Def: locally L^1 functions "fcts that are L^1 on every bdd set"

$$L^1_{loc}(m) := \left\{ f : \mathbb{R}^n \to \mathbb{C} : \forall K \in \mathcal{L} \text{ bounded}, \int_K |f| < \infty \right\}$$

Def: Let $f \in L^1_{loc}(m)$ and $\forall x \in \mathbb{R}$, $r > 0$,

$$(A_r f)(x) := \frac{1}{m(B_r(x))} \int_{B_r(x)} f \, dm.$$

"Average of f over $B_r(x)$"

Lemma: the map $(x, r) \longmapsto (A_r f)(x)$ is continuous.

Pf: Note $m(B_r(x)) = c_n r^n$, $c_n := m(B_1(0))$.

$m(\partial B_r(x)) = 0$

Thus,

(1) as $r \to r_0$, $\chi_{B_r(x)} \xrightarrow{\text{ptwise}} \chi_{B_{r_0}(x)}$ converges everywhere except (maybe) on $\partial B_{r_0}(x)$.

(2) $\forall x$ s.t. $|x - x_0| < \frac{1}{2}$, $\forall r$ s.t. $|r - r_0| < \frac{1}{2}$

$$|\chi_{B_r(x)}| < \chi_{B_{r_0+1}(x_0)}$$

Suppose $x_n \to x_0$ and $r_n \to r_0$, eventually $\chi_{B_{r_n}(x_n)}$ is dominated by $\chi_{B_{r_0+1}(x_0)}$.

WTS: $(A_{r_n} f)(x_n) \longrightarrow (A_{r_0} f)(x_0)$.

$$\Updownarrow$$

$$\int_{B_{r_n}(x_n)} f \, dm \longrightarrow \int_{B_{r_0}(x_0)} f \, dm$$

$$\|$$

$$\int \underbrace{f \chi_{B_{r_n}(x_n)}}_{=: f_n}$$

Now, $|f_n| \leq \underbrace{|f| \chi_{B_{r_0+1}(x_0)}}$ Integrable b/c $f \in L^1_{loc}$

Thus, by LDCT, the limit above holds. ∎

Def: Let $f \in L^1_{loc}(m)$, we define the HARDY-LITTLEWOOD MAXIMAL FUNCTION as

$$[Hf](x) := \sup_{r>0} (A_r f)(x)$$

FACT: Hf is measurable

HARDY-LITTLEWOOD MAXIMAL INEQUALITY: $\exists C > 0$ s.t. $\forall f \in L^1$, $\alpha > 0$,

$$m(\{x : Hf > \alpha\}) \leq \frac{C}{\alpha} \int_{\mathbb{R}^n} |f| \, dm.$$

Looks like Markov's inequality.

Lecture 19

27/11/2023

Hardy-Littlewood & Lebesgue set of a function

Vitali Covering Lemma: Let X be a metric space and $\{B_j\}_{j=1}^{N}$ be a finite collection of open balls in X. Then, there exists a collection of disjoint balls $\{B_{j_k}\}_{k=1}^{N'\leq N}$ such that

$$\bigcup_{k=1}^{N'} \underbrace{3B_{j_k}}_{\text{ball of radius } = 3 \text{ times the radius of } B_{j_k}.} \supset \bigcup_{j=1}^{N} B_j$$

Pf: (Greedy Construction) Let A_1 be the largest ball in $\{B_j\}$. Let A_2 be the largest ball disjoint from A_1. Choose A_3 to be the largest ball disjoint from both A_1 and A_2. Keep going this way. Then, we get a subcollection $\{A_k\}$ of disjoint balls.

If $\{A_k\} = \{B_j\}$, then we are done.

Otherwise, if $\exists B_j$ which has not been picked, in particular $B_j \cap \bigcup_k A_k \neq \emptyset$. Let $\bar{j}$ be the smallest index s.t.

$$\emptyset \neq A_{\bar{j}} \cap B_j.$$

Thus, radius $(B_j) \leq$ radius $(A_{\bar{j}}) \Rightarrow \underbrace{B_j \subset 3A_{\bar{j}}}$.

holds for any balls that were not picked.

$\square$

Corollary: Let $\mathcal{C}$ be an arbitrary collection of open balls in $\mathbb{R}^n$ and U be the union of the balls in $\mathcal{C}$. Let $c \leq m(U)$. Thus, there exists a finite disjoint subcollection $\{B_j\}_{j=1}^N$ s.t.

$$\sum_{j=1}^{N} m(B_j) \geq 3^{-n} c.$$

Pf: By inner regularity of m, $\exists K^{\text{compact}} \subset U$ s.t. $m(K) > c$. Extract a finite cover of K and apply Vitali to get

$$\bigcup_j 3A_j \supset K.$$

Thus

$$m(3A_j) \geq 3^n \sum m(A_j) \geq c.$$

$\square$

Def: (Hardy-Littlewood Maximal Function)

$$[Hf](x) := \sup_{r>0} [A_r|f|](x).$$

!

HARDY-LITTLEWOOD MAXIMAL INEQUALITY: For all $f \in L^1$, $\alpha > 0$, s.t.

$$m(\{x : [Hf](x) > \alpha\}) \leq \frac{3^n}{\alpha} \int |f| \, dm.$$

Pf: **VERY IMPORTANT & STANDARD**

Let $E_\alpha := \{x : [Hf](x) > \alpha\}$. Then, $\forall x \in E_\alpha$, $\exists r_x > 0$ s.t. $A_{r_x}|f|(x) > \alpha$. Clearly, $\bigcup_x B_{r_x}(x) \supset E_\alpha$. So, for any $c < m(E_\alpha)$, there exists a collection $\{x_j\}_{j=1}^N \subset E_\alpha$ s.t. $B_j := B_{r_{x_j}}(x_j)$ are disjoint and $\sum_{j=1}^N m(B_j) > 3^{-n} c$.

Then

$$c < 3^n \sum_{j=1}^N m(B_j).$$

Now,

$$A_r g \overset{\text{def}}{=} \frac{1}{m(B_r(x))} \int_{B_r(x)} g \, dm$$

gives

$$\alpha < A_{r_x} |f| = \frac{1}{m(B_{r_x}(x))} \int |f| \, dm .$$

So,

$$c < 3^n \sum_{j=1}^{N} m(B_j) \leq \frac{3^n}{\alpha} \sum_j \int_{B_j} |f| \, dm$$

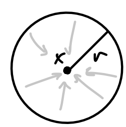
$$\leq \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f| \, dm$$

Arbitrage to get the desired inequality.

$\square$

Thm: Let $f \in L^1_{loc}$, then

$$\lim_{r \to 0} [A_r f](x) = f(x) \text{ for a.e. } x .$$



Pf: Clearly true for continuous fcts., so use their density to show the thm.

If we fix an arbitrary $L>0$ and show the statement for disk $\|x\|<L$, then we are done. Moreover, we can also assume $0<r<1$. So, we can also replace f with $f\chi_{B_{L+1}(0)} \in L^1$.

By density of continuous functions in $L^1(m)$ (Lebesgue), $\forall \varepsilon>0$, $\exists g \in L^1$ continuous s.t. $\|f-g\|_1 = \int_{\mathbb{R}^n} |f-g|\, dm < \varepsilon$.

Now, since g is continuous, $\forall x \in \mathbb{R}^n$, $\forall \delta>0$ $\exists r>0$ s.t. if $|y-x|<r$, then $|g(y)-g(x)|<\delta$. Take the average of g on $B_r(x)$, then

$$|A_r g(y) - g(x)| = \frac{1}{m(B_r(x))}\left|\int_{B_r(x)} g(y)-g(x)\, dy\right|$$

$$\leq \frac{1}{m(B_r(x))}\int_{B_r(x)} |g(y)-g(x)|\, dy$$

$$< \delta$$

So, $A_r g(x) \xrightarrow{r\to 0} g(x)$.

Finally,

$$0 \leq \limsup_{r\to 0} |A_r f(x) - f(x)| = \limsup_{r\to 0} |A_r f(x) - A_r g(x) + A_r g(x)$$

$$- g(x) + g(x) - f(x) \Big|$$

$$\leq \limsup_{r\to 0} \big| A_r f(x) - A_r g(x) \big|$$

$$+ \underbrace{\limsup_{r\to 0} \big| A_r g(x) - g(x) \big|}_{\to 0 \text{ b/c } A_r g(x) \to g(x)}$$

$$+ \big| g(x) - f(x) \big|$$

$$\leq \limsup_{r\to 0} \big| A_r f(x) - A_r g(x) \big| + \big| g(x) - f(x) \big|$$

$$\leq \limsup_{r\to 0} A_r |f-g|(x) + |f-g|(x)$$

$$\overset{(*)}{\leq} \limsup_{r\to 0} H(f-g)(x) + |f-g|(x)$$

Now, let

$$E_\alpha := \left\{ x : \limsup_{r\to 0} |A_r f - f|(x) > \alpha \right\}$$

$$\overset{(*)}{\subset} \underbrace{\left\{ x : H(f-g)(x) > \frac{\alpha}{2} \right\}}_{\text{Hardy-Littlewood Ineq.}} \cup \underbrace{\left\{ x : |f-g|(x) > \frac{\alpha}{2} \right\}}_{\text{Markov Ineq.}}$$

So,

$$m(E_\alpha) \leq \frac{3^n}{\alpha/2} \underbrace{\int_{\mathbb{R}^n} |f-g| \, dm}_{<\varepsilon} + \frac{1}{\alpha/2} \underbrace{\int_{\mathbb{R}^n} |f-g| \, dm}_{<\varepsilon} \quad \text{b/c } \|f-g\| < \varepsilon$$

$$\leq \frac{2(3^n+1)}{\alpha} \varepsilon \implies m(E_\alpha) = 0.$$

Therefore,

$$m\left(\{x : \limsup_{r\to 0} |A_r f - f|(x) > 0\}\right) = m\left(\bigcup_n E_{1/n}\right) = 0.$$

$\square$

VERY IMPORTANT !

Def: (LEBESGUE SET OF A FUNCTION) Let $f \in L^1_{loc}$, then the Lebesgue set of f is defined as

$$L_f := \left\{ x : \underbrace{\lim_{r\to 0} \frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy}_{\text{"DENSITY"}} = 0 \right\}$$

Thm: If $f \in L^1_{loc}$, then $m(L_f^c) = 0$. ! Very remarkable given the ugly nature of measurable sets....

!

LEBESGUE DIFFERENTIATION THEOREM: If $f \in L^1_{loc}$, then for all $x \in L_f$ and $\forall E_r$ that shrinks nicely to x, the following holds:

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y) - f(x)| \, dy = 0.$$

In particular, this implies that

$$\lim_{r \to 0} \frac{1}{m(E_r)} \int_{E_r} f(y) \, dy = f(x).$$

Pf: WTS: $0 \le \limsup |\cdot| = 0 \Rightarrow \liminf |\cdot| = 0 \Rightarrow \lim |\cdot| = 0$.

$$0 \le \limsup_{r \to 0} \frac{1}{m(E_r)} \left| \int_{E_r} f(y) - f(x) \, dy \right|$$

$$\le \limsup_{r \to 0} \frac{1}{m(E_r)} \int_{B_r(x)} |f(y) - f(x)| \, dy$$

$\leftarrow E_r \subset B_r(x)$

$$\le \frac{1}{\alpha} \left(\frac{1}{m(B_r(x))} \int_{B_r(x)} |f(y) - f(x)| \, dy \right) \longrightarrow 0.$$

$\square$

Def: (REGULAR MEASURES) A Borel measure ν is called <u>REGULAR</u> if

- $\forall K^{\text{compact}}, \ \nu(K) < \infty$.
- $\forall E \in \mathcal{M}, \ \nu(E) = \inf\{\nu(U) : U^{\text{open}} \supset E\}$.

A signed measure ν is regular if $|\nu|$ (tot. variation) is regular.

e.g., m is regular (but not only m)

Thm: Let ν be a regular signed measure and $\nu \overset{\text{LRN}}{=} \lambda + \rho$ (λ singular w.r.t. m, ρ abs. cont. w.r.t. m) then $\exists! f$ s.t. $\rho(E) = \int_E f \, dm$. Then, for all $\{E_r\}$ that shrinks nicely to x,

$$\lim_{r \to 0} \frac{\nu(E_r)}{m(E_r)} = f(x), \quad \text{for } m\text{-a.e. } x.$$

LECTURE 20

29/11/2023

RADON MEASURES

<u>GOAL</u>: Prove Riesz Representation Thm & unify measures & vec. spaces.

Def: A topological space is NORMAL if $\forall K^{closed}, L^{closed}$ disjoint $\exists U^{open}, V^{open}$ disjoint s.t. $K \subset U$ and $L \subset V$

URYSOHN'S LEMMA: TFAE

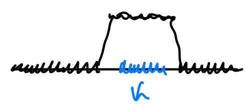
(i) X is normal

(ii) $\forall K^{closed}$ and $U^{open} \supset K$, $\exists V^{open}, L^{closed}$ s.t.

$$U^{open} \supset L^{closed} \supset V^{open} \supset K^{closed}.$$

(iii) $\forall K^{closed}, L^{closed}, K \cap L = \emptyset, \exists f \in C^0(X, [0,1])$ s.t. $f|_K \equiv 1$ and $f|_L \equiv 0$.

(iv) if $K^{closed} \subset U^{open}$, $\exists f \in C^0(X,[0,1])$ s.t. $\chi_K \le f \le \chi_U$



Pf: (i) $\Leftrightarrow$ (ii): Choose $L = U^c$.

(iii) $\Leftrightarrow$ (iv): Choose $L = U^c$

(iii) $\Rightarrow$ (i): Let $U = f^{-1}((2/3, 1])$ and $V = f^{-1}([0, 1/3))$ and find open by continuity of f.

TRICKY PART

(ii) $\Rightarrow$ (iv) Let $K_1 = K$ and $U_0 = U$. By (ii), $\exists\, K_{1/2}, U_{1/2}$ s.t. $\underbrace{K_1 \subset U_{1/2}} \subset \underbrace{K_{1/2} \subset U_0}$

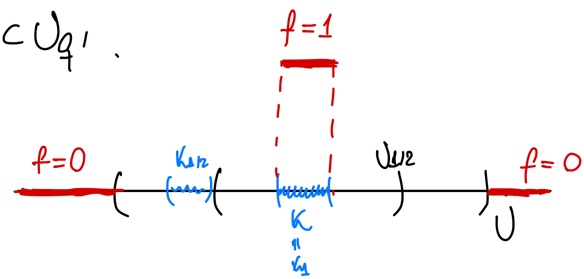
Apply (ii) again to get

$$K_1 \subset U_{3/4} \subset K_{3/4} \subset U_{1/2} \subset K_{1/2} \subset U_{1/4} \subset K_{1/4} \subset U_0$$

Upshot: for any dyatic rational $q = \frac{a}{2^n} \in [0,1]$, $U_q \subset K_q$.

Moreover, if $0 < q' < q < 1$, $K_q \subset U_{q'}$.

Now, dyatic rationals are dense in $[0,1]$.



So, define f as

$$f(x) := \sup\{q : x \in U_q\} \left(= \inf\{q : x \notin K_q\}\right)$$

Convention: $\inf \emptyset \equiv 1$
$\sup \emptyset \equiv 0$

So, $\chi_K \leq f \leq \chi_U$.

WTS: f is continuous $\Longleftrightarrow$ $f^{-1}([0,\alpha))$ is open $\forall \alpha \in (0,1)$

$f^{-1}((\beta,1])$ is open $\forall \beta \in (0,1)$

Indeed,

$$f^{-1}([0,\alpha)) \overset{\text{def}}{=} \bigcup_{q<\alpha} U_q$$ ← union of open sets, hence open.

$$f^{-1}((\beta,1]) \overset{\text{def}}{=} \bigcup_{q>\beta} X \setminus K_q$$ ← union of open, hence open.

■

Def: Hausdorff space if $\forall x \neq y \in X$ $\exists U^{\text{open}}, V^{\text{open}}$ s.t. $x \in U$, $y \in V$, $U \cap V = \emptyset$

FACT: COMPACT + HAUSDORFF $\Rightarrow$ NORMAL

Step 1: if $F^{\text{cpt}} \subset X^{\text{Haus.}}$, $x \notin F$ $\exists U^{\text{op}}, V^{\text{op}}$, $U \cap V = \emptyset$ s.t. $U \ni x$, $V \supset F$.

Pf: $\forall y \in F$ $\exists U_y^{\text{op}}, V_y^{\text{op}}$ s.t. $U_y \ni x$ $V_y \ni y$, $U_y \cap V_y = \emptyset$ (since Hausdorff). Then

$\bigcup_{y \in F} V_y$ covers F $\rightsquigarrow$ Extract finite subcover $\bigcup_i V_{y_i} =: V$.

So, V open and $V \supset F$. So,

$$\bigcap_i U_i =: U^{\text{open}} \text{ and } U \ni x.$$

$\square$

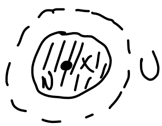
Step 2: $F^{\text{cpt}} \subset G^{\text{cpt}} \subset X^{\text{Haus.}}$ $\exists U^{\text{op}}, V^{\text{op}}$ $U \supset F$, $V \supset G$. $U \cap V = \emptyset$

Def: A set X is **locally compact** if $\forall x \in X$ there exists a compact neighborhood such that $x \in N \subset X$.

N is a neighb. of x if $\exists U^{\text{op}}$ $N \supset U \ni x$

Def: (Nice spaces) **LCH - space** $=$ Hausdorff locally compact topological space

Lemma: If $U^{\text{open}} \subset X^{LCH}$, then $\forall x \in U$, $\exists N^{\text{cpt}}$ neighborhood of x such that $N \subsetneq U$.



LECTURE 21

RADON MEASURES

04/12/2023

Recall: LCH - space $=$ Locally compact Hausdorff space

$\forall x\ \exists N^{\text{compact}}$ s.t. $x \in N$.

- Compact + Hausdorff $\Rightarrow$ Normal

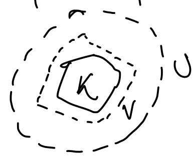
- Urysohn's Lemma: X is normal $\Leftrightarrow$ if $F^{\text{closed}} \subset U^{\text{open}}$ $\exists f \in C^0(X, [0,1])$ s.t. $\chi_F \le f \le \chi_U$.

Lemma: $U^{\text{open}} \subset X^{LCH}$ then $\forall x \in U\ \exists N^{\text{cpt.}} \ni x$ s.t. $N^{\text{cpt}} \subset U$.

Lemma: Let $K^{\text{cpt}} \subset U^{\text{open}} \subset X^{LCH}$, then $\exists V^{\text{open}}$ precompact s.t.

i.e., its closure is cpt

$$K^{\text{cpt}} \subset V \subset \overline{V} \subset U^{\text{open}}$$



Pf: By the previous lemma $\forall x \in K\ \exists N_x^{\text{cpt}} \subset U$ so $\operatorname{int} N_x$ is an open neighb. of x. So, $\bigcup_{x \in K} \operatorname{int} N_x$ covers K^{cpt} $\rightsquigarrow$ extract a finite subcover. Set $V := \bigcup_{i \in \mathbb{N}} \operatorname{int} N_{x_i}$; so V is open and precompact $\rightsquigarrow \overline{V} = \bigcup_i N_{x_i} \subset U$. $\square$

LCH-URYSOHN LEMMA: Let X be LCH, if $K^{\text{cpt}} \subset U^{\text{open}} \subset X$ then $\exists f \in C^0(X, [0,1])$ s.t. $f|_K \equiv 1$ and $f \equiv 0$ outside some cpt. subset of U.

Pf: By the lemma above, $\exists V^{\text{open}}$ precompact s.t. $K^{\text{cpt}} \subset V \subset \overline{V} \subset U \subset X^{LCH}$. But $\overline{V}$ is normal (compact & Hausdorff). So, by Urysohn's Lemma, $\exists f \in C^0(X, [0,1])$ s.t. $f|_K \equiv 1$ and $f|_{\partial V} \equiv 0$. Extend f to X by setting it $f \equiv 0$ outside $\overline{V}$.

<u>WTS</u>: f is continuous. Indeed, let $E^{\text{closed}} \subset [0,1]$, then, if $E \not\ni 0$, the preimage of E cannot sit outside of $\overline{V}$, i.e.,

$f^{-1}(E) = (f|_{\overline{V}})^{-1}(E)$, which is closed b/c $f|_{\overline{V}}$ is closed by Urysohn.

If $0 \in E$, then $f^{-1}(E) = \underbrace{(f|_{\bar{V}})^{-1}(E)}_{\text{closed}} \cup \underbrace{V^c}_{\text{closed}}$ is closed.

$\square$

IMPORTANT

Def: (RADON MEASURES) Let X be LCH and μ be Borel. Then, μ is <u>Radon</u> if

- $\forall K^{cpt}$, $\mu(K) < \infty$
- (outer regularity) $\mu(E) = \inf\{\mu(U) : U^{open} \supset E\}$ $\forall E \in \mathcal{M}$
- (inner regularity for open sets) $\mu(E) = \sup\{\mu(K) : K^{cpt} \subset E\}$ $\forall E^{open}$

a bit weaker than "regular measure"

(Radon measures are very nice)

Lemma: Any Radon measure is inner regular on any σ-finite set. (σ-compact)

(countable unions of finite-measure sets)

[**Def:** A set is σ-compact if it is the countable union of compact sets.]

AS ALWAYS

Pf: Let E be σ-finite. First, assume E is finite measure. By outer regularity of μ, $\forall \varepsilon > 0$ $\exists U^{open} \supset E$ s.t. $\mu(U) < \mu(E) + \varepsilon$. By inner regularity, $\exists F^{cpt} \subset E$ s.t. $\mu(F) > \mu(U) - \varepsilon$. So, we have that $\mu(U \setminus E) < \varepsilon$. So, $\exists V^{op} \supset U \setminus E$ s.t. $\mu(V) < \varepsilon$. Thus,

$F^{\text{closed}} \setminus V^{\text{open}} =: K$ is compact and $K \subset E$. Now,

$$\mu(K) = \mu(F) - \mu(F \cap V) > \mu(U) - \varepsilon - \mu(F \cap V)$$
$$> \mu(E) - \varepsilon - \varepsilon.$$
$$= \mu(E) - 2\varepsilon.$$

✓ Done for finite-measure sets.

If $\mu(E) = \infty$ and $E = \bigcup_i E_i$, $\mu(E_i) < \infty$. WLOG, can assume that $\mu(E_i) > i$ (i.e., the measures grow). But we can always find $K_i^{\text{cpt}} \subset E$ s.t. $\mu(K_i) > i$. Thus, the sup is ∞. $\square$

Note: If X^{LCH} is σ-compact, then every Radon measure on X is in fact regular.

Prop: If μ is a σ-finite Radon measure on X and E is Borel,

- $\forall \varepsilon > 0 \ \exists F^{\text{closed}} \subset E \subset U^{\text{open}}$ s.t. $\mu(U \setminus F) < \varepsilon$.
- $\exists A^{F_\sigma}, B^{G_\delta}$ s.t. $A \subset E \subset B$ and $\mu(B \setminus A) = 0$.

Thm: Let X be LCH and s.t. every open set is σ-compact, then any Borel measure satisfying $\mu(K^{\text{cpt}}) < \infty \ \forall K^{\text{cpt}}$ is regular (!!!)

To prove this, we need Riesz Representation Thm...

Remark: Let X be LCH and let μ be a Borel measure s.t. $\mu(K^{\text{cpt}}) < \infty$ $\forall K^{\text{cpt}}$. Then, $C_{\text{cpt}}(X) \subset L^1(\mu)$. ← b/c $f \in C_{\text{cpt}}(K) \Rightarrow \int_X f = \int_{K^{\text{cpt}}} f \leq \int \chi_K \|f\|_\infty \leq \underbrace{\mu(K) \|f\|_\infty}_{<\infty}$

Compactly supported continuous fcts. on X

Lemma: Let ℓ be a positive functional on $C_{cpt}(X^{LCH})$, then $\forall K^{cpt} \subset X \;\; \exists C_K$ s.t. $\ell(f) \le C_K \|f\|_\infty \;\; \forall f$ s.t. $\operatorname{supp} f \subset K^{cpt}$.

Pf: (for ℓ $\mathbb{R}$-valued) For $K^{cpt} \subset X$, let $U^{open} \supset K$. Let ϕ be as in the LCH-Urysohn Lemma; i.e., $\begin{cases} \phi(x) = 1 & \text{on } K \\ \phi(x) = 0 & \text{outside any } V \subset U \end{cases}$

open and precpt.

If $\operatorname{supp} f \subset K^{cpt}$, then $f \le \chi_K \|f\|_\infty \le \phi \|f\|_\infty$.

So, $\phi(x)\|f\|_\infty - f \ge 0$ and compactly supp. (on $\bar{V}$).

So, $0 \le \ell(\phi\|f\|_\infty - f) = \|f\| \underbrace{\ell(\phi)}_{= C_K} - \ell(f)$.

$$\Rightarrow \ell(f) \le C_K \|f\|. \qquad \square$$

EXTREMELY IMPORTANT !!!

Thm: (RIESZ REPRESENTATION) Let ℓ be a positive functional on $C_{cpt}(X^{LCH})$. Then $\exists!$ Radon measure μ on X^{LCH} s.t.

$$\ell(f) = \int f \, d\mu .$$

Moreover,

$f \prec U^{open}$ if $0 \le f \le 1$, $\operatorname{supp} f \subset U^{open}$

$$\forall U^{open} \subset X^{LCH}, \quad \mu(U) = \sup\{\ell(f) : f \in C_{cpt}(X) \text{ and } f \prec U\}$$

$$\forall K^{cpt} \subset X^{LCH}, \quad \mu(K) = \inf\{\ell(f) : f \in C_{cpt}(X) \text{ and } f \ge \chi_K\}$$