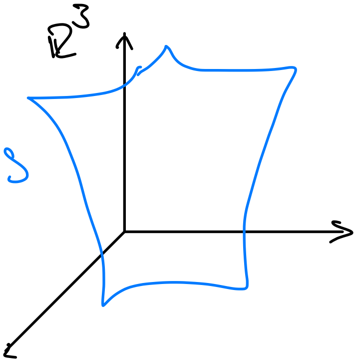
Lecture 1

07/09/2023

Introduction & Review of Differential Geometry

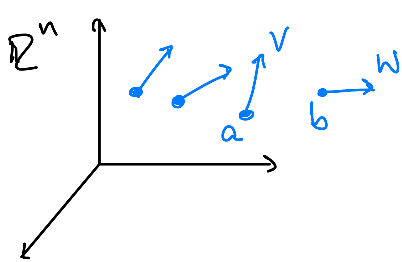
Riemannian geometry is a generalization of the geometry of surfaces.



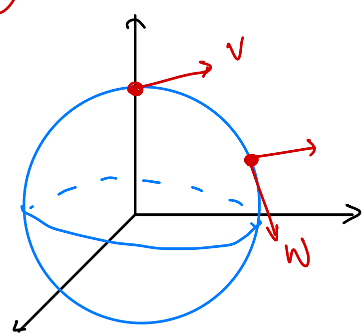
generalize to the setting of diff. manifolds.
(easy to generalize to $\mathbb{R}^n$ ...)

We will consider manifolds "on their own" (not immersed/embedded).

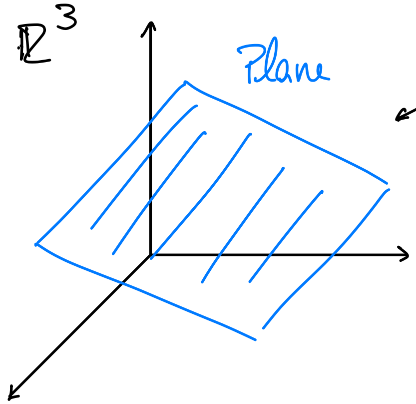
- **Scalar products:**



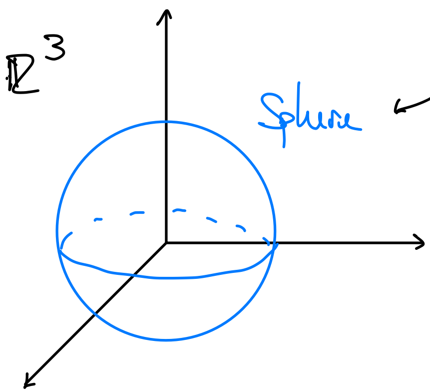
Tangent vectors



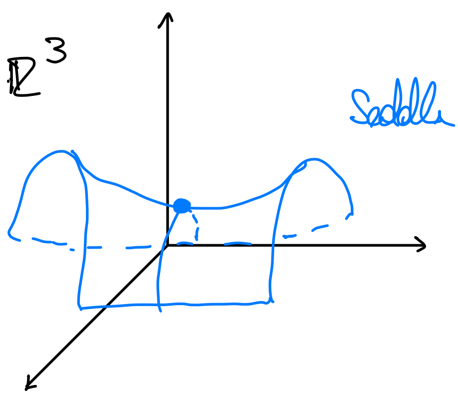
- Need to define the length of curves, area/volume, curvature



Curvature of the plane is zero.

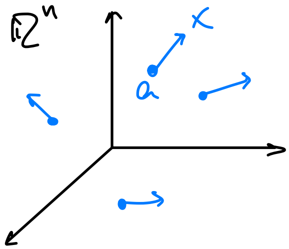


S^n has curvature 1



Negative curvature

* TANGENT VECTORS in $\mathbb{R}^n$



(1) Directed line segments.

It has a starting pt. $a = (a^1, \_\_, a^n)$ and end pt. $x = (x^1, \_\_, x^n)$.

TANGENT SPACE TO $\mathbb{R}^n$ AT a:

$$T_a\mathbb{R}^n = \text{all pairs } (a, x) = X_a$$

There is a 1-1 correspondence

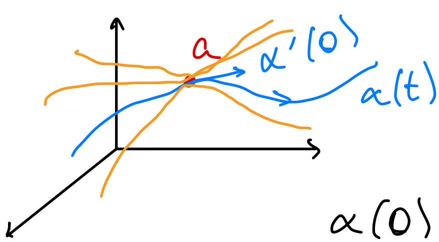
n-dim. vec. space

$$\varphi_a : T_a\mathbb{R}^n \longrightarrow V^n$$

$$\varphi_a(X_a) := (x^1 - a^1, \_\_, x^n - a^n)$$

Other equivalent ways:

- Let $\alpha(t)$ $\alpha : (-\varepsilon, \varepsilon) \to \mathbb{R}^n$ be a C^1 curve in $\mathbb{R}^n$ passing through a at $t = 0$.



Consider the following equivalence relation between curves: $\alpha(t) \sim \beta(t)$ if $\alpha(0) = \beta(0)$ and $\alpha'(0) = \beta'(0)$.

Thus, $[\alpha(t)] \underset{T_a\mathbb{R}^n}{\longleftrightarrow}$ tangent vectors.

- Consider $C^\infty(a)$ = collection of all C^∞ fcts whose domain include $a \in \mathbb{R}^n$.

$$X_a = \sum_{i=1}^{n} \partial_i E_{ia}$$

$$\boxed{\begin{aligned} V^n &= \operatorname{span}(e_1, \ldots, e_n) \\ \varphi_a &: T_a\mathbb{R}^n \longrightarrow V^n \\ E_{ia} &= \varphi_a^{-1}(e_i) \end{aligned}}$$

DIRECTIONAL DERIVATIVE OF f AT a:

$$Df(a) = \sum_{i=1}^{n} \partial_i \left.\frac{\partial f}{\partial x_i}\right|_{a = (a^1, \ldots, a^n)}$$

(X_a does not need to be unit).

So,

$$X_a^* f = \sum_{i=1}^{n} \partial_i \left(\frac{\partial f}{\partial x_i}\right)_a$$

Derivation

$$X_a^* = \sum_{i=1}^{n} \partial_i \left(\frac{\partial}{\partial x_i}\right) \text{ evaluated at } a.$$

Satisfying (1) Linearity

(2) Leibniz rule.

Def: (REGULAR SURFACES) A subset $S \subset \mathbb{R}^3$ is a <u>regular surface</u> if for any $p \in S$ there is an open subspace $U \subset \mathbb{R}^2$ and

$$\varphi : U \longrightarrow S$$

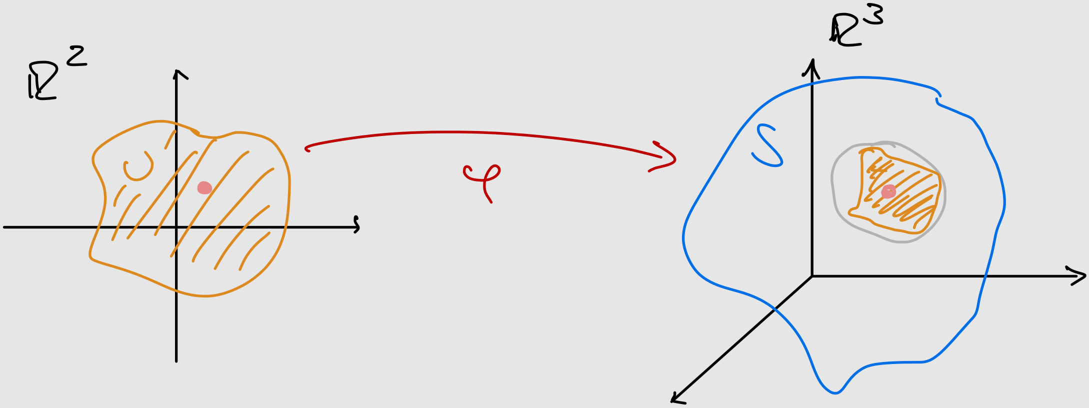
$$\varphi(u,v) = (x(u,v), y(u,v), z(u,v)),$$

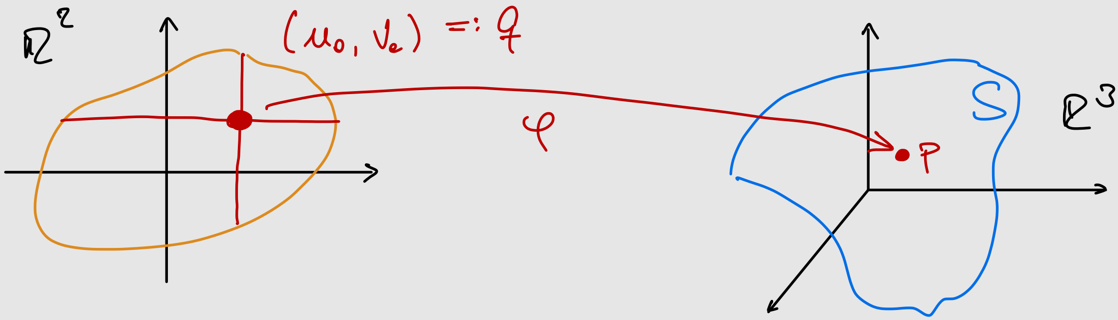
$(u,v) \in U$, such that

1. φ is differentiable (x, y, z are diff.)
2. For any $q \in U$, $d\varphi_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective.
3. $\exists$ an open set $V \subset \mathbb{R}^3$ which contains p and is s.t. $\varphi(U) = V \cap S$, $\varphi : U \to V \cap S$ is a homeomorphism

$\Leftrightarrow$ columns of $d\varphi_q$ are linearly indep.

$\Leftrightarrow$ rank $d\varphi_q = 2$.





Want to understand the derivations

$$\frac{\partial}{\partial u} = d\varphi_q E_1 = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix}$$

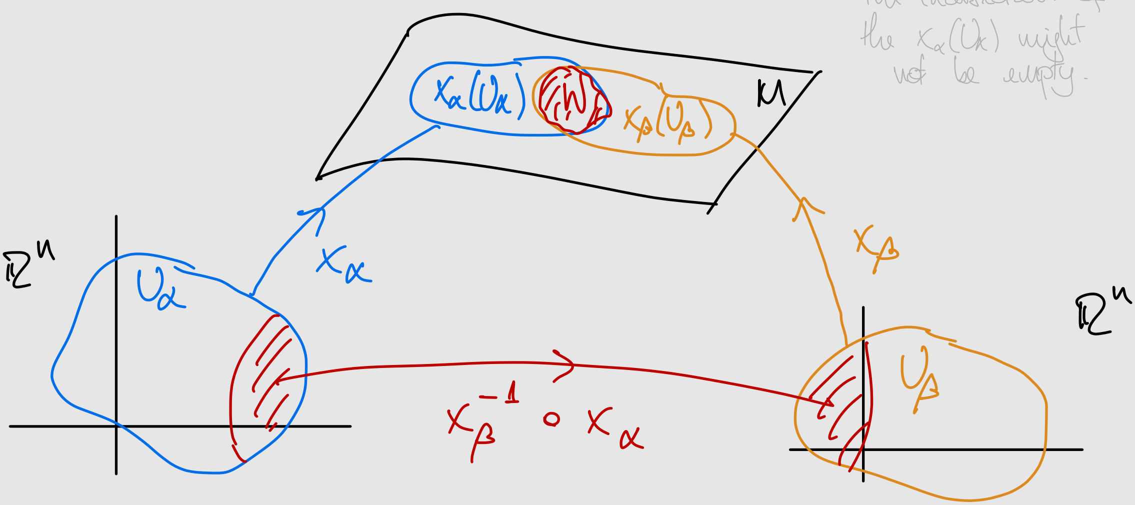
$$\frac{\partial}{\partial v} = d\varphi_q E_2 = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}$$

Def: (DIFFERENTIABLE MANIFOLDS) A differentiable manifold M^n of dimension n is a set and a family of injective maps

$$x_\alpha : U_\alpha \subset \mathbb{R}^n \longrightarrow x_\alpha(U_\alpha) \subset M$$

such that (1) $\bigcup_\alpha x_\alpha(U_\alpha) = M$.

the intersection of the $x_\alpha(U_\alpha)$ might not be empty.



(2) for any α, β with

$$x_\alpha(U_\alpha) \cap x_\beta(U_\beta) =: W \neq \emptyset$$

the sets $x_\alpha^{-1}(W)$ and $x_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and $x_\beta^{-1} \circ x_\alpha$ are differentiable.

(3) $\{(U_\alpha, x_\alpha)\}$ is maximal.

EXAMPLES OF DIFFERENTIABLE MANIFOLDS

(1) $\mathbb{R}^n$, Id.

(2) S^{n-1} in $\mathbb{R}^n$ w/ stereographic projection.

(3) The **REAL PROJECTIVE SPACE**:

$$\mathbb{RP}^n := \{\text{all straight lines of } \mathbb{R}^{n+1} \text{ through } \vec{O}\}$$

$$= \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \big/ \sim \quad \to \text{identify colinear pts.}$$

$$(x_1, \ldots, x_{n+1}) \sim \lambda\,(x_1, \ldots, x_{n+1}), \quad \lambda \in \mathbb{R}\setminus\{0\}.$$

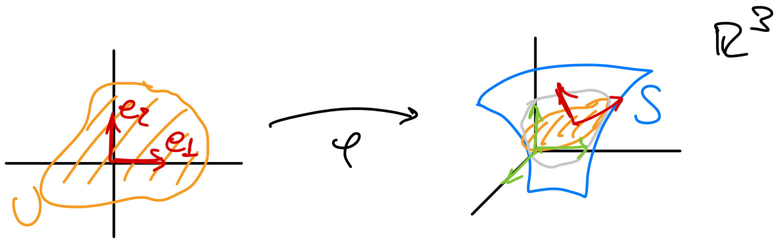
Now, we claim that $\mathbb{RP}^n$ has a "natural" differentiable structure. For that, need to cover $\mathbb{RP}^n$ w/ sets and then define the coördinate charts so that the transition fcts are differentiable as maps from/to Euclidean spaces. (next lecture)

LECTURE 2

REVIEW & EXAMPLES OF DIFFERENTIABLE MANIFOLDS

12/09/2023

* RECAP: A surface in $\mathbb{R}^3$ is a set $S \subset \mathbb{R}^3$



$$\varphi(u,v) = (x(u,v), y(u,v), z(u,v))$$

1) φ differentiable, i.e., $d\varphi$ exists.

2) φ homeo. onto the image

3) $d\varphi$ is full - rank.

$$d\varphi_q = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix}$$

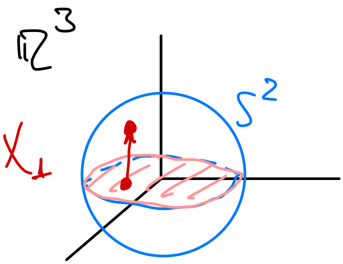
Now,

$$d\varphi_q e_1 = \begin{pmatrix} \frac{\partial x}{\partial u} \\ \frac{\partial y}{\partial u} \\ \frac{\partial z}{\partial u} \end{pmatrix} ; \qquad d\varphi_q e_2 = \begin{pmatrix} \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial v} \end{pmatrix}$$

$$\frac{\partial}{\partial u} := d\varphi_q e_1 \qquad \frac{\partial}{\partial v} := d\varphi_q e_2$$

EXAMPLE: ROUND SPHERE

$$S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$$



Claim: S^2 is a regular surface of $\mathbb{R}^3$.

Pf: Consider "Inverse to a projection operator"

$$X_1(x, y) := \left(x, y, \sqrt{1-(x^2+y^2)} \right),$$

where $(x, y) \in U$, U is an open disk. Note that X_1^{-1} is a projection. Compute:

$$dX_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ \frac{-x}{\sqrt{1-(x^2+y^2)}} & \frac{-y}{\sqrt{1-(x^2+y^2)}} \end{pmatrix}.$$

Full rank ✓

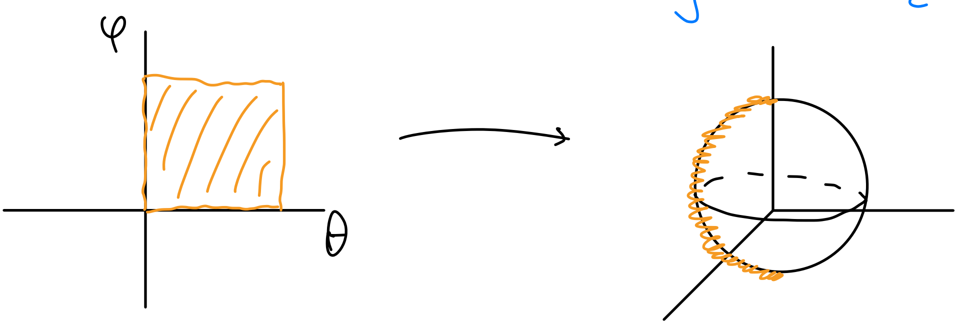
Transition maps are automatically differentiable

$\square$

GEOGRAPHICAL COORDINATES:

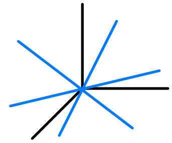
$$V = \{(\theta, \varphi) : 0 < \theta < \pi,\ 0 < \varphi < 2\pi\}$$

$$X(\theta, \varphi) = (\underbrace{\sin\theta\cos\varphi}_{= x},\ \underbrace{\sin\theta\sin\varphi}_{= y},\ \underbrace{\cos\theta}_{= z})$$



[ASIDE: def. of diff. manifold was motivated by the def. of regular surface.]

* REAL PROJECTIVE SPACE $\mathbb{R}P^n$



$$\mathbb{R}P^n := \left(\mathbb{R}^{n+1} \setminus \{0\}\right) \Big/ \sim$$

identify colinear pts.

where

$$(x_1, \ldots, x_{n+1}) \sim (\lambda x_1, \ldots, \lambda x_{n+1}), \quad \lambda \in \mathbb{R} \setminus \{0\}.$$

Claim: There is a natural differentiable structure in $\mathbb{R}P^n$

COVER $\mathbb{R}P^n$ BY COORDINATE CHARTS: Define the following atlas $\{(V_i, \varphi_i)\}_{i=1}^{n+1}$

$$V_i := \{ [x_1, \ldots, x_n] : x_i \neq 0 \}$$

$$\varphi_i : V_i \longrightarrow \mathbb{R}^n$$

$$\varphi_i([x_1, \ldots, x_{n+1}]) := \left(\frac{x_1}{x_i}, \ldots, \widehat{\frac{x_i}{x_i}}, \ldots, \frac{x_{n+1}}{x_i} \right)$$

$$\varphi_i^{-1}(y_1,\dots,y_n) = [y_1,\dots,\underset{\uparrow \text{ i-th position}}{1},\dots,y_n].$$

Now, need to show it satisfies the usual conditions:

(1) φ_i is injective, onto its image:

<u>Injective</u>: if $\varphi_i(y_1,\dots,y_n) = \varphi_i(\tilde{y}_1,\dots,\tilde{y}_n)$

then $[y_1,\dots,1,\dots,y_n] = [\tilde{y}_1,\dots,1,\dots,\tilde{y}_n]$

$\Rightarrow \lambda = 1 \Rightarrow y_i = \tilde{y}_i \ \forall i$, hence injective.

<u>Onto</u>: Take $x \in V_i$

$$x = [x_1,\dots,x_i,x_{i+1},\dots,x_{n+1}].$$

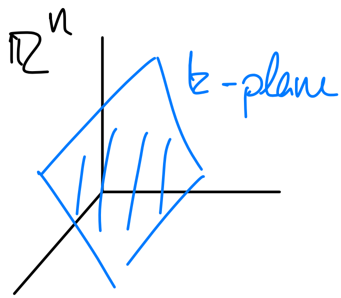
Take $y_j = \dfrac{x_j}{x_i}$. Now, WLOG $i > j$,

$$\varphi_i^{-1}(V_i \cap V_j) = \{(y_1,\dots,y_n) \in \mathbb{R}^n : y_j \neq 0\}.$$

So,

$$\varphi_j^{-1} \circ \varphi_i(y_1,\dots,y_n) = \varphi_j^{-1}([y_1,\dots,y_{i-1},1,y_i,\dots,y_n])$$
$$= \left(\frac{y_1}{y_j},\dots,\frac{y_{j-1}}{y_j},\dots,\frac{y_{i-1}}{y_j},\frac{1}{y_j},\frac{y_i}{y_j},\dots,\frac{y_n}{y_j}\right).$$

* **GRASSMANIAN MANIFOLDS $G(k,n)$**: This is the generalization of projective spaces. It is the set of k-planes through the origin.



Def: $G(k,n) := F(k,n) / \sim$

- $F(k,n)$ = set of k-frames
- k-frame = sequence of k-linearly independent vectors

Take

$$X_1 = (x_1^1, \_\_\_\_\_, x_1^n)$$

$$X_2 = (x_2^1, \_\_\_\_\_, x_2^n)$$

$$\vdots$$

$$X_k = (x_k^1, \_\_\_\_\_, x_k^n)$$

This can be naturally identified with a $(k \times n)$ - matrix of rank k (since the vectors are linearly independent)

$$\begin{pmatrix} x_1 \\ \vdots \\ x_k \end{pmatrix}_{k \times n} = X$$

But this is just an <u>open</u> <u>subset</u> <u>of</u> <u>$M^{k \times n}$</u>, where $M^{k \times n}$ = space of all $k \times n$ matrices.

Now, two frames X and Y

$$X = (x_1, \ldots, x_k), \quad Y = (y_1, \ldots, y_k)$$

determine the same plane iff

$$y_i = \sum_{j=1}^{k} \alpha_{ij} x_j,$$

where $a = (\alpha_{ij})$ is a non-singular $k \times k$

matrix such that $Y = aX$. So, we cover $G(k,n)$ by the following coördinate charts:

$J := (j_1, \ldots, j_k)$ ordered subset of $(1, \ldots, n)$

$X_J :=$ $(k \times k)$ submatrix of $(k \times n)$ X that has rank k.

We define a complementary submatrix by taking X and removing the columns that corresponds to $j_1, \ldots, j_k$. Let U_J be an open set in $F(k,n)$ consisting of matrices for which X_J is <u>not</u> singular. Take $\pi : F(k,n) \longrightarrow G(k,n)$, $U_J = \pi(\tilde{U}_J)$ $y \in \tilde{U}_J$ $\exists!$ $k \times n$ matrix X in which the submatrix X_J is Id. For $J = (1, \ldots, k)$,

$$X = \left(\begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{matrix} \middle| \begin{matrix} X_{k,k+2} & \cdots & X_{1,n} \\ \vdots & & \vdots \\ X_{k,k+1} & \cdots & X_{k,n} \end{matrix} \right).$$

* MAPS BETWEEN MANIFOLDS:

Thm: Let M_1^n and M_2^m be differentiable manifolds and $\varphi : M_1^n \to M_2^m$ a map between them. We say that φ is differentiable at $p \in M_1^n$ if given a parametrization

$$Y : V \subset \mathbb{R}^m \longrightarrow M_2 \text{ at } \varphi(p)$$

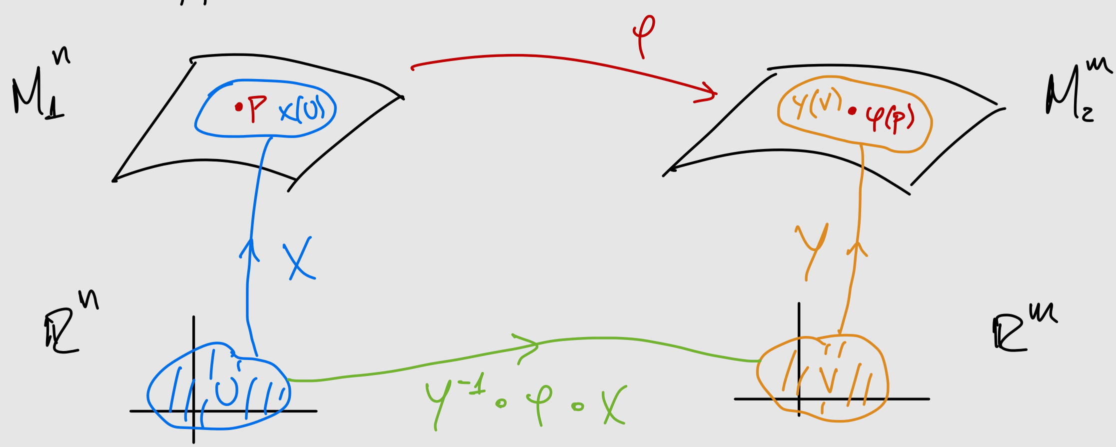
there exists a parametrization

$$X : U \subset \mathbb{R}^n \longrightarrow M_1 \text{ at } p$$

such that $\varphi(X(U)) \subset Y(V)$ and the map

$$Y^{-1} \circ \varphi \circ X : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

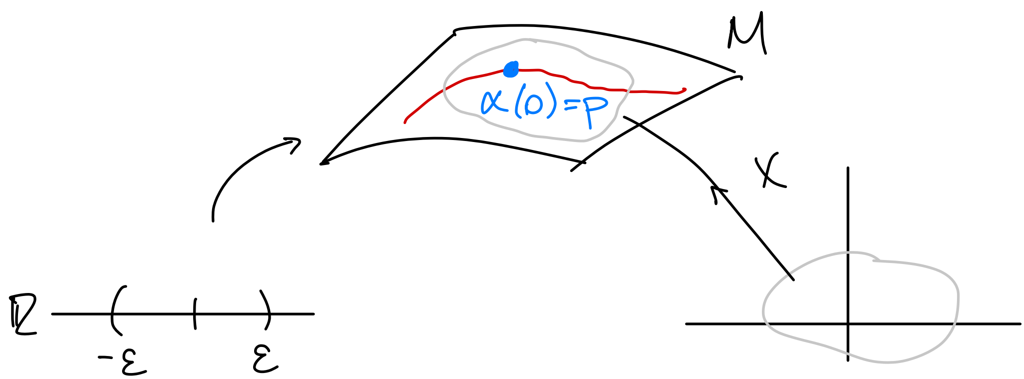
is differentiable.



* **DIFFERENTIABLE CURVES**: A curve

$$\alpha : (-\varepsilon, \varepsilon) \longrightarrow M$$

which is differentiable is called a diff. curve.



In local coördinates $(x_1(t), \ldots, x_n(t))$, the curve is $X^{-1} \circ \alpha(t)$.

THE TANGENT VECTOR TO THE CURVE α AT $t=0$:

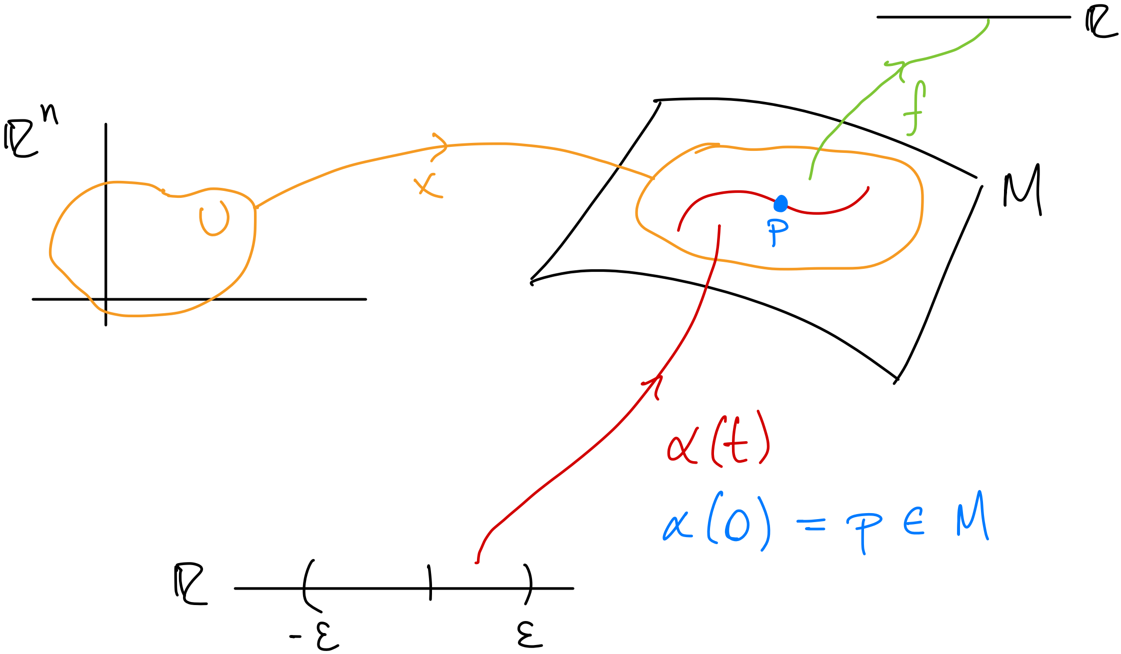
is a function $\alpha'(0) : D \longrightarrow \mathbb{R}$ s.t.

$$\boxed{\alpha'(0)(f) = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0}}$$

* **TANGENT SPACE T_pM:**

T_pM = the set of all tangent vectors to M at the pt. $p \in M$.

Choose $x : U \to M^n$



NOTE: $f \circ \alpha(t) = f \circ x \circ x^{-1} \circ \alpha$, but

$$x^{-1} \circ \alpha(t) = (x_1(t), \ldots, x_n(t))$$

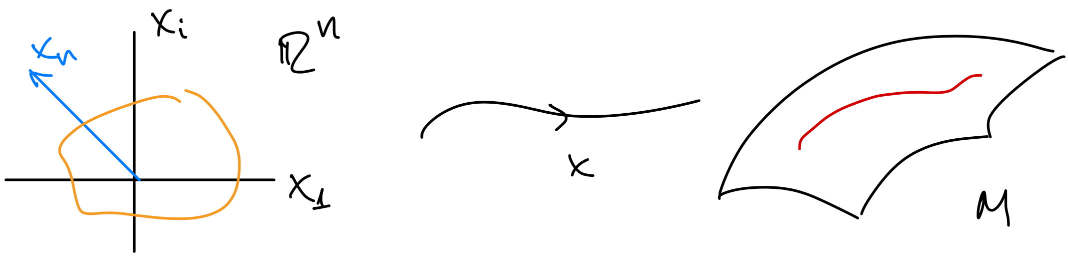
$$\alpha'(0)(f) = \frac{d}{dt}(f \circ \alpha)\Big|_{t=0} = \frac{d}{dt} f(x_1(t), \ldots, x_n(t))\Big|_{t=0}$$

$$= \sum_{i=1}^{n} x_i'(0) \left.\frac{\partial f}{\partial x_i}\right|_0$$

$$= \left(\underbrace{\sum_i x_i'(0) \left(\frac{\partial}{\partial x_i}\right)\Big|_0 \Big)}\, f$$

$$\Rightarrow \alpha'(0) = \sum_i x_i'(0) \left.\frac{\partial}{\partial x_i}\right|_0$$

So, what are the $\left.\frac{\partial}{\partial x_i}\right|_0$?



So, $\left\{ \left.\frac{\partial}{\partial x_i}\right|_0 \right\}_{i=1}^{n}$ are a basis for T_pM.

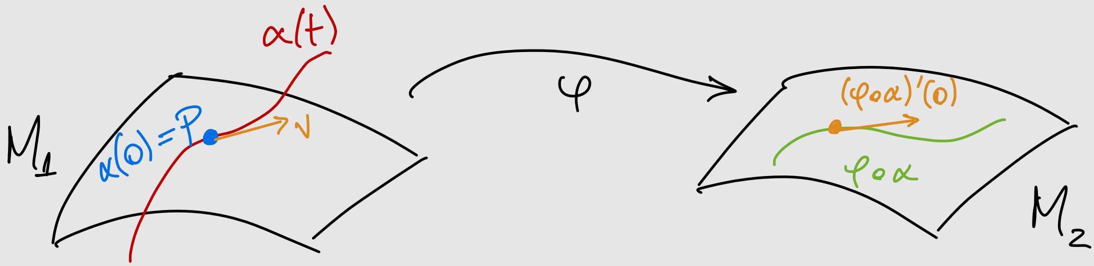
$\Rightarrow$ $\dim T_pM = n$.

* THE DIFFERENTIAL:

Def: (DIFFERENTIAL) Let M_1^n, M_2^m be differentiable manifolds and let $\varphi : M_1 \to M_2$ be a differentiable mapping.

For every $p \in M_1$, each $v \in T_pM$ is such that $\alpha(0) = p$, $\alpha'(0) = v$. So,

$$\boxed{d\varphi(v) = (\varphi \circ \alpha)'(0)}$$



Note that $d\varphi$ does <u>not</u> depend on the curve.

Def: Let M, N be diff. manifolds. A map $\varphi : M \to N$ is a

- **DIFFEOMORPHISM** if φ is differentiable and bijective
- **LOCAL DIFFEOMORPHISM** at $p \in M$ if $\exists U \ni p$ and $\exists V \ni \varphi(p)$ s.t. $\varphi : U \to V$ is a diffeo.

Thm: Let $\varphi : M \to N$ be differentiable. Let $p \in M$ be s.t. $d\varphi_p : T_pM \longrightarrow T_{\varphi(p)}N$ is an isomorphism. Then φ is a local diffeo. at p. (By the Inverse Function Thm)

Def: Let M, N be differentiable manifolds and $\varphi : M \to N$ differentiable.

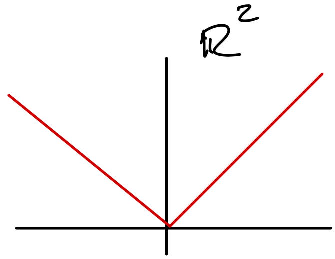
- φ is an **IMMERSION** if $d\varphi_p : T_pM \to T_{\varphi(p)}N$ is injective $\forall p \in M$.

If it is also a homeomorphism onto $\varphi(M)$, then φ is an **EMBEDDING**.

EXAMPLE:

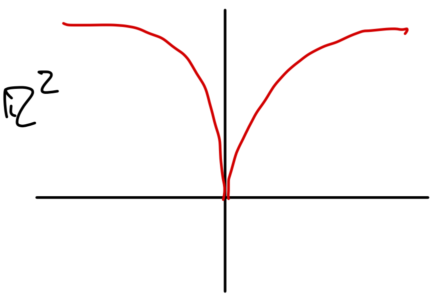
1) $\alpha : \mathbb{R} \longrightarrow \mathbb{R}^2$

$$\alpha(t) = (t, |t|)$$



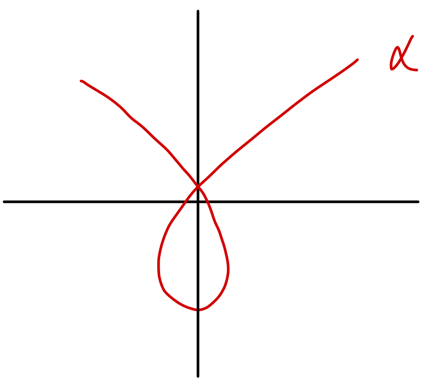
NOT IMMERSION (not even differentiable)

2)



NOT IMMERSION b/c not full-rank at all pts.

3)



$$\alpha(t) = (t^3 - 4t,\ t^2 - 4)$$

IS IMMERSION.

NOT EMBEDDING.

Lecture 3

14/09/2023

Tangent Bundles, Riemannian Metric

Def: (Tangent bundle TM) Let M^n be a differentiable manifold. Then

$$TM := \{ (p,v) : p \in M, v \in T_pM \}$$

$\dim TM = 2n$

Atlas for TM: Let $\{(U_\alpha, x_\alpha)\}$ be an atlas for M. Then, define

$$\boxed{V_\alpha := U_\alpha \times \mathbb{R}^n}$$

$$y_\alpha : \underbrace{U_\alpha}_{} \times \underbrace{\mathbb{R}^n}_{} \longrightarrow \underbrace{TM}_{\text{elements are of the form } (p,v)}$$

There is a basis for T_pM associated with x_α:

$$\left\{ \frac{\partial}{\partial x_1^\alpha}, \text{—}, \frac{\partial}{\partial x_n^\alpha} \right\}.$$

So, we set

$$y_\alpha (x_1^\alpha, \text{—}, x_n^\alpha, u_1, \text{—}, u_n)$$
$$:= \left(x_\alpha (x_1^\alpha, \text{—}, x_n^\alpha), \sum_{i=1}^n u_i \frac{\partial}{\partial x_i^\alpha} \right)$$

NOTE: the map $(u_1, \text{—}, u_n) \longmapsto \sum_{i=1}^n u_i \frac{\partial}{\partial x_i^\alpha}$ is just the differential of x_α at $x_1^\alpha, \text{—}, x_n^\alpha$

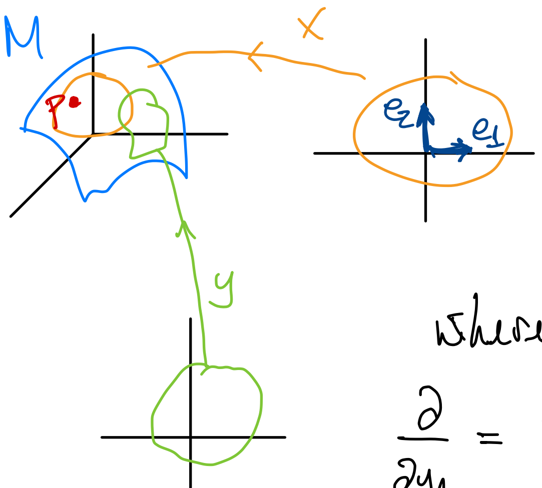
$$(dx_\alpha)_{(x_1^\alpha, \text{—}, x_n^\alpha)}.$$

If $(p, v) \in y_\alpha(V_\alpha) \cap y_\beta(V_\beta)$, then

$$(p, v) = \left(x_\alpha (q_\alpha), dx_\alpha (v_\alpha) \right) = \left(x_\beta (q_\beta), dx_\beta (q_\beta) \right)$$

TRANSITION MAPS: $y_\beta^{-1} \circ y_\alpha = \left(x_\beta^{-1} \circ x_\alpha, d(x_\beta^{-1}) \circ dx_\alpha \right)$
$$= \left(x_\beta^{-1} \circ x_\alpha, d(x_\beta^{-1} x_\alpha) \right)$$

* **ORIENTATION**: Consider regular surfaces



$$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right\} \to \left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right]$$

$$\left\{\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\} \to \left[\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right]$$

where:

$$\frac{\partial}{\partial y_1} = \frac{\partial x_1}{\partial y_1}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial y_1}\frac{\partial}{\partial x_2}$$

$$\frac{\partial}{\partial y_2} = \frac{\partial x_1}{\partial y_2}\frac{\partial}{\partial x_1} + \frac{\partial x_2}{\partial y_2}\frac{\partial}{\partial x_2}$$

$$\begin{pmatrix} \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial y_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} \\ \frac{\partial x_1}{\partial y_2} & \frac{\partial x_2}{\partial y_2} \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial x_2} \end{pmatrix}$$

and we need that $\boxed{\det d(x^{-1} \circ y) > 0}$.

Then the orientations are "consistent"

Def: (ORIENTABILITY) Let M be a differentiable manifold. Then, we say that M is **orientable** if M has a differentiable structure $\{(U_\alpha, x_\alpha)\}$ s.t. $\forall$ pair α, β s.t.

$$x_\alpha(U_\alpha) \cap x_\beta(U_\beta) \neq \emptyset$$

the differential of the transition map has positive determinant.

If it is not possible to find such charts, then M is <u>nonorientable</u>

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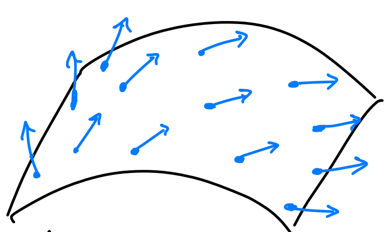
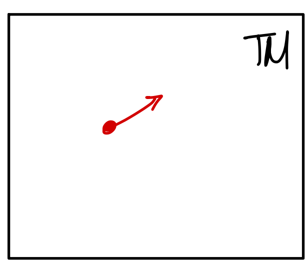
* **VECTOR FIELDS**:



M

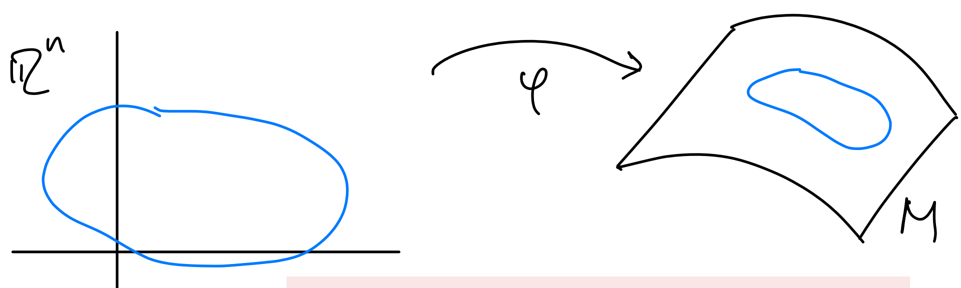


Def: (VECTOR FIELDS) A vector field X on M is a correspondance that associates to each $p \in M$, a vector $X(p) = X_p \in T_pM$,

$$X : M \longrightarrow TM.$$

If X is differentiable, then it is called a differentiable vec. field.

In (messy) local coördinates:



$\left\{\frac{\partial}{\partial x_i}\right\}_{i=1}^{n}$

$$\boxed{X = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}}$$

X differentiable $\iff$ $a_i(p)$ are all differentiable

Collection of differentiable vector fields on M is denoted $\mathfrak{X}(M)$.

Another way: as acting on fcts.

$$X : \mathcal{D} \to \{\text{fcts.}\}$$ might not be diff'able

X differentiable $\iff$ $X(f)$ is differentiable.

* **LIE BRACKETS**: Let $X, Y \in \mathfrak{X}(M)$, then the Lie bracket between X and Y is:

$$[X, Y](f) := (XY - YX)(f)$$

This is a derivation.

Lemma: Given $X, Y \in \mathfrak{X}(M)$, $\exists!\ Z \in \mathfrak{X}(M)$ s.t. $\forall f \in C^\infty(M)$, $Z(f) = (XY - YX)(f)$.

Pf: Compute in local coord. and see that 2^{nd} derivatives cancel out. $\square$

PROPERTIES:

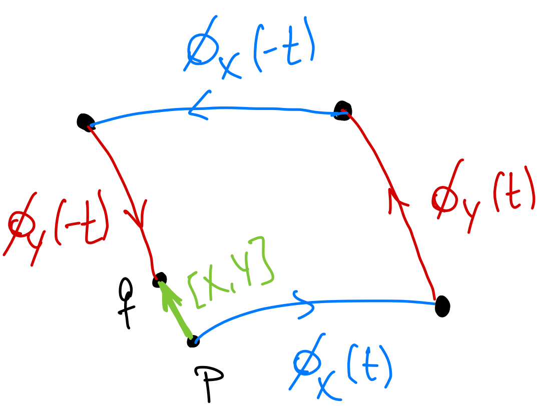
- $[X,Y] = -[Y,X]$ (anticommutativity)

- $[aX+bY, Z] = a[X,Z] + b[Y,Z]$
(linearity)

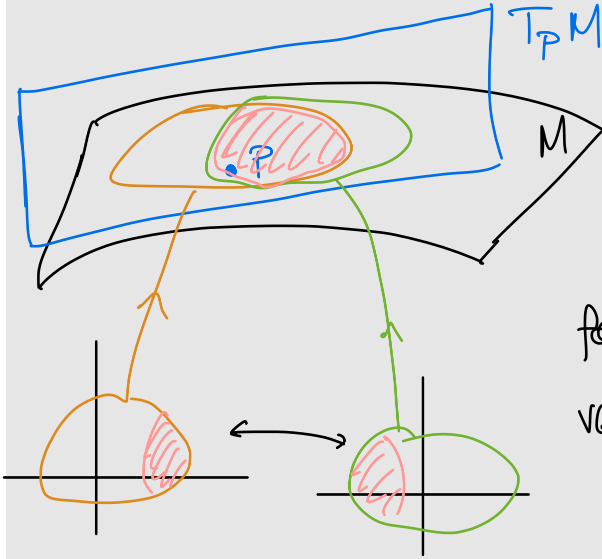
- $[[X,Y],Z] + [[Z,X],Y] + [[Y,Z],X] = 0$
(Jacobi identity)

- $[fX, gY] = fg[X,Y] + f X(g) Y - g Y(f) X$
$\forall f,g \in C^\infty(M)$.

↳ Measures how much X varies along Y



* **RIEMANNIAN METRIC:**



Def: (RIEMANNIAN METRIC)

A Riemannian metric g on a diff. manifold M is a (smoothly varying) inner product on the tangent spaces of M that satisfies the following properties: $g_p : T_pM \times T_pM \to \mathbb{R}$

- $g_p(v, w) = g_p(w, v) \quad \forall\, v, w \in T_pM$
- $g_p(v, v) \geq 0 \quad \forall\, v \in T_pM$
- $g_p(v, v) = 0 \iff v = 0$

Every smooth manifold admits (many) Riemannian metrics

Lecture 4

19/09/2023

Riemannian Metrics

Recall: We define a Riemannian metric on a smooth manifold M as: $g_p : T_pM \times T_pM \to \mathbb{R}$ smooth varying inner product on the tangent spaces of M s.t.

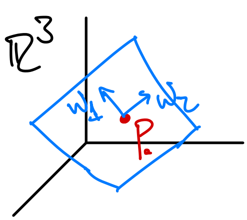
(i) $g_p(v, w) = g_p(w, v) \quad \forall v, w \in T_pM$

(ii) $g_p(v, v) \geq 0 \quad \forall v \in T_pM$

(iii) $g_p(v, v) = 0 \iff v = 0$.

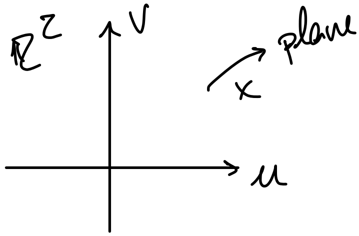
In local coördinates, $g_{ij} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$ defined on a chart U.

EXAMPLE: Plane in $\mathbb{R}^3$ that passes through $P_0 := (x_0, y_0, z_0) \in \mathbb{R}^3$ and contains orthonormal vectors



$$w_1 = (a_1, a_2, a_3); \quad w_2 = (b_1, b_2, b_3).$$

Now, we want to write g_{ij}. For that, parametrize the plane by



$$x(u,v) := P_0 + u w_1 + v w_2$$

So,

$$g_{11} = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle$$

$$g_{12} = g_{21} = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle$$

$$g_{22} = \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle$$

Moreover,

$$\frac{\partial}{\partial u} = dx \cdot e_1, \quad \frac{\partial}{\partial v} = dx \cdot e_2$$

So,

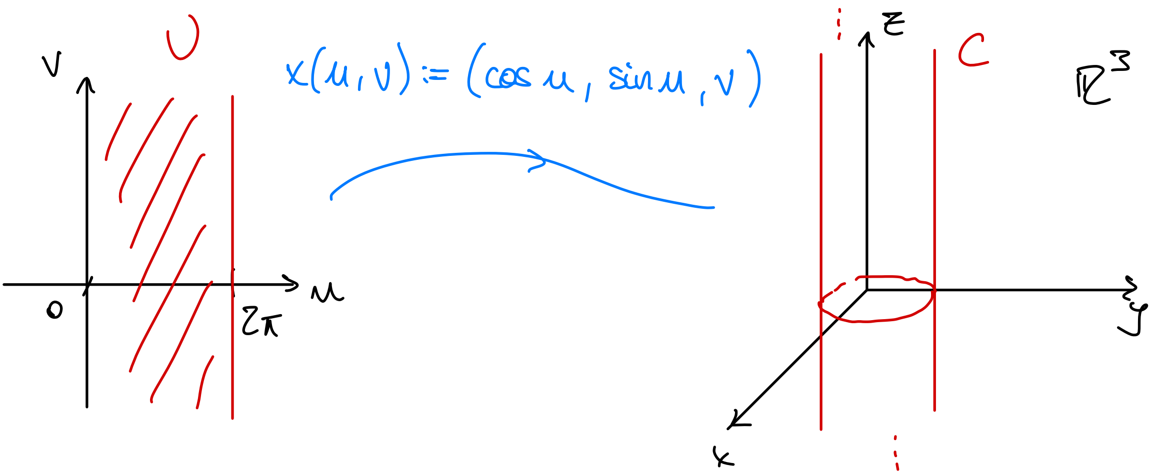
$\frac{\partial}{\partial u} = w_1$, $\frac{\partial}{\partial v} = w_2$

$$dx = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} \end{pmatrix} = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{pmatrix}$$

$$\Rightarrow \boxed{g = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}.$$

! EXAMPLE: Right circular cylinder

$$C := \{ (x,y,z) \in \mathbb{R}^3 : x^2 + y^2 = 1 \}$$



Find g_{ij}:

$$\begin{cases} g_{11} = ? & \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \rangle \\ g_{21} = g_{12} = ? & \langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \rangle \end{cases}$$

$$g_{22} = ? \qquad \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle$$

Same as for abstract manifolds

$$\frac{\partial}{\partial u} = dx\, e_1 , \qquad \frac{\partial}{\partial v} = dx\, e_2$$

$$dx = \begin{array}{c} \begin{pmatrix} -\sin u & 0 \\ \cos u & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{array}{cc} \underbrace{\qquad\quad}_{\frac{\partial}{\partial u}} & \underbrace{\quad}_{\frac{\partial}{\partial v}} \end{array} \end{array}$$

$$g_{11} = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = \sin^2 u + \cos^2 u = 1$$

$$g_{21} = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = 0$$

$$g_{22} = \left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right) = 1$$

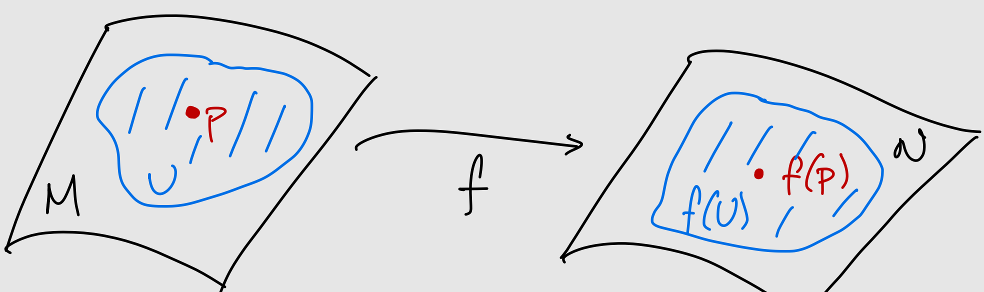
$$\Rightarrow \quad g_{ij} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Def: (ISOMETRIES) Let M, N be Riemannian manifolds. A diffeomorphism $f: M \to N$ is called an <u>ISOMETRY</u> if the

$$\langle u, v \rangle_p = \langle df_p u, df_p v \rangle_{f(p)}$$

$\forall p \in M$ and $\forall u, v \in T_p M$.

Def: (LOCAL ISOMETRY) Let M, N be Riemannian manifolds. Then a differentiable map $f: M \to N$ is a <u>LOCAL</u> <u>ISOMETRY</u> at $p \in M$ if there exists a neighborhood $U \subset M$ of p s.t. $f|_U : U \to f(U)$ is an isometry.



Def: M is locally isometric to N if for all $p \in M$, there exists a neighborhood U for which $f|_U$ is an isometry onto $f(U)$.

— // —

HIGHER DIMENSIONS: Take some examples:

(1) $M = \mathbb{R}^n$.

ith position

$$\frac{\partial}{\partial x_i} = (0, \_\_\_, 0, 1, 0, \_\_\_\_\_, 0)$$

Then $g_{ij} = \delta_{ij}$.

(2) Metric induced by immersion: Suppose that $f: M^n \longrightarrow N^{n+k}$ is an immersion. Then $df_p : T_pM \longrightarrow T_{f(p)}N$

$$\langle u, v \rangle_p \overset{def}{=} (df_p u, df_p v)_{f(p)}.$$

* Need immersion to have positive-definiteness !

(3) Metrics induced by inclusions: take the inclusion $S^n \hookrightarrow \mathbb{R}^{n+1}$,

$$S^n = \left\{ (x_1, \dots, x_{n+1}) : \sum_{i=1}^{n} x_i^2 = 1 \right\}.$$

Round sphere / Standard sphere

(4) Metric induced by products: Let M_1, M_2 be Riemannian manifolds. Consider the product manifold $M_1 \times M_2$. Recall the canonical proj.:

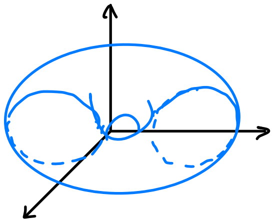
$$\pi_1 : M_1 \times M_2 \longrightarrow M_1$$

$$\pi_2 : M_1 \times M_2 \longrightarrow M_2$$

Then, the induced metric is defined as

$$\langle u, v \rangle_{(p,q)} \overset{\text{def}}{=} \langle d\pi_1 u, d\pi_1 v \rangle_p + \langle d\pi_2 u, d\pi_2 v \rangle_q$$

Example: Torus in $\mathbb{R}^3$



$$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n \text{ times}}.$$

We have the following natural projections

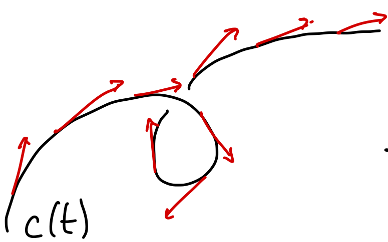
ith circle

$$\pi_i : T^n = S^1 \times \cdots \times S^1 \longrightarrow S^1$$

and define the metric from there. ■

---//---

Def: Let $c : I_t \longrightarrow \mathbb{R}$ be a piecewise smooth parametrized curve (I open in $\mathbb{R}$). Let $v(t)$ be a vector field along the curve $c(t)$. Then, $v(t)$ is the <u>velocity vector field</u> defined as



$$v(t) := \frac{dc}{dt} = dc\left(\frac{d}{dt}\right).$$

Then, the <u>speed</u> is defined as $\left|\frac{dc}{dt}\right| = \left\langle \frac{dc}{dt}, \frac{dc}{dt} \right\rangle^{1/2}$.

We can define the length of the curve $c(t)$ as:

$$L_{a,b}(c) = \int_a^b \left| \frac{dc}{dt} \right| dt .$$

Def: Let $\gamma : [a,b] \to (M^n, g)$ be a piecewise smooth curve. The length of γ (w.r.t. g) is

$$L_g(\gamma) := \int_a^b \underbrace{g_{\gamma(t)}(\gamma'(t), \gamma'(t))^{1/2}}_{\|\gamma'(t)\|} dt$$

---

* **EXISTENCE OF RIEMANNIAN METRICS**

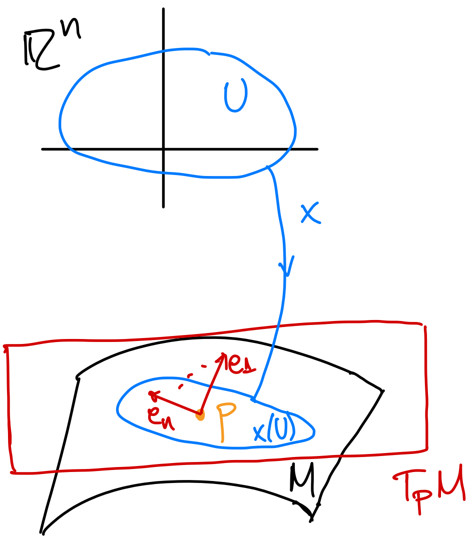
Prop: Every smooth manifold M admits (many) Riemannian metrics

Pf: Take an atlas $\{(V_\alpha, \varphi_\alpha)\}$ for M. On each V_α, define

$$\langle u, v \rangle_p^\alpha := \langle d\varphi_p^\alpha u, d\varphi_p^\alpha v \rangle_{\varphi(p)} . \qquad \square$$

* VOLUME:

Def: Let M^n be an oriented manifold and take



$x : U \subset \mathbb{R}^n \longrightarrow x(U) \subset M$ that belongs to the family of charts consistent w/ the orientation of M.

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis for T_pM.

Then, $$\frac{\partial}{\partial x_i} = \sum_j a_{ij}\, e_j$$

and we have that

$$g_{ik} = \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle$$

$$= \left\langle \sum_j a_{ij}\, e_j , \sum_l a_{kl}\, e_l \right\rangle$$

$$= \sum_{j,l} a_{ij}\, a_{kl} \langle e_j, e_l \rangle$$

$$= \sum_j a_{ij} a_{kj} .$$

So, let $A := (a_{ij})$. Then

$$\boxed{(g_{ij}) = A^T A} .$$

$$\Rightarrow \det (g_{ij}) = (\det A)^2$$

$$\Rightarrow \det A = \sqrt{\det (g_{ij})}$$

Then, the volume of the parallelopiped that $\left\{ \frac{\partial}{\partial x_i} \right\} = \{ e_i \}$ spans is equal to

$$\boxed{\text{volume} = \det (a_{ij}) = \sqrt{\det (g_{ij})}} .$$

Now, suppose there is another chart

$$y : V \subset \mathbb{R}^n \longrightarrow y(V) \subset M$$

s.t. $x(U) \cap y(V) \neq \emptyset$ and is consistent w/ the orientation of M. Then, similarly to x, we have the parametrization $\left\{\frac{\partial}{\partial y_i}\right\}$ for T_pM and the metric

$$h_{ij} = \left\langle \frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j} \right\rangle .$$

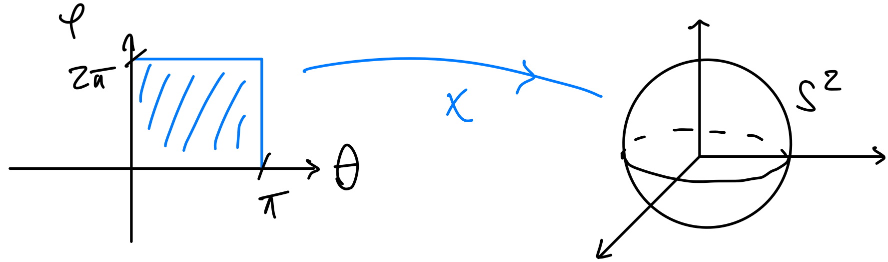
Then,

$$\begin{aligned}
\sqrt{\det(g_{ij})} &= \mathrm{vol}\left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}\right) \\
&= J \ \mathrm{vol}\left(\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\right) \\
&= J \sqrt{\det(h_{ij})} ,
\end{aligned}$$

where $J = \det\left(dy^{-1} \circ dx\right)$ is obtained from the Change of Variables formula !

Simple example

Ex: Calculate the volume (i.e., area) of S^2



where $x(\theta, \varphi) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$.

First, calculate

$$dx = \begin{pmatrix} \cos\theta\cos\varphi & -\sin\theta\sin\varphi \\ \cos\theta\sin\varphi & \sin\theta\cos\varphi \\ -\sin\theta & 0 \end{pmatrix}$$

$$\underbrace{\quad\quad}_{\frac{\partial}{\partial\theta}} \qquad \underbrace{\quad\quad}_{\frac{\partial}{\partial\varphi}}$$

Then,

$$g_{11} = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\theta} \right\rangle = 1$$

$$g_{21} = g_{12} = \left\langle \frac{\partial}{\partial\theta}, \frac{\partial}{\partial\varphi} \right\rangle = 0$$

$$g_{22} = \left\langle \frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\varphi} \right\rangle = \sin^2\theta$$

So, integrate now

$$\int_0^{2\pi}\int_0^{\pi} \sin\theta \; d\theta \, d\varphi = 4\pi .$$

$\blacksquare$

---

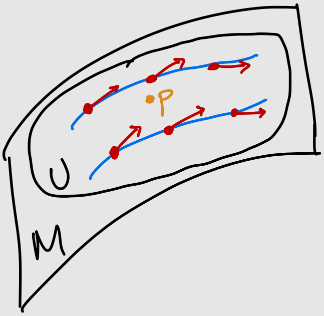
* **LIE BRACKETS:** Let $X, Y \in \mathfrak{X}(M)$. Then recall that we can apply those to smooth functions on M $f \in C^\infty(M)$; i.e., $X(Yf)$ and $Y(Xf)$.

Claim: There exists a unique vector field $Z \in \mathfrak{X}(M)$ such that

$$Zf = (XY - YX)f \quad \forall f \in C^\infty(M).$$

"Lie bracket $[X,Y](f)$".

Thm: Let $X \in \mathfrak{X}(M)$ and $p \in M$. Then, there exists



- a neighborhood $U \subset M$ of p,
- an interval $(-\delta, \delta)$, $\delta > 0$,
- differentiable map

$$\phi_t : (-\delta, \delta) \times U \longrightarrow M$$

such that the curve $t \longmapsto \phi_t(q)$, $t \in (-\delta, \delta)$, is the <u>unique</u> curve satisfying

$$\frac{\partial \phi}{\partial t} = X(\phi_t(q)), \quad \phi_0(q) = q .$$

Existence and uniqueness for manifolds

<u>NOTATION</u>:

(i) $\alpha : (-\delta, \delta) \longrightarrow M$ s.t. $\begin{cases} \alpha'(t) = X(\alpha(t)) \\ \alpha(0) = q \end{cases}$

is called the TRAJECTORY OF X

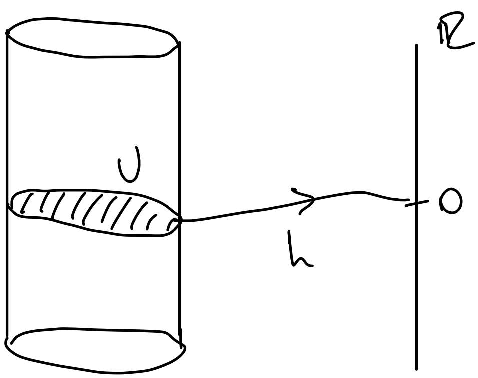
(ii) $\phi_t(q) =$ LOCAL FLOW OF X

Prop: (FISHERMAN'S DERIVATIVE) Let $X, Y \in \mathfrak{X}(M)$ and let ϕ_t be the local flow of X in a neighborhood U of p. Then

$$[X,Y]_p = \lim_{t\to 0} \frac{[Y - d\phi_t(Y)]\,\phi_t(p)}{t}.$$

Hadamard

Lemma from Calculus: Let $h: (-\delta,\delta)\times U \to \mathbb{R}$ be differentiable and s.t. $h(0,q) = 0 \ \forall q \in U$.



Then, there exists a diff'ble $g: (-\delta,\delta)\times U \to \mathbb{R}$ s.t.

$$h(t,q) = t\,g(t,q)$$

$$g(0,q) = \frac{\partial h}{\partial t}(t,q)\Big|_{t=0}.$$

Just integrate

Use this lemma here

Pf: (Fisherman's derivative) For $f \in C^\infty(M)$

$$X(f) = \lim_{t \to 0} \frac{(f \circ \phi_t)(q) - f(q)}{t}.$$

precisely this limit

$$X(f)(q) = \frac{\partial \phi_t}{\partial t} f = d\phi_t\left(\frac{\partial}{\partial t}\right) f = \underbrace{\frac{\partial}{\partial t}(f \circ \phi_t)}$$

Now, let $F := Yf$. So,

$$XY(f) \overset{(1)}{=} \lim_{t \to 0} \frac{(Yf \circ \phi_t)(q) - Yf(q)}{t}.$$

Moreover, by the Lemma above,

$$(d\phi_t\, Y)(f)_{\phi_t(p)} = Y(f \circ \phi_t)(p)$$

$$= Yf(p) + t\, Yg(t, p),$$

where

$$g(t,p) \overset{(4)}{=} \frac{(f \circ \phi_t)(p) - f(p)}{t} \quad \text{and}$$

$$g(0,q) \overset{(3)}{=} X(f)(q).$$

From this, the RHS of the formula is:

$$\lim_{t\to 0} \frac{1}{t}\left[Y - d\phi_t(Y)\right] f(\phi_t(p))$$

$$= \lim_{t\to 0} \frac{1}{t}\left[Y f(\phi_t(p)) - Y f(p) - t\, Y g(t,p)\right]$$

$$= \lim_{t\to 0} \frac{1}{t}\left[Y f(\phi_t(p)) - Y f(p)\right] - Y g(0,p)$$

$$= \overbrace{XY(f)(p)}^{\text{by } (1)} - \overbrace{YX(f)(p)}^{\text{by } (3)}$$

$$= [X,Y](f)(p).$$

Finally, need to prove (4): define

$$h(t,q) = f(\phi_t(q)) - f(q)$$

$$h(0,q) = f(q) - f(q) = 0$$

$\Rightarrow \exists\, g \in C^\infty(M)$ s.t.

$$(f \circ \phi_t)(q) - f(q) = t\, g(t,q)$$

$$\Rightarrow g(t,q) = \frac{(f\circ\phi_t)(q) - f(q)}{t}$$

and $g(0,q) = X(f)(q)$.

$\square$

LECTURE 5

21/09/2023

AFFINE CONNECTIONS

Def: (AFFINE CONNECTION)

An affine connection on the tangent bundle TM of a smooth manifold M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M) \text{ satisfying:}$$

(i) $\nabla_{fX+gY} Z = f\, \nabla_X Z + g\, \nabla_Y Z$ (C^∞-bilinear in $\nabla_{(\cdot)}$)

(ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$ ($\mathbb{R}$-bilinear in $\nabla(\cdot)$)

(iii) $\nabla_X (fY) = f\nabla_X Y + \underbrace{X(f)}\, Y$ (Leibniz rule)

$X(f) = df(X)$

NOTE: $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ "connects" nearby tangent spaces so that we can differentiate vector fields as if they were fcts. on M w/ values in a fixed vector space

In local coordinates, say we have

$$X = \sum_i x_i \frac{\partial}{\partial x_i}, \quad Y = \sum_i y_i \frac{\partial}{\partial y_i} \in \mathfrak{X}(M).$$

Then,

$$\nabla_X Y = \nabla_{\sum_i x_i \frac{\partial}{\partial x_i}} \left(\sum_j y_j \frac{\partial}{\partial y_j} \right)$$

bilinearity in $\nabla_\bullet$

$$= \sum_i x_i \nabla_{\frac{\partial}{\partial x_i}} \sum_j y_j \frac{\partial}{\partial y_j}$$

$$= \sum_{i,j} x_i \nabla_{\frac{\partial}{\partial x_i}} y_j \frac{\partial}{\partial y_j}$$

Leibniz Rule

$$= \sum_{i,j} x_i y_j \boxed{\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial y_j}} + \sum_{i,j} x_i \frac{\partial y_j}{\partial x_i} \frac{\partial}{\partial y_j}$$

This is a vector field

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial y_j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

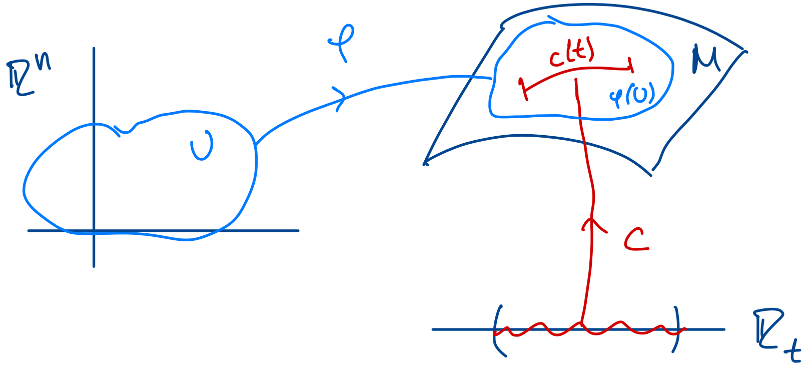
Thm: Let M be a smooth manifold with ∇.
Then there exists a unique correspondence which associates to $V(t)$ along $c(t): I \to M$ another vector field $\frac{DV}{dt}$ (called the covariant derivative) satisfying:

(i) $$\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$$

(ii) $$\frac{D}{dt}(fV) = \frac{df}{dt} V + f \frac{DV}{dt}$$

(iii) If V is induced by a vec. field $Y \in \mathfrak{X}(M)$, then $$\frac{DV}{dt} = \nabla_{\frac{dc}{dt}} Y.$$

Pf: Consider a segment of $c(t) \subset \varphi(U)$.



Express $c(t)$ in local coördinates:

$$\varphi^{-1} \circ c(t) = \left(x_1(t), \_\_\_\_, x_n(t)\right).$$

Then

$$V(t) = \sum_i v^j \frac{\partial}{\partial x_j},$$

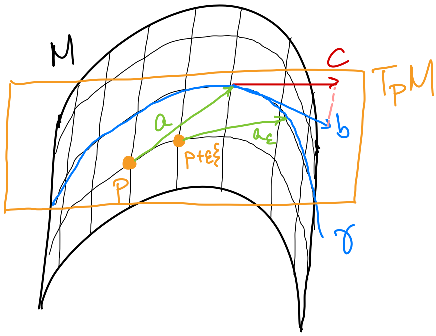
$v^j = v^j(t)$. Then, compute:

$$\frac{DV}{dt} \sum_j v^j \frac{\partial}{\partial x_j} = \sum_j \frac{dv_j}{dt} \frac{\partial}{\partial x_j} + \sum_j v^j \underbrace{\frac{D}{dt}\left(\frac{\partial}{\partial x_j}\right)}_{(*)}$$

$$(*):\ \frac{D}{dt}\left(\frac{\partial}{\partial x_j}\right) \overset{\text{def}}{=} \nabla_{\frac{dc}{dt}} \frac{\partial}{\partial x_j} = \nabla_{\sum_i \frac{dx_i}{dt}\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$

bilinearity

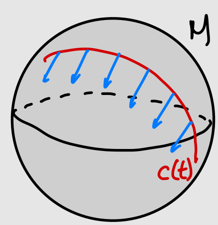
$$= \sum_i \frac{dx_i}{dt} \underbrace{\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}}_{\sum_k \Gamma_{ij}^k \frac{\partial}{\partial x_k}}$$



$$= \sum_j \frac{dv^j}{dt} \frac{\partial}{\partial x_j} + \sum_{i,j} \frac{dx_i}{dt} v^j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$$

Def: (PARALLEL VECTOR FIELDS) The vector field V along a curve $c(t): I \to M$ is parallel if

$$V'(t) = 0 \iff \frac{DV}{dt} = 0.$$



Consequence of ODE theory

Prop: Let (M, ∇) be a smooth manifold w/ an affine connection and $c(t) : I \to M$ a diffable. curve. Let $V_0 \in T_{c(t_0)} M$, then there exists a unique vector field $V(t)$ such that

$$\frac{DV}{dt} = 0 \quad \text{and} \quad V(t_0) = V_0 .$$

Such vector field $V(t)$ is called the <u>parallel</u> <u>transport</u> of V_0 along $c(t)$.

Pf: Assume $c(t) \subset \varphi(U)$. Then, as before

$$0 = \frac{DV}{dt} = \sum_k \frac{dv^k}{dt} \frac{\partial}{\partial x_k} + \sum_{i,j,k} \frac{dx_i}{dt} v^j \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

n 1st order linear ODEs

$$= \sum_k \left(\frac{dv^k}{dt} + \sum_{i,j} v^j \frac{dx_i}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}$$

$= 0 . \implies \exists!$ solution on $\varphi(U)$ with $v^k(t_0) = V_0^k$.

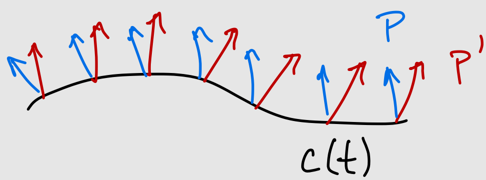
ODE theory

$\square$

* Riemannian (or Levi-Civita) Connections

Let (M, g) be a Riemannian manifold and ∇ an affine connection.

Def: (Compatibility w/ g) A connection ∇ is said to be compatible with g if for any smooth curve $c(t) : I \to M$ and any pair of parallel vec. fields P, P',



$$\langle P, P' \rangle = \text{const.}$$

$$\Longleftrightarrow \quad g(P, P') = \text{const.}$$

Prop: Let (M, g) be a Riemannian manifold. Then, ∇ is compatible with g iff

$$\frac{d}{dt} g(V, W) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right),$$

V, W smooth vec. fields along $c(t)$.

Pf: $(\Leftarrow)$ Suppose the "product rule" holds.

WTS: ∇ is compatible w/ g.

WTS: $g(P, P') = \text{const.}$ where P, P' are parallel along $c(t)$.

Indeed

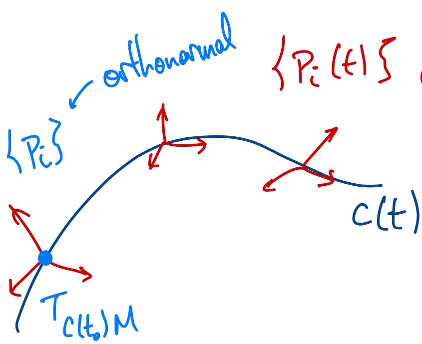
$$\frac{d}{dt} g(P, P') = g\Big(\underbrace{\frac{DP}{dt}}_{=0}, P'\Big) + g\Big(P, \underbrace{\frac{DP'}{dt}}_{=0}\Big)$$

$$= 0$$

since parallel

$(\Rightarrow)$ Suppose ∇ is compatible w/ g

$\{P_i(t)\}$ orthonormal



$$V := \sum v^i P_i$$

$$W := \sum w^i P_i$$

$$g(V, W) = g\left(\sum v^i P_i, \sum w^i P_i\right)$$

$$= \sum v^i w^i$$

$$\Rightarrow \frac{d}{dt} \sum v^i w^i = \sum \frac{dv^i}{dt} w^i + \frac{dw^i}{dt} v^i$$

$$= g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right).$$

∎

LECTURE 6

GEODESICS

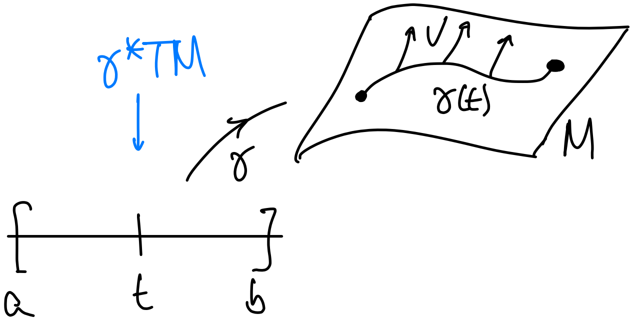
26/09/2023

Vector field along a curve:

$\gamma : [a,b] \to M$

$V : [a,b] \to TM$

such that



$V_{\gamma(t)} \in T_{\gamma(t)} M \quad \forall t \in [a,b]$; i.e., V is a section of $\gamma^* TM$.

$$V' = \frac{DV}{dt} := \nabla_{\gamma'} V$$ is defined locally extending V

Recall:

$$\boxed{\nabla \text{ compatible with } g} \iff \boxed{\frac{d}{dt} g(V,W) = g\left(\frac{DV}{dt}, W\right) + g\left(V, \frac{DW}{dt}\right).}$$

for all V, W smooth vec. fields along diffable curve $c(t)$.

!

Corollary:

∇ is compatible with metric g $\iff$ $$X g(Y,Z) = g(\nabla_X Y, Z) - g(Y, \nabla_X Z) \quad \forall X, Y, Z \in \mathfrak{X}(M)$$

Def: (SYMMETRIC ∇) An affine connection is said to be symmetric when

$$\nabla_X Y - \nabla_Y X = [X, Y]$$

$\forall X, Y \in \mathfrak{X}(M)$

In coördinates: $\left\{\frac{\partial}{\partial x_i}\right\}$, we have

$$\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$$

$$\Rightarrow \quad \Gamma_{ij}^{k} = \Gamma_{ji}^{k}$$

Thm: (LEVI-CIVITA) Given a Riemannian manifold (M^n, g), there exists a unique affine connection on TM such that

- $\nabla_X Y - \nabla_Y X = [X, Y]$ (torsion-free or symmetric)
- $X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ (compatible w/ g, or $\nabla g = 0$)

Pf: We have that

(1) $X g(Y,Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$

(2) $Y g(Z,X) = g(\nabla_Y Z, X) + g(Z, \nabla_Y X)$

(3) $Z g(X,Y) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$

So,

$$Xg(Y,Z) + Yg(Z,X) + Zg(X,Y)$$

$$= g([X,Z],Y) + g([Y,Z],X)$$

$$+ g([X,Y],Z) + 2g(\nabla_Y X, Z)$$

Thus,

$$g(\nabla_Y X, Z) = \frac{1}{2}\Big[Xg(Y,Z) + Yg(Z,X) - Zg(X,Y)$$

$$- g([X,Z],Y) - g([Y,Z],X)$$

$$- g([X,Y],Z) \Big].$$

"Koszul Formula"

The above uniquely defines ∇. $\blacksquare$

Note that ∇ determines Γ_{ij}^{k} and also vice-versa.

In coordinates, suppose

$$X = \frac{\partial}{\partial x_j}, \quad Y = \frac{\partial}{\partial x_i}, \quad Z = \frac{\partial}{\partial x_k}.$$

Then, substituting X and Y in Koszul formula and solving for Z, we find that $\Gamma_{ij}^{k} : U \to \mathbb{R}$

$$\Gamma_{ij}^{m} = \frac{1}{2} \sum_{k} \left(\frac{\partial}{\partial x_i} \underset{\substack{\uparrow \\ g\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)}}{g_{jk}} + \frac{\partial}{\partial x_j} \underset{\substack{\uparrow \\ g\left(\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_i}\right)}}{g_{ki}} - \frac{\partial}{\partial x_k} \underset{\substack{\uparrow \\ g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)}}{g_{ij}} \right) g^{km}$$

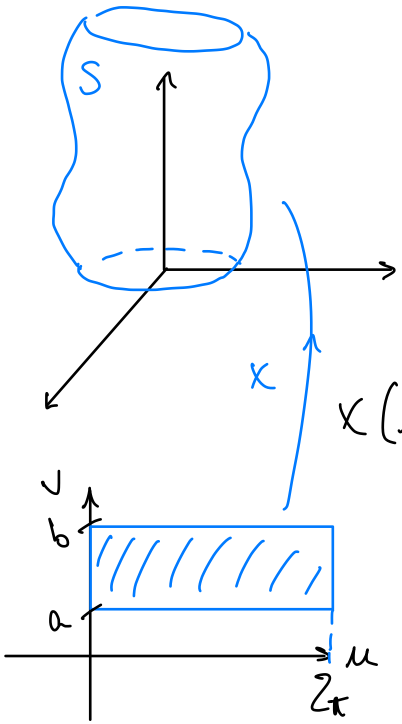
where $g_{ij} = g\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$ and

(g^{km}) is the inverse matrix to (g_{ij})

Note: $\left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0 \quad \forall i, j$, so last 3 terms vanish.

"Fun" Example: Computing Christoffel Symbols

Consider a surface of revolution S.



Let $\begin{cases} x = f(v) \\ z = g(v) \end{cases}$

$$X(u,v) = (f(v)\cos u,\, f(v)\sin u,\, h(v))$$

Compute Γ^1_{11}:

First,

$$dX = \begin{pmatrix} \underbrace{\begin{matrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{matrix}}_{\frac{\partial}{\partial u}} & \underbrace{\begin{matrix} f'(v)\cos u \\ f'(v)\sin u \\ h'(v) \end{matrix}}_{\frac{\partial}{\partial v}} \end{pmatrix}$$

So,

$$g_{11} = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial u}\right) = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial u} \right\rangle = f(v)^2$$

$$g_{21} = g_{12} = g\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}\right) = \left\langle \frac{\partial}{\partial u}, \frac{\partial}{\partial v} \right\rangle = 0$$

$$g_{22} = g\left(\frac{\partial}{\partial v}, \frac{\partial}{\partial v}\right) = \left\langle \frac{\partial}{\partial v}, \frac{\partial}{\partial v} \right\rangle = f'(v)^2 + h'(v)^2$$

So,

$$(g_{ij}) = \begin{pmatrix} f(v)^2 & 0 \\ 0 & f'(v)^2 + h'(v)^2 \end{pmatrix}$$

Now,

$$\frac{\partial}{\partial u} g_{11} = 0 \qquad \frac{\partial}{\partial v} g_{11} = 2f(v)f'(v)$$

$$\frac{\partial}{\partial u} g_{22} = 0 \qquad \frac{\partial}{\partial v} g_{22} = 2f'(v)f''(v) + 2h'(v)h''(v)$$

Invert (g_{ij}) :

$$g^{11} = \frac{1}{f(v)^2}, \quad g^{22} = \frac{1}{f'(v)^2 + h'(v)^2}, \quad g^{21} = g^{12} = 0$$

Thus,

$$\Gamma_{11}^{1} \overset{\text{def}}{=} \frac{1}{2}\left[\frac{\partial}{\partial x_1} g_{11} + \frac{\partial}{\partial x_1} g_{11} - \frac{\partial}{\partial x_1} g_{11}\right] g^{11}$$

$$+ \frac{1}{2}\left[\frac{\partial}{\partial x_1} g_{12} + \frac{\partial}{\partial x_1} g_{21} - \frac{\partial}{\partial x_2} g_{11}\right] g^{21}$$

$$= 0. \qquad \blacksquare$$

---

* **GEODESICS** !!

Def : (GEODESICS) A *geodesic* is a curve $\gamma(t)$ such that $\dot{\gamma}(t)$ is parallel. Equivalently, If $\dot{\gamma} = \sum_i \dot{\gamma}_i(t) \frac{\partial}{\partial x_i}$, then

Say $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$

$\dot{\gamma}(t) = \frac{d\gamma}{dt} = \sum_i \frac{d\gamma_i}{dt} \frac{\partial}{\partial x_i}$

$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right)$

$$\frac{D\dot{\gamma}}{dt} = \sum_i \gamma_i''(t) \frac{\partial}{\partial x_i}$$

$$+ \sum_{j,k} \gamma_i'(t)\, \gamma_j'(t)\, \Gamma_{ij}^{k}(\gamma(t)) \frac{\partial}{\partial x_k} = 0$$

i.e., $\forall i$

$$\gamma_i''(t) + \sum_{j,k} \gamma_j' \gamma_k' \Gamma_{jk}^{i} = 0$$

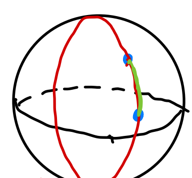
"Geodesic ODE"

(System of n coupled 2^{nd} order nonlinear ODEs)

NOTE: Not all geodesic minimize distances!

Take S^2. Geodesics are big circles on the sphere.

Green portion minimizes the Geodesic



distance, but the other part of the big circle is also a geodesic between the blue pts. ∎

NOTE: If $\gamma(t)$ is geodesic, then

$$\frac{d}{dt}\, g\left(\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = 2\, g\left(\frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right) = 0$$

In local coördinates: $\gamma(t) = (\gamma_1(t), \_\_, \gamma_n(t))$

$$\frac{d\gamma}{dt} = \sum_i \frac{d\gamma_i}{dt}\frac{\partial}{\partial x_i}.$$ Then

$$\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = \frac{D\dot{\gamma}}{dt} = \frac{D}{dt}\left(\sum_i \frac{d\gamma_i}{dt}\frac{\partial}{\partial x_i}\right)$$

$$= \sum_i \frac{d^2\gamma_i}{dt^2}\frac{\partial}{\partial x_i} + \sum_i \frac{d\gamma_i}{dt}\frac{D}{dt}\left(\frac{\partial}{\partial x_i}\right)$$

$$= \sum_i \frac{d^2\gamma_i}{dt^2}\frac{\partial}{\partial x_i} + \sum_i \frac{d\gamma_i}{dt}\nabla_{\frac{d\gamma}{dt}}\left(\frac{\partial}{\partial x_i}\right)$$

$$= \sum_i \frac{d^2\gamma_i}{dt} \frac{\partial}{\partial x_i} + \sum_i \frac{d\gamma_i}{dt} \nabla_{\left(\sum_j \frac{d\gamma_j}{dt} \frac{\partial}{\partial x_j}\right)} \frac{\partial}{\partial x_i}$$

$$= \sum_i \frac{d^2\gamma_i}{dt} \frac{\partial}{\partial x_i} + \sum_{i,j} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$$

$$= \sum_i \frac{d^2\gamma_i}{dt} \frac{\partial}{\partial x_i} + \sum_{i,j,k} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$$

$$= \sum_k \left(\frac{d^2\gamma_k}{dt} + \sum_{i,j} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} \Gamma_{ij}^k \right) \frac{\partial}{\partial x_k}$$

geodesic

$$\overset{!}{=} 0$$

Geodesic Equation: $\frac{d^2\gamma_k}{dt} + \sum_{i,j} \frac{d\gamma_i}{dt} \frac{d\gamma_j}{dt} \Gamma_{ij}^k = 0$

n 2^{nd} order coupled nonlinear ODEs

Let $\frac{d\gamma_k}{dt} =: y_k = \frac{d^2\gamma_k}{dt^2} = \frac{dy_k}{dt}$

Immediate consequence from ODE theory:

Thm: On a Riemannian manifold (M^n, g), given $p \in M$ and $v \in T_pM$, there exists a unique maximal geodesic $\gamma_v : (T_-, T_+) \to M$ with $\gamma_v(0) = p$ and $\dot{\gamma}_v(0) = p$. Moreover, such γ_v depends smoothly on its initial conditions $(p, v) \in TM$.

EXISTENCE & UNIQUENESS

HOMOGENEITY OF GEODESICS: If the unique maximal geodesic $\gamma_v : (-\delta, \delta) \to M$ s.t. $\gamma_v(0) = p \in M$ and $\dot{\gamma}_v(0) = v \in T_pM$, then, the geodesic γ_{av} with $\dot{\gamma}_{av}(0) = av$, $a \in \mathbb{R}_+$, is defined on the interval $\left(-\frac{\delta}{a}, \frac{\delta}{a}\right)$ and $\gamma_{av}(t) = \gamma_v(at)$.

Pf: Let $h : \left(-\frac{\delta}{a}, \frac{\delta}{a}\right) \to M$ be a curve such that

$h(t) = \gamma_v(at)$, $\gamma_v(0) = p \in M$. Then, by the Chain Rule

$$\frac{dh}{dt}(0) = a\,\gamma_v'(at)\Big|_{t=0} = av$$

$$\frac{D}{dt}\left(\frac{dh}{dt}\right) = \nabla_{h'(t)} h'(t) = a^2 \nabla_{\dot\gamma_v(at)} \dot\gamma_v(at)$$

$$= 0$$

$$\Rightarrow h(t) = \gamma_{av}(t).$$

∎

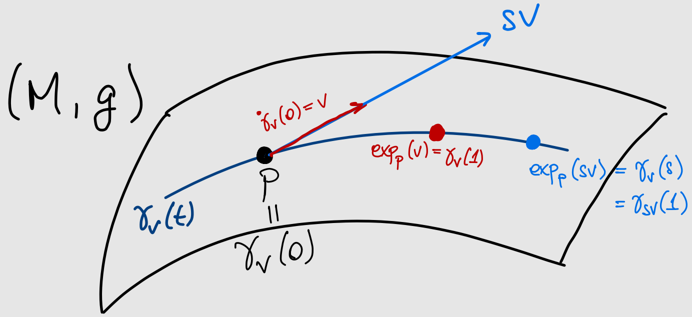
---

* THE EXPONENTIAL MAP

Def: The Riemannian exponential map at $p \in M$ is

$$\begin{aligned} \exp_p : \mathcal{O}_p \subset T_pM &\longrightarrow M \\ v &\longmapsto \gamma_v(1) \end{aligned}$$

where O_p is the open neighborhood of $0 \in T_pM$ such that $\gamma_v(t)$ is defined up to $\gamma_v(1)$ whenever $v \in O_p$.



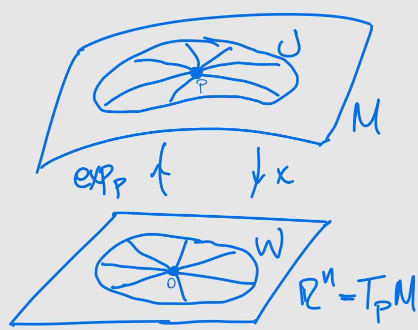
Prop: $d(\exp_p v)_0 = v$ for all $v \in T_pM$; i.e., $d(\exp_p)_0 = id$.

In particular, there are open subsets $W \subset T_pM$ and $U \subset M$, with $0 \in W$ and $p \in U$, s.t. $\exp_p\big|_W : W \to U$ is a diffeomorphism

By the Inverse Function Theorem.

This means that the map

$$\left(\exp_p\big|_W^{-1} : U \to \mathbb{R}^n \right)$$



defines a local chart. These are called "geodesic normal coördinates."

Pf: $$d(\exp_p v)_0 v = \frac{d}{dt}(\exp_p)(tv)\Big|_{t=0}$$

Chain Rule:

$$df_p v = \frac{d}{dt} f(\gamma_v(t))\Big|_{t=0}$$

$$= \frac{d}{dt}\gamma_{tv}(1)\Big|_{t=0}$$

$$= \frac{d}{dt}\gamma_v(t)\Big|_{t=0}$$

$$= \dot{\gamma}_v(0) = v. \quad \blacksquare$$

Lecture 7

28/09/2023

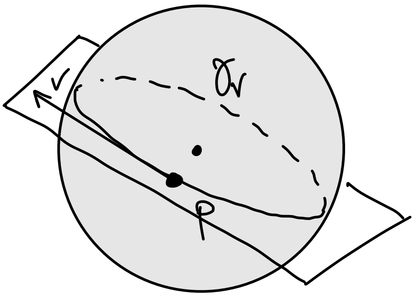
Gauss' Lemma

Recall: exponential map is a local diffeo.

EXAMPLES: (GEODESICS & EXP. MAPS)

1) $\mathbb{S}^2 \subset \mathbb{R}^3$. Note that

Connection on $\mathbb{S}^2$ is inherited from $\mathbb{R}^3$ and it's just the directional derivative



$$\mathbb{S}^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^2 = 1\}$$

Parametrized by

$$c(\theta) = (\cos\theta, \sin\theta, 0)$$

$$\Rightarrow \frac{dc}{d\theta} = (-\sin\theta, \cos\theta, 0)$$

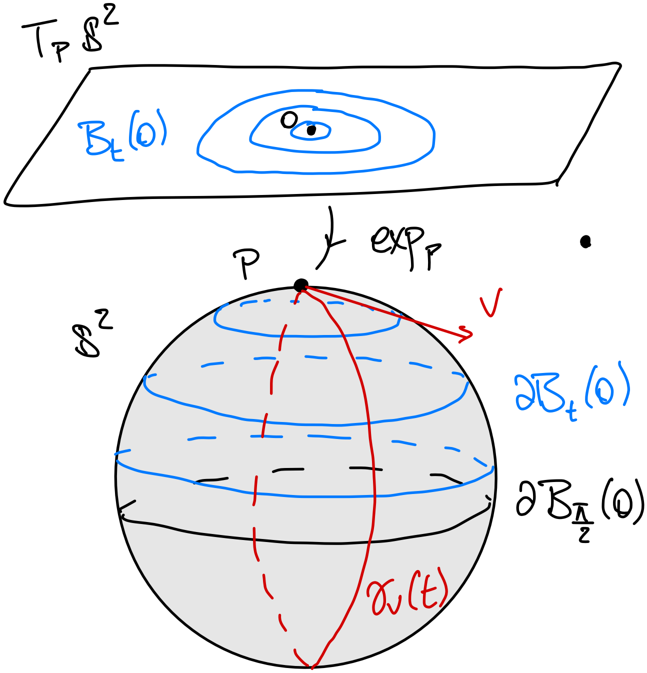
$$\frac{D}{d\theta}\left(\frac{dc}{d\theta}\right) = (-\cos\theta, -\sin\theta, 0)^T \overset{!}{=} 0$$

Thus, geodesics on the sphere are "great circles"

$$\gamma_v(t) = (\cos t)\, p + (\sin t)\, v$$

The exponential is then $\exp_p : T_p\mathbb{S}^2 \to \mathbb{S}^2$

$$\exp_p(B_r(0)) = B_r(p), \quad \forall r \in (0, \pi)$$



"Injectivity radius" of $S^n(1)$ is π.

- $\exp_p\big|_{B_\pi(0)} : B_\pi(0) \to S^2 \setminus \{p\}$ is a diffeomorphism.

$$g = dt^2 + \sin^2 t \, d\theta^2$$

i.e., $g = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2 t \end{pmatrix}$ on $\left\{ \frac{\partial}{\partial t}, \frac{\partial}{\partial \theta} \right\}$

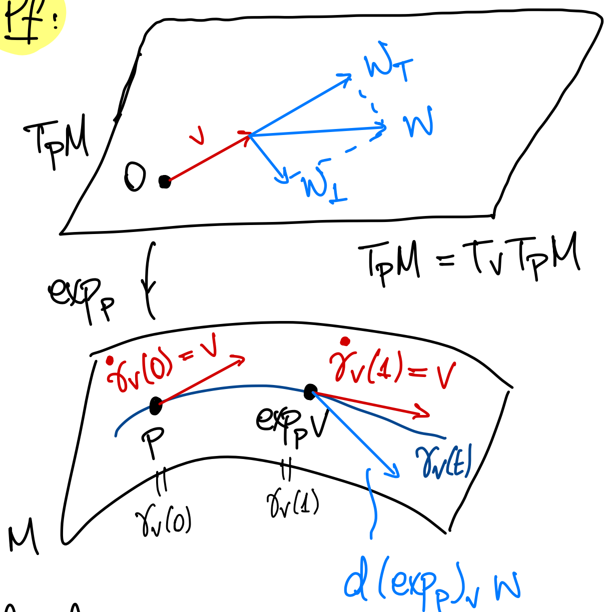
GAUSS' LEMMA: $\exp_p$ is a radial isometry.

More precisely,

$$\langle (d\exp_p)_v v, (d\exp_p)_v w \rangle = \langle v, w \rangle$$

$\forall v, w \in T_pM = T_v T_pM$.

Pf:



Write $w = w_\perp + w_T$

where

$$\begin{cases} w_T = \alpha v \\ \langle w_\perp, v \rangle = 0 \end{cases}$$

Clearly,

$$d(\exp_p)_v v = \frac{d}{dt}(\exp_p)((t+1)v)\Big|_{t=0}$$

$$= \frac{d}{dt}(\exp_p)(tv)\Big|_{t=1}$$

$$= \frac{d}{dt}\gamma_v(t)\Big|_{t=1} = \dot\gamma_v(1)$$

$$= \underbrace{P_p^{\gamma_v(1)}(v)}_{\text{Parallel transport of } v \in T_pM \text{ along } \gamma_v \text{ to } \gamma_v(1)}$$

$P_p^{\gamma_v(1)} : T_pM \to T_{\gamma_v(1)}M$

Thus,

$$\langle d(\exp_p)_v v, d(\exp_p)_v w\rangle = \langle d(\exp_p)_v v, d(\exp_p)_v(\alpha v)\rangle$$
$$+ \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle$$

$$= \alpha \langle P_p^{\gamma_v(1)} v, P_p^{\gamma_v(1)} v\rangle$$

$$+ \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle$$

$$= \langle v, \underbrace{\alpha v}_{w_T}\rangle + \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle$$

$$= \langle v, w\rangle + \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle .$$

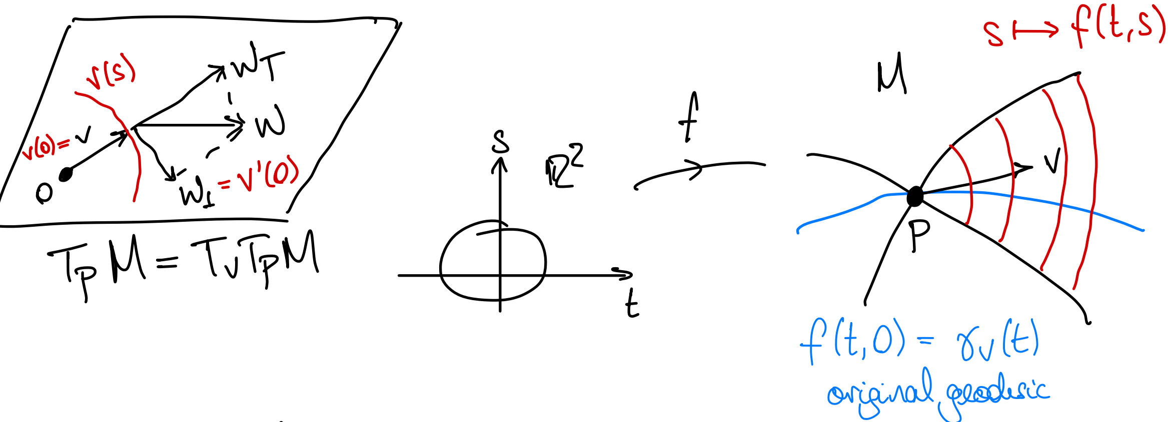
So, we need to show that

$$\langle d(\exp_p)_v v, d(\exp_p)_v w_\perp\rangle = 0 .$$

Let $v(s) = (\cos s)v + (\sin s)w_\perp$. So that

$v(0) = v, \quad v'(0) = w_\perp, \quad \|v(s)\| = \text{const}.$

and define $f(t,s) := \exp_p(t v(s)) = \gamma_{v(s)}(t)$.



So, compute:

$$\begin{cases} d(\exp_p)_v v = \dfrac{\partial}{\partial t} \exp_p(t v(s)) \Big|_{\substack{t=1 \\ s=0}} = \dfrac{\partial f}{\partial t}(1,0) \\[2ex] d(\exp_p)_v w_\perp = \dfrac{\partial}{\partial s} \exp_p(t v(s)) \Big|_{\substack{t=1 \\ s=0}} = \dfrac{\partial f}{\partial s}(1,0) \end{cases}$$

$$\Rightarrow \langle d(\exp_p)_v v, d(\exp_p)_v w_\perp \rangle = \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (1,0)$$

So,

$$\frac{\partial}{\partial t} \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle = \left\langle \frac{D}{dt} \frac{\partial f}{\partial s}, \frac{\partial f}{\partial s} \right\rangle + \underbrace{\left\langle \frac{\partial f}{\partial t}, \frac{D}{dt} \frac{\partial f}{\partial s} \right\rangle}_{\substack{\| \\ \text{b/c geodesic}}}$$

$$0$$

$$= \left\langle \frac{D}{ds}\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle$$

metric compatibility

$$\overset{\downarrow}{=} \frac{1}{2}\frac{\partial}{\partial s}\left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial t} \right\rangle = 0$$

since $t \mapsto f(t,s) = \gamma_{v(s)}(t)$ are geodesics

Therefore, $t \mapsto \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (t,0)$ is constant, and computing it at $t=0$, we find:

$$\frac{\partial f}{\partial s}(t,0) = \frac{\partial}{\partial s}(\exp_p)(t v(s))\Big|_{s=0}$$

$$= d(\exp_p)(t \underbrace{v(0)}_{v})(t \underbrace{v'(0)}_{w_\perp})$$

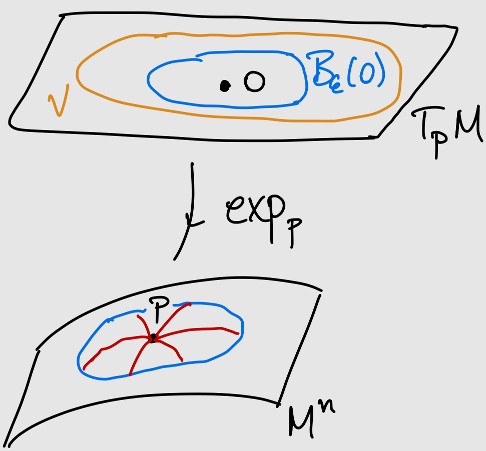
$$= d(\exp_p)_{tv}\, t w_\perp$$

$$\Rightarrow \lim_{t\to 0} \frac{\partial f}{\partial s}(t,0) = \lim_{t\to 0} d(\exp_p)_{tv}\, t w_\perp = 0$$

$$\Rightarrow \left\langle \frac{\partial f}{\partial t}, \frac{\partial f}{\partial s} \right\rangle (1,0) = 0 .$$

$\blacksquare$

Def: (NORMAL NEIGHBORHOOD) If $\exp_p$ is a diffeo. on some neighborhood V of the origin of T_pM, then $\exp_p(V)$ is called a <u>normal neighborhood</u> of p.



$$\exp_p(B_\varepsilon(0)) = \underbrace{B_\varepsilon(p)}$$

"Normal ball / Geodesic ball"

$$\partial B_\varepsilon(p) = S^{n-1}$$

LECTURE 8

03/10/2023

GEODESICS & CURVATURE

Prop: Let $p \in M$ and V be a normal neighborhood of p, $B \subset V$ be a normal ball centered at p and

$\gamma : [0,1] \to M$ a geodesic s.t. $\gamma(0) = p$.

If $c : [0,1] \to M$ is any piecewise differentiable curve joining $\gamma(0)$ and $\gamma(1)$, then

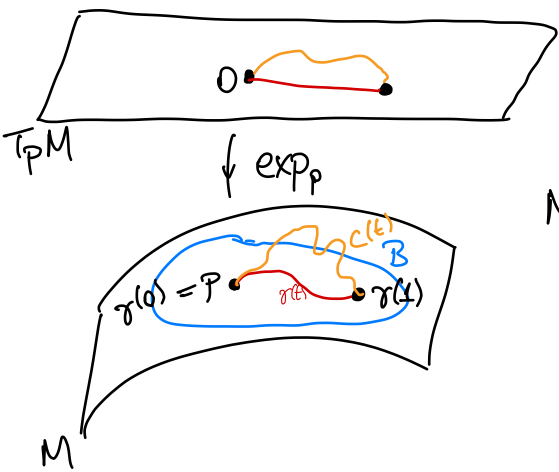
$$\underbrace{L}_{L = \text{length}}(\gamma) \leq L(c)$$

and if $L(\gamma) = L(c)$, then $\gamma([0,1]) = c([0,1])$.

Pf: Suppose $c([0,1]) \subset B$. Then

$$c(t) = \exp_p(r(t)v(t))$$

for $t \neq 0$ and $|v(t)| = 1$.



Now, the length of c is:

$$L(c) \overset{\text{def}}{=} \int_0^1 \left| \frac{dc}{dt} \right| dt$$

Set $\exp_p(r(t)v(t)) =: f(r(t), t)$ and differentiate:

$$\frac{dc}{dt} = \frac{\partial f}{\partial r} r'(t) + \frac{\partial f}{\partial t}$$

chain rule $\Rightarrow$ $\dfrac{\partial f}{\partial r} = d(\exp_p)_{r(t)v(t)} \, v(t)$

$$\frac{\partial f}{\partial t} = d(\exp_p)_{r(t)v(t)}\, r(t)v'(t)$$

where $|v(t)| = 1$, $\langle v, v' \rangle = 0$

$$\left|\frac{\partial f}{\partial r}\right| = \sqrt{\langle d(\exp_p)_{r(t)v(t)} v(t), d(\exp_p)_{r(t)v(t)} v(t) \rangle}$$

$$\overset{\text{Gauss' Lemma}}{=} |v(t)| = 1$$

Thus

$$\left\langle \frac{\partial f}{\partial r}, \frac{\partial f}{\partial t} \right\rangle$$

Gauss' Lemma & $\langle v, v' \rangle = 0$

$$= \langle d(\exp_p)_{r(t)v(t)} v(t), d(\exp_p)_{r(t)v(t)} r(t)v'(t) \rangle = 0$$

Then,

$$\left|\frac{dc}{dt}\right|^2 = |r'(t)|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq |r'(t)|^2$$

So,

$$\int_\varepsilon^1 \left|\frac{dc}{dt}\right| dt \geq \int_\varepsilon^1 |r'(t)|\, dt$$

$$\geq \int_{\varepsilon}^{1} r'(t)\, dt$$

$$= r(1) - r(\varepsilon)$$

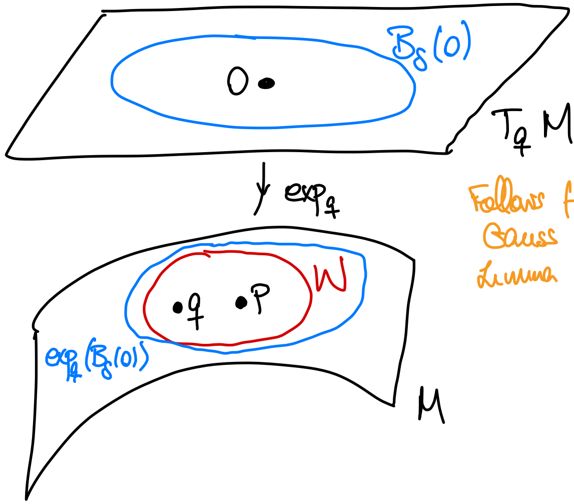
Take $\varepsilon \to 0$ and we get $= r(1)$
$= L(\gamma)$.

Lastly, if $L(c) = L(\gamma)$, then the inequalities must be equalities. Thus, $\frac{\partial f}{\partial t} = 0$ and $|r'(t)| = r'(t) \Rightarrow \gamma([0,1]) = c([0,1])$. ∎

---

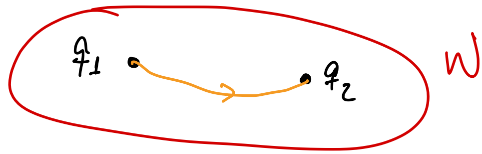
* **TOTALLY NORMAL NEIGHBORHOODS**: For each $p \in M$ there exists a neighborhood W of p and $\delta > 0$ s.t. for every $q \in W$, $\exp_q$ is a diffeomorphism on the $B_\delta(0) \subset T_qM$ and $\exp_q(B_\delta(0)) \supset W$.

This such W is called *totally normal*.



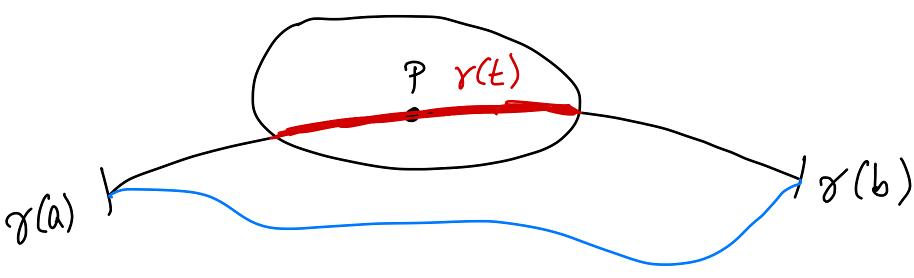
Given any two pts. $q_1, q_2 \in W$, there exist a unique geodesic $\gamma(t)$ connecting q_1 and q_2

Follows from Gauss Lemma



Proof of existence of such totally normal neighborhood is in do Carmo.

Corollary: If a ptwise differentiable curve $\gamma:[a,b]\to M$ with parameter t proportional to its arclength has length smaller than the length of any other ptwise diff. curve connecting $\gamma(a)$ and $\gamma(b)$, then γ is a geodesic.

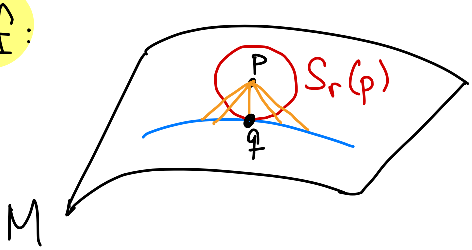


Def: (STRONGLY CONVEX) A subset $S \subset M$ is **strongly convex** if for any pts. $q_1, q_2 \in \bar{S}$, there exists a unique minimizing geodesic $\gamma: [a,b] \to M$ connecting q_1 and q_2 ($q_1 = \gamma(a)$ & $q_2 = \gamma(b)$) s.t. $\gamma((a,b)) \subset S$.

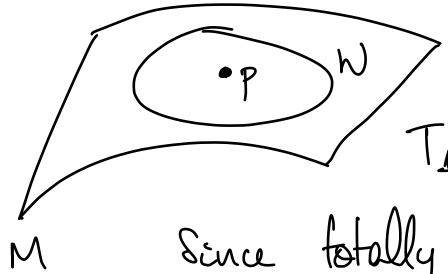
[GOAL: show that every pt. in a manifold has a strongly convex neighborhood.]

Lemma: For any $p \in M$, $\exists c > 0$ such that any geodesic in M that is tangent at $q \in M$ to $S_r(p)$ of radius $r < c$ stays out of the geodesic ball in some neighborhood of q.

Pf:



Everything will take place in a totally normal neighborhood W of p.



Consider

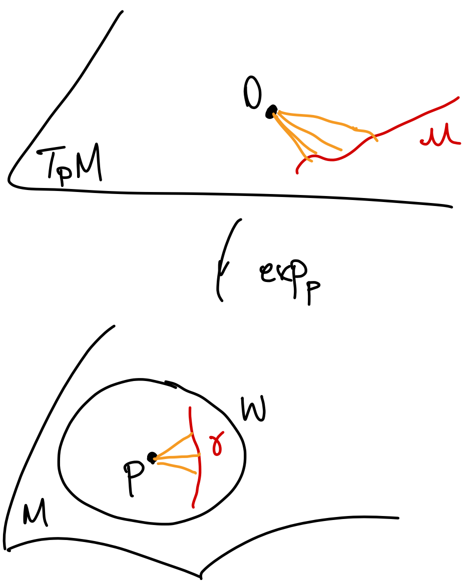
$$T_1 W := \{ (q, v) : q \in W,\ v \in T_q M,\ |v| = 1 \}.$$

Since totally normal, geodesic

$$t \xmapsto{\ \gamma\ } \gamma(t, q, v)$$

s.t. $\gamma(0) = q$, $\gamma'(0) = v$, $|v| = 1$. Let

$$u(t, q, v) := \exp_p^{-1}\big(\gamma(t, q, v)\big)$$



Set

$$F(t, q, v) := |u(t, q, v)|^2$$

Differentiate

$$\frac{\partial F}{\partial t} = 2 \left\langle \frac{\partial u}{\partial t}, u \right\rangle$$

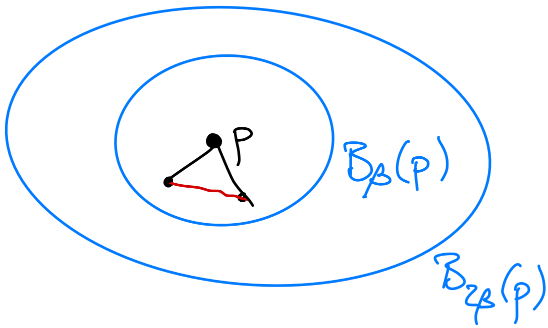
$$\frac{\partial^2 F}{\partial t^2} = 2 \left\langle \frac{\partial^2 u}{\partial t^2}, u \right\rangle + \left| \frac{\partial u}{\partial t} \right|^2$$

But $u(t, q, v) = tv \Rightarrow \dfrac{\partial^2 F}{\partial t^2} = 2|v| = 2 > 0$.

$\blacksquare$

Thm: (Existence of Strongly Convex Neighborhoods) For any $p \in M$ there exists $\beta > 0$ s.t. $B_\beta(p)$ is strongly convex.

Pf: Of course, $\beta < \frac{c}{2}$.



■

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* **CURVATURE**: The curvature tensor R is a $(3,1)$-tensor and can be seen as a section of $TM^* \otimes TM^* \otimes TM^* \otimes TM$. We can define

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$$

$$R(X,Y)Z := \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

!!

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_k} = \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k}.$$

Prop: $R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ is a tensor; i.e., $(R(X,Y)Z)_p$ only depends on X_p, Y_p, Z_p and we may consider this R as a section of $TM^* \otimes TM^* \otimes TM^* \otimes TM$.

"(3,1) - tensor"

Pf: Follows from

$(X,Y) \mapsto R(X,Y)Z$ is $C^\infty(M)$-bilinear

$Z \mapsto R(X,Y)Z$ is $C^\infty(M)$-linear ■

LOWERING INDICES: we get a (4,0)-tensor

$$R : \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \times \mathfrak{X}(M) \to C^\infty(M)$$

$$R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle$$

Properties:

(1) 1st Bianchi Identity:

$$R(X,Y)Z + R(Z,X)Y + R(Y,Z)X = 0$$

(consequence of Jacobi's identity)

(2) $$R(X,Y,Z,T) = -R(Y,X,Z,T)$$

(3) $$R(X,Y,Z,T) = -R(X,Y,T,Z)$$

(4) $$R(X,Y,Z,T) = R(Z,T,X,Y)$$

symmetric

$$R(\underbrace{X, Y}_{\text{skew}}, \underbrace{Z, T}_{\text{skew}})$$

Pf: (3) Let's show that $R(X,Y,Z,Z) = 0$

$$R(X,Y,Z,Z) = \langle R(X,Y)Z, Z \rangle$$

$$= \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, Z \rangle$$

$$= 0 \quad \left\{ \begin{aligned} \langle \nabla_Y \nabla_X Z, Z \rangle &= Y \langle \nabla_X Z, Z \rangle - \langle \nabla_X Z, \nabla_Y Z \rangle \\ &= \tfrac{1}{2} YX \langle Z, Z \rangle - \langle \nabla_X Z, \nabla_Y Z \rangle \\ \langle \nabla_X \nabla_Y Z, Z \rangle &= \tfrac{1}{2} XY \langle Z, Z \rangle - \langle \nabla_Y Z, \nabla_X Z \rangle \\ \langle \nabla_{[X,Y]} Z, Z \rangle &= \tfrac{1}{2} [X,Y] \langle Z, Z \rangle \end{aligned} \right.$$

So, we can write

$$\begin{aligned} 0 &= R(X, Y, Z+T, Z+T) \\ &= R(X, Y, T, Z) + \cancelto{0}{R(X, Y, Z, Z)} \\ &\quad + R(X, Y, Z, T) + \cancelto{0}{R(X, Y, T, T)} \end{aligned}$$

$$\Rightarrow \quad R(X, Y, Z, T) = R(X, Y, T, Z).$$

$\blacksquare$

In (dreaded) coordinates, $\qquad \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right] = 0$

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) \frac{\partial}{\partial x_k} \overset{\text{def}}{=} \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_k} - \nabla_{\frac{\partial}{\partial x_i}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_k}$$

$$\overset{\text{def}}{=} \nabla_{\frac{\partial}{\partial x_j}} \left(\sum_l \Gamma_{ik}^l \frac{\partial}{\partial x_l} \right)$$

$$-\nabla_{\frac{\partial}{\partial x_i}}\left(\sum_l \Gamma_{jk}^l \frac{\partial}{\partial x_l}\right)$$

$$= \sum_l \Gamma_{ik}^l \boxed{\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_l}} + \sum_l \frac{\partial \Gamma_{ik}^l}{\partial x_j} \frac{\partial}{\partial x_l}$$

$$- \sum_l \Gamma_{jk}^l \boxed{\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_l}} - \sum_l \frac{\partial \Gamma_{jk}^l}{\partial x_i} \frac{\partial}{\partial x_l}$$

$$= \cdots$$

$$R_{ijk}^l = \sum_s \Gamma_{ik}^s \Gamma_{js}^l - \sum_s \Gamma_{jk}^s \Gamma_{is}^l + \frac{\partial}{\partial x_j}\Gamma_{ik}^l - \frac{\partial}{\partial x_i}\Gamma_{jk}^l$$

So that

$$R(X,Y)Z = \sum_l R_{ijk}^l \, a_i b_j c_k \frac{\partial}{\partial x_l}$$

if $X = \sum a_i \frac{\partial}{\partial x_i}$, $Y = \sum b_j \frac{\partial}{\partial x_j}$,

$Z = \sum c_k \frac{\partial}{\partial x_k}$.

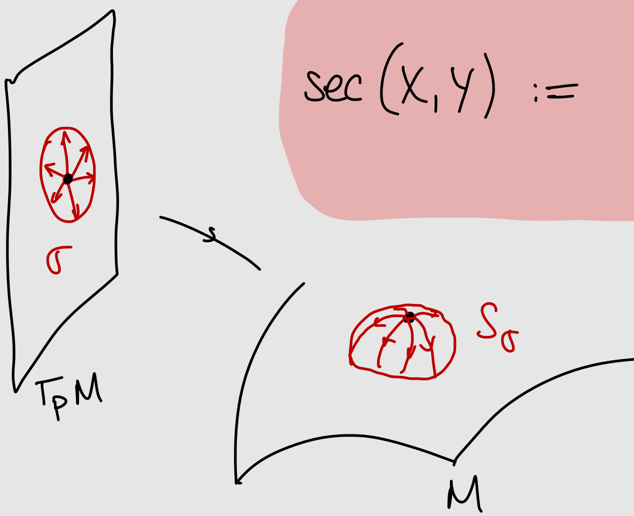
Lecture 9

05/10/2023

Sectional Curvature

Def: (Sectional Curvature) For $p \in M$ and $\sigma = \mathrm{span}\{X, Y\} \subset T_pM$, we define the sectional curvature of X and Y at p as

$$\sec(X,Y) := \frac{\langle R(X,Y)Y, X\rangle}{\|X\|^2\|Y\|^2 - \langle X, Y\rangle} = \sec(\sigma)$$



Prop: $\sec(X,Y)$ only depends on $\sigma \subset \mathrm{span}\{X,Y\} \subset T_pM$.

Prop: R_p is determined by $\sec : Gr_2 T_pM \to \mathbb{R}$

That is, the curvature at a pt. p is determined by the sectional curvature.

Lemma: Let V be a vector space w/ $\dim V \geq 2$, with an inner product $\langle \cdot, \cdot \rangle$. Let

$$R : V \times V \times V \to V$$

$$R' : V \times V \times V \to V$$

be multilinear maps satisfying

$$(X, Y, Z, T) = \langle R(X,Y)Z, T \rangle$$

$$(X, Y, Z, T)' = \langle R'(X,Y)Z, T \rangle .$$

Define

$$\sec(\sigma) := \frac{(X, Y, X, Y)}{|X \wedge Y|^2},$$

$$\sec'(\sigma) := \frac{(X, Y, X, Y)'}{|X \wedge Y|^2}.$$

If $\forall \sigma \subset V$ we have $\sec(\sigma) = \sec'(\sigma)$, then $R = R'$.

Pf: WTS: $(X,Y,Z,T) = (X,Y,Z,T)'$.

We know that $(X,Y,X,Y) = (X,Y,X,Y)'$ $\forall X,Y \in V$ by assumption. So,

$$\underbrace{(X+Z,Y,X+Z,Y)} = \underbrace{(X+Z,Y,X+Z,Y)'}$$

$$\text{LHS} = (X,Y,Z,Y) + (X,Y,Z,T) + (X,T,Z,Y) + (X,T,Z,T)$$

$$\text{RHS} = (X,Y,Z,Y)' + (X,Y,Z,T)' + (X,T,Z,Y)' + (X,T,Z,T)'$$

By 1st Bianchi Identity, we get

$$3\left[(X,Y,Z,T) - (X,Y,Z,T)'\right] = 0. \quad \blacksquare$$

* SPACES OF CONSTANT SECTIONAL CURVATURE :

Examples of (complete) Riem. manifold with $\sec \equiv k$:

	SIMPLY-CONNECTED	THEIR QUOTIENTS
• $k > 0$	$S^n(1/\sqrt{k})$	$\mathbb{R}P^n$, lens space ...
• $k = 0$	$\mathbb{R}^n$	T^n, Klein bottle ...
• $k > 0$	$\mathbb{H}^n(1/\sqrt{-k})$	Hyperbolic surfaces...

Lemma: Let $p \in M$ and

$$R' : T_pM \times T_pM \times T_pM \longrightarrow T_pM$$

s.t.

$$\langle R'(X,Y)W, Z \rangle = \langle X, W \rangle \langle Y, Z \rangle - \langle Y, W \rangle \langle X, Z \rangle$$

Then, M has constant sectional curvature at p (i.e., $\sec_p \equiv k_0$) iff $R = k_0 R'$.

Let $f: A \to M$ be a parametrized surface $f(s,t)$. Let $V(s,t)$ be a vector field along this surface.

Claim: $\dfrac{D}{dt}\dfrac{DV}{ds} - \dfrac{D}{ds}\dfrac{DV}{dt} = R\left(\dfrac{\partial f}{\partial s}, \dfrac{\partial f}{\partial t}\right)V.$

Pf: Take local coördinates $\varphi: U \to M$ and $\left\{\dfrac{\partial}{\partial x_i}\right\}$ on T_pM, $p \in U$, so that

$$V = \sum_i v^i \frac{\partial}{\partial x_i}.$$

Then,

$$\frac{DV}{ds} = \frac{D}{ds}\sum_i v^i \frac{\partial}{\partial x_i}$$

$$= \sum_i v^i \frac{D}{ds}\left(\frac{\partial}{\partial x_i}\right) + \sum_i \frac{\partial v^i}{\partial s}\frac{\partial}{\partial x_i}.$$

So,

$$\frac{D}{\partial t}\frac{D}{\partial s} V = \sum_i v^i \frac{D}{\partial t}\frac{D}{\partial s}\left(\frac{\partial}{\partial x_i}\right)$$

$$+ \sum_i \frac{\partial v^i}{\partial t}\frac{D}{\partial s}\left(\frac{\partial}{\partial x_i}\right)$$

$$+ \sum_i \frac{\partial v^i}{\partial s}\frac{D}{\partial t}\left(\frac{\partial}{\partial x_i}\right)$$

$$+ \sum_i \frac{\partial^2 v^i}{\partial t \partial s}\frac{\partial}{\partial x_i}$$

Similar for $\frac{D}{\partial s}\frac{D}{\partial t} V$. So,

$$\frac{D}{\partial t}\frac{D}{\partial s} V - \frac{D}{\partial s}\frac{D}{\partial t} V$$

$$= \sum_i v^i \left(\frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial}{\partial x_i} - \frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial}{\partial x_i}\right)$$

$$\frac{\partial f}{\partial s} = \sum_j \frac{\partial x_i}{\partial s}\frac{\partial}{\partial x_j}, \quad \frac{\partial f}{\partial s} = \sum_k \frac{\partial x_k}{\partial t}\frac{\partial}{\partial x_k}$$

But,

$$\frac{D}{ds}\left(\frac{\partial}{\partial x_i}\right) = \nabla_{\sum_j \frac{\partial x^j}{\partial s}\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \sum_j \frac{\partial x_j}{\partial s} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$$

$$\frac{D}{\partial t}\frac{D}{\partial s}\left(\frac{\partial}{\partial x_i}\right) = \sum \frac{\partial^2 x_j}{\partial t\, \partial s} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$$

$$+ \sum_{j,k} \frac{\partial x_j}{\partial s}\frac{\partial x_k}{\partial t} \nabla_{\frac{\partial}{\partial x_k}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i}$$

Same for the other term of the subtraction....

At the end, we have:

$$\frac{D}{\partial t}\frac{D}{\partial s} V - \frac{D}{\partial s}\frac{D}{\partial t} V$$

$$= \sum_{i,j,k} V^i \frac{\partial x_j}{\partial s}\frac{\partial x_k}{\partial t} \overbrace{\left(\nabla_{\frac{\partial}{\partial x_k}} \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} - \nabla_{\frac{\partial}{\partial x_j}} \nabla_{\frac{\partial}{\partial x_k}} \frac{\partial}{\partial x_i} \right)}^{= R\left(\frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k}\right)\frac{\partial}{\partial x_i}}$$

Multilinearity

$$= R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right) V$$

∎

LECTURE 10

RICCI CURVATURE

10/10/2023

Def: (RICCI TENSOR) The Ricci tensor of (M^n, g) is the bilinear symmetric tensor

$$\mathrm{Ric} : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow C^\infty(M)$$

given by

orthonormal bases of $v^\perp$

$$\mathrm{Ric}(X,Y)_p := \frac{1}{n-1} \underbrace{\sum_{i=1}^{n} \langle R(e_i, X)Y, e_i \rangle}_{\mathrm{tr}\, R(\cdot, X)Y}$$

(by definition of trace)

In particular, $\mathrm{Ric}(V) = \mathrm{Ric}(V,V) = \mathrm{tr}\, R_V$

since $R_V : \mathfrak{X}(M) \to \mathfrak{X}(M)$, $R_V(X) = R(X,V)V$

Geometrically, $\mathrm{Ric}(V) = \frac{1}{n-1} \sum_{i=1}^{n-1} \sec(V, e_i)$ is an "average" of sectional curvatures that contain V.

* **JACOBI FIELDS**: Studies how fast the geodesics "spread out".

Lemma: The Jacobi field along $\gamma(t)$ with

$$J(0) = 0 \quad \text{and} \quad J'(0) = w$$

is

$$J(t) := d\left(\exp_{\gamma(0)}\right)_{t\dot{\gamma}(0)} tw$$

Unique Jacobi field along $\gamma(t)$ w/ these initial conditions

Similar formulation: can also write the unique Jacobi field along $\gamma(t)$ w/ arbitrary initial condition $J(0)$ and $J'(0)$

$$J(t) = \frac{\partial}{\partial s} \exp_{\alpha(s)} tw(s) \Big|_{s=0},$$

where $\begin{cases} \alpha(s) \text{ is a curve s.t. } \alpha(0) = \gamma(0),\ \alpha'(0) = J(0) \\ w(s) \text{ is a vector field along } \alpha(s) \text{ with} \\ w(0) = \gamma'(0) \text{ and } w'(0) = J'(0) \end{cases}$

NOTE:

$$\frac{D}{\partial t}\frac{\partial}{\partial t}\left(\exp_p tv(s)\right) = 0$$

$$\frac{D}{\partial s}\frac{\partial}{\partial s}\left(\exp_p tv(s)\right) = 0$$

Now,

$f(t,s) = \exp_p tv(s)$

$$\frac{D}{\partial s}\frac{D}{\partial t}\frac{\partial f}{\partial t} = \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} - R\left(\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right)\frac{\partial f}{\partial t}$$

$$= \frac{D}{\partial t}\frac{D}{\partial s}\frac{\partial f}{\partial t} + R\left(\frac{\partial f}{\partial t}, \frac{\partial f}{\partial s}\right)\frac{\partial f}{\partial t}$$

$$= 0$$

$$\Longrightarrow \boxed{\frac{D^2}{dt^2} J + R(\gamma', J)\,\gamma' = 0}$$

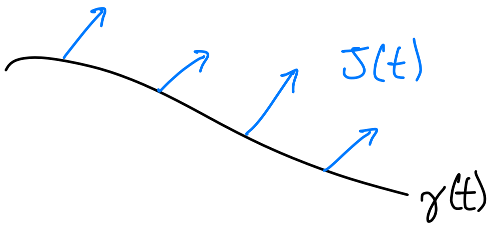
JACOBI EQUATION

$\rightsquigarrow$ $\underbrace{2^{nd} \text{ order}}$, linear system of ODEs

Demands 2 initial conditions

$$J(0),\ J'(0) = \frac{DJ}{dt}(0)$$

Def: The Jacobi field $J(t)$ is the <u>unique</u> vector field that solves Jacobi equation.



$\{e_i(t)\}$ orthonormal basis from the parallel transport along $\gamma(t)$.

Then, we can write

$$J(t) = \sum_i f_i(t)\, e_i(t)$$

Since e_i are orthonormal

$$\frac{DJ}{dt} = \sum_i f_i'(t)\, e_i(t)$$

$$\frac{D^2J}{dt} = \sum_i f_i''(t)\, e_i(t)$$

$$R(\gamma', J)\,\gamma' = \sum_j (\text{coeffs})\, e_j(t)$$

$$(\text{coeffs}) = \langle R(\gamma', J)\gamma', e_j \rangle$$

Now, $\quad a_{ij} = \langle R(\gamma', e_i)\gamma', e_j \rangle$

$$R(\gamma', J)\gamma' = \sum \langle R(\gamma', J)\gamma', e_j \rangle e_j$$

$$= \sum_{i,j} f_i \langle R(\gamma', e_i)\gamma', e_j \rangle e_j$$

$$= \sum_{i,j} f_i a_{ij} e_j$$

$$\Rightarrow \boxed{f_j'' + \sum_i a_{ij} f_i = 0, \quad j = 1, \ldots, n}$$

2^{nd} order linear system of n ODEs.

Examples: $\gamma'(t)$ is Jacobi since $\frac{D\gamma'}{dt} = 0$, so $\dfrac{D^2\gamma'}{dt^2} = 0$ and $R(\gamma',\gamma')\gamma' = 0$ by antisymmetry.

$\boxed{t\gamma'(t)}$ is also Jacobi.

—————— // ——————

MANIFOLDS OF CONSTANT SECTIONAL CURVATURE

Say $\sec \equiv \kappa \geq 0$. Then the Jacobi field orthogonal to γ (which is parametrized by arclength $|\gamma'| = 1$) is

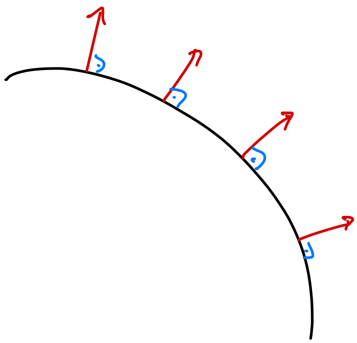
$$\boxed{R(\gamma', J)\gamma' = \kappa J}.$$

Compute: $Z \in \mathcal{X}(M)$

$$\langle R(\gamma', J)\gamma', Z\rangle = \kappa\Big[\overbrace{\langle\gamma',\gamma'\rangle}^{=1 \text{ since arclength}}\langle J, Z\rangle - \underbrace{\cancel{\langle\gamma', J\rangle}}_{J\perp\gamma'}\langle\gamma', Z\rangle\Big]$$

$$= \kappa\langle J, Z\rangle.$$

Take $W(t)$ parallel along γ and orthogonal to γ with $|W(t)| = 1$. Then, the Jacobi field along W is



$\gamma(t)$

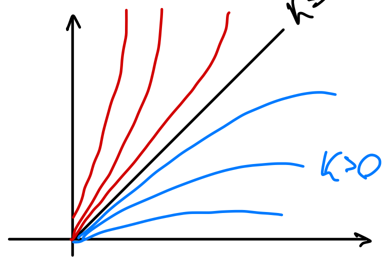
$$J(t) = a(t)\,W(t)$$

$$\frac{D^2 J}{dt} + R(\gamma', J)\gamma' = 0$$

$$\Downarrow$$

$$a''(t)\,w(t) + \kappa a\, w(t) = 0$$

$$\boxed{a'' + \kappa a = 0\,, \quad a(0) = 0}$$

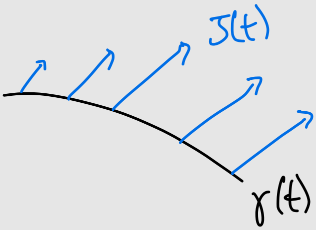


$$\Rightarrow \quad J(t) = \begin{cases} \dfrac{\sin(t\sqrt{\kappa})}{\sqrt{\kappa}}\, w(t)\,, & \kappa > 0 \\ t\,w(t)\,, & \kappa = 0 \\ \dfrac{\sinh(t\sqrt{-\kappa})}{\sqrt{-\kappa}}\,, & \kappa < 0 \end{cases}$$

Proposition: Let $\gamma : [0,1] \to M$ be a geodesic and let $J(t)$ be the Jacobi field along γ with

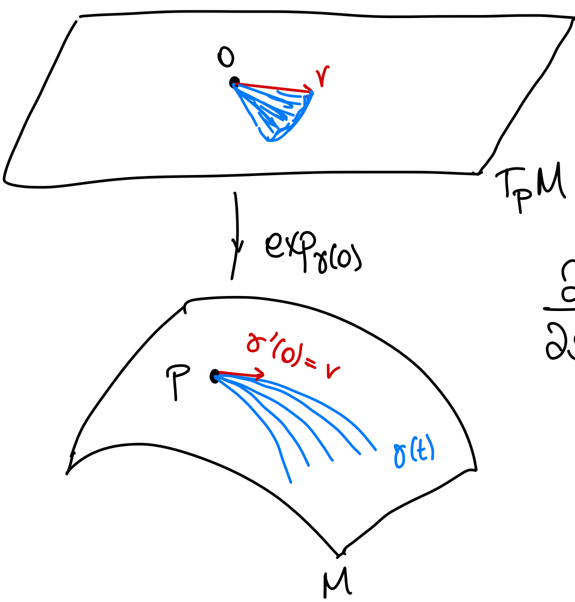
$$J(0) = 0, \qquad \frac{DJ}{dt}(0) = w. \qquad \gamma'(0) = v$$

Thus



$$\boxed{J(t) = d\left(\exp_{\gamma(0)}\right)_{t\gamma'(0)} tw.}$$

Pf: Clearly, $J(t)$ is the variational field of a variation of γ by geodesics, so it is a Jacobi field. Indeed,



$$\frac{\partial}{\partial s}\left(\exp_{\gamma(0)}\left(t(v+sw)\right)\right)\Big|_{s=0}$$

$$= d\left(\exp_{\gamma(0)}\right)_{tv} tw$$

$$= J(t).$$

∎

* JACOBI & CURVATURE

Prop: Let $p \in M$ and $\gamma : [0,a] \to M$ a geodesic with $\gamma(0) = p$ and $\gamma'(0) = v$. Let

$$w \in T_v(T_pM) \simeq T_pM, \quad |w| = 1.$$

Let J be the Jacobi field along γ given by

$$J(t) = d(\exp_p)_{tv}\, tw.$$

Then, the Taylor expansion:

$$\|J(t)\|^2 = t^2 - \frac{1}{3}\langle R(v,w)v, w\rangle t^4 + O(t^5)$$

Curvature controls the length of Jacobi fields.

Pf: Compute some derivatives:

$$\langle J,J\rangle(0) = 0$$

$$\langle \underset{\shortparallel \atop 0}{J}, J\rangle'(0) = 2\langle J, \underset{\shortparallel \atop 0}{J'}\rangle(0) = 0$$

$$\langle J,J\rangle''(0) = 2\overbrace{\langle J', J'\rangle}^{=|w|^2=1}(0) + 2\langle J'', \underset{\shortparallel \atop 0}{J}\rangle(0) = 2.$$

Also, $J''(0) = -R(\overset{=0}{J}, \gamma')\gamma'(0) = 0$ so

$$\langle J, J\rangle'''(0) = 6\langle \underset{\overset{\shortparallel}{0}}{J'}, J''\rangle(0) + 2\langle J''', \underset{\overset{\shortparallel}{0}}{J}\rangle(0)$$

$$= 0.$$

What about $\langle J, J\rangle''''(0)$?

<u>Claim</u>: $\nabla_{\gamma'}(R(\gamma', J)\gamma')(0) = R(\gamma', J')\gamma'(0)$

Let W be any vector field along γ and compute

$$\left\langle \frac{D}{dt} R(\gamma', J)\gamma', W\right\rangle = \frac{d}{dt} \underbrace{\langle R(\gamma', J)\gamma', W\rangle}_{= \langle R(W, \gamma')\gamma', J\rangle}$$

$$- \langle R(J, \gamma')\gamma', W'\rangle$$

$$= \left\langle \frac{D}{dt} R(W, \gamma')\gamma', J\right\rangle$$

$$+ \overbrace{\cancel{\langle R(W, \gamma')\gamma', J'\rangle}}^{\langle R(J', \gamma')\gamma', W\rangle}$$

$$- \cancel{\langle R(J, \gamma')\gamma', W'\rangle} \quad = 0 \text{ at zero}$$

At zero, we only have that

$$\frac{D}{dt} R(J,\gamma')\gamma' \Big|_{t=0} = R(\gamma', J')\gamma' \Big|_{t=0}$$

So,

$$\langle J, J\rangle''''(0) = 2\underbrace{\langle J'''', J\rangle(0)}_{=0} + 8\langle J''', J'\rangle(0) + 6\underbrace{\langle J'', J''\rangle(0)}_{=0}$$

$$= 8\langle J''', J'\rangle(0)$$

$$= -8\langle R(v,w)v, w\rangle .$$

$\square$

Def: (CONJUGATE) Let $\gamma(t)$ be a geodesic in M. Then, a point $q = \gamma(t_*)$ is conjugate to the point $p = \gamma(0)$ along $\gamma(t)$ if there exists a Jacobi field $J(t)$ along $\gamma(t)$ such that

$$J(0) = 0 \quad \text{and} \quad J(t_*) = 0$$

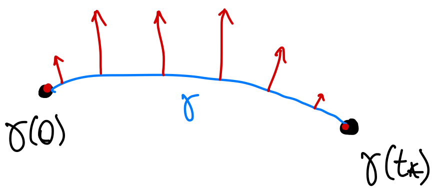
Ex: Antipodal points on S^n.

LECTURE 11

12/10/2023

CONJUGATE LOCUS & 2^{nd} FUND. FORM

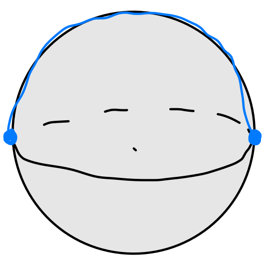
Recall: Let $\gamma(t)$ be a geodesic of M. Then $q = \gamma(0)$ and $p = \gamma(t_*)$ are conjugate if there exists a nonzero Jacobi field $J(t)$ such that



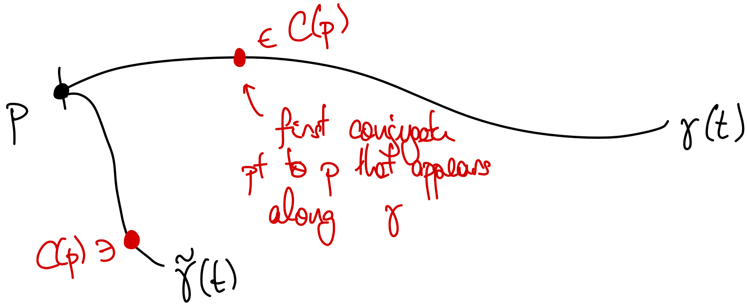
$$0 = J(0) = J(t_*).$$

$$\boxed{\text{Multiplicity of conjugate pts.}} = \underbrace{\boxed{\text{Maximal \# of linearly independent such Jacobi fields.}}}_{= n-1}$$

Ex: $\mathbb{S}^n$ has $n-1$ such fields.



Def: For $p \in M$, the set of all first conjugate pts. $C(p)$ is called the conjugate locus.



Thm: If M has constant non-positive sectional curvature ($\sec \equiv K \le 0$), then $C(p) = \emptyset$.

Pf: Let $\gamma(t)$ be a geodesic. Suppose

$$J(0) = J(a) = 0 \underset{\text{WTS}}{\Longrightarrow} J(t) \equiv 0.$$

It suffices to show that $\|J(t)\|^2 = \text{constant}$.

So, differentiate:

$$\frac{d}{dt}\|J(t)\|^2 = \frac{d}{dt}\langle J, J\rangle = 2\langle J', J\rangle$$

$$\frac{d^2}{dt^2}\|J(t)\|^2 = -2\underbrace{\langle J'', J\rangle}_{\langle R(\gamma', J)\gamma', J\rangle} + 2\langle J', J'\rangle$$

$$= -2\langle R(\gamma', J)\gamma', J\rangle + 2\langle J', J'\rangle$$

$$\geq 0 \quad (\text{since } \sec \leq 0)$$

But, since

$$\langle J', J\rangle(0) = \langle J', J\rangle(a) = 0 . \quad (*)$$

Then,

$$\frac{d^2}{dt^2} \| J(t) \|^2 = 0 \overset{(*)}{\Rightarrow} \frac{d}{dt} \| J(t) \|^2 = 0$$

$$\overset{(*)}{\Rightarrow} J(t) \equiv 0 .$$

$\square$

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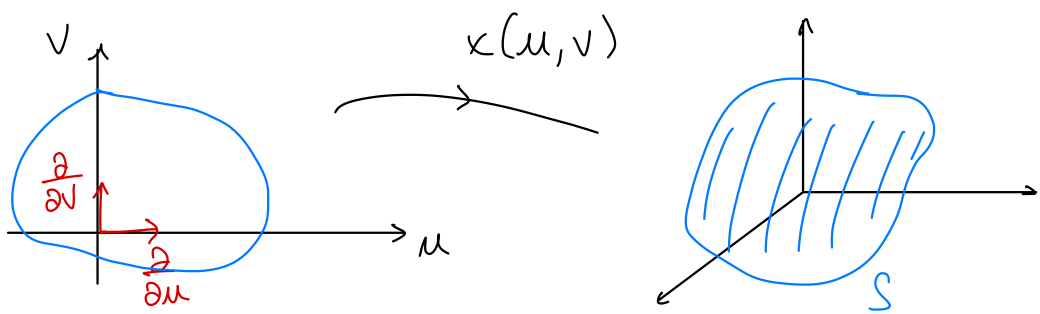
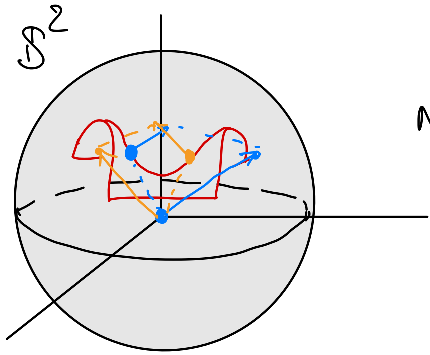
* **GAUSS MAP**: Suppose $S \subset \mathbb{R}^3$ with the following parametrization:



where $\left\{\frac{\partial}{\partial v}, \frac{\partial}{\partial u}\right\}$ is the associated basis. Let $\hat{N}$ be the normal vector field of $S \subset \mathbb{R}^3$. Then,

$$\hat{N} = \frac{\frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}}{\left\| \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} \right\|}$$

GAUSS MAP: $N : S \to \mathbb{S}^2$



N translates the normal vectors to S to the origin.

Eigenvalues tell us sec

$$dN : T_p S \to T_{N(p)} \mathbb{S}^2 \simeq T_p S$$

since the planes are parallel

What does dN_p do?

$$dN_p \circ \alpha'(0) = (N \circ \alpha)'(0)$$

Ex: Plane $ax + by + cz + d = 0$. Then

$$\hat{N} = \frac{(a, b, c)}{\sqrt{a^2+b^2+c^2}} \Rightarrow dN = 0 \quad (\text{i.e., } \hat{N} \text{ is constant})$$

Sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$.

Let $\alpha(t) = (x(t), y(t), z(t))$ be a parametrized curve.

We must have that

$$x(t)^2 + y(t)^2 + z(t)^2 = 1$$

$$\Rightarrow 2x \frac{dx}{dt} + 2y \frac{dy}{dt} + 2z \frac{dz}{dt} = 0.$$

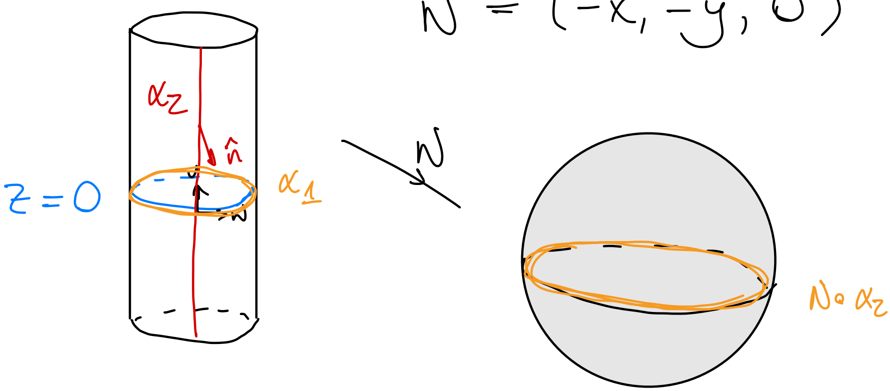
Choose $\hat{N} = (-x, -y, -z)$. So,

$$dN_p \circ v = -v$$

$$\Rightarrow \text{Eigenvalues of } dN_p \text{ are } \underbrace{\lambda_1 = \lambda_2 = -1}_{\text{All positive curvature}}$$

Cylinder $\quad C = \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2=1\}$

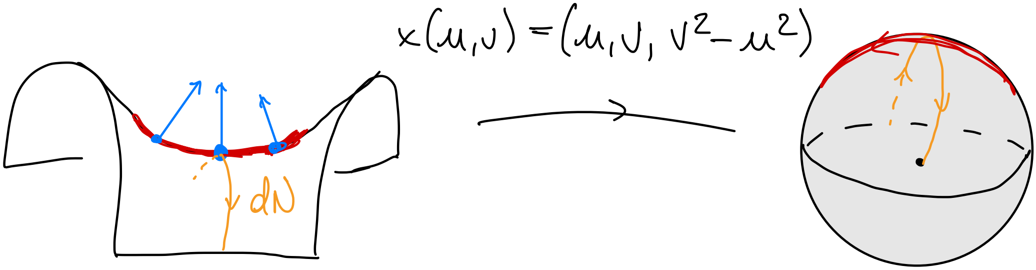
$$\hat{N} = (-x,-y,0)$$



$$\left.\begin{array}{l} \text{For } \alpha_2, \quad dN_p\, v = 0 \\ \text{For } \alpha_1, \quad dN_p\, v = -v \end{array}\right\} \underbrace{\lambda_1 = 0}_{\text{No curvature}},\ \underbrace{\lambda_2 = -1}_{\text{Positive curvature}}$$

Hyperbolic Paraboloid $\quad H = \{(x,y,z)\in\mathbb{R}^3 : y^2 - x^2 = z^2\}$

$$x(u,v) = (u, v, v^2 - u^2)$$



Boring calculation $\Longrightarrow$ Eigenvalues of dN are $\underbrace{\lambda_1 = 2}_{\text{Negative curvature}},\ \underbrace{\lambda_2 = -2}_{\text{Positive curv}}$.

Lecture 12

17/10/2023

2nd Fundamental Form & Fundamental Equations

Suppose $f: M^n \longrightarrow \bar{M}^{n+m}$ is a differentiable immersion. Then, we can induce a Riem. metric on M using f:

$$\forall v_1, v_2 \in T_pM, \quad \langle v_1, v_2 \rangle = \langle df_p(v_1), df_p(v_2) \rangle .$$

We also have an induced Levi-Civita connection $\nabla_X Y$ on M with this induced metric. Suppose that $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$.

For $X, Y \in \mathfrak{X}(M)$, define

Tangential component (not transpose)

$$\nabla_X Y := \left(\bar{\nabla}_{\bar{X}} \bar{Y} \right)^T ,$$

$\bar{X}, \bar{Y}$ are the local extensions of these vector fields to the ambient space $\bar{M}$.

<u>Claim</u>: ∇ is Levi-Civita. Pf: Check the properties.

For $p \in M$, we can split

$$T_p\overline{M} = T_pM \oplus (T_pM)^\perp.$$

By doing this, for $v \in T_pM$, we can split it into normal & tangential components:

$$v = v^T + v^N.$$

Clearly, the projections

$$(p, v) \longmapsto (p, v^T)$$

$$(p, v) \longmapsto (p, v^N)$$

are differentiable.

Def: For $X, Y \in \mathfrak{X}(M)$,

$$\boxed{B(X, Y) := \overline{\nabla}_{\overline{X}}\overline{Y} - \nabla_X Y}$$

vec. field normal to M

REMARKS: (i) B does not depend on the 'choice of extension" $\bar{X}, \bar{Y} \in \mathfrak{X}(\bar{M})$ for $X, Y \in \mathfrak{X}(M)$.

Pf: Suppose $\bar{X}_1$ is another extension of X. Then

$$(\bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y) - (\bar{\nabla}_{\bar{X}_1} \bar{Y} - \nabla_X Y) = \bar{\nabla}_{\bar{X}} \bar{Y} - \bar{\nabla}_{\bar{X}_1} \bar{Y} = \bar{\nabla}_{\underbrace{\bar{X} - \bar{X}_1}_{= 0 \text{ on } M}} \bar{Y} = 0$$

Suppose $\bar{Y}_1$ is another extension of Y. Then

$$(\bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y) - (\bar{\nabla}_{\bar{X}} \bar{Y}_1 - \nabla_X Y) = \bar{\nabla}_{\underbrace{\bar{X}}_{\bar{X} \text{ is tangent to } M}} \underbrace{(\bar{Y} - \bar{Y}_1)}_{\text{Normal to } M \text{ and constant on } M} = 0 \quad \text{on } M.$$

$\blacksquare$

Prop: $B(X,Y) = \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y$ is <u>bilinear</u> and <u>symmetric</u>.

Pf: • Linear in the first argument:

$$B(fX, Y) = \bar{\nabla}_{\bar{fX}} \bar{Y} - \nabla_{fX} Y$$

$\bar{f}$ is the local extension

$$= \underbrace{\bar{f}}_{= f \text{ on } M} \bar{\nabla}_{\bar{X}} \bar{Y} - f \nabla_X Y$$

$$= f(B(X,Y)).$$

$$B(X_1 + X_2, Y) = \bar{\nabla}_{\bar{X}_1 + \bar{X}_2} \bar{Y} - \nabla_{X_1 + X_2} Y$$

Linearity of connection

$$= B(X_1, Y) + B(X_2, Y)$$

• B is symmetric (i.e., $B(X,Y) = B(Y,X)$):

$$B(X,Y) = \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y$$

$$= \bar{\nabla}_{\bar{Y}} \bar{X} + \underbrace{[\bar{X}, \bar{Y}]}_{= [X,Y] \text{ on } M} - \nabla_Y X - [X,Y]$$

$$= \bar{\nabla}_{\bar{Y}} \bar{X} - \nabla_Y X$$

$$= B(Y,X).$$

■

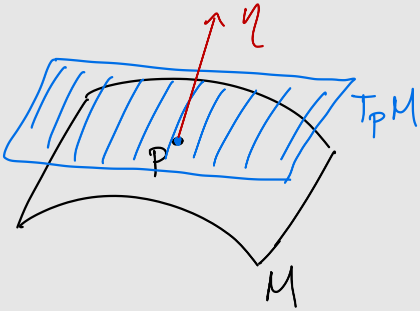
Upshot: Since $B(X,Y)$ is bilinear, the value of $B(X,Y)(p)$ depends only on the values of $X(p)$, $Y(p)$.

Indeed, if

$$X = \sum a_i \frac{\partial}{\partial x_i} \quad , \quad Y = \sum b_j \frac{\partial}{\partial y_j}$$

$$B(X,Y) = \sum_{i,j} a_i b_j \, B\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_j}\right).$$

Def: (2nd FUNDAMENTAL FORM) Let $p \in M$ and $\eta \in (T_pM)^\perp$



Define $H_\eta : T_pM \times T_pM \to \mathbb{R}$

as $H_\eta(x,y) := \langle B(x,y), \eta \rangle$, symmetric & bilinear

$x, y \in T_pM$.

The 2nd Fundamental Form (quadratic form)

$$\boxed{\mathrm{II}_\eta(x) := H_\eta(x,x)}.$$

Since H_η is bilinear, it is associated with a self-adjoint operator $S_\eta : T_pM \longrightarrow T_pM$

$$\langle S_\eta(x), y \rangle = H_\eta(x,y) = \langle B(x,y), \eta \rangle.$$

Prop: Let $p \in M$, $x \in T_pM$, $\eta \in (T_pM)^\perp$. Let N be the local extension of η normal to M. Then

$$\boxed{S_\eta(x) = -\left(\bar{\nabla}_x N\right)^T}$$ Tangential component

Pf: Let $y \in T_pM$ and let $\bar{X}, \bar{Y}$ be local extensions of x, y tangent to M. Then $\langle N, Y\rangle = 0$ and so

$$\langle S_\eta(x), y\rangle = \langle B(X,Y)(p), N\rangle$$

$$= \langle \bar{\nabla}_{\bar{X}} \bar{Y} - \nabla_X Y, N\rangle(p)$$

$$= \langle \bar{\nabla}_{\bar{X}} \bar{Y}, N\rangle(p)$$

$$= \underbrace{\cancel{\bar{X}\langle \bar{Y}, N\rangle}}_{0=}(p) - \langle Y, \bar{\nabla}_{\bar{X}} N\rangle(p)$$

$$= -\langle \bar{\nabla}_x N, y\rangle$$

$$\Longrightarrow \boxed{S_\eta(x) = -\bar{\nabla}_x N.}$$

∎

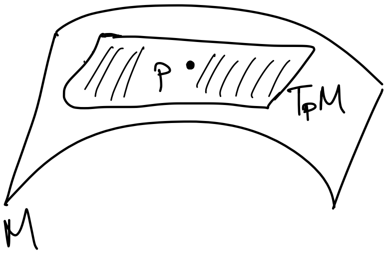
EXAMPLE: (Codimension of the immersion is 1)

Then $f: M^n \longrightarrow \bar{M}^{n+1}$ and $f(M^n)$ is called a hypersurface (can have self-intersections).

Let $p \in M$ and $\eta \in (T_pM)^\perp$, $|\eta| = 1$. So,

$$S_\eta : T_pM \to T_pM,$$

S_η is <u>symmetric</u> & <u>linear</u>.



So, there exists an orthonormal basis for T_pM of eigenvectors of S_η:

$e_1, \_\_\_, e_n \longleftarrow$ eigenvectors of S_η

$\lambda_1, \_\_\_, \lambda_n \longleftarrow$ eigenvalues of S_η

$$S_\eta(e_i) = \lambda_i e_i, \quad 1 \le i \le n.$$

Suppose M and $\overline{M}$ are both orientable. Then, choose η so that

- $\{e_1, \_, e_n\}$ is consistent w/ orientation of M
- $\{e_1, \_\, e_n, \eta\}$ is consistent w/ orientation of $\overline{M}$

Then

e_i = PRINCIPAL DIRECTIONS

$\lambda_i := k_i$ PRINCIPAL CURVATURES OF f

$\det S_\eta = \lambda_1 \cdots \lambda_n = k_1 \cdots k_n \leftarrow$ GAUSS-KROENECKER CURVATURE OF f.

$\frac{1}{n}(\lambda_1 + \cdots + \lambda_n) \leftarrow$ MEAN CURVATURE OF f

<u>Important case</u>: $\overline{M} = \mathbb{R}^{n+1}$. Let N be a local extension of η, $|\eta| = 1$, η normal to M.

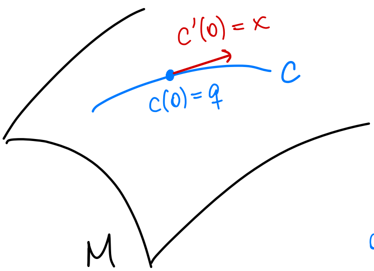
Consider

$$\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}. \quad \text{(unit sphere)}$$

Define the <u>Gauss Spherical Mapping</u> $g : M^n \to \mathbb{S}^n$ by translating the origin of the field N to the origin of $\mathbb{R}^{n+1}$ and taking for $q \in M^n$

$g(q)$ = endpoint of the translation of $N(q)$

Note: $T_q M^n$ and $T_{g(q)} \mathbb{S}^n$ are parallel, so we can identify them. Then, the differential of the Gauss map is



$$dg_q : T_q M^n \longrightarrow T_q M^n$$

$$dg_q(x) = -S_\eta(x).$$

$\langle N, N\rangle = 1$

$$dg_q(x) = \frac{d}{dt}(N \circ c(t))\Big|_{t=0} = \bar{\nabla}_x N = (\bar{\nabla}_x N)^T = -S_\eta(x)$$

where $c:(-\varepsilon,\varepsilon) \to M$ is a curve w/ $c(0)=q$ and $c'(0)=x$.

GAUSS EQUATION: Let $p \in M$, x, y be orthonormal vectors in $T_p M$. Then,

$$\boxed{\sec(x,y) - \overline{\sec}(x,y) = \langle B(x,x), B(y,y)\rangle - |B(x,y)|^2.}$$

Can compute sec using either connections ∇ or $\bar{\nabla}$

Relates the curvature of ambient w/ the curvature of immersed space.

Pf: Local extension to M: X, Y

Local extension to $\bar{M}$: $\bar{X}, \bar{Y}$

$$\sec(x,y) - \overline{\sec}(x,y) = \langle \nabla_Y \nabla_X X - \nabla_X \nabla_Y X - (\bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X} - \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{X}), Y\rangle (p)$$

$$+ \langle \cancel{\nabla_{[X,Y]} X - \bar{\nabla}_{[\bar{X},\bar{Y}]} \bar{X}}, \bar{Y} \rangle (p)$$

$0 =$

$$= \langle \nabla_Y \nabla_X X - \bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X}, Y \rangle (p)$$

$$- \langle \nabla_X \nabla_Y X - \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{X}, Y \rangle (p)$$

Deal with this piece by piece:

$\bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X} \rightsquigarrow$ Find $\bar{\nabla}_{\bar{X}} \bar{X}$ and then differentiate

Recall that $B(X,X) = \bar{\nabla}_{\bar{X}} \bar{X} - \nabla_X X$

$$\Rightarrow \bar{\nabla}_{\bar{X}} \bar{X} = B(X,X) + \nabla_X X$$

Now, $\bar{M}^{n+m}, M^n \rightsquigarrow m = \operatorname{codim} M = \dim \bar{M} - \dim M$.

Choose orthonormal fields normal to M. Denote them $E_1, \_\_, E_m$. Recall that $B(X,X)$ is normal to M, so we can write

$\langle B(X,X), E_i \rangle$

$$B(X,X) = \sum_i H_{E_i}(X,X)\, E_i$$

$$\Rightarrow \bar{\nabla}_{\bar{X}} \bar{X} = B(X,X) + \nabla_X X$$

$$= \sum_i H_{E_i}(X,X) E_i + \nabla_X X$$

Thus,

$$\bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X} = \bar{\nabla}_{\bar{Y}} \left[\sum_i H_{E_i}(X,X) E_i + \nabla_X X \right]$$

$$= \sum_i H_{E_i}(X,X) \bar{\nabla}_{\bar{Y}} E_i + \bar{Y}\, H_{E_i}(X,X) E_i$$

$$+ \bar{\nabla}_{\bar{Y}} \nabla_X X .$$

So, at p,

$$\langle \bar{\nabla}_{\bar{Y}} \bar{\nabla}_{\bar{X}} \bar{X}, Y \rangle \overset{(A)}{=} - \overbrace{\sum_i H_{E_i}(X,X) H_{E_i}(Y,Y)}^{= \langle B(X,X), B(Y,Y) \rangle}$$

$$+ \langle \nabla_Y \nabla_X X, Y \rangle$$

$$\langle \bar{\nabla}_{\bar{X}} \bar{\nabla}_{\bar{Y}} \bar{X}, Y \rangle \overset{(B)}{=} - \sum_i H_{E_i}(X,Y) H_{E_i}(X,Y)$$

$$+\langle D_X D_Y X, Y\rangle .$$

Using (A) & (B), we get Gauss' Equation. $\blacksquare$

REMARK: In the case of a hypersurface

$f: M \to \overline{M}^{n+1}$, $p \in M$, $\eta \in (T_pM)^\perp$,

$\{e_1, \ldots, e_n\}$ orthonormal basis of T_pM

$\Rightarrow$ S_η is diagonal, $S_n e_i = \lambda_i e_i$

$B(x,y) = H_\eta(x,y)$

$$\Rightarrow \boxed{\begin{aligned}\langle S_\eta(x), y\rangle &= \langle H_\eta(x,y), \eta\rangle \\ &= H_\eta(x,y)\end{aligned}}$$

$H(e_i, e_i) = \lambda_i$

$H(e_i, e_j) = 0 \quad \forall i \neq j.$

Thus, by Gauss' equation:

$$\sec(e_i, e_j) - \overline{\sec}(e_i, e_j)$$

$$= \underbrace{\langle B(e_i, e_i), B(e_j, e_j) \rangle}_{= \lambda_i \lambda_j} - \underbrace{\langle B(e_i, e_j), B(e_i, e_j) \rangle}_{= 0}$$

Upshot: $\boxed{\sec(e_i, e_j) - \overline{\sec}(e_i, e_j) = \lambda_i \lambda_j .}$

Ex: If $\overline{M} = \mathbb{R}^3$, M surface in $\mathbb{R}^3$, then

$\overline{\sec}(e_i, e_j) = 0$, $\sec(e_i, e_j) = \lambda_1 \lambda_2$.

EXAMPLE: (CURVATURE OF $\mathbb{S}^n \subset \mathbb{R}^{n+1}$)

$$\mathbb{S}^n = \{ x \in \mathbb{R}^{n+1} : |x| = 1 \}, \quad N(x) = x$$

Gauss map $= -\mathrm{id}$

Differential of Gauss map $= -\mathrm{id}$ $\Rightarrow$ All eigenvals are -1

$\Rightarrow$ Product of any 2 eigenvalues is $= 1$ $\Rightarrow$ $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ has curvature 1

Def: An immersion $f: M \to \bar{M}$ is geodesic at $p \in M$ if $\forall \eta \in (T_pM)^\perp$ the second fundamental form $H_\eta = 0$ at p.

If an immersion is geodesic at all $p \in M$, it is called totally geodesic ← e.g. linear subspaces of Euclidean spaces.

LECTURE 13 — ISOMETRIC IMMERSIONS

19/10/2023

Thm: An immersion $f: M \to \bar{M}$ is geodesic at a point $p \in M$ iff every geodesic γ at $p \in M$ is also a geodesic at $p \in \bar{M}$.

Pf: We have $\gamma(t)$ and $\gamma'(t)$. But, when we compute the covariant derivative, we can do it w.r.t. 2 connections; namely:

$$\nabla_{\gamma'(t)} \gamma'(t) \quad \text{or} \quad \bar{\nabla}_{\gamma'(t)} \gamma'(t).$$

Let γ be s.t. $\gamma(0) = p$ and $\gamma'(0) = x$ and let $\eta \in (T_pM)^\perp$. Extend x to X, η to N so that $\rightsquigarrow$ $\langle X, N \rangle = 0$.

Then

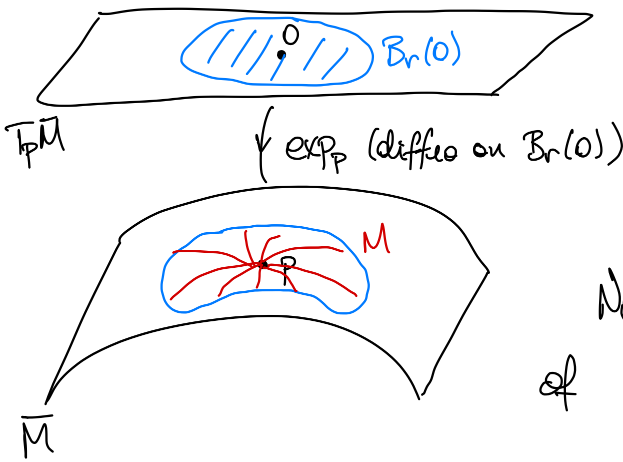
$$
\begin{aligned}
\mathrm{II}_\eta(x) \overset{\text{def}}{=} H_\eta(x,x) &= \langle S_\eta(x), x \rangle \\
&= \langle -\bar{\nabla}_X N, X \rangle (p) \\
&= -\underbrace{X\langle N, X\rangle}_{=0} + \langle N, \bar{\nabla}_X X \rangle (p) \\
&= \langle N, \bar{\nabla}_X X \rangle (p) \\
&= 0 \iff \bar{\nabla}_X X \text{ does not have a normal component.}
\end{aligned}
$$

■

——— // ———

GEOMETRIC INTUITION ABOUT sec:

Let $(\bar{M}, g)$ be a Riem. manifold. Take a normal neighborhood $B_r(0) \subset T_p\bar{M}$.



Let M be the Riem. manifold given by the image of $B_r(0)$ via the $\exp_p$ map.

Note that M is a submfd of $\bar{M}$.

Thus, by Gauss' eq. $\sec(\sigma_p) - \overline{\sec}(\sigma_p) = 0$.

$\Rightarrow$ They are equal!

---

* FUNDAMENTAL EQUATIONS

1) GENERALIZED GAUSS' EQUATION

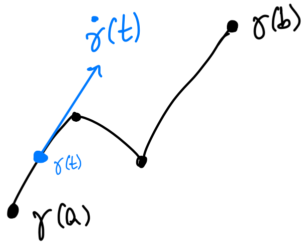
$$\langle \bar{R}(X,Y)Z, T\rangle - \langle R(X,Y)Z, T\rangle$$

$$= \langle B(X,T), B(Y,Z)\rangle - \langle B(Y,T), B(X,Z)\rangle.$$

* TOPOLOGY & GEOMETRY

Def: (GEODESICALLY COMPLETE) A Riem. manifold M is geodesically complete if $\forall p \in M$, the map $\exp_p$ is defined $\forall v \in T_pM$. ← i.e., all geodesics can be extended to $(-\infty, \infty)$

Def: Let $\gamma : [a,b] \to (M^n, g)$ be a piecewise smooth curve. The length of γ (w.r.t. g) is



$$L_g(\gamma) := \int_a^b \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))}\, dt.$$

Def: Given points $p \in M$ and $q \in M$, the distance (w.r.t. g) is defined as

$$d(p,q) := \inf \left\{ L_g(\gamma) : \begin{array}{l} \gamma : [a,b] \to M \text{ piecewise} \\ \text{smooth w/ } \gamma(a) = p,\ \gamma(b) = q \end{array} \right\}$$

Prop: (M^n, d) is a metric space and its topology agrees with the manifold topology. Proof on RGB notes

! **VERY IMPORTANT**

Can take normal balls as basis for both topologies.

Thm: (Hopf–Rinow, 1931) Let (M,g) be a Riem. manifold. The following are equivalent:

(i) $\exists p_0 \in M$ s.t. $\exp_{p_0}$ is defined on all of $T_{p_0}M$.

(ii) $K \subset M$ closed and bounded $\Rightarrow$ K compact (Heine–Borel Property)

(iii) (M,d) is a complete metric space (i.e., Cauchy seq. converge)

(iv) M is geodesically complete (i.e., can extend geodesics to $(-\infty,\infty)$ $\Leftrightarrow$ $\forall p \in M$ $\exp_p$ defined on all of T_pM)

(v) There is a sequence of nested compact sets $K_n \subset M$, with $K_n \subset K_{n+1}$, s.t. $\bigcup_n K_n = M$.

* If any, hence all, of the above holds, then for any $q \in M$, there exists a geodesic γ connecting p to q with $L(\gamma) = d(p,q)$.

Not equivalent to the rest. Take $B_r(0) \subset \mathbb{R}^n$ (open). Any 2 pts. are joined by geodesics but not complete.

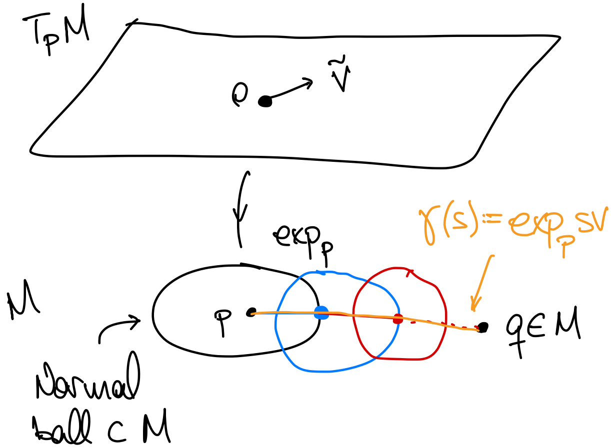
Lecture 14

24/10/2023

Hopf - Rinow & Algebraic Topology

Pf: (Hopf - Rinow)

$((i) \Rightarrow *)$ $\quad d(p,q) = r$. Say $v = \dfrac{\tilde{v}}{\|\tilde{v}\|}$



WTS: $\exp_p rv = q$

$\Updownarrow$

$d(\exp_p rv, q) = 0$

Consider

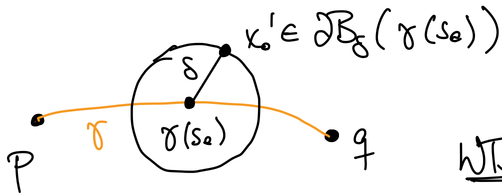
$$A = \{ s \in [0, r] \mid d(\exp_p v, q) = r - s \}$$

WTS: $A = [0, r]$.

Since d is bounded above, $\sup A$ exists.

WTS: $\sup A = r$ and $r \in A$.

Suppose $\sup A = s_0 < r$. We will show that for some small $\delta > 0$, the sup is attained: $s_0 + \delta$



$d(\gamma(s_0+\delta), q) = r - s_0 - \delta$

WTS : $\boxed{x_0' = \gamma(s_0+\delta)}$.

Claim : $\gamma(s_0+\delta) \underset{(*)}{=} x_0' \implies d(\gamma(s_0+\delta), q) = r - s_0 - \delta$

Note that

$$d(\gamma(s_0), q) \geq \delta + \underbrace{\min_{x \in B_\delta(\gamma(s_0))} d(q, x)}$$

and

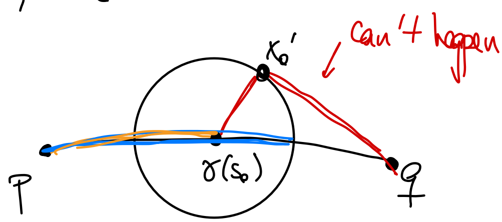
$$d(\gamma(s_0), q) \leq d(\gamma(s_0), \gamma(s_0+\delta)) + \overbrace{d(\gamma(s_0+\delta), q)}^{\parallel\,(*)}$$

Thus : $r - s_0 = \delta + d(\gamma(s_0+\delta), q)$

$$\implies d(\gamma(s_0+\delta), q) = r - s_0 - \delta$$

Assuming $\gamma(s_0+\delta) = x_0'$, $s_0 + \delta \in A$.

Claim : $\gamma(s_0+\delta) = x_0'$.



By triangle inequality

$$d(p,q) \leq d(p, x_0') + d(x_0', q)$$

$$d(p, x_0) \geq d(p, q) - d(x_0', p)$$

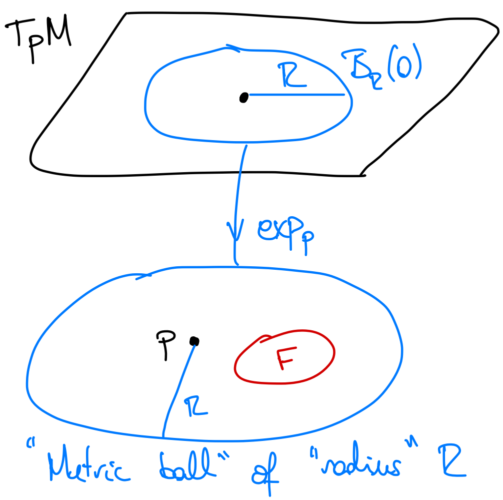
$$\overset{\shortparallel}{s_0 + \delta} \Rightarrow d(p, x_0) = s_0 + \delta$$

$$\Rightarrow x_0' = \gamma(s_0 + \delta).$$

$\square$

((i) $\Rightarrow$ (ii)) Suppose $F \subset M$ is closed & bounded.

WTS: F is a closed subset of a compact set (since Hausdorff, this means that F is compact).



$B_R(0) \subset T_pM$

$\Rightarrow \overline{B_R(p)}$ is closed (since $\exp_p$ is diffeo)

$\Rightarrow F$ compact.

$\square$

((ii) $\Rightarrow$ (iii)) Let (x_n) be Cauchy. Then $\{x_n\}$ is bdd.

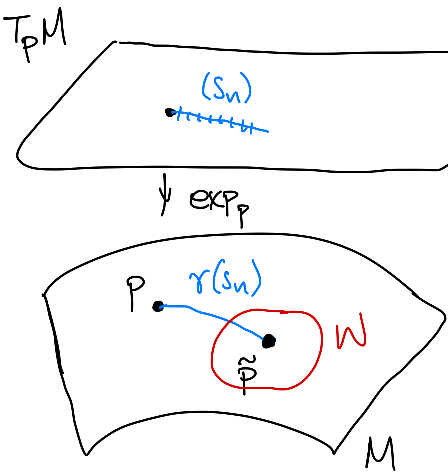
So, $\overline{\{x_n\}}$ is closed & bdd, hence compact by assumption

$\Rightarrow (x_n)$ has a convergent subsequence

$\overset{\text{Cauchy}}{\Longrightarrow} (x_n)$ converges

$\square$

$\big((iii) \Rightarrow (iv)\big)$ Suppose not: $\exists p \in M$ s.t. $\exp_p$ is <u>not</u> defined on all of T_pM.



(s_n) Cauchy. So, $\gamma(s_n)$ converges to some $\tilde{p} \in M$

$d(\gamma(s_n), \gamma(s_m)) \leq |s_n - s_m| \varepsilon$

$\forall n, m \geq N$

W is a totally normal neighborhood.

$\Downarrow$

$\gamma(s_0)$ is defined. $\rightarrow\!\leftarrow$

$\square$

$\big((ii) \Leftrightarrow (v)\big)$ $(\Leftarrow)$ Let A be closed and bdd.

<u>Claim</u>: $\exists\, n$ s.t. $A \subset K_n$

Otherwise, for each n, $\exists q_n \in A$ s.t. $q_n \notin K_n$

which means A is not bdd $\rightarrow\!\leftarrow$

Thus, A is compact.

$\blacksquare$

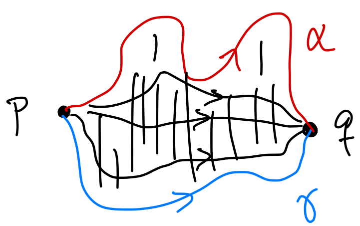
Diversion into Algebraic Topology

topological spaces

Def: (Fundamental Group) Say $f, \tilde{f} : X \to Y$ then f is homotopic to $\tilde{f}$ if $\exists\, F : X \times [0,1] \to Y$ s.t.

$$F(x,0) = f(x)$$
$$F(x,1) = \tilde{f}(x)$$

Path homotopy: two paths γ, α. Define the equivalence class of $\alpha \sim \gamma$ if they are homotopic: $[\gamma]$



Define the FUNDAMENTAL GROUP as the group of these homotopy classes of closed loops:

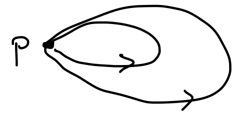
concatenation

$$\pi_1(\cdot) = G = \{ [\gamma] \} \text{ w/ operation } [\gamma] * [\beta].$$

Need to be loops

1 - Homotopy equivalence

↖ At a pt. p : $\pi_1(\cdot, p)$

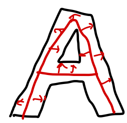


2 - Covering spaces

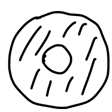
3 - $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$

$$\left(\pi_1(S^1) = \mathbb{Z} \Rightarrow \pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z} \right)$$

4 - Seifert - van Kampen.

__Q__: Are these equivalent: A  ?

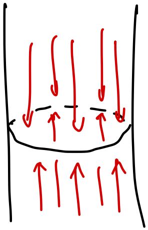
__A__: Yes, they are homotopy equivalent.

__Q__: Are these equiv.   ? Yes, homeomorphic even.

__Def__: (Deformation Retract) A deformation retraction of X onto a subspace A is a 1-parameter family of maps $f_t : X \to X$, $t \in [0,1]$, such that for

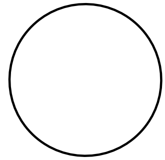
$$f_0 = id, \quad f_1(X) = A, \quad f_t|_A = id$$

Ex:



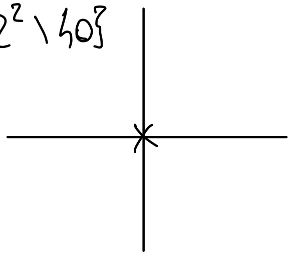
cylinder

Deformation retract $\rightsquigarrow$



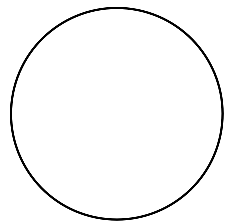
circle

$\mathbb{R}^2 \setminus \{0\}$

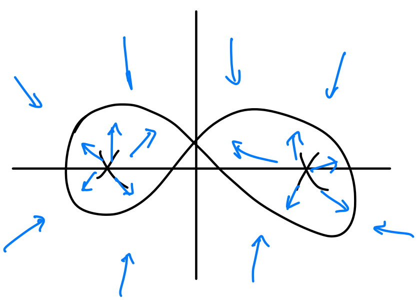


Punctured plane

Deformation retracts $\rightsquigarrow$



circle



Twice punctured plane

$\rightsquigarrow$

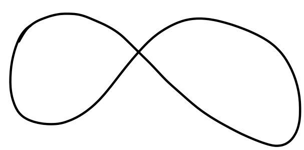


Figure 8

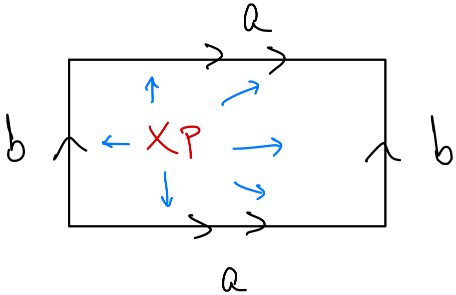
$$\boxed{A \text{ deformation retract of } X} \implies \boxed{\pi_1(A,a) \simeq \pi_1(X,a)}$$

Ex: $\pi_1(S^1) \simeq \mathbb{Z}$, $\pi_1(\text{figure } 8) \simeq \mathbb{Z} * \mathbb{Z}$

Free group onto generators (huge)

$$\pi_1(\mathbb{R}^2 \setminus \{p, q\}) = \mathbb{Z} * \mathbb{Z}.$$

Ex: Punctured Torus $T^2 \setminus \{p\}$



← Deformation retracts to a figure 8

$$\Downarrow$$

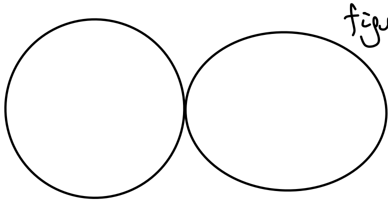
$$\pi_1(T^2 \setminus \{p\}) = \mathbb{Z} * \mathbb{Z}.$$

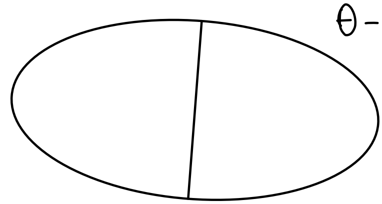
$\square$

Def: (Homotopy Equivalence) A map $f: X \to Y$ is called a homotopy equivalence if $\exists\, g: X \to Y$ s.t.

$$fg = id, \quad gf = id.$$

Obs: If there is a homotopy equivalence between two spaces, their fund. groups are the same.

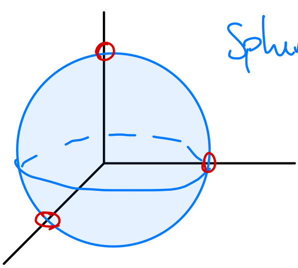




These are all homotopy equivalent!

$$\Rightarrow \pi_1(G) = \mathbb{Z} * \mathbb{Z}$$

Ex: $X = \mathbb{R}^3 \setminus \{\text{nonnegative axes}\}$



Sphere w/ 3 punctured holes

The same as twice punctured plane (by stereographic projection)

$$\Rightarrow \pi_1(X) = \mathbb{Z} * \mathbb{Z}.$$

$\square$

Thm: (Seifert - van Kampen) Let $X = U_1 \cup U_2$ where U_1, U_2 are both open and $U_1 \cap U_2 \neq \emptyset$ and connected.

$$\begin{tikzcd}
 & U_1 \arrow[dr, "\psi_1"] & \\
U_1 \cap U_2 \arrow[ur, "\varphi_1"] \arrow[dr, "\varphi_2"'] & & X \\
 & U_2 \arrow[ur, "\psi_2"'] &
\end{tikzcd}$$

← Inclusion maps

Then,

$$\begin{tikzcd}
 & \pi_1(U_1) \arrow[dr, "\psi_{1*}"] & \\
\pi_1(U_1 \cap U_2) \arrow[ur, "\varphi_{1*}"] \arrow[dr, "\varphi_{2*}"'] & & \pi_1(U_1 \cup U_2) \\
 & \pi_1(U_2) \arrow[ur, "\psi_{2*}"'] &
\end{tikzcd}$$

↙ Induced maps

This means this is the identity : $aba^{-1}b^{-1} = id$

$\Rightarrow ab = ba$

$\Rightarrow$ Commutative

$\langle ab : \overbrace{aba^{-1}b^{-1}} \rangle$

↖ "Relations"

generators ↓ relations ↓

$$\pi_1(U_1) = \langle G_1 : R_1 \rangle$$

$$\pi_1(U_2) = \langle G_2 : R_2 \rangle$$

"Presentations of these fundamental groups"

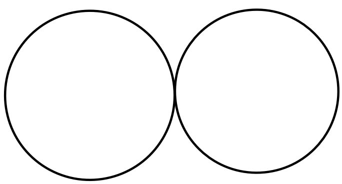
$$\pi_1(U_1 \cap U_2) = \langle G : R \rangle$$

$$\pi_1(U_1 \cup U_2) = \langle \underbrace{G_1 \cup G_2}_{\text{Combine generators}} : R_1 \cup R_2 \cup R_S \rangle$$

Words in the intersection of U_1 and U_2 must match in the dictionaries of U_1 & U_2.

$$R_S = \langle \varphi_{1*} S = \varphi_{2*} S ,\ S \text{ is a word in } U_1 \cap U_2 \rangle$$

Ex:

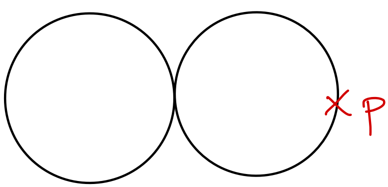


$= X$

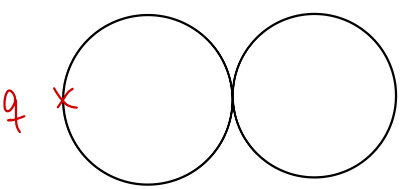
Can deform. retract to a circle

generated by an integer

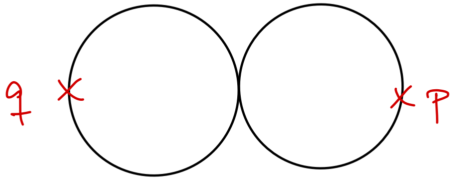
is generated by a pt. since $\pi_1(S^1) = \mathbb{Z}$



$= U_1$ $\quad \pi_1(U_1) = \langle \alpha : \emptyset \rangle$



$= U_2$ $\quad \pi_1(U_2) = \langle \beta : \emptyset \rangle$



$= U_1 \cap U_2$

Thus, $\pi_1(X) = \langle a, \beta : \emptyset \rangle =$ Free group of 2 generators

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Lecture 15

26/10/2023

Covering Spaces

Use van Kampen to find the fund. group of a whole from knowing the fund. group of the parts.

Ex: S^n, $n \geq 2$.

Simply connected $\Rightarrow \pi_1(S^n, p) = \{0\}$ (trivial)

$S^n = U_1 \cup U_2$, where $U_1 = S^n \setminus \{N\}$ ← North pole

$U_2 = S^n \setminus \{S\}$ ← South pole

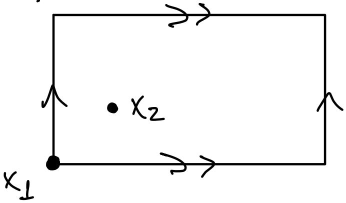
U_1, U_2 are homeo. to $\mathbb{R}^n$ so has a homotopy type of a pt. $\Rightarrow \pi_1(\mathbb{R}^n) = \{0\}$

$U_1 \cap U_2$ path connected $\overset{\text{van}}{\underset{\text{Kampen}}{\Longrightarrow}} \pi_1(S^n) = \{0\}$. $\square$

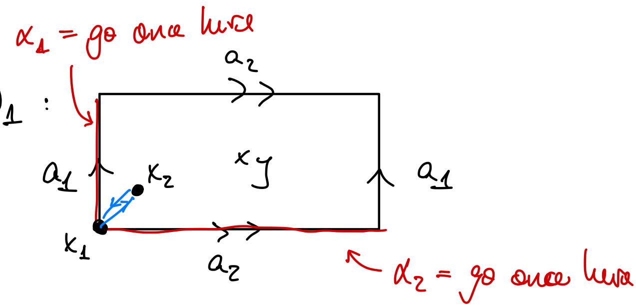
Ex: $T^2 = S^1 \times S^1$. Let $U_1 := T^2 \setminus \{y\}$

Step 1:

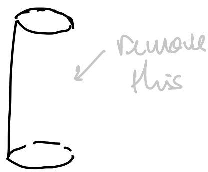
T^2:



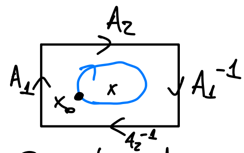
U_1:



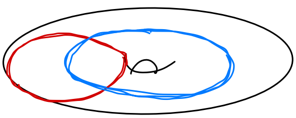
U_2:



$\Longrightarrow \quad U_1 \cap U_2 =$



Punctured open ball



Step 2:

$$\pi_1(U_1, x_1) = \langle [\alpha_1], [\alpha_2] : \emptyset \rangle$$

free group

free product onto gen.

$$\pi_1(U_1, x_2) = \langle A_1, A_2 : \emptyset \rangle$$

$$\pi_1(U_2) = \{0\} \quad \leftarrow \text{it's a ball}$$

Step 3:

$$\pi_1(U_1 \cap U_2) = \langle \gamma : \emptyset \rangle$$

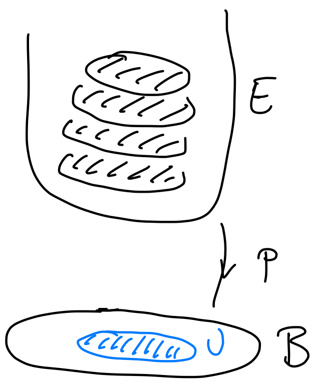
Step 4: van Kampen

$$\pi_1(T^2, x_0) = \langle A_1 A_2 : A_1 A_2 A_1^{-1} A_2^{-1} \rangle$$

$$= \mathbb{Z} \oplus \mathbb{Z}$$

$\square$

Def: Let $p: E \to B$ be a continuous surjective map.



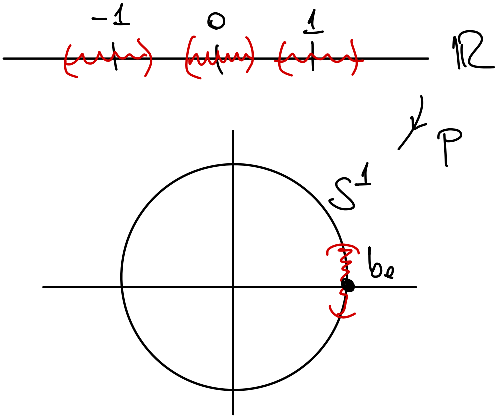
The subset $U \subset B$ is said to be EVENLY COVERED by p if there exists $\{V_\alpha\}$ s.t.

$$p^{-1}(U) = \bigsqcup_\alpha V_\alpha$$

with $(p|_{V_\alpha}): V_\alpha \longrightarrow U$ homeomorphic. This collection $\{V_\alpha\}$ is called a partition of $p^{-1}(U)$ into slices.

Def: Let $p: E \to B$ be a continuous & surjective map s.t. every pt. $b \in B$ has a neighborhood that is evenly covered by p. Then p is a covering map and E is a covering space.

Ex: $p: \mathbb{R} \to S^1$, $p(t) = (\cos 2\pi t, \sin 2\pi t)$ is a covering map $\Rightarrow \mathbb{R}$ is a covering space of S^1.

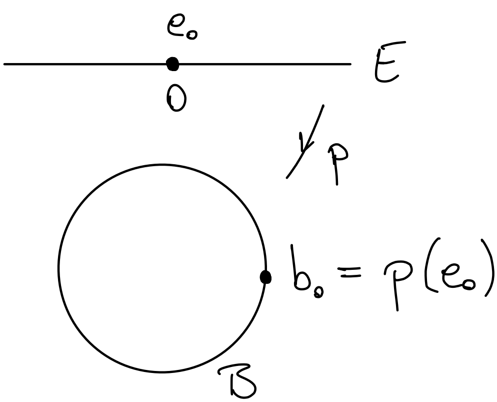


$\square$

* **LIFTINGS**: With a covering space, we can lift paths/homotopies from B to E.

Can lift when the diagram commutes

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, dashed, "\tilde{f}"] \arrow[r, "f"'] & B \arrow[ul, phantom, "\circlearrowright"] \end{tikzcd}$$

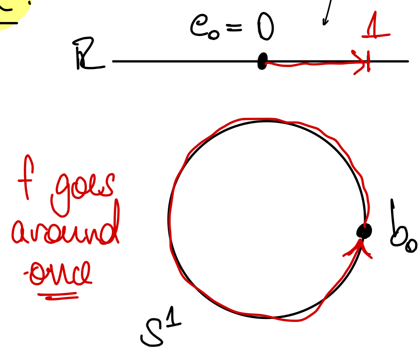


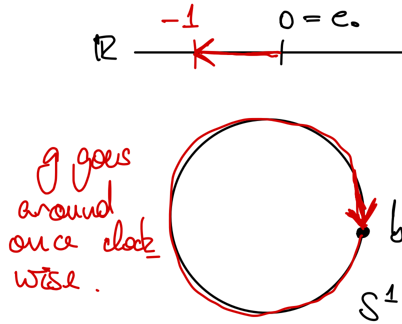
For any path $f : [0,1] \to B$ s.t. $f(0) = b_0$ $\exists!$ f that begins at e_0

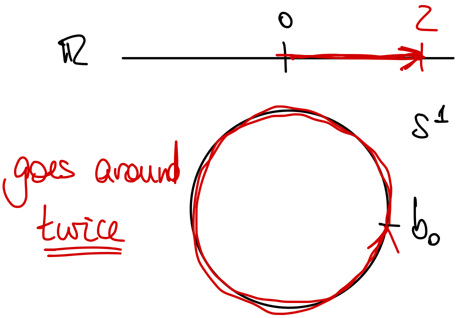
Compactness and Lebesgue number lemma

Ex:

Not closed anymore (really shows how $\pi_1(S^1)=\mathbb{Z}$)





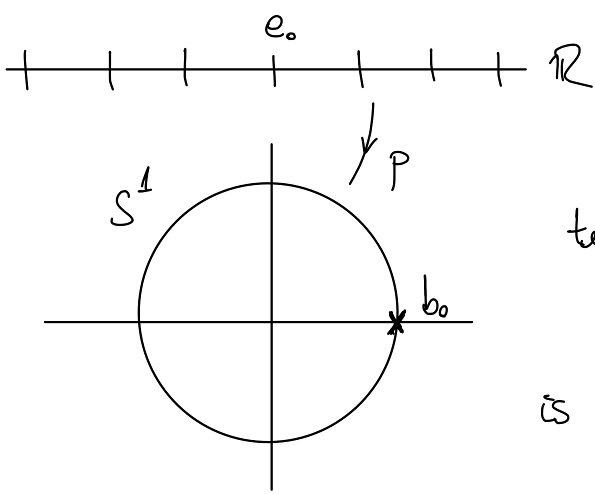


$$\boxed{\pi_1(S^1) = \mathbb{Z}}$$

LIFTING CORRESPONDENCE: Let $p: E \to B$ be a covering map. Let $b_0 \in B$ and $p(e_0) = b_0$. Given $[f] \in \pi_1(B, b_0)$, let $\tilde{f}$ be the lift of f starting at e_0. Then let $\Phi: \pi_1(B, b_0) \to p^{-1}(b_0)$

$$\Phi([f]) := \tilde{f}(1).$$

- E path connected $\Rightarrow$ Φ onto
- E simply connected $\Rightarrow$ Φ bijection



Inverse image of p is $\mathbb{Z} \subset \mathbb{R}$.

Since $\mathbb{R}$ is simply connected,

$$\Phi : \pi_1(S^1, b_0) \to \mathbb{Z}$$

is a <u>bijection</u>.

<u>COVERING TRANSFORMATIONS</u>: Let $p: E \to B$ be a covering space. Consider the set of equivalences of this covering space with itself.

These equivalences are homeos h s.t. the diagram commutes and $p \circ h = p$.

$$\begin{tikzcd} E \arrow[rr, "h"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowleft"] & E \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

Denote the covering space $\underbrace{\mathcal{C}(E, p, B)}_{\text{group under composition}}$ (also called "deck transformations")

Induced maps:

$$p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$$

$$p_*(\pi_1(E, e_0)) =: H_0$$

$$N(H_0) = \{ g \in G : g H_0 g^{-1} = H_0 \}$$ NORMALIZER OF H_0 IN G

Note

$$\mathcal{C}(E, p, B) \simeq N(H_0) / H_0$$

E simply-connected $\Rightarrow$ $\mathcal{C}(E, p, B) = \pi_1(B, b_0)$.

LECTURE 16

31/10/2023

COMPARISON GEOMETRY

Thm: (Cartan – Hadamard) Let (M^n, g) be a complete Riemannian manifold with $\sec \leq 0$. Then for any $p \in M$, $\exp_p : T_pM \to M$ is a covering map; so $\pi_k M = \{1\}$ $\forall k \geq 2$. In particular, if $\pi_1 M = \{1\}$, then $M^n \underset{\text{diffeo}}{\cong} \mathbb{R}^n$.

i.e., M is simply-connected

Pf: By Rauch I, given any geodesic $\gamma: \mathbb{R} \to M$ and a Jacobi field $J: \mathbb{R} \to M$ along γ with $J(0) = 0$, we have $\|J(t)\| \geq t\|J'(0)\| > 0$. So, there are no conjugate pts. along γ. Thus, $\exp_p : T_pM \to M$ has non-singular differential everywhere; i.e., ← conjugate pts. are critical pts. of $\exp_p$.

$$d(\exp_p)_v : T_vT_pM \to T_{\exp_p v} M$$

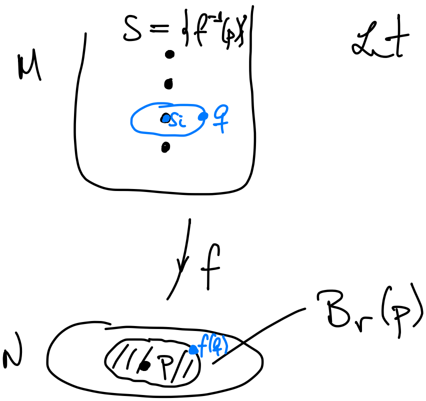
is invertible for all $v \in T_pM$ (b/c

$$0 \neq J(t) = d(\exp_p)_{\underbrace{tJ(0)}_{v}} tJ'(0) \quad \forall t \neq 0).$$

Since $\exp_p : T_pM \to M$ is a local diffeo, it is a covering map. If $\pi_1 M = \{1\}$, then $\exp_p$ is a homeomorphism (by topology), and since it is smooth and nonsingular, it is a diffeomorphism. $\square$

> **Thm** (Ambrose) Let $f: M \to N$ (← complete?) be surjective map that is a local isometry to a complete Riem. manifold. Then, f is a covering map.

Pf: <u>WTS</u>: every $p \in N$ has an evenly covered neighborhood.



Let $S := \{f^{-1}(p)\}$.

<u>Claim</u>: S is discrete.

This is the case b/c f is a local isometry, so, in particular, a local diffeo.

Let $\mathcal{F} := \{q \in M : d(s,q) < r\}$. <u>Claim</u>: $f^{-1}(B_r(p)) = \mathcal{F}$

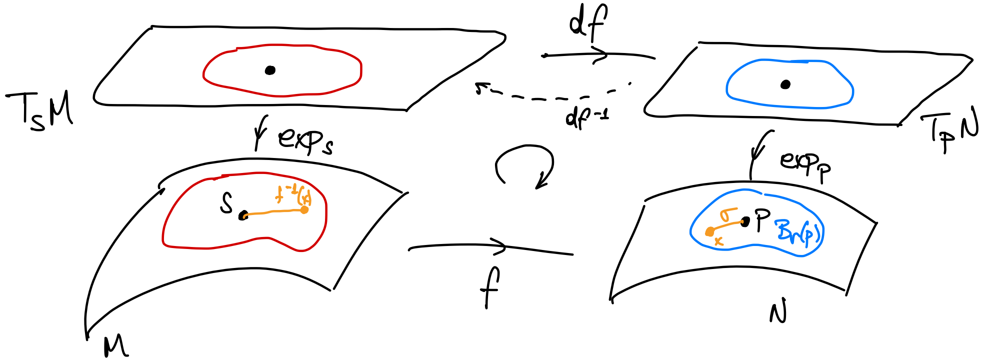
($\supset$) Take $q \in \mathcal{F} \Rightarrow \exists s_i$ s.t. $d(q, s_i) < r$. Then

$$d(f(q), \underbrace{f(s_i)}_{p}) \leq d(q, s_i) < r. \quad \text{So } f(q) \in B_r(p)$$

$$\Downarrow$$

$$q \in f^{-1}(B_r(p)).$$

($\subset$) Conversely, note that this diagram commutes



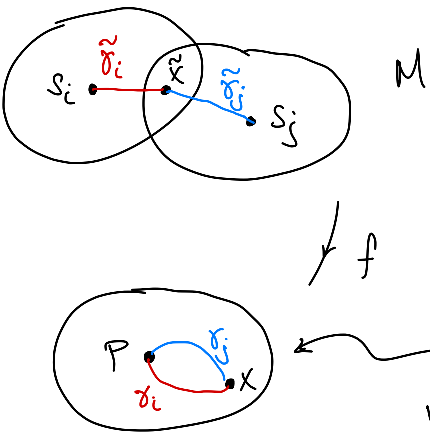
Let $q \in f^{-1}(B_r(p)) \Rightarrow \exists\, x \in B_r(p)$ s.t. $q = f^{-1}(x)$.

Let $\tilde{\sigma}$ be the lift of σ (geodesic) starting at $f^{-1}(x)$. Then set $s_i := \tilde{\sigma}(1) \Rightarrow d(s_i, f^{-1}(x)) < r \Rightarrow q \in \mathcal{F}$.

Thus, $\mathcal{F} = \bigcup_i B_r(s_i)$.

<u>Claim</u>: this is a <u>disjoint</u> union.

Suppose not: say $\exists\, \tilde{x} \in B_r(s_i) \cap B_r(s_j)$



Contradicts uniqueness of dist. minimizing geodesics on normal balls.

$\square$

* **HYPERBOLIC SPACES**: Half-space model

$$\mathbb{H}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_n > 0\}$$

$$g_{ij}(x_1, \ldots, x_n) := \frac{\delta_{ij}}{x_n^2} \quad \text{(metric)}$$

$(\mathbb{H}^n_+, g_{ij})$ is simply-connected, complete, hyperbolic space of dimension n.

Claim: $\sec_{(\mathbb{H}^n_+, g)} \equiv -1$.

Pf: More generally, consider $g_{ij} = \dfrac{\delta_{ij}}{F^2}$.

← Some nice enough fct.

The inverse of the metric is

$$g^{ij} = F^2 \delta_{ij}.$$

Let $\log F := f$ and denote $\dfrac{\partial f}{\partial x_i} = f_i$. Then

$$\frac{\partial g_{ik}}{\partial x_j} = \delta_{ik}\left(\frac{\partial}{\partial x_j}(F^{-2})\right) = -\frac{2}{F^3}\delta_{ik}\frac{\partial F}{\partial x_j}$$

$\log F = f$

$\dfrac{1}{F}\dfrac{\partial F}{\partial x_j} = f_j$

$$= -\frac{2}{F^3}\delta_{ik} \cdot F f_j = -\frac{2}{F^2}\delta_{ij} f_j.$$

Christoffel Symbols:

$$\Gamma_{ij}^{k} = \frac{1}{2} \sum_{m} \left(\frac{\partial}{\partial x_i} g_{jm} + \frac{\partial}{\partial x_j} g_{mi} - \frac{\partial}{\partial x_m} g_{ij} \right) g^{mk}$$

$$= \frac{1}{2} \left(\frac{\partial}{\partial x_i} g_{jk} + \frac{\partial}{\partial x_j} g_{ki} - \frac{\partial}{\partial x_k} g_{ij} \right) F^2$$

$$= - \delta_{jk} f_i - \delta_{ki} f_j + \delta_{ij} f_k$$

if k, i, j are all different $\Rightarrow \Gamma_{ij}^{k} = 0$.

otherwise, compute:

$$\Gamma_{ij}^{i} = -f_j, \quad \Gamma_{ii}^{j} = f_j, \quad \Gamma_{ij}^{j} = -f_i, \quad \Gamma_{ii}^{i} = -f_i$$

To compute $\sec\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)$, we need to compute

$$\Big\langle \underbrace{R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_i}}, \frac{\partial}{\partial x_j} \Big\rangle = R_{ijij}$$

$$R\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right)\frac{\partial}{\partial x_i} = \sum R_{ijk}^{s} \frac{\partial}{\partial x_j}$$

$$R^s_{ijk} = \sum_l \Gamma^l_{ii} \Gamma^s_{jl} - \sum_l \Gamma^l_{ji} \Gamma^j_{il}$$

$$+ \frac{\partial}{\partial x_j} \Gamma^l_{ii} - \frac{\partial}{\partial x_i} \Gamma^s_{ji}$$

$$\frac{\partial}{\partial x_j} \Gamma^j_{ii} = f_{jj} \quad , \quad \frac{\partial}{\partial x_i} \Gamma^j_{ji} = - f_{ii}$$

$$\Rightarrow F^2 R_{ijij} = - \sum_l f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj}$$

Thus :

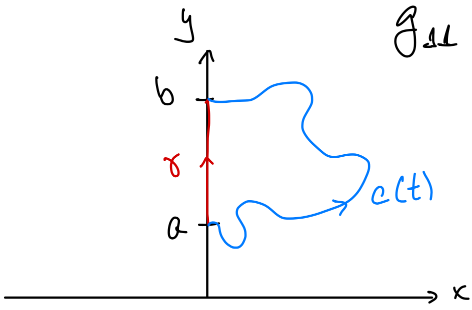
$$\sec\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{R_{ijij}}{\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle \left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle}$$

$$= \left(- \sum_l f_l^2 + f_i^2 + f_j^2 + f_{ii} + f_{jj} \right) F^2$$

Take $F^2 = x_n^2$ ... and get $\boxed{\sec_{(\mathbb{H}^n_{+}, g)} \equiv -1}$.

$\square$

* **DIMENSION 2:** $\mathbb{H}^2_+ = \{(x,y)\in\mathbb{R}^2 : y>0\}$



$$g_{11} = g_{22} = \frac{1}{y^2}, \quad g_{12} = g_{21} = 0$$

Claim: $\gamma(t) = (0,t)$ between $(0,a)$ and $(0,b)$ is a geodesic.

Pf: Take another $c(t) : [a,b] \to \mathbb{H}^2_+$ w/ $c(a) = (0,a)$
$c(b) = (0,b)$

$$L_g(c) = \int_a^b \sqrt{g(\dot{c}(t), \dot{c}(t))}\, dt$$

$$= \int_a^b \frac{1}{y}\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\geq \int_a^b \left(\frac{1}{y}\frac{dy}{dt}\right) dt$$

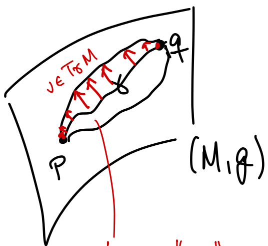
$$\geq L_g(\gamma).$$

ISOMETRIES OF THE PLANE: Möbius transformations $z \mapsto \dfrac{az+b}{cz+d}$

$ad-bc=1.$

* **CALCULUS OF VARIATIONS**: Variations of energy

Fix $p, q \in M$ and



$$X = \left\{ \gamma \in \underbrace{W^{1,2}([0,L], M)} : \begin{matrix} \gamma(0) = p \\ \gamma(L) = q \end{matrix} \right\}$$

This is a Hilbert manifold locally modeled on the Hilbert space $W^{1,2}([0,L], \mathbb{R}^n)$

These "v" are called VARIATIONS (just vec. fields along γ)

Given $\gamma \in X$, we can identify

$$T_\gamma X = \left\{ v \in W^{1,2}([0,L], TM) : \begin{matrix} \text{vector field along } \gamma \\ \text{w/ } v(0) = 0, \ v(L) = 0 \end{matrix} \right\}$$

Define the **ENERGY FUNCTIONAL** $E : X \to \mathbb{R}$

$$E(\gamma) = \frac{1}{2} \int_0^L g(\dot\gamma, \dot\gamma) \, dt$$

Alternatively, can consider the length functional
$$L(\gamma) = \int_0^L \|\dot\gamma\| \, dt$$
on curves $\gamma \in W^{1,1}$.

Then $\gamma \in X$ is a critical point of E, i.e. $\underbrace{\delta E(\gamma) = 0}$, iff γ is a geodesic.

$\delta E(\gamma) : T_\gamma X \to \mathbb{R}$

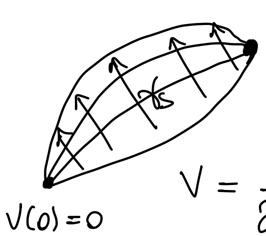
Indeed: First and Second Variations of energy (next time)

Lecture 17

02/11/2023

Variation of Energy

First Variation:



$$\delta E(\gamma)(V) = \frac{d}{ds} E(\gamma_s)\Big|_{s=0}$$

$$V = \frac{\partial}{\partial s}\gamma_s\Big|_{s=0}$$

Variational Field

$$= \frac{1}{2}\int_0^L \frac{d}{ds} g(\dot\gamma_s, \dot\gamma_s)\Big|_{s=0} dt$$

$\left(\text{e.g., } \gamma_s(t) = \exp_{\gamma_0(t)} sV(t)\right)$

$$= \int_0^L g\left(\frac{D}{ds}\dot\gamma_s\Big|_{s=0}, \dot\gamma_s\right) dt$$

$$= \int_0^L g\left(\frac{DV}{dt}, \dot\gamma\right) dt$$

by parts

$$\overset{\text{by parts}}{=} \underbrace{g(V, \dot\gamma)\Big|_0^L}_{\substack{=0 \text{ b/c} \\ \text{boundary conditions} \\ \text{are } V(0)=V(L)=0}} - \int_0^L g\left(V, \frac{D\dot\gamma}{dt}\right) dt$$

$$= -\int_0^L g\left(V, \frac{D\dot\gamma}{dt}\right) dt$$

Therefore: $\delta E(\gamma_s)(V) = \frac{d}{ds} E(\gamma_s)\Big|_{s=0} = 0$ for all variations γ_s if and only if $\frac{D\dot\gamma}{dt} = 0$ (i.e., γ is geodesic)

Fundamental Lemma of Calculus of Variations: $\int \phi \psi = 0 \ \forall \psi \Leftrightarrow \phi = 0$

i.e., $\langle \phi, \psi \rangle_{L^2} = 0 \ \forall \psi \Leftrightarrow \phi = 0$

(and hence $\|\dot\gamma\| = \text{const.}$)

SECOND VARIATION: Suppose γ is a geodesic, then the "Hessian" of E at γ is

$$\delta^2 E(\gamma)(V,V) = \frac{d^2}{ds^2} E(\gamma_s)\Big|_{s=0} = \frac{1}{2}\int_0^L \frac{d^2}{ds^2} g(\dot\gamma_s, \dot\gamma_s)\Big|_{s=0} dt$$

$\delta^2 E : T_\gamma X \times T_\gamma X \to \mathbb{R}$ symmetric bilinear form called the "Index Form"

or $\delta^2 E : T_\gamma X \to T_\gamma X$ symmetric endomorphism

$$= \int_0^L \frac{d}{ds} g\left(\frac{D}{ds}\dot\gamma_s, \dot\gamma_s\right)\Big|_{s=0} dt$$

$$= \int_0^L g\left(\frac{D^2}{ds^2}\dot\gamma_s\Big|_{s=0}, \dot\gamma\right) + g\left(\frac{D}{ds}\dot\gamma_s\Big|_{s=0}, \frac{D}{ds}\dot\gamma_s\Big|_{s=0}\right) dt$$

$V = \frac{\partial}{\partial s}\gamma_s$

$$= \int_0^L g\left(\frac{D}{ds}V', \dot{\gamma}\right) + g(V', V')\, dt$$

$$V' = \frac{DV}{dt} = \frac{D}{dt}\frac{\partial}{\partial s}\gamma_s = \frac{D}{ds}\frac{\partial}{\partial t}\gamma_s = \frac{D}{ds}\dot{\gamma}_s$$

$$\overset{\text{Jacobi}}{=} \int_0^L g\left(\frac{D}{dt}\frac{D}{ds}V + R(V,\dot{\gamma})V, \dot{\gamma}\right) + g(V', V')\, dt$$

$$= \int_0^L g\left(\frac{D}{dt}\frac{D}{ds}V, \dot{\gamma}\right) - g\left(R(V,\dot{\gamma})\dot{\gamma}, V\right) + g(V', V')\, dt$$

Int. by parts (x2)

$$= \underbrace{g\left(\frac{D}{ds}V, \dot{\gamma}\right)\Big|_0^L}_{=0 \text{ b/c } V(0)=V(L)=0} - \int_0^L g\left(\frac{D}{ds}V, \cancel{\frac{D}{dt}\dot{\gamma}}^{\,0}\right) dt$$

$$+ \underbrace{g(V', V)\Big|_0^L}_{=0 \text{ b/c } V(0)=V(L)=0} - \int_0^L g(V'', V) + g\left(R(V,\dot{\gamma})\dot{\gamma}, V\right) dt$$

$$= -\int_0^L g\left(V'' + R(V,\dot{\gamma})\dot{\gamma}, V\right) dt$$

This vanishes iff V is a <u>Jacobi field</u>

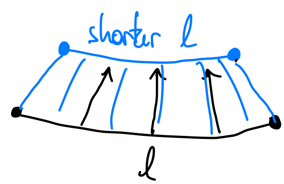
$$V'' + R(V, \dot\gamma)\dot\gamma = 0$$

NOTE: If $\sec_M > 0$, then $g(V'' + R(V,\dot\gamma)\dot\gamma, V) > 0$ so using a parallel vector field V along a geodesic γ, we get

$$\delta^2 E(\gamma)(V,V) = -\int_0^L \underset{\overset{\parallel}{0}}{g(V'',V)} + \underbrace{g(R(V,\dot\gamma)\dot\gamma, V)}_{>0}\, dt$$

i.e., γ is <u>unstable</u>; small variations of γ <u>decrease</u> its energy (and its length)

$\sec > 0$



REMARKS ABOUT ENERGY v. LENGTH OF CURVES:

- Critical points of E come <u>parametrized w/ constant speed</u>, i.e., $\delta E(\gamma) = 0$ implies $\|\dot\gamma\| = $ constant., while the length functional is <u>invariant</u> under reparametrizations of γ;

in particular, critical points need not have constant speed.

- Apply Cauchy-Schwartz inequality $\left(\int_0^L \phi\psi\right)^2 \leq \int_0^L \phi^2 \int_0^L \psi^2$ with $\phi = 1$ to get

$$L_g(\gamma)^2 = \left(\int_0^L \|\dot\gamma\|\, dt\right)^2 \leq L \int_0^L \|\dot\gamma\|^2\, dt = 2L \cdot E(\gamma).$$

and "=" iff $\|\dot\gamma\| = 1$

So, if γ is a unit speed minimal geodesic from p to q and β is a curve from p to q, then

$$E(\gamma) = \frac{1}{2\ell} L(\gamma)^2 \leq \frac{1}{2\ell} L(\beta)^2 \leq E(\beta)$$

with $E(\gamma) = E(\beta) \iff \beta$ is a unit speed and hence $L(\beta) = L(\gamma)$.

Upshot:

Maybe not minimal

γ is a critical pt. of $E \iff \gamma$ is a unit speed geodesic

$\underbrace{\gamma \text{ is a minimizer of } E}_{\text{w/ boundary conditions } E : X \to \mathbb{R},} \iff \gamma$ is a unit speed min. geodesic

realizes distance

$X = \{\gamma \in W^{1,2}([0,\ell], M) : \gamma(0) = p \text{ and } \gamma(\ell) = q\}$

Lecture 18 — Comparison Geometry (ctd.)

14/11/2023

If $\gamma : [0,\ell] \to M$ is a unit speed, then given any variation

$$\boxed{\delta L(\gamma)(V) = \frac{1}{\ell}\,\delta E(\gamma)(V) = \frac{1}{\ell}\left(g(V,\dot\gamma)\Big|_0^\ell - \int_0^\ell g\left(V, \frac{D\dot\gamma}{dt}\right)dt \right)}$$

Similar for $\delta^2 L(\gamma)$ if $\delta L(\gamma) = 0$.

!

Thm: (Myers, 1941) If (M^n, g) is a complete Riemannian manifold w/ $Ric_M \geq \kappa(n-1)$, with $\kappa > 0$, then

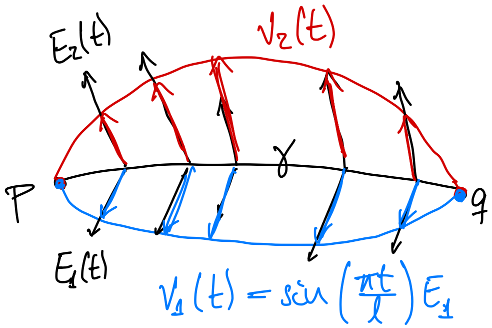
$$\operatorname{diam}(M^n, g) \leq \frac{\pi}{\sqrt{\kappa}}.$$

In particular, (M^n, g) is compact and $\pi_1 M$ is finite.

Pf: Suppose M^n has $Ric \geq \kappa(n-1) > 0$ and let $\gamma : [0,\ell] \to M$ be a unit speed geodesic; i.e., $\delta E(\gamma) = 0$. If γ is min. (i.e., $\operatorname{dist}_g(\gamma(0), \gamma(\ell)) = \ell$) then $\delta^2 E(\gamma)(V,V) \geq 0$ for all V along γ w/ $V(0) = 0$ and $V(\ell) = 0$. Let $\{E_i\}$ be a parallel o.n.b. of vector fields along γ; i.e.,

$g(E_i, \dot\gamma) = 0$, $g(E_i, E_j) = \delta_{ij}$ and set

$V_i(t) = \sin\left(\frac{\pi t}{\ell}\right) E_i(t)$, so $V_i(0) = 0$ and $V_i(\ell) = 0$.



Thus,

$$\delta^2 E(\gamma)(V_i, V_i) = -\int_0^\ell g(V_i'', V_i) + g(R(V_i, \dot\gamma)\dot\gamma, V_i)\, dt$$

$$= \int_0^\ell \sin\left(\frac{\pi t}{\ell}\right)^2 \left(\frac{\pi^2}{\ell^2} - g(R(E_i, \dot\gamma)\dot\gamma, E_i)\right) dt$$

$$V_i'(t) = \frac{\pi}{\ell}\cos\left(\frac{\pi t}{\ell}\right) E_i(t) + \sin\left(\frac{\pi t}{\ell}\right) \underbrace{E_i'(t)}_{=0}$$

$$V_i''(t) = -\frac{\pi^2}{\ell^2}\sin\left(\frac{\pi t}{\ell}\right) E_i(t) + \frac{\pi}{\ell}\cos\left(\frac{\pi t}{\ell}\right) \underbrace{E_i'(t)}_{=0}$$

Thus, adding from $i=1$ to $i=n-1$:

$$0 \le \sum_{i=1}^{n-1} \delta^2 E(\gamma)(V_i, V_i) = \sum_{i=1}^{n-1} \int_0^l \sin\left(\frac{\pi t}{\ell}\right)^2 \left(\frac{\pi^2}{l^2} - g(R(E_i, \dot\gamma)\dot\gamma, E_i)\right) dt$$

$$= \int_0^l \sin\left(\frac{\pi t}{\ell}\right)^2 \left((n-1)\frac{\pi^2}{\ell^2} - \underbrace{\sum_{i=1}^{n-1} g(R(E_i, \dot\gamma)\dot\gamma, E_i)}_{Ric(\dot\gamma, \dot\gamma) \ge K(n-1)}\right) dt$$

$$\leq \int_0^{\ell} \sin\left(\frac{\pi t}{\ell}\right)^2 (n-1) \underbrace{\left(\frac{\pi^2}{\ell^2} - k\right)}_{<0 \text{ if } \ell > \frac{\pi}{\sqrt{k}}} \, dt \quad \ldots$$

So, such minimizing unit speed geodesic $\gamma : [0, \ell] \to M$ must have length $\ell \leq \frac{\pi}{\sqrt{k}}$, for otherwise we get a contradiction above. ∎

RIGIDITY IN MYERS THEOREM

(originally due to Shiu-Yuen Cheng w/ diff. proof. student of S.S. Chern)

Thm: Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Ric} \geq k(n-1) > 0$ and

$$\mathrm{diam}(M^n, g) = \mathrm{diam}\left(S^n\left(1/\sqrt{k}\right)\right) = \frac{\pi}{\sqrt{k}}.$$

Then $(M^n, g) \underset{\text{isom.}}{\cong} S^n\left(1/\sqrt{k}\right)$.

M compact and $\partial M = \emptyset$.

Thm: (SYNGE, 1936) Let (M^n, g) be a closed Riem. mnfd w/ $\sec > 0$.

If n is even, then M orientable $\Rightarrow \pi_1 M \cong \{1\}$

M non-orientable $\Rightarrow \pi_1 M \cong \mathbb{Z}_2$

If n is odd, then M is orientable.

LECTURE 19

CARTAN'S THEOREM

(Curvature is the only local invariant in a Riem. mnfd)

16/11/2023

See pages 21 and 22 of RGB's notes.

LECTURE 20

SPACE FORMS

21/11/2023

Thm: (Killing-Hopf) If M^n is simply-connected and has $\sec \equiv k = \text{const.}$, then

$$M^n \underset{\text{isom.}}{\cong} \begin{cases} \mathbb{S}^n(1/\sqrt{k}) & \text{if } k > 0 \\ \mathbb{R}^n & \text{if } k = 0 \\ \mathbb{H}^n(1/\sqrt{-k}) & \text{if } k < 0 \end{cases}$$

"Constant curvature model spaces"

Pf: Case 1: $k = 0$ or $k = -1$. Denote $\mathbb{H}^n$, $\mathbb{R}^n$ by E.

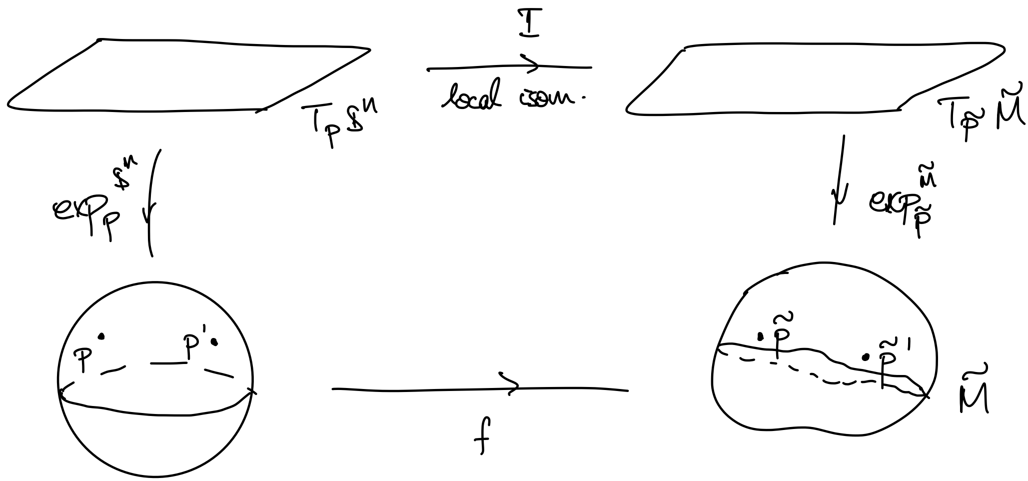
Consider

$$\begin{array}{ccc} T_pE & \xrightarrow{I} & T_{\tilde{p}}\tilde{M} \\ \downarrow \exp_p & & \downarrow \exp_{\tilde{p}} \\ p \in E & \xrightarrow{f} & \tilde{M} \ni \tilde{p} \end{array}$$

But the exp. maps are diffeos. So, define $f := \exp_{\tilde{p}} \circ I \circ \exp_p^{-1}$.

By Cartan, f is a local isometry (and diffeom.), hence f is an isometry.

<u>Case 2</u>: $K=1$, $\tilde{M}$ is complete, simply connected. Want to construct $S^n \to \tilde{M}$ diffeo & local isometry.



Define $f: S^n \setminus \{p\} \to \tilde{M}$, $\tilde{p}' = f$, $I' = df_{p'}$, as

$f := \exp_{\tilde{p}} \circ I \circ \exp_p^{-1}$. By Cartan, f is a local isometry. Now, $p' \in S^n \setminus \{p, -p\}$ and set

$$f' := \exp_{\tilde{p}} \circ I' \circ \exp_p^{-1}.$$

So, $S^n \setminus \{-p'\} \to \tilde{M}$, $f(p') = \tilde{p}' = f'(p')$

$$df_{p'} = I' = df'_{p'}$$

Set

$$g(r) = \begin{cases} f(r) & r \in S^n \setminus \{-p\} \\ f'(r) & r \in S^n \setminus \{-p'\} \end{cases}$$

$\Rightarrow$ g is a diffeo, local isometry $\Rightarrow$ isometry. ∎

— // —

Def: INDEX FORM $\equiv \delta^2 E(\gamma)$ ("Hessian" of energy functional)

$$I_{t_0}(V,V) := \int_0^{t_0} \langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle \, dt$$

Morse

INDEX LEMMA: Let $\gamma : [0,a] \to M$ be a geodesic with no conjugate points on $(0,a]$ to $\gamma(0)$. Let J be a Jacobi field along γ with $\langle J, \gamma' \rangle = 0$.

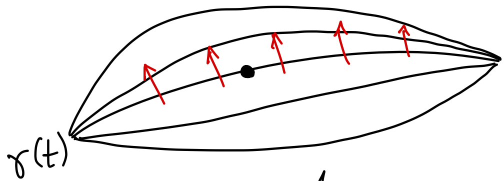
Let V be a variational field (piecewise diff.) along γ w/ $\langle V, \gamma' \rangle = 0$, $V(0) = J(0) = 0$ $(V(t_0) = J(t_0),\ t_0 \in (0,a])$.

Then, $I_{t_0}(J,J) \leq I_{t_0}(V,V)$. Moreover,

$$I_{t_0}(J,J) = I_{t_0}(V,V) \iff V = J \text{ on } [0, t_0].$$

Pf: $J(0) = 0$. Note that Jacobi fields form a $(n-1)$-dim. vector space. Can write a basis $\{J_i(t)\}_{i=1}^{n-1}$ for it and then

since no conjugate pts.



$$J = \sum_{i=1}^{n-1} a_i J_i(t)$$

Then, $V(t) = \sum_{i=1}^{n-1} f_i(t) J_i(t)$. So, compute:

$$I_{t_0}(V,V) = \langle V', V'\rangle - \langle R(\gamma', V)\gamma', V\rangle .$$

$$V' = \sum_i f_i' J_i + \sum_i f_i J_i'$$

$$\langle V', V'\rangle = \boxed{\begin{aligned} &\Big\langle \sum_i f_i' J_i, \sum_j f_j' J_j \Big\rangle \\ &+ \Big\langle \sum_i f_i J_i', \sum_j f_j' J_j \Big\rangle \\ &+ \Big\langle \sum_i f_i' J_i, \sum_j f_j J_j' \Big\rangle \\ &+ \Big\langle \sum_i f_i J_i', \sum_j f_j J_j' \Big\rangle . \end{aligned}}$$

$$\langle R(\gamma', V)\gamma', V\rangle = \Big\langle R\big(\gamma', \sum_i f_i J_i\big)\gamma', \sum_j f_j J_j \Big\rangle$$

$$= \Big\langle \sum_i f_i R(\gamma', J_i)\gamma', \sum_j f_j J_j \Big\rangle$$

$$= \boxed{-\left\langle \sum_i f_i J_i'', \sum_j f_j J_j \right\rangle}$$

So, putting everything together, we obtain:

$$+ \quad = \left\langle \sum_i f_i' J_i, \sum_j f_j' J_j \right\rangle + \frac{d}{dt}\left\langle \sum_i f_i J_i, \sum_j f_j J_j' \right\rangle.$$

Indeed,

$$h(t) = \langle J_i', J_j \rangle - \langle J_i, J_j' \rangle, \quad h(0) = 0$$

$$h'(t) = -\langle R(\gamma', J_i)\gamma', J_j \rangle + \langle R(\gamma', J_j)\gamma', J_i \rangle$$

$$= 0.$$

$$\Longrightarrow h(t) \equiv 0.$$

Thus,

$$I_{t_0}(V,V) = \left\langle \sum_i f_i J_i, \sum_j f_j J_j' \right\rangle (t_0) + \int_0^{t_0} \left| \sum_i f_i' J_i \right|^2 dt$$

$$\geq I_{t_0}(J,J) = \left\langle \sum_i a_i J_i(t), \sum_j a_j J_j(t) \right\rangle.$$

Now, if $I_{t_0}(J,J) = I_{t_0}(V,V)$, then

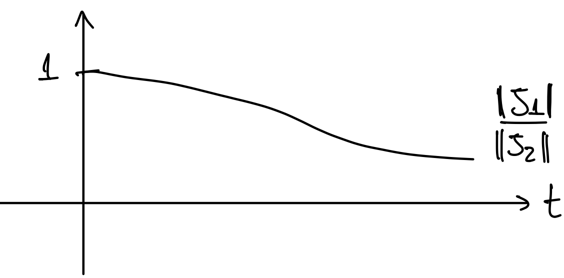
$$\sum_i f_i' J_i = 0, \quad f_i' = 0 \Rightarrow f_i(t_0) = a_i \rightsquigarrow f_i = a_i$$

$\to V = J$ on $[0,t_0]$.

∎

RAUCH COMPARISON:

> **Thm:** (Rauch I) Suppose J_i are solutions to $J_i'' + R_i J_i = 0$ with $R_1 \geq R_2$ and $J_i(0) = 0$, $\|J_1'(0)\| = \|J_2'(0)\|$. Then $\|J_1\| \leq \|J_2\|$ up to first zero of J_1.



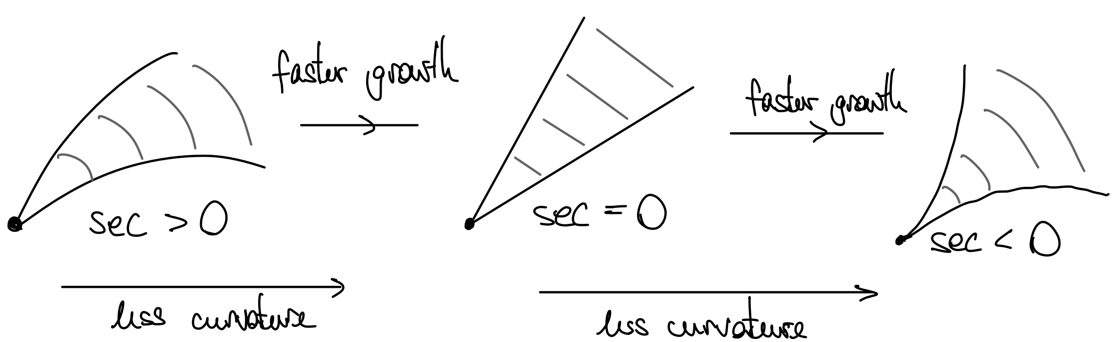
We had an infinitesimal version of this before

$\|J'(0)\| = 1$

$$\|J\| = t - \frac{1}{6}\langle R(J), J\rangle t^2 + O(t^3)$$

So $R_1 \geq R_2 \Rightarrow \|J_1\| < \|J_2\|$ for $t \approx 0$

Recall $R_1 \geq R_2$ iff $\langle (R_1 - R_2)v, v\rangle \geq 0 \ \forall v$



Pf: (Rauch I) Assume $\langle J_1, \gamma' \rangle = 0$, $\langle J_2, \gamma' \rangle = 0$. Then

$$\lim_{t \to 0} \frac{\|J_1(t)\|^2}{\|J_2(t)\|^2} \overset{\text{l'Hôpital } \times 2}{=} \lim_{t \to 0} \frac{\cancel{\langle J_1'', J_1 \rangle}^{=0} + \|J_1'\|^2}{\underbrace{\cancel{\langle J_2'', J_2 \rangle}}_{=0} + \|J_2'\|^2} = 1.$$

WTS: $\|J_1\|^2 \le \|J_2\|^2 \iff \dfrac{\|J_2\|^2}{\|J_1\|^2} \ge 1$

Show: $\dfrac{d}{dt}\left(\dfrac{\|J_2\|^2}{\|J_1\|^2} \right) \ge 0$. Fix t_0, then

$$\|J_2'(t_0)\|^2 = 2 \langle J_2'(t_0), J_2(t_0) \rangle.$$

Suppose $\|J_2(t_0)\|^2 \neq 0$. Let

$$u_1(t) := \frac{J_1(t)}{\|J_1(t_0)\|}, \qquad u_2(t) := \frac{J_2(t)}{\|J_2(t_0)\|}$$

Then

$$2 \langle u_2'(t_0), u_2(t_0) \rangle = \langle u_2, u_2 \rangle'(t_0) = \int_0^{t_0} \langle u_2, u_2 \rangle'' \, dt$$

$$= 2 \int_0^{t_0} \langle u_2', u_2' \rangle - \langle u_2, R(\gamma', u_2)\gamma' \rangle \, dt$$

$$= 2\, I_{t_0}(u_2, u_2).$$

WTS: $I_{t_0}(u_1, u_1) \leq I_{t_0}(u_2, u_2)$. But we can take bases for the variations of each field (and parallel transport them)

LECTURE 21

23/11/2023

MORSE & RAUCH

APPLICATIONS OF RAUCH:

> **Cor 1**: Let (M^n, g) be a complete Riem. mnfd such that $0 < k \leq \sec \leq K$. Then the distance d between consecutive conjugate points along geodesics in (M^n, g) is
> $$\frac{\pi}{\sqrt{K}} \leq d \leq \frac{\pi}{\sqrt{k}}.$$

Pf: Let $\gamma : [0, L] \to M$ be a geodesic, $J : [0, L] \to M$ a Jacobi field with $J(0) = 0$. Let $\tilde{J}$ be a Jacobi field on the round sphere $S^n\left(\frac{1}{\sqrt{K}}\right)$ with $\tilde{J}(0) = 0$ and $\|\tilde{J}'(0)\| = \|J'(0)\|$. Then, by Rauch II, $\|J(t)\| \geq \|\tilde{J}(t)\| > 0$ $\forall t \in \left(0, \frac{\pi}{\sqrt{K}}\right)$ b/c

$$\tilde{J}(t) = \tilde{J}'(0) \cdot \frac{\sin(t\sqrt{K})}{\sqrt{K}}, \quad \text{so } d \geq \frac{\pi}{\sqrt{K}}.$$

Similarly, if $d > \frac{\pi}{\sqrt{K}}$, then by Rauch II, the round sphere $S^n\left(\frac{1}{\sqrt{K}}\right)$ would only have conjugate pts. after a distance $\frac{\pi}{\sqrt{K}}$, a contradiction.

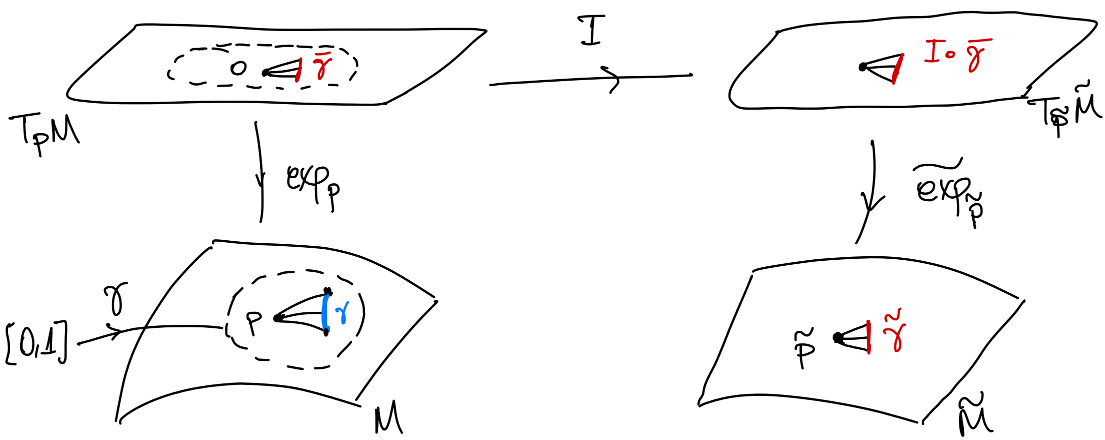
$\square$

———— // ————

Comparing Lengths:

> **Thm:** Let (M^n, g) and $(\tilde{M}^n, \tilde{g})$ be Riem. mnfds. Suppose that for all $p \in M$ and $\tilde{p} \in \tilde{M}$, $\sigma \subset T_pM$ and $\tilde{\sigma} \subset T_{\tilde{p}}\tilde{M}$ such that $\widetilde{\sec}_{\tilde{p}}(\tilde{\sigma}) \geq \sec_p(\sigma)$. Then $L_g(\gamma) \geq L_{\tilde{g}}(\tilde{\gamma})$.

Pf:

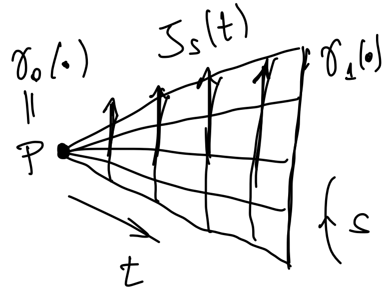


$$\sec_M \leq \sec_{\tilde{M}}$$

$\underline{\text{WTS}}$: $L_g(\gamma) \geq L_{\tilde{g}}\left(\widetilde{\exp}_{\tilde{p}} \circ I \circ \exp_p^{-1} \circ \gamma\right)$.

Take a variation of geodesics $\gamma_s(t) = \exp_p t\bar{\gamma}(s)$.

For fixed s, $t \mapsto \gamma_s(t)$ is a geodesic and $J_s(t) = \frac{\partial}{\partial t}\gamma_s(t)$ is a Jacobi field along $t \mapsto \gamma_s(t)$ with $J_s(0) = 0$ and $J_s(1) = \dot{\gamma}(s)$.



Since $\sec_M \leq \sec_{\tilde{M}}$, by Rauch I, $\|J_s(t)\| \geq \|\tilde{J}_s(t)\|$.

$\tilde{\gamma}_s(t) = \widetilde{\exp}_{\tilde{p}}\, t I(\bar{\gamma}(s))$

$\rightsquigarrow \tilde{J}_s(t) = \frac{\partial \tilde{\gamma}_s}{\partial s} = d(\widetilde{\exp}_{\tilde{p}})_{tI(\bar{\gamma}(s))}\, t\, I(\bar{\gamma}'(s))$

$\tilde{J}_s(0) = 0$ and $\tilde{J}_s(1) = \tilde{\gamma}'(s)$

So,

$\|J_s(0)\| = \|\tilde{J}_s(0)\| = 0$

$\|J_s'(0)\| = \|\tilde{J}_s'(0)\|$

$\Rightarrow \text{length}(\gamma) \geq \text{length}(\tilde{\gamma})$. $\square$

LECTURE 22

TOPONOGOV COMPARISON

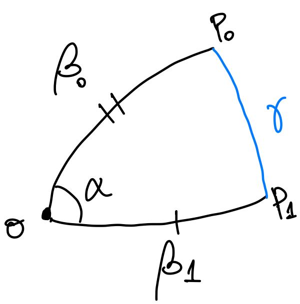
28/11/2023

if $K>0$, assume all lengths are $< \frac{\pi}{\sqrt{K}}$ can be negative

TOPONOGOV COMPARISON (HINGE VERSION): If (M^n, g) has $\sec \geq K$, $o, p_0, p_1 \in M$, and β_i is a minimal geodesic from o to p_i, then $\ell(\gamma) \leq \ell(\tilde{\gamma})$; where $\gamma, \tilde{\gamma}$ are the minimal geodesics that close the hinge:

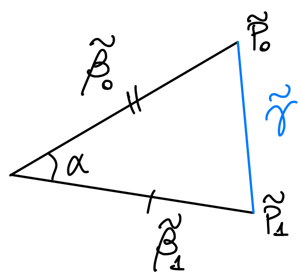
$$\ell(\gamma) = \mathrm{dist}_g(p_0, p_1)$$
$$\ell(\tilde{\gamma}) = \mathrm{dist}_g(\tilde{p}_0, \tilde{p}_1)$$

ORIGINAL TRIANGLE



$\sec \geq K$

COMPARED TRIANGLE W/ HINGE



$\sec \equiv K$

$\ell(\beta_i) = \ell(\tilde{\beta}_i)$
$\alpha = \tilde{\alpha}$

REFERENCES:

- "On Toponogov's Comparison Theorem for Alexandrov spaces", Urs Lang, Victor Schroeder.
- "Toponogov's Theorem and Applications" by Wolfgang Meyer.

Useful for HW04

* "Critical points of distance functions and applications to geometry" by J. Cheeger

HW: A pt. $q \in M^n$ is critical w.r.t. p if $\forall v \in T_q M$, there exists a minimizing geod. γ from q to p s.t.

$$|\langle \gamma'(0), v\rangle| \le \frac{\pi}{2}$$

<u>Q2</u>:

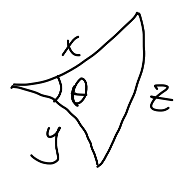
Let q_1 be critical w.r.t. p. Let q_2 be s.t. $\mathrm{dist}_g(p, q_2) \ge \alpha\, \mathrm{dist}_g(p, q_1)$ for some $\alpha > 1$. Let γ_1, γ_2 be min. geod. from p to q_1, q_2, respectively. Let θ be an angle between $\gamma_1'(0), \gamma_2'(0)$.
If $\underline{\sec_M \ge -1}$ (& M is compact), then

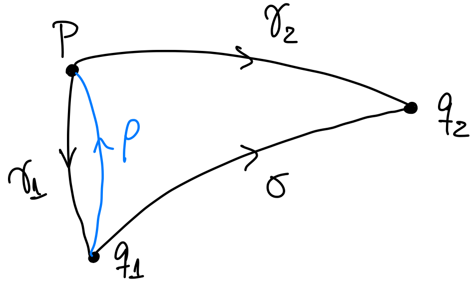
Toponogov comparing w/ hyperb. space twice

$$\cos\theta \le \frac{\tanh\left(\frac{\mathrm{diam}_M}{\alpha}\right)}{\tanh(\mathrm{diam}_M)}.$$

<u>Law of Cosines for Hyperbolic Triangles</u>:

$$\cosh z = \cosh x \cosh y - \sinh x \sinh y \cos\theta$$





Apply Toponogov to the hinge $\{\sigma, \rho\}$ and the hinge $\{\gamma_1, \gamma_2\}$ using the hyp. law of cosines.

Thm: (Bishop–Cheeger–Gromov Volume Comparison) Let (M^n, g) be a Riem. mnfd. with $\mathrm{Ric} \geq (n-1)\kappa$ and $\overline{M}$ be the simply-connected Riem. mnfd. with $\sec_{\overline{M}} \equiv \kappa$. Then, $\forall p \in M$, we have $\mathrm{vol}(B_r(p)) \leq \mathrm{vol}(\overline{B_r})$, where $B_r(p) \subset M$ and $\overline{B_r} \subset \overline{M}$ are balls of radius r. Moreover, equality holds if and only if $B_r(p) \underset{\text{isom.}}{\cong} \overline{B_r}$.

HW04 Q5: Apply the above theorem and use:

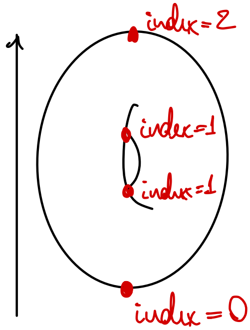
"The Comparison Geometry of Ricci Curvature" by Shun-Hui Zhu

HW04 Q4: Bochner formula, use:

"Comparison Geometry for Ricci curvature" Xiaozhe Dai and Guofang Wei.

HW04 Q1?: Analogous proof to Synge–Weinstein

MORSE THEORY: (see Milnor's book for details)



index = dim. of space for which Hess is negative def. (when we "can go down")

Critical pts. $\underset{\text{dim.}}{\overset{\text{fin.}}{\rightsquigarrow}}$ gluing cells.

hight function $\rightsquigarrow$ Morse fct.

MORSE INDEX THEOREM: The index λ of the Hessian

$$\delta^2 E(\gamma): \underline{T_\gamma \Omega} \times \underline{T_\gamma \Omega} \longrightarrow \mathbb{R}$$

vector space of tangent vec. fields along γ (space of variational vec. fields)

$$\delta^2 E(\gamma)(V,V) := \int_0^L \langle V', V' \rangle - \langle R(\gamma', V)\gamma', V \rangle \, dt$$

is defined to be the maximum dimension of a subspace of $T_\gamma \Omega$ on which $\delta^2 E(\gamma)$ is negative definite.

> **Thm**: (MORSE INDEX) The index λ of $\delta^2 E(\gamma)$ is equal to the number of points $\gamma(t)$, $t \in (0,L)$, such that $\gamma(t)$ is conjugate to $\gamma(0)$ along γ. Each such conjugate pt. is counted with its multiplicity. The index λ is always finite.

Prop: $V \in T_\gamma \Omega$ belongs to the kernel of $\delta^2 E(\gamma)$ if and only if V is a Jacobi field along γ.