

# Notes on Random Matrix Theory Applied to Log-Gas Systems

## Abstract

We present a physically motivated review of seminal results in random matrix theory from the perspective of the log-gas model in statistical physics. We explore Gaussian, circular, and Cauchy random matrix ensembles whose eigenvalues' p.d.f.'s describe physical systems in which charges are confined to an interval under an external harmonic potential, and repel each other via a logarithmic Coulomb kernel. Lastly, we examine three concrete examples of log-gas systems with periodic boundary conditions and how they relate to circular random matrix ensembles.

**Outline.** In Section 1, we provide a measure-theoretic introduction to random matrix theory. We state and prove Wigner's semicircle law theorem, introduce the Stieltjes transform, and discuss Fredholm determinants related to gap probabilities. We begin Section 2 by introducing the concept of a log-gas using postulates from statistical physics as the starting point. We then compute several densities and Boltzmann factors for specific systems, such as log-gases on a line and on a circle. After that, in Sections 2.1, 2.2, and 2.3, we discuss the Gaussian, circular, and Cauchy random matrix ensembles, respectively, while deriving log-gas systems corresponding to each one of them. Finally, in Section 3, we impose periodic boundary conditions on circular log-gas systems, and study three cases of these systems: transverse periodic boundary conditions, the metal wall, and doubly periodic boundary conditions.

## 1 Preliminaries from random matrix theory

In this section, we provide some useful definitions and key results in random matrix theory largely following Chapters 1, 2, and 3 of [2]. In particular, we present a combinatorial proof of Wigner's theorem, and introduce the Stieltjes transform and Fredholm determinants in the context of Gaussian matrix ensembles. A more extensive introduction to this topic from the points of view of measure-theoretic probability and dynamical analysis may be found in [2] and [5], respectively.

Consider two independent families of centered i.i.d. real-valued random variables  $\{X_{i,j}\}_{1 \leq i < j}$  and  $\{Y_i\}_{i \geq 1}$  such that  $\mathbb{E}[X_{1,2}^2] = 1$  and, for all  $k \in \mathbb{N}$ ,

$$r_k := \max(\mathbb{E}|X_{1,2}|^k, \mathbb{E}|Y_1|^k) < \infty. \quad (1)$$

Denote by  $W_N$  the symmetric  $N \times N$  matrix with entries

$$W_N(j, i) = W_N(i, j) = \begin{cases} X_{i,j}/\sqrt{N}, & \text{if } i < j \\ Y_i/\sqrt{N}, & \text{if } i = j \end{cases}. \quad (2)$$

We call such matrices *Wigner matrices*, and if  $\{X_{i,j}\}$  and  $\{Y_i\}$  are Gaussian random variables, we say they are *Gaussian Wigner matrices* [2]. In particular, Gaussian Wigner matrices rescaled by  $\sqrt{N}$  with  $\mathbb{E}Y_1^2 = 1$  are of special significance and are referred to as *Gaussian orthogonal ensemble* (GOE) matrices.

Now, let  $\{\lambda_i^N\}$  be the (real) eigenvalues of a Wigner matrix  $W_N$  ordered such that  $\lambda_1^N \leq \lambda_2^N \leq \dots \leq \lambda_N^N$ . We define the *empirical distribution* of the eigenvalues of  $W_N$  as the probability measure

$$L_N := \frac{1}{N} \sum_{i=1}^N \delta_{\lambda_i^N} \quad (3)$$

on  $\mathbb{R}$ ; where, as usual,  $\delta_x$  is the unique measure such that  $\int_{\mathbb{R}} f d\delta_x = f(x)$  for all  $f \in C_c(\mathbb{R})$ .

Next, we define the *semicircle distribution* (or *law*) as the probability distribution  $\sigma(x) dx$  on  $\mathbb{R}$  with density

$$\sigma(x) := \frac{1}{2\pi} \sqrt{4 - x^2} \mathbf{1}_{\{|x| \leq 2\}}. \quad (4)$$

Now, we can state Wigner's theorem, which can be considered the starting point of random matrix theory [2].

**Theorem 1** (Wigner). *For a Wigner matrix  $W_N$ , the empirical distribution  $L_N$  of its eigenvalues converges weakly in probability to the semicircle distribution  $\sigma(x) dx$ . That is, for all  $\varepsilon > 0$  and all  $f \in C_c(\mathbb{R})$ ,*

$$\lim_{N \rightarrow \infty} \mathbb{P}(|\langle L_N, f \rangle - \langle \sigma, f \rangle| > \varepsilon) = 0. \quad (5)$$

There are a multitude of ways to prove Wigner's theorem; we provide here an overview of a combinatorial proof that follows very closely the original argument that Wigner presented in [7]. But, before we can start working through the proof, we need to unravel some properties of the semicircle distribution. Define moments

$$\mu_k := \langle \sigma, x^k \rangle, \quad k \in \mathbb{N}. \quad (6)$$

Recall the Catalan numbers

$$C_k := \frac{1}{k+1} \binom{2k}{k} = \frac{(2k)!}{k!(k+1)!}, \quad k \in \mathbb{N}. \quad (7)$$

We now claim that, for all  $k \in \mathbb{N}$ ,

$$\mu_{2k} = C_k \quad \text{and} \quad \mu_{2k+1} = 0. \quad (8)$$

Indeed,  $\mu_{2k+1} = 0$  by symmetry. Meanwhile,

$$\begin{aligned} \mu_{2k} &= \int_{-2}^2 x^{2k} \sigma(x) dx = \frac{2 \cdot 2^{2k}}{\pi} \int_{-\pi/2}^{\pi/2} \sin^{2k}(\theta) \cos^2(\theta) d\theta \\ &= \frac{2 \cdot 2^{2k}}{\pi} \int_{-\pi/2}^{\pi/2} \sin^{2k}(\theta) d\theta - (2k+1) \mu_{2k}; \end{aligned} \quad (9)$$

i.e.,

$$\mu_{2k} = \frac{2 \cdot 2^{2k}}{\pi(2k+2)} \int_{-\pi/2}^{\pi/2} \sin^{2k}(\theta) d\theta = \frac{4(2k-1)}{2k+2} \mu_{2k-2}. \quad (10)$$

So, combining the above with the fact that  $\mu_0 = 1$ , one finds (8).

As a first combinatorial interpretation of the Catalan numbers, say that a  $\mathbb{Z}$ -valued sequence  $\{S_n\}_{0 \leq n \leq \ell}$  is a *Bernoulli walk* of length  $\ell$  if  $S_0 = 0$  and  $|S_{t+1} - S_t| = 1$  for  $t \leq \ell - 1$ . Then,  $C_k$  counts the number of *Dyck paths* of length  $2k$ ; that is, the number of nonnegative Bernoulli walks of length  $2k$  that end at 0. Let  $\beta_k$  denote the number of such Dyck paths.

**Proposition 1.** *We have that  $\beta_k = C_k < 4^k$ . Moreover, the generating function  $\hat{\beta}(z) := 1 + \sum_{k=1}^{\infty} z^k \beta_k$  is such that*

$$\hat{\beta}(z) = \frac{1 - \sqrt{1 - 4z}}{2z}, \quad \text{for } |z| < 1/4. \quad (11)$$

*Proof.* Let  $B_k$  be the number of Bernoulli walks  $\{S_n\}$  of length  $2k$  such that  $S_{2k} = 0$ , and denote by  $\tilde{B}_k$  the number of Bernoulli walks  $\{S_n\}$  of length  $2k$  such that  $S_{2k} = 0$  and  $S_t < 0$  for some  $t < 2k$ . Then,  $\beta_k = B_k - \tilde{B}_k$ . By reflection at the first hitting of  $-1$ , we have that  $\tilde{B}_k$  is equal to the number of Bernoulli walks  $\{S_n\}$  of length  $2k$  with  $S_{2k} = -2$ . Thus,

$$\beta_k = B_k - \tilde{B}_k = \binom{2k}{k} - \binom{2k}{k-1} = C_k, \quad (12)$$

as desired.

Now, we aim to evaluate  $\hat{\beta}(z)$ . Considering the first return time of the Bernoulli walk  $\{S_n\}$  to 0, we find that

$$\beta_k = \sum_{j=1}^k \beta_{k-j} \beta_{j-1}, \quad k \in \mathbb{N}, \quad (13)$$

where we take  $\beta_0 = 1$ . By the known fact that  $C_k < 4^k$  (see for example [1]), we have that  $\beta_k < 4^k$ , hence  $\hat{\beta}(z)$  is

well-defined and analytic for  $|z| < 1/4$ . Substituting (13) into the definition of  $\hat{\beta}(z)$ , we find that

$$\hat{\beta}(z) - 1 = \sum_{k=1}^{\infty} z^k \sum_{j=1}^k \beta_{k-j} \beta_{j-1} = z \sum_{k=0}^{\infty} z^k \sum_{j=0}^{\infty} \beta_{k-j} \beta_j. \quad (14)$$

On the other hand,

$$\hat{\beta}(z)^2 = \sum_{k,l=0}^{\infty} z^{k+l} \beta_k \beta_l = \sum_{m=0}^{\infty} \sum_{p=0}^m z^m \beta_{m-p} \beta_p. \quad (15)$$

Combining these last two equations, we find that

$$\hat{\beta}(z) = 1 + z\hat{\beta}(z)^2, \quad (16)$$

from which the claimed formula follows (using the fact that  $\hat{\beta}(0) = 1$  to pick the correct branch of the square-root).  $\square$

*Remark 1.* Note that the power series expansion around zero of  $\hat{\beta}(z)$  as in Equation (11) is, for  $|z| < 1/4$ ,

$$\hat{\beta}(z) = \frac{2 \sum_{k=1}^{\infty} \frac{z^k (2k-2)!}{k!(k-1)!}}{2z} = \sum_{k=0}^{\infty} \frac{(2k)!}{k!(k+1)!} z^k = \sum_{k=0}^{\infty} C_k z^k, \quad (17)$$

providing an alternative way of finding that  $\beta_k = C_k$ .

As a brief aside before we move on to the proof of Wigner's theorem, the following two propositions exemplify the *Stieltjes transform*, which is a very useful tool in random matrix theory and probability.

**Proposition 2.** *The Stieltjes transform  $S(z)$  of the empirical measure  $L_N$  of the eigenvalues  $\{\lambda_i\}$  of an  $N \times N$  matrix  $A$  is  $S(z) = \frac{1}{N} \text{tr}((A - z\mathbf{1})^{-1})$ .*

*Proof.* By the definition of the Stieltjes transform of a probability measure,

$$S(z) \stackrel{\text{def}}{=} \int \frac{1}{x-z} L_N(dx) = \frac{1}{N} \sum_{i=1}^N \frac{1}{\lambda_i - z} = \frac{1}{N} \text{tr}((A - z\mathbf{1})^{-1}), \quad (18)$$

as desired.  $\square$

**Proposition 3.** *The Stieltjes transform  $S(z)$  of the semicircle law  $\sigma(x) dx$  is*

$$S(z) = \frac{-z + \sqrt{z^2 - 4}}{2}, \quad \text{for } z \in \mathbb{C}^+. \quad (19)$$

*Proof.* By definition,

$$S(z) \stackrel{\text{def}}{=} \int \frac{1}{x-z} \sigma(dx) = - \int \frac{1}{z(1 - \frac{x}{z})} \sigma(dx) = - \int \frac{1}{z} \sum_{n=0}^{\infty} \left(\frac{x}{z}\right)^n \sigma(dx), \quad (20)$$

for  $|z| > R$  large enough to ensure uniform convergence of the series. Then,

$$S(z) = - \sum_{n=0}^{\infty} \int_{-R}^R \frac{x^n}{z^{n+1}} \sigma(dx) = - \sum_{n=0}^{\infty} \frac{\mu_n}{z^{n+1}}, \quad (21)$$

where  $\mu_k := \langle \sigma, x^k \rangle$ ,  $k \in \mathbb{N}$ , as before. So, by Equation (8),

$$S(z)^2 = \left( \sum_{k=0}^{\infty} \frac{C_k}{z^{2k+1}} \right)^2 = \sum_{p=0}^{\infty} \sum_{q=0}^p \frac{C_q C_{p-q}}{z^{2p+1}} = \sum_{p=1}^{\infty} \frac{C_p}{z^{2p+1}} = -zS(z) - 1. \quad (22)$$

Thus,  $S(z)$  must satisfy

$$S(z)^2 + zS(z) + 1 = 0. \quad (23)$$

The two solutions are

$$S(z) = \frac{-z \pm \sqrt{z^2 - 4}}{2}, \quad (24)$$

but, since  $z \in \mathbb{C}^+$ , we must have  $S(z) \in \mathbb{C}^+$ , hence  $S(z)$  must be the solution with the plus sign.  $\square$

Now, we provide an overview of the proof of Wigner's theorem (see Theorem 1). Define the probability distribution  $\bar{L}_N := \mathbb{E}L_N$  via  $\langle \bar{L}_N, f \rangle = \mathbb{E}\langle L_N, f \rangle$ , for all  $f \in C_c(\mathbb{R})$ , and let  $\mu_k^N := \langle \bar{L}_N, x^k \rangle$ ,  $k \in \mathbb{N}$ . Wigner's theorem follows from the next two lemmas, which we state here without proof since they require somewhat involved combinatorics arguments (for details, we refer the reader to [7] and Chapter 2 of [2]).

**Lemma 1.** For all  $k \in \mathbb{N}$ ,

$$\lim_{N \rightarrow \infty} \mu_k^N = \mu_k. \quad (25)$$

*Proof.* See Lemma 2.1.6 of [2].  $\square$

**Lemma 2.** For all  $k \in \mathbb{N}$  and  $\varepsilon > 0$ ,

$$\lim_{N \rightarrow \infty} \mathbb{P}(|\langle L_N, x^k \rangle - \langle \bar{L}_N, x^k \rangle| > \varepsilon) = 0. \quad (26)$$

*Proof.* See Lemma 2.1.7 of [2].  $\square$

*Proof (Theorem 1).* Assuming the two lemmas above, we only need to verify that

$$\lim_{N \rightarrow \infty} \langle L_N, f \rangle = \langle \sigma, f \rangle \quad (27)$$

in probability for all  $f \in C_c(\mathbb{R})$ . Indeed, by Chebyshev's inequality, for  $B > 0$  and  $k \in \mathbb{N}$ ,

$$\mathbb{P}(\langle L_N, |x|^k \mathbf{1}_{\{|x| > B\}} \rangle > \varepsilon) \leq \frac{1}{\varepsilon} \mathbb{E}[\langle L_N, |x|^k \mathbf{1}_{\{|x| > B\}} \rangle] \leq \frac{1}{\varepsilon} \langle \bar{L}_N, |x|^k \mathbf{1}_{\{|x| > B\}} \rangle \leq \frac{1}{\varepsilon B^k} \langle \bar{L}_N, x^{2k} \rangle. \quad (28)$$

So, by Lemma 1,

$$\limsup_{N \rightarrow \infty} \mathbb{P}(\langle L_N, |x|^k \mathbf{1}_{\{|x| > B\}} \rangle > \varepsilon) \leq \frac{1}{\varepsilon B^k} \langle \sigma, x^{2k} \rangle = \frac{C_k}{\varepsilon B^k} \leq \frac{4^k}{\varepsilon B^k}. \quad (29)$$

Thus, since the left side above is increasing in  $k$ , picking  $B = 5$  gives that

$$\limsup_{N \rightarrow \infty} \mathbb{P}(\langle L_N, |x|^k \mathbf{1}_{\{|x| > B\}} \rangle > \varepsilon) = 0. \quad (30)$$

From now on, we assume that  $f \in C_c(\mathbb{R})$  is supported on  $[-5, 5]$ . Fix such  $f$  and fix  $\delta > 0$ . By Weierstrass' approximation, there exists a polynomial  $Q_\delta(x) := \sum_{i=0}^L c_i x^i$  such that

$$\sup_{|x| \leq B} |Q_\delta(x) - f(x)| \leq \frac{\delta}{8}. \quad (31)$$

Then,

$$\begin{aligned} \mathbb{P}(|\langle L_N, f \rangle - \langle \sigma, f \rangle| > \delta) &\leq \underbrace{\mathbb{P}\left(|\langle L_N, Q_\delta \rangle - \langle \bar{L}_N, Q_\delta \rangle| > \frac{\delta}{4}\right)}_{=: T_1} + \underbrace{\mathbb{P}\left(|\langle \bar{L}_N, Q_\delta \rangle - \langle \sigma, Q_\delta \rangle| > \frac{\delta}{4}\right)}_{=: T_2} \\ &\quad + \underbrace{\mathbb{P}\left(|\langle L_N, Q_\delta \mathbf{1}_{\{|x| > B\}} \rangle| > \frac{\delta}{4}\right)}_{=: T_3}. \end{aligned} \quad (32)$$

Finally, by Lemma 2, it follows that  $T_1 \xrightarrow{N \rightarrow \infty} 0$ ; by Lemma 1, we know that  $T_2 = 0$  for large enough  $N$ ; and, lastly, Equation (30) implies that  $T_3 \xrightarrow{N \rightarrow \infty} 0$ .  $\square$

The remainder of this section is dedicated to a brief discussion of Fredholm determinants following Chapter 3 of [2] in view of their importance to the study of gap probabilities. Loosely speaking, the term “gap probability” refers to the probability that a certain set does *not* contain any eigenvalues of a given random matrix. As it turns out, this gap probability is given by a Fredholm determinant. This means that the asymptotic analysis of gap probabilities involves the study of such determinants. Since this theory can involve several different aspects of functional analysis, we only provide key definitions and facts about Fredholm determinants.

Let  $X$  be a locally compact space and  $\mathcal{B}_X$  its Borel  $\sigma$ -algebra. Let  $\nu$  be a  $\mathbb{C}$ -valued measure on  $(X, \mathcal{B}_X)$  with

$$\|\nu\|_1 := \int_X |\nu(dx)| < \infty. \quad (33)$$

Usually,  $X = \mathbb{R}$  and  $\nu$  is a scalar multiple of the Lebesgue measure on a bounded interval. We say that a  $\mathcal{B}_X$ -measurable  $\mathbb{C}$ -valued function  $K(x, y)$  defined on  $X \times X$  is a *kernel* if

$$\|K\| := \sup_{(x, y) \in X \times X} |K(x, y)| < \infty. \quad (34)$$

The *trace* (with respect to  $\nu$ ) of a kernel  $K(x, y)$  is defined as

$$\mathrm{tr} K := \int K(x, x) d\nu(x). \quad (35)$$

We can define a *composition* (with respect to  $\nu$ ) of two kernels  $K(x, y)$  and  $L(x, y)$  as

$$(K \star L)(x, y) := \int K(x, z) L(z, y) d\nu(z). \quad (36)$$

Note that the trace and composition operations are well-defined since  $\|K\| < \infty$  and  $\|\nu\|_1 < \infty$ . Moreover,  $K \star L$  is itself a kernel. By Fubini's theorem, for any kernels  $K, L, M$ , we have

$$(K \star L) \star M = K \star (L \star M) \quad \text{and} \quad \mathrm{tr}(K \star L) = \mathrm{tr}(L \star K). \quad (37)$$

*Remark 2.* As a warning, we are not requiring kernels to be continuous here. This means that even if two kernels  $K, \tilde{K}$  satisfy  $K = \tilde{K}$ ,  $(\nu \times \nu)$ -a.e., we still might have  $\mathrm{tr}(K) \neq \mathrm{tr}(\tilde{K})$ .

Now, to define the Fredholm determinant associated to a kernel  $K(x, y)$ , for  $n > 0$ , set

$$\Delta_n = \Delta_n(K, \nu) := \int \cdots \int \det_{i,j=1}^n K(\xi_i, \xi_j) d\nu(\xi_1) \cdots d\nu(\xi_n), \quad (38)$$

with  $\Delta_0 := 1$ . Note that, as a consequence of Hadamard's inequality, we have that

$$\left| \int \cdots \int \det_{i,j=1}^n K(\xi_i, \xi_j) d\nu(\xi_1) \cdots d\nu(\xi_n) \right| \leq \|\nu\|_1^n \|K\|^n n^{n/2} < \infty, \quad (39)$$

hence  $\Delta_n(K, \nu)$  is well-defined. Then, the *Fredholm determinant* associated to the kernel  $K(x, y)$  is defined as

$$\Delta(K) = \Delta(K, \nu) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \Delta_n(K, \nu). \quad (40)$$

As shown by Lemmas 2.5.12 and 3.4.9 of [2], the series defining the Fredholm determinant above converges absolutely by Stirling's approximation.

*Remark 3.* Consider two families of maps  $\{f_i\}_{i=1}^N, \{g_i\}_{i=1}^N$  taking values on  $X$  such that  $\max_i \sup_x f_i(x) < \infty$  and  $\max_i \sup_y g_i(y) < \infty$ . Let

$$K(x, y) = \sum_{i=1}^N f_i(x) g_i(y). \quad (41)$$

Then,  $K(x, y)$  is a kernel. Moreover, one finds that

$$\Delta(K) = \det_{i,j=1}^N \left( \delta_{ij} - \int f_i(x) g_j(x) d\nu(x) \right) \quad (42)$$

is the Fredholm determinant associated to  $K(x, y)$  [Lemma 3.4.2, 2]. Because of the form of the right-hand side of Equation (42), it is common to use the notation  $\det(\mathbb{1} - K)$  for the Fredholm determinant of  $K(x, y)$ .

Lastly, Fredholm determinants are extensively used in numerical computations and eigenvalue  $N \rightarrow \infty$  asymptotics because we can write gap probabilities  $\mathbb{P}(\{\lambda_i^N\} \not\subset \Lambda)$  of eigenvalues  $\{\lambda_i^N\}$  on a measurable  $\Lambda \subset \mathbb{R}$  as Fredholm determinants with different kernels depending on the type of ensemble we are considering. For more details on the relation between gap probabilities of different random matrix ensembles and their various Fredholm determinants, we direct the reader to Chapters 3, 4, and 5 of [2], and [3].

## 2 Log-gases and random matrix ensembles

In this section, we introduce the concepts of log-gases and Boltzmann factors as they were conceived in statistical physics in [4] and [3]. Then, following the first three chapters of [6], we consider Gaussian, circular, and Cauchy random matrix ensembles (alongside general facts about them) to study how the p.d.f.'s of their eigenvalues relate to certain types of log-gases.

We begin this discussion with some basic theory from statistical mechanics to introduce Boltzmann factors, log-gases, and show how the Boltzmann factor of a log-gas (or, more generally, of a one-component Coulomb system) is computed. This canonical formulation of statistical physics applies to mechanical systems with  $N \in \mathbb{N}$  particles that are free to roam in a fixed domain  $\Omega \subset \mathbb{R}^2$ , which is in equilibrium at an absolute temperature  $T > 0$  [6]. A foundational postulate tells us that the p.d.f. for the event that these  $N$  particles are located at positions  $\vec{r}_1, \dots, \vec{r}_N$  is

$$\frac{1}{Z_N} e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}, \quad (43)$$

where  $\beta := 1/(k_B T)$  with  $k_B$  denoting Boltzmann's constant<sup>1</sup>,  $U(\vec{r}_1, \dots, \vec{r}_N)$  is the *total potential energy* of the system, and  $Z_N$  is the *normalization constant* given by

$$Z_N = \int_{\Omega} \dots \int_{\Omega} e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)} d\vec{r}_1 \dots d\vec{r}_N. \quad (44)$$

The term  $e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)}$  in (43) is called the *Boltzmann factor* of the system, and  $\mathcal{Z}_N := Z_N/N!$  is referred to as the *canonical partition function*.

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<sup>1</sup> Note that the quantity  $\beta := 1/(k_B T)$  is proportional to the inverse temperature in a statistical system. For this reason,  $\beta$  is commonly referred to simply as inverse temperature.

Now, the total potential energy  $U$  for log-potential Coulomb systems (i.e., log-gases) is computed using 2-dimensional electrostatics, and  $\Omega \subset \mathbb{R}^2$  is required to be either 1- or 2-dimensional. In particular, the particles can be seen as infinitely-long charged parallel lines perpendicular to the confining domain  $\Omega$  [6]. The vacuum electrostatic potential  $\Phi$  at  $\vec{r} = (x, y)$  due to a 2-dimensional unit charge located at  $\vec{r}' = (x', y')$  is the solution to the *Poisson equation*

$$\Delta_{\vec{r}} \Phi(\vec{r}, \vec{r}') = -2\pi \delta(\vec{r} - \vec{r}'), \quad (45)$$

where

$$\Delta_{\vec{r}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \quad (46)$$

From elementary PDE theory, we have that

$$\Phi(\vec{r}, \vec{r}') = -\log(|\vec{r} - \vec{r}'|). \quad (47)$$

We say that a Coulomb system is *one-component* if all  $N$  particles are of the same charge  $q \in \mathbb{R}$ . Now, to prevent all particles from repelling each other to the boundary, we introduce a neutralizing background charge density  $-q\rho_b(\vec{r})$  subject to the *electroneutrality condition*  $\int_{\Omega} \rho_b(\vec{r}) d\vec{r} = N$  [6]. Thus, the total potential energy  $U$  is given by

$$U = U_1 + U_2 + U_3, \quad (48)$$

where

$$U_1 := -q^2 \sum_{1 \leq j < k \leq N} \log |\vec{r}_k - \vec{r}_j| \quad (49)$$

is the energy of the particle-particle interaction,

$$U_2 := q^2 \sum_{j=1}^N V(\vec{r}_j) \quad \text{with} \quad V(\vec{r}_j) := \int_{\Omega} \log |\vec{r} - \vec{r}_j| \rho_b(\vec{r}) d\vec{r} \quad (50)$$

is the energy of the particle-background interaction, and, lastly,

$$U_3 := -\frac{q^2}{2} \int_{\Omega} \int_{\Omega} \rho_b(\vec{r}') \rho_b(\vec{r}) \log |\vec{r}' - \vec{r}| d\vec{r} d\vec{r}' = -\frac{q^2}{2} \int_{\Omega} \rho_b(\vec{r}') V(\vec{r}') d\vec{r}' \quad (51)$$

is the energy of the background-background interaction [6]. Note that the factor of  $1/2$  in  $U_3$  compensates for double counting of the potential energy implicit in the double integration. From this expression for the total potential energy  $U$ , we conclude that the Boltzmann factor for a one-component log-potential Coulomb system (i.e., a log-gas) is

$$e^{-\beta U_3} \prod_{l=1}^N e^{-\gamma V(\vec{r}_l)} \prod_{1 \leq j < k \leq N} |\vec{r}_k - \vec{r}_j|^{-\gamma}, \quad (52)$$

where  $\gamma := q^2/k_B T$  [6]. The following proposition (see [1.4.1, 6]) illustrates that, for certain “simple” geometries and background densities, the potentials  $U_3$  and  $V(\vec{r})$  are easily evaluated.

**Proposition 4.** *The Boltzmann factor of a log-gas with  $N \in \mathbb{N}$  particles of charge  $q \equiv 1$  confined to a circle of radius  $R > 0$  with a uniform neutralizing background is given by*

$$R^{-N\beta/2} \prod_{1 \leq k < j \leq N} |e^{i\theta_k} - e^{i\theta_j}|^{\beta}, \quad (53)$$

where the positions of the particles are in polar coordinates.

*Proof.* Since  $\vec{r}$  and  $\vec{r}'$  are both confined to a circle of radius  $R$ , we can specify their positions using polar coordinates

by  $|\vec{r} - \vec{r}'| = R|e^{i\theta} - e^{i\theta'}|$ . Using this change of coordinates, we can have

$$V(\vec{r}) = \frac{N}{2\pi R} \int_0^{2\pi} \log |Re^{i\theta} - Re^{i\theta'}| R d\theta' = N \log R + \frac{N}{2\pi} \int_0^{2\pi} \log |e^{i\theta'} - 1| d\theta'. \quad (54)$$

Now, note that the last integral in the equation above vanishes, hence  $V(\vec{r}) = N \log R$ . Thus, we find that  $U_3 = -\frac{1}{2}N^2 \log R$  (since  $q \equiv 1$  here). Substituting these expressions into the Boltzmann factor (52), and noting that  $\gamma = \beta$ , we find the desired result.  $\square$

*Remark 4.* As seen above, the Boltzmann factor is proportional to the p.d.f. for the location of particles in these statistical systems. In fact, more generally, the Boltzmann factor appears in the definition of all statistical quantities related to the equilibrium state of a given system. For instance, the *canonical average* of a function  $f(\vec{r}_1, \dots, \vec{r}_N)$  is defined as

$$\langle f \rangle := \frac{1}{Z_N} \int_{\Omega} \dots \int_{\Omega} f(\vec{r}_1, \dots, \vec{r}_N) e^{-\beta U(\vec{r}_1, \dots, \vec{r}_N)} d\vec{r}_1 \dots d\vec{r}_N. \quad (55)$$

For  $f(\vec{r}, \vec{r}_1, \dots, \vec{r}_N) = \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j)$ , the canonical average is called the *one-point correlation function* (or *particle density*) and is given by

$$\rho_{(1)}(\vec{r}) := \left\langle \sum_{j=1}^N \delta(\vec{r} - \vec{r}_j) \right\rangle = \frac{N}{Z_N} \int_{\Omega} \dots \int_{\Omega} e^{-\beta U(\vec{r}, \vec{r}_2, \dots, \vec{r}_N)} d\vec{r}_2 \dots d\vec{r}_N, \quad (56)$$

where the equality above is true for a system of identical particles; i.e., when the Boltzmann factor is a symmetric function of the coordinates of the particles [6].

Now, since we require the particles to be confined under a harmonic potential, the background charge density  $-q\rho_b(y)$  satisfies

$$\frac{x^2}{2} + C = \int_{\mathbb{R}} \rho_b(y) \log |x - y| dy, \quad (57)$$

where  $C \in \mathbb{R}$  is a constant. Note that it is impossible to solve (57) for  $|x| \rightarrow \infty$  because, in the limit, the right-hand side is of order  $N \log |x|$ ; i.e., it is of a different order from the left-hand side. To solve this issue, we instead need to solve the integral equation for  $\rho_b(y)$  with support on a finite interval  $(-\alpha, \alpha) \subset \mathbb{R}$  and  $x$  confined to this interval; that is, we seek a solution to

$$\frac{x^2}{2} + C = \int_{-\alpha}^{\alpha} \rho_b(y) \log |x - y| dy, \quad x \in (-\alpha, \alpha). \quad (58)$$

The solution of the equation above is explicitly computed in Section 1.4 of [6] by expanding in eigenfunctions of a shifted integral operator with  $\log$  as its kernel. As shown in Proposition 1.4.3 of [6], the solution of the integral equation (58) is

$$\rho_b(y) = \frac{\alpha}{\pi} \sqrt{1 - \left(\frac{y}{\alpha}\right)^2} - \frac{1}{\pi \log 2} \left( \frac{1}{4} - \frac{N}{\alpha^2} \log \alpha + \frac{C}{\alpha^2} + \frac{\log 2}{2} \right) \frac{\alpha}{\sqrt{1 - \left(\frac{y}{\alpha}\right)^2}}. \quad (59)$$

Note that the overall behavior of the solution greatly depends on  $C$ . Unless we set

$$C \stackrel{!}{=} N \log \alpha - \frac{\alpha^2}{4} - \frac{\alpha^2 \log 2}{2}, \quad (60)$$

the density  $\rho_b(y)$  has an inverse square root singularity at  $y = \pm\alpha$ , which is not physically sensible. However, picking  $C$  as in (60), the term with the problematic pole vanishes. Then, with this choice of  $C$ , we can determine  $\alpha$  using the neutrality condition  $\int_{-\alpha}^{\alpha} \rho_b(y) dy = N$  [6]. In this sense, as shown in Proposition 1.4.4 of [6], the Boltzmann factor of the log-gas with unit charges ( $q \equiv 1$ ) situated at positions  $x_1, \dots, x_N$  confined to  $[-\sqrt{2N}, \sqrt{2N}]$ , with a



neutralizing background charge density

$$-\rho_b(y) = -\frac{\sqrt{2N}}{\pi} \sqrt{1 - \frac{y^2}{2N}}, \quad (61)$$

is

$$\exp \left[ -\frac{\beta N^2}{4} \log \left( \frac{N}{2} \right) + \frac{3\beta N^2}{8} \right] \exp \left( -\frac{\beta}{2} \sum_{j=1}^N x_j^2 \right) \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta. \quad (62)$$

As a final remark before we move on to particular random matrix ensembles and their log-gases, using simple asymptotic analysis, it is possible to show that the *global density*  $\tilde{\rho}(x)$  of the last system above, defined as

$$\tilde{\rho}(x) := \lim_{N \rightarrow \infty} \rho_b(\sqrt{2N}x) \sqrt{\frac{2}{N}}, \quad (63)$$

obeys

$$\tilde{\rho}(x) = \begin{cases} \frac{1}{2\pi} \sqrt{4 - x^2}, & |x| \leq 2 \\ 0, & |x| > 2 \end{cases}, \quad (64)$$

which is precisely the density of the Wigner semicircle law as in Equation (4).

In the remaining of this section, we introduce the Gaussian, circular, and Cauchy random matrix ensembles along with key results about them and their connections to log-gas systems.

## 2.1 Gaussian ensembles

In this section, motivated by log-gas systems with  $\beta = 1, 2, 4$  (where  $\beta := 1/(k_B T)$  as in (43)), we define the Gaussian orthogonal, unitary, and symplectic ensembles (abbreviated as GOE, GUE, and GSE, respectively) using the joint p.d.f. of the eigenvalues of its elements. It turns out that, for  $\beta = 1, 2, 4$ , these joint p.d.f.'s can be written directly. However, for general  $\beta > 0$ , this task becomes more laborious, and one needs to rewrite the general  $\beta$ -Gaussian matrices in tridiagonal form.

**Definition 1.** Let  $H$  be a Hermitian matrix with real ( $\beta = 1$ ), complex ( $\beta = 2$ ), or real quaternion ( $\beta = 4$ ) entries, and decompose it using its eigenvalues and eigenvectors as  $H = U\Lambda U^*$ , where  $\Lambda$  is a diagonal matrix with the diagonal consisting of the eigenvalues of  $H$ , and  $U$  is a unitary matrix with real ( $\beta = 1$ ), complex ( $\beta = 2$ ), or real quaternion ( $\beta = 4$ ) entries consisting of the corresponding eigenvectors of  $H$ . We say that such  $N \times N$  Hermitian matrix  $H = U\Lambda U^*$  is in the GOE, GUE, or GSE if the joint p.d.f. of its eigenvalues  $\{\lambda_j\}_{1 \leq j \leq N}$  is given by

$$\frac{1}{G_{\beta,N}} \exp \left( -\frac{\beta}{2} \sum_{j=1}^N \lambda_j^2 \right) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad (65)$$

with  $\beta = 1, 2, 4$ , respectively, where  $G_{\beta,N}$  is a normalization constant.  $\diamond$

*Remark 5.* As shown in Proposition 1.9.4 of [6], the normalization constant of the joint p.d.f. in (65) is given by

$$G_{\beta,N} = \beta^{-N/2 - N\beta(N-1)/4} (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)\beta/2)}{\Gamma(1 + \beta/2)}, \quad (66)$$

for  $\beta = 1, 2, 4$ .

*Remark 6.* In order for the decomposition  $H = U\Lambda U^*$  to be unique, the eigenvalues of  $H$  must be ordered and the first component of its eigenvectors must be real and positive. However, since the joint p.d.f. (65) is a symmetric function of the eigenvalues  $\{\lambda_j\}_{1 \leq j \leq N}$ , we may remove the ordering constraint from the eigenvalues. In particular,  $G_{\beta,N}$  is the normalization constant *without* the ordering requirement [6].

*Remark 7.* By comparing (65) and (52), it is immediately clear that the joint p.d.f. of the eigenvalues of the GOE, GUE, and GSE are identical to the Boltzmann factor for the one-component log-potential Coulomb system confined to a line, where the location of the eigenvalues correspond to the positions of the charged particles in the log-gas analogy [6].

Now, the p.d.f. (65) for the eigenvalues of the GOE, GUE, and GSE is obtained when  $\beta = 1, 2, 4$ , respectively. Next, we present the p.d.f. of the eigenvalues for general  $\beta > 0$  Gaussian ensembles by reducing the matrices in consideration to their tridiagonal form [6]. A canonical tool from linear algebra is the similarity transformation of a  $\mathbb{R}$ -symmetric matrix into tridiagonal form using a sequence of reflection matrices called *Hauseholder transformations* [6]. More precisely, for a  $\mathbb{R}$ -symmetric matrix  $M = (m_{ij})_{1 \leq i, j \leq N}$ , one can define  $\mathbb{R}$ -symmetric orthogonal matrices  $U_1, \dots, U_{N-2}$  such that

$$U_{N-2}U_{N-3} \cdots U_1 M U_1 U_2 \cdots U_{N-2} \quad (67)$$

is a symmetric tridiagonal matrix [6]. In particular, the matrices  $U_1, \dots, U_{N-2}$  are of the form

$$U_j = \begin{bmatrix} \mathbf{1}_j & 0_{j \times (N-j)} \\ 0_{(N-j) \times j} & V_{(N-j) \times (N-j)} \end{bmatrix}, \quad (68)$$

where  $V_{(N-j) \times (N-j)}$  is  $\mathbb{R}$ -symmetric orthogonal [6]. Following Propositions 1.9.1 and 1.9.4 of [6], we define Gaussian ensembles for general  $\beta > 0$ .

**Definition 2.** Fix  $\beta > 0$ . Let  $\mathbf{N}[0, 1]$  denote the standard normal distribution, and let  $\tilde{\chi}_k$  be the square root of the Gamma distribution  $\Gamma[k/2, 1]$ <sup>2</sup>. Define the *general  $\beta$ -Gaussian ensemble* as the set of symmetric tridiagonal matrices

$$T_\beta := \begin{bmatrix} \mathbf{N}[0, 1] & \tilde{\chi}_{\beta(N-1)} & & & & \\ \tilde{\chi}_{\beta(N-1)} & \mathbf{N}[0, 1] & \tilde{\chi}_{\beta(N-2)} & & & \\ & \tilde{\chi}_{\beta(N-2)} & \mathbf{N}[0, 1] & \tilde{\chi}_{\beta(N-3)} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \tilde{\chi}_{2\beta} & \mathbf{N}[0, 1] & \tilde{\chi}_\beta \\ & & & \tilde{\chi}_\beta & \mathbf{N}[0, 1] \end{bmatrix}. \quad (69)$$

◇

Finally, using certain special Dirichlet integrals as in [Proposition 1.9.4, 6], one can show that the joint p.d.f. of the eigenvalues of an  $N \times N$  matrix in the  $\beta$ -Gaussian ensemble defined above is given by

$$\frac{1}{\tilde{G}_{\beta, N}} \exp \left( - \sum_{j=1}^N \frac{\lambda_j^2}{2} \right) \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad \text{where} \quad \tilde{G}_{\beta, N} = (2\pi)^{N/2} \prod_{j=0}^{N-1} \frac{\Gamma(1 + (j+1)\beta/2)}{\Gamma(1 + \beta/2)}. \quad (70)$$

## 2.2 Circular ensembles

In this section, we explore several features of the so called circular random matrix ensembles as they were conceived by the physicist Freeman Dyson [4]. Dyson noticed that Gaussian ensembles are not uniquely determined by the

<sup>2</sup> The p.d.f. of the Gamma distribution  $\Gamma[k/2, 1]$  is defined as

$$\frac{1}{\Gamma(k/2)} u^{k/2-1} e^{-u} \mathbf{1}_{\{u>0\}}.$$

Note that this is realized by the sum of the squares of  $k$  independent normal distributions  $\mathbf{N}[0, 1/\sqrt{2}]$  [6]. This means that the p.d.f. of  $\tilde{\chi}_k$  is given by

$$\frac{2}{\Gamma(k/2)} u^{k-1} e^{-u^2} \mathbf{1}_{\{u>0\}}.$$

invariance of probability measures on the space of matrices under conjugation. This motivated Dyson to develop a theory of unitary random matrices that possessed the same invariances as Hermitian matrices used to model quantum Hamiltonians [6]. In particular, as we will discuss shortly, unitary random matrices can naturally model Floquet operators and scattering matrices in quantum mechanics. If there is no time reversal symmetry, the unitary matrices simply form the group  $U(N)$ , where, as it is well-known, the Haar measure is the unique uniform measure [6]. On the other hand, if there is time reversal symmetry, we need to impose further constraints on the matrices and appropriately modify the Haar measure. In either case, again for inverse temperatures  $\beta = 1, 2, 4$ , the p.d.f. of the eigenvalues of matrices in the circular ensemble corresponds to the p.d.f. of the positions of charged particles in a log-gas system confined to a circle [6].

First, we consider unitary random matrices related to scattering phenomena of plane waves inside a cavity of arbitrary shape with random scattering imperfections. In this sense, the plane waves enter and leave the cavity through a wave guide that allows  $N \in \mathbb{N}$  distinct plane wave states (sometimes called *channels*) to enter. Denote the amplitudes of the  $N$  incoming states by  $\vec{I}$ , and the amplitudes of the  $N$  outgoing wave states by  $\vec{O}$ . The  $N \times N$  scattering matrix  $S$  is then defined so that

$$S\vec{I} = \vec{O}. \quad (71)$$

In order to observe conservation of flux, we impose  $|\vec{I}|^2 = |\vec{O}|^2$ . This implies that  $SS^* = \mathbb{1}$ , hence  $S$  is unitary [6]. More generally,  $S$  is given by the limit of the evolution operator of the system:

$$S = \lim_{t_0 \rightarrow -\infty} \lim_{t \rightarrow +\infty} U(t, t_0), \quad (72)$$

where the *evolution operator*  $U(t, t_0)$  is

$$U(t, t_0) := \mathcal{T} \exp \left( -\frac{i}{\hbar} \int_{t_0}^t H(s) ds \right), \quad (73)$$

with  $\mathcal{T}$  denoting the time ordering and  $H$  being the Hamiltonian of the system in consideration [6]. Again, this implies that  $S$  is unitary. However, the specific form of  $S$  depends on whether the system has time reversal symmetry.

**Definition 3.** A time-dependent Hamiltonian  $H(t)$  has a time reversal symmetry  $T$  if  $T$  is anti-unitary and

$$T^{-1}H(t)T = H(-t). \quad (74)$$

◇

**Proposition 5.** Let  $T$  be a time reversal symmetry of a time-dependent Hamiltonian  $H$ . Then,

$$T^{-1}U(t, t_0)T = U(-t, -t_0). \quad (75)$$

*Proof.* See Proposition 2.1.2 of [6]. □

Now, assume that the statistical properties of the scattering matrix of an irregular shaped cavity are determined entirely by a global time reversal symmetry  $T$  of a time-dependent Hamiltonian  $H$ . Taking the limits  $t_0 \rightarrow -\infty$  and  $t \rightarrow +\infty$  in Equation (75), we find that

$$T^{-1}ST = S^*. \quad (76)$$

As shown in Section 2.1.1 of [6], if the time reversal symmetry  $T$  is such that  $T^2 = \mathbb{1}$ , then  $S$  must be symmetric. In particular, the subgroup of unitary matrices which, under a similarity transformation, map  $S$  to a symmetric matrix is precisely the real orthogonal group [6]. On the other hand, if  $T^2 = -\mathbb{1}$ , then  $S$  is a self-dual quaternion matrix. In this case, the subgroup of unitary matrices which, under a similarity transformation, map self-dual quaternion matrices to self-dual quaternion matrices is precisely the symplectic group [6].

We now turn our attention to random Floquet operators. Consider evolution operators of the form  $U(t) := U(t, 0)$ ,

as defined in (73), for a  $\tau$ -periodic Hamiltonian; i.e.,  $H(t) = H(t+\tau)$ ,  $\forall t \geq 0$ . Then, for  $n \in \mathbb{Z}$ , we have  $U(n\tau) = F^n$ , where  $F := U(\tau)$  [6]. The operator  $F$  is called the *Floquet operator*. Note that, since  $U$  is unitary,  $F$  is also unitary. As it was the case above with scattering matrices, the form of the Floquet operator  $F$  is again determined by the existence (or non-existence) of time reversal symmetry [6]. More precisely, evaluating (75) for  $t = \tau$  and  $t_0 = 0$ , we find that  $T^{-1}FT = F^{-1} = F^*$ . This means that, analogously to scattering matrices, if  $T^2 = \mathbb{1}$ , then  $F$  must be symmetric; and, if  $T^2 = -\mathbb{1}$ , then  $F$  must be a self-dual quaternion unitary matrix [6].

Now, in order to abide the time reversal symmetry constraints on scattering and Floquet operators, we seek a theory of unitary, symmetric, and self-dual quaternion random matrices. This was provided by Dyson in the seminal paper [4] from 1962. Contrary to the theory of Gaussian random matrices presented in Section 2.1, the theory of circular random matrices did not appear from an explicit formula for its elements; rather, it was developed by imposing the existence of a uniform measure on each of these spaces [6].

In this sense, we define the *Haar volume form*, denoted  $(d_H U)$ , by requiring it to have the following homogeneity property: for any fixed unitary matrix  $V \in \mathbf{U}(N)$ ,

$$(d_H UV) = (d_H VU) = (d_H U). \quad (77)$$

That is, the Haar form is invariant under right and left actions of  $\mathbf{U}(N)$ ; thus, the Haar form gives a uniform measure on  $\mathbf{U}(N)$  [6]. Moreover, it is a well-known theorem that the Haar form specified by (77) is unique up to a normalization constant for all compact groups [6]. As shown in Proposition 2.2.1 of [6], we have the following formula for the Haar form for  $\mathbf{U}(N)$ .

**Proposition 6.** *The Haar form for the unitary group  $\mathbf{U}(N)$  is*

$$(d_H U) = \frac{1}{C} (U^* dU), \quad (78)$$

where  $C$  is a normalization constant.

We can now explicitly define an ensemble of unitary random matrices.

**Definition 4.** The *circular unitary ensemble* (CUE) is the group of unitary matrices endowed with the Haar volume form

$$(d_H U) = \frac{1}{C} (U^* dU). \quad (79)$$

◇

Defining symmetric and self-dual quaternion unitary random matrix ensembles requires a bit more work since these spaces do not form groups. In particular, this means that the homogeneity condition (77) has to be modified. For the symmetric unitary case, recall that two symmetric unitary matrices  $S_1, S_2$  are related by  $S_2 = V^T S_1 V$  for some unitary matrix  $V \in \mathbf{U}(N)$ . This is because any symmetric unitary matrix  $S$  can be written as  $S = U^T U$  for some  $U \in \mathbf{U}(N)$ . Thus, the required homogeneity property to obtain a uniform measure in this space is

$$(d_H S) = (d_H V^T S V), \quad (80)$$

for all  $V \in \mathbf{U}(N)$  [6]. Then, as shown in Section 2.2.2 of [6], the Haar form  $\delta S$  for the symmetric unitary case is

$$\delta S = (U^T)^* dS U^*. \quad (81)$$

An analogous reasoning can be used to find the Haar form in the self-dual quaternion unitary case [6]. First, we decompose self-dual quaternion unitary matrices  $\tilde{S}$  as

$$\tilde{S} = Z_{2N}^{-1} U^T Z_{2N} U =: U^D U \quad (82)$$

for  $U \in \mathrm{U}(2N)$ , and  $Z_{2N}$  a  $(2N) \times (2N)$  block diagonal matrix with each  $2 \times 2$  diagonal block given by

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad (83)$$

This means that two self-dual quaternion unitary matrices  $\tilde{S}_1, \tilde{S}_2$  are related by  $\tilde{S}_2 = V^D \tilde{S}_1 V$  for some  $V \in \mathrm{U}(2N)$  [6]. So, the homogeneity property required for a uniform measure in this case is

$$(\mathrm{d}_H \tilde{S}) = (\mathrm{d}_H Z_{2N}^{-1} V^T Z_{2N} \tilde{S} V). \quad (84)$$

Thus, as shown in Section 2.2.2 of [6], the Haar form in the self-dual quaternion unitary case is given by

$$\delta \tilde{S} = (Z_{2N}^{-1} U^T Z_{2N})^* \mathrm{d} \tilde{S} U^*, \quad (85)$$

**Definition 5.** The *circular orthogonal ensemble* (COE) is the space of symmetric unitary random matrices equipped with the volume form (80). Similarly, the *circular symplectic ensemble* (CSE) is the space of self-dual quaternion unitary matrices equipped with the volume form (85).  $\diamond$

Finally, we provide the p.d.f. of the eigenvalues corresponding to the three circular ensembles defined in this section. As thoroughly described in Section 2.2.3 of [6], the volume forms for the COE, CUE, and CSE correspond, respectively, to the unitary portion of the generators of the matrix Lie algebras  $\mathfrak{gl}(N, \mathbb{R})$ ,  $\mathfrak{gl}(N, \mathbb{C})$ , and  $\mathfrak{u}^*(2N)$ , and, as such, are invariant under Hermitian conjugation. As shown in Proposition 2.2.4 of [6], these Lie algebras are related, respectively, to the following compact quotient spaces via the Hermitian conjugation map:

$$\mathrm{U}(N)/\mathrm{O}(N), \quad \mathrm{U}(N) \times \mathrm{U}(N)/\mathrm{U}(N), \quad \mathrm{U}(2N)/\mathrm{Sp}(2N). \quad (86)$$

Lastly, as shown in Propositions 2.2.5 and 2.8.7 of [6], one can use the decompositions mentioned above to rewrite elements of the COE, CUE, and CSE to obtain the following result.

**Proposition 7.** *With each eigenvalue written as  $\lambda_j = e^{i\theta_j}$ , the eigenvalue p.d.f. for the COE, CUE, and CSE is*

$$\frac{1}{C_{\beta, N}} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \quad -\pi < \theta_j < \pi \quad (87)$$

with  $\beta = 1, 2, 4$ , respectively. The normalization constant  $C_{\beta, N}$  is given by

$$C_{\beta, N} = (2\pi)^N \frac{\Gamma(\beta N/2 + 1)}{\Gamma(\beta/2 + 1)}. \quad (88)$$

By comparing Proposition 7 with Proposition 4, we immediately see that the eigenvalue p.d.f. of the circular ensembles COE, CUE, and CSE are directly proportional to the Boltzmann factor of the one-component log-potential Coulomb gas confined to a circle [6].

*Remark 8.* Following Definition 5 to numerically generate elements of the COE and CSE seems like a hopeless task. Luckily, using some results by Hurwitz, one can promptly rewrite these elements into pieces that are easy to specify in a computer. See Section 2.3 of [6] for an in-depth description of these decomposition methods.

## 2.3 Cauchy ensembles

In this section, we use a stereographic-like matrix projection to reinterpret the circular ensembles presented in the previous section. Recall the usual *stereographic projection* given by

$$\mathbb{R} \ni \lambda \longmapsto \frac{1 + i\lambda}{1 - i\lambda} =: e^{i\theta} \in S^1 \subset \mathbb{C}, \quad -\pi < \theta < \pi. \quad (89)$$

Note that we can use the stereographic projection map above to write

$$\prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta d\theta_1 \cdots d\theta_N = 2^{N+\beta N(N-1)/2} \prod_{j=1}^N (1 + \lambda_j^2)^{-\beta(N-1)/2-1} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta d\lambda_1 \cdots d\lambda_N. \quad (90)$$

*Remark 9.* In the theory of orthogonal polynomials, functions of the form

$$w(x) = \frac{1}{(1+x^2)^\alpha} \quad (91)$$

appearing in the first product of the right-hand side of (90) are called a *Cauchy weights*, hence the name of the matrix ensemble presented in this section.

**Definition 6.** Define the *Cauchy ensemble* as the collection of random matrices whose eigenvalues have p.d.f.

$$\frac{1}{C} \prod_{j=1}^N (1 + \lambda_j^2)^{-\alpha} \prod_{1 \leq j < k \leq N} |\lambda_k - \lambda_j|^\beta, \quad (92)$$

where  $C$  is a normalization constant. ◇

Note that, for  $\alpha = \beta(N-1)/2 + 1$  and  $\beta = 1, 2, 4$ , Equation (90) gives that the p.d.f. of the Cauchy ensemble is realized by the stereographic projection of the p.d.f. of the eigenvalues of the circular ensembles onto the real line [6]. In fact, unitary matrices  $H \in \mathbf{U}(N)$  with the eigenvalue p.d.f. (92) for  $\alpha = \beta(N-1)/2 + 1$  and  $\beta = 1, 2, 4$ , can be obtained out of their circular counterparts  $U$  via the *Cayley transform*

$$H = i(\mathbf{1}_N - U)(\mathbf{1}_N + U)^{-1}. \quad (93)$$

Moreover, note that, from the inverse Cayley transform

$$U = (\mathbf{1}_N + iH)(\mathbf{1}_N - iH)^{-1}, \quad (94)$$

the eigenvalues  $e^{i\theta_j}$  of  $U$  are related to the eigenvalues  $\lambda_j$  of  $H$  by the stereographic projection (89) [6]. Thus, using Equation (90), we have that the joint p.d.f. of the matrices  $H$  can be written as

$$\frac{1}{C} (\det(\mathbf{1}_N + H^2))^{-\beta(N-1)/2-1}, \quad (95)$$

for  $H$  real symmetric ( $\beta = 1$ ), complex unitary ( $\beta = 2$ ), and self-dual quaternion real ( $\beta = 4$ ) [6].

Lastly, as shown in Section 2.5 of [6], for general  $\beta > 0$ , the eigenvalue p.d.f. (92) of the Cauchy ensemble represents the Boltzmann factor for a one-component log-gas confined to a line ( $x$ -axis) and subject to a one-body potential

$$V(\lambda) = \frac{\alpha}{\beta} \log(1 + \lambda^2). \quad (96)$$

This potential corresponds to an external charge of strength  $-2\alpha/\beta$  located at  $(0, 1)$  on the  $xy$ -plane [6]. For  $2\alpha/\beta = N - 1 + 2/\beta$  (i.e., when (90) holds), since the particle density is uniform in the circular ensemble by symmetry, we find that the density in this particular Cauchy ensemble is

$$\rho(\lambda) = \frac{N}{\pi(1 + \lambda^2)}; \quad (97)$$

in particular, it is independent of  $\beta$  [6].

### 3 Log-gases with periodic boundary conditions

In this section, we follow Section 2.7 of [6] to present a way of reinterpreting the p.d.f.'s in the circular random matrix ensembles as log-gas systems on a line with periodic boundary conditions. Lastly, we explore three concrete examples of log-gas systems with different types of periodic boundary conditions.

Recall from Proposition 7 that the p.d.f. of the eigenvalues of the CUE, COE, and CSE

$$\frac{1}{C_{\beta,N}} \prod_{1 \leq j < k \leq N} |e^{i\theta_k} - e^{i\theta_j}|^\beta, \quad C_{\beta,N} = (2\pi)^N \frac{\Gamma(\beta N/2 + 1)}{\Gamma(\beta/2 + 1)}, \quad -\pi < \theta_j < \pi, \quad (98)$$

with  $\beta = 1, 2, 4$ , respectively, can be interpreted as the Boltzmann factor for a log-gas system on a circle [6]. An equivalent interpretation of the p.d.f. (98) is that of a log-gas that is defined on a line with periodic (instead of free) boundary conditions [6]. More precisely, suppose the line is in the  $x$ -direction and is of length  $L$ . To specify a 2-dimensional Coulomb system here, we need to solve Poisson's equation

$$\Delta_{\vec{r}} \Phi(\vec{r}, \vec{r}') = -2\pi \delta(\vec{r} - \vec{r}'), \quad (99)$$

where

$$\Delta_{\vec{r}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (100)$$

subject to the semiperiodic boundary condition  $\Phi((x+L, y), (x', y')) = \Phi((x, y), (x', y'))$ . As shown in Section 2.7 of [6], using the intuitive facts that the solution ought to depend only on  $x - x'$  and  $y - y'$ , and that, if  $\vec{r} \sim \vec{r}'$ , we have  $\Phi(\vec{r}, \vec{r}') \sim -\log |\vec{r} - \vec{r}'|$  (which is the solution to Poisson's equation with free boundary conditions) in conjunction with a theorem from complex analysis that asserts that the real part of an analytic function satisfies Laplace's equation, we find that

$$\Phi(\vec{r}, \vec{r}') = -\log \left( \frac{L}{\pi} \left| \sin \left( \frac{\pi(x - x' + i(y - y'))}{L} \right) \right| \right). \quad (101)$$

Finally, requiring the particles to be confined to  $[0, L]$  in the  $x$ -direction, we have

$$-\log \left( \frac{L}{\pi} \left| \sin \left( \frac{\pi(x - x')}{L} \right) \right| \right) = -\log \left( \frac{L}{2\pi} \left| e^{2\pi i x/L} - e^{2\pi i x'/L} \right| \right), \quad (102)$$

which is the same as if we solved Poisson's equation (99) with  $(\vec{r}, \vec{r}')$  confined to a circle [6].

Next, we discuss three examples of particular periodic boundary conditions to illustrate the different types of log-gases that they represent: transverse periodic boundary conditions, the metal wall, and doubly periodic boundary conditions.

**Example 1** (Transverse periodic boundary conditions). Suppose that we have a log-gas system in which the particles again interact via the pair potential (101) but, instead of being confined to  $[0, L]$  in the  $x$ -direction, now the particles are confined to the full line in the  $y$ -direction. Then, as shown in Section 2.7.1 of [6], up to an additive constant, the pair potential is

$$-\log \left| \sinh \left( \frac{\pi(y - y')}{L} \right) \right|. \quad (103)$$

Thus, the Boltzmann factor for this system is of the form

$$\prod_{j=1}^N e^{-\beta c' y_j^2/2} \prod_{1 \leq j < k \leq N} \left| \sinh \left( \frac{\pi(y_k - y_j)}{L} \right) \right|^\beta, \quad -\infty < y_j < \infty, \quad (104)$$

where  $N \in \mathbb{N}$  is the number of particles,  $c' > 0$ , and the external attractive harmonic potential above is introduced to make sure the particles do not repel each other to infinity [6]. The Boltzmann factor (104) above with  $\beta = 1$

appears in the theory of non-intersecting paths, and the partition function of the  $\beta = 2$  case appears in Chern-Simon field theory [6].

The quadratic term in the Boltzmann factor (104) comes from the particle-background interaction

$$c'y^2 + C = \int_{\mathbb{R}} \rho_b(x) \log \left| \sinh \left( \frac{\pi(y-x)}{L} \right) \right| dx, \quad (105)$$

where  $\rho_b$  is the neutralizing background charge density to prevent particles from escaping to  $\pm\infty$  [6]. Now, a small caveat is that Equation (105) cannot be solved as  $y \rightarrow \infty$  since each side of the equation is of a different order in that limit. However, it is possible to construct a solution that is valid as  $N \rightarrow \infty$  when  $x, y$  are confined to an interval [6]. Note that  $\log |\sinh(\pi(y-x)/L)| \sim \pi|y-x|/L$  for  $|y-x| \rightarrow \infty$ . This means that  $\pi|y-x|/L$  satisfies the 1-dimensional Poisson equation

$$\frac{d^2}{dy^2} \left( \frac{\pi|y-x|}{L} \right) = \frac{2\pi}{L} \delta(x-y). \quad (106)$$

So, differentiating both sides of Equation (105) twice with respect to  $y$ , we obtain the asymptotic solution

$$\rho_b(x) = \frac{Lc'}{\pi}, \quad |x| < \frac{N\pi}{2Lc'}. \quad (107)$$

As pointed out in Section 2.7.1 of [6], an immediate consequence of this result is that, assuming local charge neutrality for Coulomb systems<sup>3</sup>, for large  $N$  and fixed  $y$ , the leading order density of  $\{y_j\}_{j=1}^N$  in (104) is the constant  $Lc'/\pi$  for  $|x| < N\pi/2Lc'$ , and zero elsewhere.

As a final remark, by writing  $x_j := e^{-2\pi(y_j - y_0)/L}$ , we can rewrite the Boltzmann factor (104) as

$$\prod_{n=1}^N w(x_n) \prod_{1 \leq j < k \leq N} |x_k - x_j|^\beta, \quad (108)$$

where

$$w(x) := e^{-\beta c(\log x)^2}, \quad x > 0. \quad (109)$$

This Boltzmann factor appears often when studying the conductance of different materials [6]. ♠

**Example 2** (Metal wall). Consider the same log-gas with periodic boundary conditions in the  $x$ -direction as we had before but now suppose that a perfect metal conductor occupies the  $\{(x, y) \in \mathbb{R}^2 : y < 0\}$  region. In this case, the pair potential satisfies Poisson's equation (99) subject to the boundary condition

$$\Phi((x, y), (x', y')) = 0 \quad \text{for } y = 0. \quad (110)$$

Using the method of images, we find that the solution is given by

$$\Phi(\vec{r}, \vec{r}') = -\log \left| \frac{\sin(\pi(x - x' + i(y - y'))/L)}{\sin(\pi(x - x' + i(y + y'))/L)} \right|. \quad (111)$$

As usual, we can interpret the solution above as there being a charge of opposite sign located at  $(x', -y')$ . Because of this image effect of the metal wall, it is natural – from a log-gas point of view – to introduce a uniform background charge density  $-\lambda$  that is independent of the particle density [6]. We now prove the following special case of a result of Section 2.7.2 of [6].

**Proposition 8.** *The Boltzmann factor for the log-gas system with  $N$  mobile unit charges confined to the line  $y = d$*

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<sup>3</sup>That is, on average, there is the same number of positive and negative charges in a given neutral material.



near a metal wall as above is given by

$$e^{-\beta(\pi\lambda^2 Ld - 2\pi\lambda N)} \left( \frac{\pi}{L \sinh(2\pi d/L)} \right)^{N\beta/2} \prod_{1 \leq j < k \leq N} \left| \frac{\sin(\pi(x_k - x_j)/L)}{\sin(\pi(x_k - x_j + 2id)/L)} \right|^\beta. \quad (112)$$

*Proof.* The total potential energy of this system is given by  $U = U_1 + U'_1 + U_2 + U_3$ , where

$$U_1 = - \sum_{1 \leq j < k \leq N} \log \left| \frac{\sin(\pi(x_k - x_j)/L)}{\sin(\pi(x_k - x_j + 2id)/L)} \right| \quad (113)$$

is the particle-particle energy;

$$U'_1 := \frac{1}{2} \sum_{j=1}^N \lim_{x' \rightarrow x_j} (\Phi((x_j, d), (x', d)) - \log |x_j - x|) = -\frac{N}{2} \log \left( \frac{\pi}{L \sinh(2\pi d/L)} \right) \quad (114)$$

is the self-energy;

$$U_2 = \lambda \sum_{j=1}^N \int_0^L \log \left| \frac{\sin(\pi x/L)}{\sin(\pi(x + 2id)/L)} \right| dx = -2\pi\lambda Nd \quad (115)$$

is the particle-background energy; and

$$U_3 = \pi\lambda^2 Ld \quad (116)$$

is the background-background energy.  $\square$

Finally, note that, as  $d \rightarrow 0$ , the Boltzmann factor (112) becomes independent of  $\{x_j\}_{j=1}^N$ . Meanwhile, as  $d \rightarrow +\infty$ , the Boltzmann factor for a perfect log-gas is recovered.  $\spadesuit$

**Example 3** (Doubly periodic boundary conditions). Suppose now that we also have a periodic boundary condition of period  $W$  in the  $y$ -direction in addition to the one of period  $L$  in the  $x$ -direction. In this case, the pair potential with doubly periodic conditions must obey the charge neutral Poisson equation

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = -2\pi\delta(x - x')\delta(y - y') + \frac{2\pi}{LW}. \quad (117)$$

*Remark 10.* Without the charge neutrality assumption, such doubly periodic potential is not possible since the integral of the right-hand side of Equation (117) over the fundamental rectangle is zero [6].

In order to solve Equation (117), we introduce the Jacobi  $\theta_1$  function as defined in [1] and Section 2.7.3 of [6]:

$$\theta_1(z; q) := -i \sum_{n \in \mathbb{Z}} (-1)^n q^{(n-1/2)^2} e^{2i(n-1/2)z} = 2q^{1/4} \sin z \prod_{n \in \mathbb{N}} (1 - q^{2n} e^{2iz})(1 - q^{2n} e^{-2iz})(1 - q^{2n}). \quad (118)$$

Note that, if  $q = e^{i\pi\tau}$ ,  $\text{Im } \tau > 0$ , then  $\theta_1$  is entire [1]. Moreover,  $\theta_1(z; q) = 0$  if and only if  $z = \pi m + \pi n$ ,  $m, n \in \mathbb{Z}$ , and  $\theta_1(z; q) \sim z\theta'_1(0; q)$  as  $z \rightarrow 0$  [1]. So, combining these facts, as shown in Section 2.7.3 of [6], we have that

$$\tilde{\Phi}(z, z') := -\log \left( \frac{L|\theta_1(\pi(z - z')/L; q)|}{\pi\theta'_1(0; q)} \right), \quad q := e^{-\pi W/L}, \quad (119)$$

satisfies Equation (99) for  $0 \leq x, x' < L$  and  $0 \leq y, y' < W$  with  $\tilde{\Phi}(z, z') \sim -\log |z - z'|$  as  $|z - z'| \rightarrow 0$ . Now, since  $\theta_1(z + \pi; q) = -\theta_1(z; q)$  and  $\theta_1(z + \pi\tau; q) = -q^{-1}e^{-2iz}\theta_1(z; q)$ , as shown in [6], the function  $\tilde{\Phi}$  in Equation (119) is such that

$$\begin{aligned} \tilde{\Phi}((x + L, y), (x', y')) &= \tilde{\Phi}((x, y), (x', y')) \\ \tilde{\Phi}((x, y + W), (x', y')) &= \tilde{\Phi}((x, y), (x', y')) - \frac{\pi(2y + W)}{L}. \end{aligned} \quad (120)$$

Thus, the potential

$$\Phi(z, z') := \frac{\pi y^2}{LW} + \tilde{\Phi}(z, z') \quad (121)$$

is doubly periodic and satisfies the charge neutral Poisson equation (117), as desired [6].

Finally, for a log-gas system confined to the  $x$ -axis, the quadratic term in Equation (121) does not contribute, and, in this case, the pair potential is simply given by Equation (119) [6]. Moreover, as proven in Section 2.7.3 of [6], the Boltzmann factor for this system of  $N$  unit charges confined to the interval  $[0, L]$  in the  $x$ -axis under a smeared out neutralizing background is

$$\left( \frac{\pi \theta'_1(0; q)}{L} \right)^{N\beta/2} \left( q^{1/4} \prod_{n \in \mathbb{N}} (1 - q^{2n}) \right)^{-\beta N^2/2} \prod_{1 \leq j < k \leq N} \left| \theta_1 \left( \frac{\pi(x_k - x_j)}{L}; q \right) \right|^\beta. \quad (122)$$

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