# RIEMANNIAN POSITIVE MASS THEOREM

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(ref. Dan Lee's Geometric Relativity, 2019)

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**MAIN THEOREM:** (RPMT) Let $(M,g)$ be a complete asymptotically flat manifold with nonnegative scalar curvature. Then, the ADM mass of each end of $M$ is nonnegative.

Rigidity: If the ADM mass of any end of $(M,g)$ is zero, then $(M,g)$ is isometric to Euclidean space.

due to Schoen & Yau

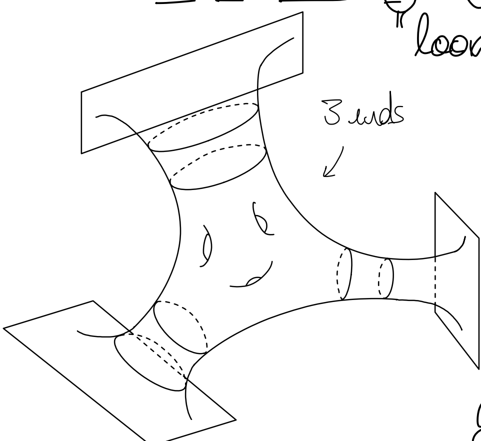
Proof: dim $=3 \longrightarrow$ Schoen & Yau 1979 "Plateau problem" (using minimal surfaces and Gauss–Bonnet essentially, by contradiction)

dim $< 8 \longrightarrow$ Same proof technique as Schoen's & Yau's

dim $\geq 8 \longrightarrow$ spinors (follows Witten's proof)

- What do we mean by asymptotically flat? Riem. mnflds that "look" Euclidean as we go off to infinity.

↳ Can be made formal using weighted Hölder / Sobolev spaces...




EXAMPLE: Schwarzschild Space of Mass $m>0$: $(M^n, g_m)$

$d\Omega^2 := d\theta^2 + \sin^2\theta\, d\phi^2$

$$g_m = \left(1+\frac{m}{2\rho^{n-2}}\right)^{4/n-2}\left(d\rho^2+\rho^2 d\Omega^2\right),$$

as $\rho \nearrow \infty$, "$g_m \to g_{Euc}$".

Rmk: $g_m$ is just a conformal factor away from $g_{Euc}$ ("Isotropic coordinates" in Physics)

Note: We are usually concerned with the mass of the ENDS of an asymptotically flat mnfd.

Note: $g_m$ has an explicit dependence on mass $m$. Often, it's not clear how to make the mass explicitly appear in the metric.

$\Rightarrow$ GOAL: understand what mass is!

- What really is mass? Newton: gravitational potential $U: \mathbb{R}^3 \to \mathbb{R}$ ($U(x) \to 0$ as $|x| \to \infty$) ($\ddot{x} = -\operatorname{grad} U$) determines the distribution of matter in the universe via $\Delta U = 4\pi\rho$ ($G=1$, etc...) $\leftarrow \rho: \mathbb{R}^3 \to \mathbb{R}$ MASS DENSITY FUNCTION

If $\rho$ is supported on $|x|<R$, then on $|x|>R$, $U$ is harmonic.

$\boxed{\text{Newton's Shell Thm} + \text{Spherical Harmonic exp}} \Rightarrow U(x) = -\frac{m}{|x|} + O(|x|^{-2})$.

$$\int_{\mathbb{R}^3} \rho(x)\,dx \overset{\text{Poisson}}{=} \underbrace{\int_{\mathbb{R}^3} \frac{1}{4\pi}\Delta U\,dx = \lim_{r\to\infty}\int_{S_r} \frac{1}{4\pi}\frac{\partial U}{\partial r}\,d\mu_{S_r}}_{m} = \lim_{r\to\infty}\int \frac{1}{4\pi}\left(\frac{m}{|x|^2} + O(|x|^{-3})\right)d\mu_{S_r} = m$$

$\Rightarrow$ Mass = Asymptotic behavior of $U$

$m = \int_{\mathbb{R}^3} \rho(x)\,dx$ Useful consequence of linearity of Laplacian

- Why scalar curvature? Test particle can only feel total mass when it's close enough to infinity!

Intuition: scal measures the volume defect of small (geodesic) balls in $(M,g)$ compared to a space form; e.g.: scal $>0$ at a pt. $\Rightarrow$ small geodesic ball around that pt. has smaller volume than an Euclidean ball of same radius

In GR, scalar curvature is the mass density fct $\rho$ up to a const. (e.g., dim=3 $\rightsquigarrow$ $\mathrm{scal}_g = 16\pi\rho$)

$\rho$ constraints $g$ via $\mathrm{scal}_g = 16\pi\rho$; asymptotic flatness is "bdry condition"

$\Rightarrow$ Want: Mass as asymptotic integral involving $g$

$\bar{\cdot} \to$ computed using background Euclidean metric

Def: (ADM mass) $(M^n,g)$ asymp. flat w/ ends $M_1,\dots,M_\ell$, $\omega_{n-1} \to \mathrm{vol}(S^{n-1})$

$$m_{ADM}(M_j,g) := \lim_{r\to\infty} \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\rho} \left[\overline{\operatorname{div}} g - d(\overline{\operatorname{tr}} g)\right]\bar{\nu}\,d\mu_{S_\rho}$$

$\bar{\nu} \to$ outward ptg normal to $S_\rho$

Rmk: Can write in coörds & w/ $G := \mathrm{Ric} - \frac{1}{2}\mathrm{scal}\, g$ After linearizing $\mathrm{scal}_g$ at $g_{Euc} =: \bar{g}$

RECENT DEVELOPMENTS: Almost rigidity questions (Dong & Song, 2023)

If we have a sequence $(M_i^3, g_i)$ of asymptotically flat 3-mnfds with $\mathrm{scal} \geq 0$ and $m_{ADM}(g_i) \longrightarrow 0$, then "how" do these mnfds converge to Euclidean space?

In some Gromov-Hausdorff sense

By rigidity in PMT, we know they must converge to 3-Euclidean space.

As it turns out, these mnfds need not converge "smoothly" to $\mathbb{R}^3$. Get bounded volumes that shrink as things get flatter.