CONTACT STRUCTURES & WEINSTEIN^3

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REFERENCES:
- Thurston, 3-dim. Geometry & Topology
- Hutchings, "Taube's proof of the Weinstein conjecture in dimension three".

MOTIVATION FOR CONTACT STRUCTURES: Contact structures can be thought of as odd-dimensional counterparts to symplectic structures. More precisely, contact structures arose in the context of Hamiltonian systems with some sort of symmetry.

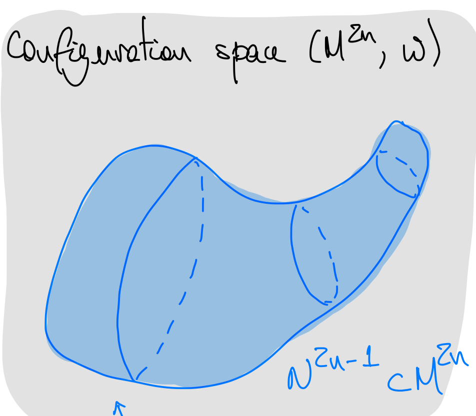
(M,ω) symplectic $\Rightarrow$ $\dim M$ is even (b/c ω non-degenerate)

The symmetry of the system (e.g. conservation of momentum, conservation of energy, etc.) makes the configuration space have less degrees of freedom. This amounts to the dimension of the configuration space to decrease.

Since $\dim M$ is even, in systems w/ symmetries, the motion becomes confined to odd-dimensional spaces.

Q: How to study this (since odd-dim. spaces do not admit symplectic structures)?

A: Contact structures.



Because of some symmetry, configuration space reduces to a submfd.

History: Contact structures first studied by Hamilton/Jacobi/Huygens. Later by Lie, Cartan, and Darboux.

Focus the discussion on 3-manifolds from now on ↓

Def: (PLANE FIELD) A plane field ξ on a manifold M is a subbundle of TM such that $\xi_p = T_pM \cap \xi$ is a 2-dimensional subspace of T_pM for each $p \in M$.

Ex 1: Let $M^3 := \Sigma \times S^1$, Σ is a surface. Let (x, θ) be local coordinates for $\Sigma \times S^1$. Then, for each $p = (x, \theta)$, set

$$\xi_p := T_x\Sigma \subset T_pM.$$

← plane field on M

Ex 2: Let $\alpha \in \Omega^1(M)$; i.e., at each $p \in M$, $\alpha_p : T_pM \to \mathbb{R}$ linear. Then, $\ker \alpha_p$ is either a plane or all of T_pM. Assuming α never has all of T_pM as its kernel, then $\xi := \ker \alpha$ is a plane field. → Rmk: In the previous example, $\alpha = d\theta$ defines ξ.

Proposed by Weinstein (1979)

Def: (CONTACT STRUCTURE) A plane field ξ is a contact structure if for any 1-form α with $\xi = \ker \alpha$, we have

"CONTACT FORM" locally or globally defined

$$\alpha \wedge d\alpha \neq 0$$

Rmk: From calculus, $\alpha \wedge d\alpha \neq 0 \iff d\alpha|_{\xi} \neq 0$.

Obs: The plane field in the first example above is NOT a contact structure ← That plane field was defined by $\alpha = d\theta$. But $d\alpha = d(d\theta) = 0$.

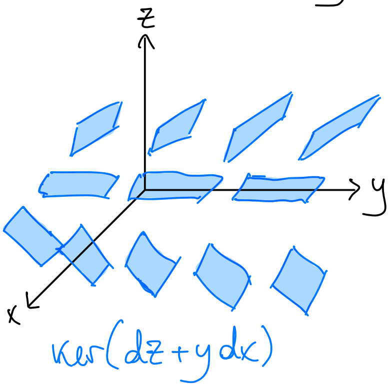
Ex 3: Take $\mathbb{R}^3$ with standard Cartesian coordinates (x,y,z) and the 1-form

→ Sometimes called "STANDARD CONTACT STRUCTURE ON $\mathbb{R}^3$"

$$\alpha = dz + x\,dy.$$

Note that $d\alpha = dx \wedge dy \longrightarrow \alpha \wedge d\alpha = dz \wedge dx \wedge dy \neq 0$

i.e., α is a contact form on $\mathbb{R}^3$ and $\xi := \ker\alpha$ is a contact structure



$\xrightarrow{\text{Compute}}$ ξ: at (x,y,z),

$$\xi = \operatorname{span}\left\{\frac{\partial}{\partial x},\ x\frac{\partial}{\partial z} - \frac{\partial}{\partial y}\right\}$$

Ex 4: Again, take $\mathbb{R}^3$ but w/ cylindrical coordinates (r,θ,z) and

$$\alpha = dz + r^2 d\theta.$$

Then $\alpha \wedge d\alpha = 2r\,dr \wedge d\theta \wedge dz \neq 0 \implies \tilde{\xi} := \ker\alpha$ is a contact structure

At (r,θ,z), $\tilde{\xi} = \operatorname{span}\left\{\frac{\partial}{\partial r},\ r^2\frac{\partial}{\partial z} - \frac{\partial}{\partial \theta}\right\}$.

→ if $r=0$ (i.e., on the z-axis), $\tilde{\xi}$ is horizontal. As we move out, the planes twist clockwise

Def: Two contact structures ξ_1 and ξ_2 on M are contactomorphic if there is a diffeomorphism $f: M \to M$ s.t. $f_*(\xi_1) = \xi_2$.

↖ orientation preserving

e.g.: the structures from Exs 3 and 4 are contactomorphic

Ex 5: Take the unit 3-sphere S^3 and

$$\beta = (x_1\,dy_1 - y_1\,dx_1 + x_2\,dy_2 - y_2\,dx_2)\big|_{S^3}$$

Then $\xi := \ker\beta$ is a contact structure on S^3.

Note: $(S^3\setminus\{N\}, \xi|_{S^3\setminus\{N\}})$ is contactomorphic to $(\mathbb{R}^3, \tilde{\xi})$ (can see this using stereographic coordinates).

REEB VECTOR FIELDS: Much like symplectic forms defining Hamiltonian vector fields in symplectic manifolds, contact forms have a "special" vector field associated to them.

Consider (M,ω) and $H \in C^\infty(M,\mathbb{R})$. Then, Hamiltonian vec. field is defined as $X_H := -(\omega^\#)^{-1}(dH)$.
convention
Flow of X_H determines the motion.

French Mathematician Georges Reeb

Def: (REEB VECTOR FIELD) A contact form α on M determines a vector field R_α on M, called the Reeb vector field, characterized by

$$\underbrace{d\alpha(R_\alpha,\cdot) = 0,}_{\iff R_\alpha \in \ker d\alpha} \quad \alpha(R_\alpha) = 1$$

Rmk: If α is a contact form for some contact structure, the Reeb vector field R is unique since $\underbrace{\dim\ker d\alpha = 1}_{d\alpha|_{T_pM^{2n}} \text{ is a skew-symmetric form of maximal rank } 2n-1.}$ and we need $\alpha(R) = 1$.

Rmk: Different contact forms whose kernels produce the same contact structure will output different Reeb vector fields (i.e., the dynamical system will be different).

Ex 6: Consider the following family of contact forms on the 3-sphere $S^3 \subset \mathbb{R}^4$:

$$\alpha_t := (x_1 dy_1 - y_1 dx_1) + (1+t)(x_2 dy_2 - y_2 dx_2),$$

where $t \geq 0$ is a real parameter. The Reeb vector field of α_t is

$$R_{\alpha_t} = \left(x_1 \frac{\partial}{\partial y_1} - y_1 \frac{\partial}{\partial x_1}\right) + \frac{1}{1+t}\left(x_2 \frac{\partial}{\partial y_2} - y_2 \frac{\partial}{\partial x_2}\right).$$

Note that the flow of R_{α_0} defines the Hopf fibration. In particular, all orbits of R_{α_0} are closed.

(If $t \in \mathbb{R}_{\geq 0} \setminus \mathbb{Q}$, however, R_{α_t} has only 2 periodic orbits ....)

WEINSTEIN CONJECTURE: Before moving on, we need a technical definition:

Def: (CLOSED ORBIT) Let M be a closed manifold and let $X \in \mathfrak{X}(M)$. A closed orbit of X is a map

$$\gamma : \mathbb{R}/c\mathbb{Z} \longrightarrow M,$$

for some $c > 0$, satisfying

$$\frac{d}{dt}\gamma(t) = X(\gamma(t)).$$

GOAL: Understand when vector field have closed orbits. Are there cases in which all vector fields have closed orbits?

Rmk: In some special cases such as the 3-torus, it is fairly easy to construct vector fields with no closed orbits.

However, things get complicated very soon: do all vector fields in S^3 have closed orbits? As it turns out, the answer is NO.

Examples of vector fields of increasing regularity on S^3 with no closed orbits were produced by

$$\text{Schweizer } (1974) \rightsquigarrow C^1$$
$$\text{Harrison } (1998) \longrightarrow C^2$$
$$\text{Kuperberg } (1994) \rightsquigarrow C^\infty$$

WEINSTEIN CONJECTURE (1978): Let M be a closed oriented odd-dimensional manifold with a contact form α. Then, the associated Reeb vector field R_α has a closed orbit.

↳ Remains open in several cases but it was proven for all closed 3-manifolds by Clifford Taubes in 2007:

Thm: (TAUBES, 2007) If M is a closed oriented 3-manifold with a contact form α, then the Reeb vector field R_α has a closed orbit.

Taubes proof used some fancy techniques like Seiberg-Witten theory.

Ex 7: Let Q be a smooth manifold. From a classical construction, there is a canonical 1-form ω on the cotangent bundle T^*Q. Let $\pi: T^*Q \to Q$ be the standard projection. If $q \in Q$ and $p \in T^*_q Q$, then $\omega: T_{(q,p)} T^*Q \to \mathbb{R}$ is given by

$$T_{(q,p)} T^*Q \xrightarrow{\pi_*} T_q Q \xrightarrow{p} \mathbb{R}.$$

Explicitly, if $q^1, \ldots, q^n, p_1, \ldots, p_n$ are local coordinates in T^*Q, we can write

$$\omega = \sum_{i=1}^{n} p_i \, dq^i .$$

Important object when studying classical mechanics

It follows that $d\omega$ is a symplectic form on T^*Q. Now, suppose we equip Q with a Riem. metric g. This induces a metric on T^*Q and we can consider the unit cotangent bundle

$$ST^*Q := \{ p \in T^*Q : \|p\| = 1 \}.$$

The restriction of ω to ST^*Q gives a <u>contact form</u>. It turns out that the Reeb vector field $R_{\omega|_{ST^*Q}}$ agrees with the geodesic flow under the identification $T^*Q \simeq TQ$.

isom. b/c g is Riemannian (in part., non-degenerate)

Thus, closed orbits in ST^*Q from $R_{\omega|_{ST^*Q}}$ are equivalent to closed geodesics in Q!

If Q is compact, then ST^*Q is also compact, and we can apply Weinstein's conjecture.

In this case, this is equivalent to Lyusternik-Fet affirming that every compact Riemannian manifold has at least one closed geodesic.