SIMPLICIAL COMPLEXES AND TRIANGULATIONS

Toronto, January 23rd 2024

(Hempel Ch. 1 and Schultens Intro. to 3-mnflds)

TOP
|
PDIFF $\simeq$ PL
|
DIFF

GOAL: Study PL-manifolds (piecewise linear).

Transition maps are pl-fcts

Also called pdiff (piecewise diffable)

NOTATION:

$$M \underset{\text{homeo}}{\simeq} B^n = \{x \in \mathbb{R}^n : \|x\| \leq 1\} \Rightarrow M = n\text{-cell}$$

$$M \underset{\text{homeo}}{\simeq} S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\} \Rightarrow M = (n-1)\text{-sphere}$$

Def: (n-mnfd) Hausdorff, 2nd countable topological space s.t. each pt. has an open nighb. homeo. to $\mathbb{R}^n$ or $\mathbb{H}^n_+$.

Upper half space $\mathbb{H}^n_+ = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n \geq 0\}$

Def: $\partial M = \{$pts. in M whose neighborhoods are homeo. to $\mathbb{H}^n_+\}$

Rmk: $\partial(\partial M) = \emptyset$. $\rightsquigarrow$ follows from $\partial(\partial \mathbb{H}^n_+) = \emptyset$

Def: $\text{int}\, M := M \setminus \partial M$.

Def: M is closed if compact and $\partial M = \emptyset$.

Def: M is open if M has no compact component and $\partial M = \emptyset$.

Def: (SIMPLEX; SIMPLICES) Let V be a $\mathbb{R}$-vec. space and let $\{v_0, \dots, v_k\}$ be lin. indep. vectors in V. The (convex) set

$$\left\{ a_0 v_0 + \dots + a_k v_k : a_0, \dots, a_k \geq 0, \sum_{j=1}^{k} a_j = 1 \right\}$$

is a k-simplex. **Notation**: $[s]$, $[v_0, \dots, v_k]$.

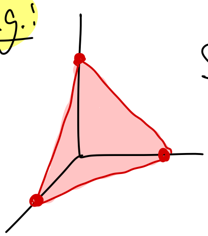
Open k-simplex:

$$(s) = (v_0, \dots, v_k) = \left\{ a_0 v_0 + \cdots + a_k v_k : \begin{array}{l} a_0, \dots, a_k > 0 \\ \sum_j a_j = 1 \end{array} \right\}$$

Rmk: $\dim [s] = k = \dim (s)$

Topologically: every k-simplex is a k-mnfd "w/ corners".

e.g.:



Standard 2-simplex in $\mathbb{R}^3$

$$\Delta^2 = \left\{ (t_0, t_1, t_2) \in \mathbb{R}^3 : t_j \geq 0, \sum_j t_j = 1 \right\}$$

Def: (Barycenter) For $v = [v_0, \dots, v_k]$ k-simplex,

$\{a_0, \dots, a_k\}$ = "Barycentric coord. of v".

NOTATION

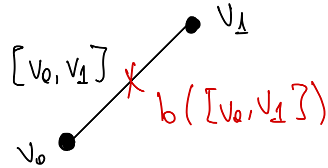
Barycenter of $[v_0, \dots, v_k] := b([v_0, \dots, v_k])$

"CENTER OF MASS"

$$= \frac{v_0}{k+1} + \cdots + \frac{v_k}{k+1} \in V$$

e.g.: $b([v_0]) = v_0$ Barycenter of a 0-simplex is itself

$$b([v_0, v_1]) = \frac{1}{2}(v_0 + v_1)$$



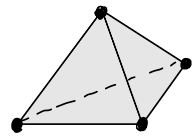
Def: (FACES) An ℓ-face of a k-simplex $[v_0, \ldots, v_k]$ is an ℓ-simplex of the form:

$$[v_{j_0}, \ldots, v_{j_\ell}], \quad 0 \leq \ell \leq k,$$

where $\{v_{j_0}, \ldots, v_{j_\ell}\}$ is a lin. indep. subset of $\{v_0, \ldots, v_k\}$.

e.g.! 0-dimensional face = VERTEX
1-dimensional face = EDGE
2-dimensional face = FACE
3-dimensional face = CELL

$\dim(\ell\text{-face}) = k - \ell$



Hempel

Def: (SIMPLICIAL COMPLEX) A simplicial complex K is a (locally) finite collection of closed simplexes (embedded) in some $\mathbb{R}^n$ s.t.

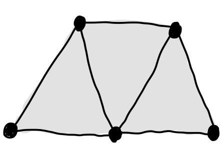
Hempel's convention

(i) if $\sigma \in K$ and τ is a face of σ, then $\tau \in K$.

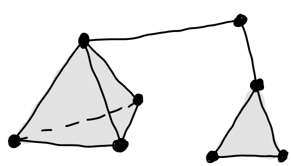
(ii) if $\sigma, \tau \in K$, then $\sigma \cap \tau$ is a face of both σ and τ.

$\dim K$ = dim. of highest dimensional simplex in K.

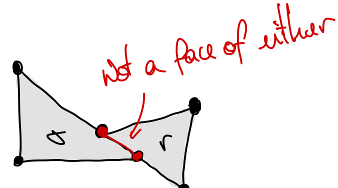
$$|K| := \bigcup_{\sigma \in K} \sigma = \text{"Underlying space of } K\text{"}$$
(set of pts. in K)



2-dim simplicial complex

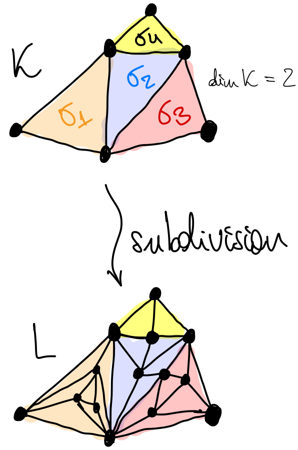


3-dim. simplicial complex

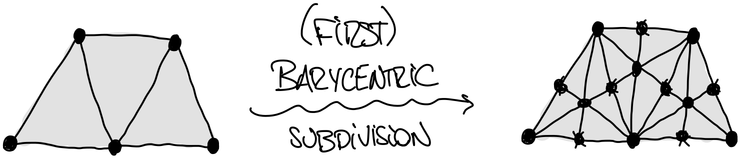


NOT a simplicial complex (violates (ii))

Subdivision of K = a simplicial complex L s.t. $|L| = |K|$ (as sets) and each simplex in L lies in some simplex of K.



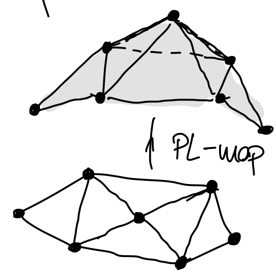
e.g.: Barycentric subdivision (informally)



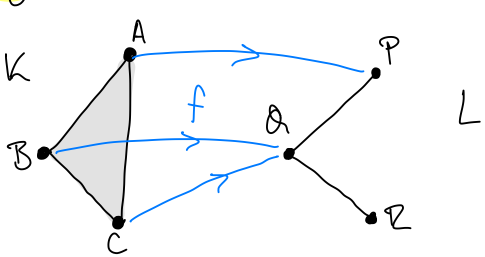
SECOND BARYCENTRIC SUBDIVISION $\longrightarrow$ ...

Can proceed inductively adding barycenters...

Def: (PL-map) For simplicial complexes K_1, K_2, a map $f : |K_1| \longrightarrow |K_2|$ is <u>piecewise</u> <u>linear</u> if there exist subdivisions L_1 of K_1 and L_2 of K_2 with respect to which f is simplicial, i.e., f takes vertices of L_1 to vertices of L_2 and maps each simplex of L_1 linearly (w.r.t. barycentric coordinates) onto a simplex of L_2.

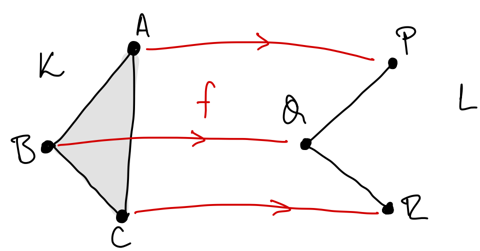


e.g.: SIMPLICIAL MAP:



<u>Simplicial</u>: $f(ABC) = PQ \in L$ etc.

e.g.: NON-SIMPLICIAL MAP:



~~Not~~ simplicial: $f(CA) = RP \notin L$

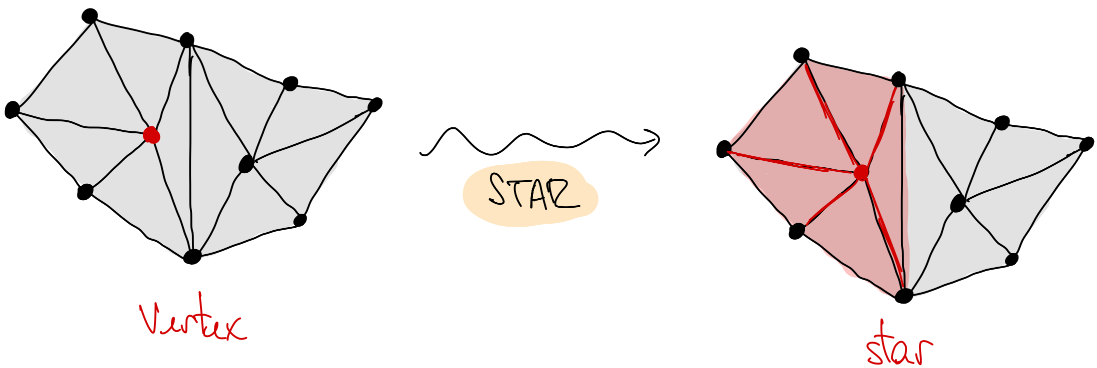
Def: (STAR) If v is a <u>vertex</u> of a simplicial complex K, then the star of v is the set

$$st(v) := \{\sigma \in K : v \in \sigma\}.$$

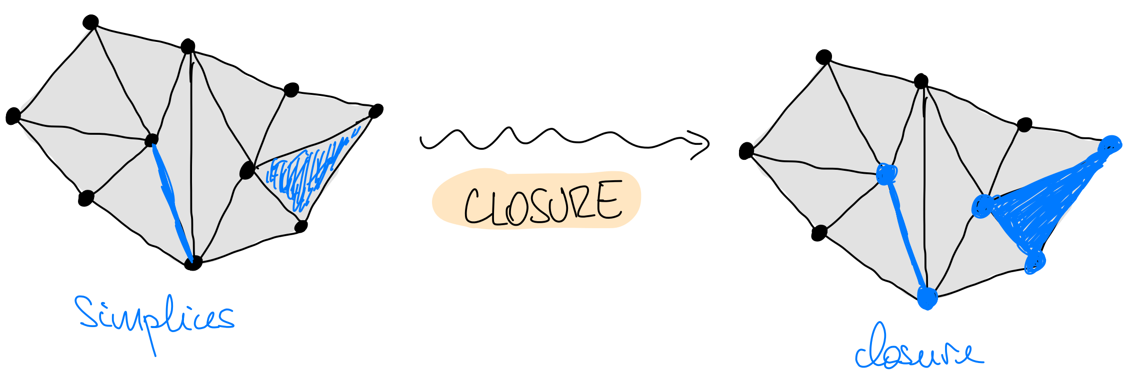
Points in $st(v)$ are denoted by $|st(v)|$

More generally, the star of a <u>subcomplex</u> K' of K is

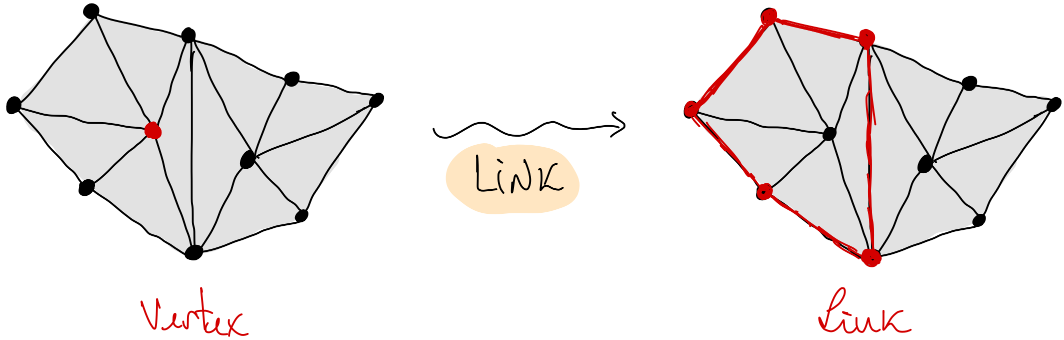
$$st(K') := \{\sigma \in K : K' \cap \sigma \neq \emptyset\}.$$



Def: (CLOSURE) For a collection of simplices S in a simplicial complex K, the <u>closure</u> of S, $cl(S)$, is the smallest simplicial subcomplex of K that contains each simplex in S.



Def: (LINK) The *link* of a collection S of simplexes in K is $lk(S,K) := cl(st(S)) \setminus st(cl(S))$



Def: (TRIANGULATION) A triangulation of a space X is a *pair* (T,h), where T is a simplicial complex and $h: |T| \to X$ is a homeomorphism.

Triangulations (T_1,h_1) and (T_2,h_2) are compatible $\overset{\text{def}}{\Longleftrightarrow}$ $h_2^{-1} \circ h_1 : |T_1| \longrightarrow |T_2|$ is piecewise linear.

Prop: Every compact 1-mnfd admits triangulation (unique up to homeo.)

Pf: Classification of compact 1-mnfds $\Rightarrow M \underset{\text{homeo}}{\simeq} S^1$ (or a finite union of circles...)

Triangulate each circle by an n-gon, $\underbrace{n \geq 3}$, $\triangle$ Unique up to homeo (can have ⬠, etc.)

Topological type of triangulation of circle is defined by the # of 1-simplices

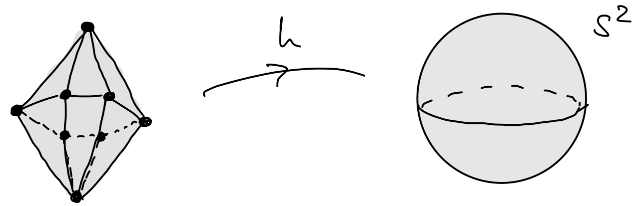
$\square$

Radó (1925): Every compact 2-mnfd admits triangulation.

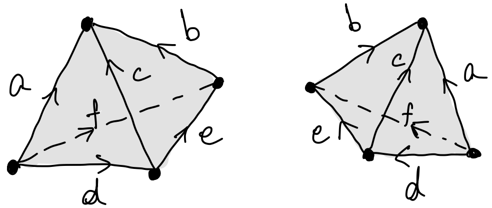
Bing & Moise (1952): Every 3-mnfd admits triangulation (very hard to prove...)
Paper on Annals

False in dimensions ≥ 4.

EXAMPLES: 1) S^2:



2) S^3:

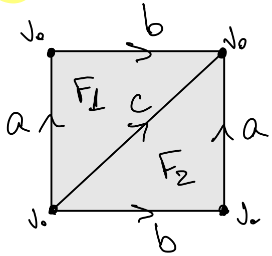


obs: 2-faces are identified if all 3 edges are identified.

3) NON-EXAMPLE: Torus T^2

"pseudo-triangulation"

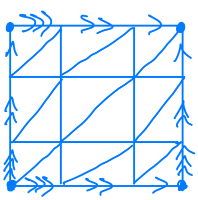
CW-complex



Not a triangulation b/c this is not a simplicial complex!

F_1 and F_2 intersect at $\underbrace{a \cup b \cup c}_{\text{Not a simplex}}$ → Intersection of any pair of simplices must be a simplex

Fix by subdivision



This is a simplicial complex, so this is a triangulation of the torus.

Rmk: pseudo-triangulations are fun to compute homology groups: $H_0(T^2) = \mathbb{Z}$

$H_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$

$H_2(T^2) = \mathbb{Z}$

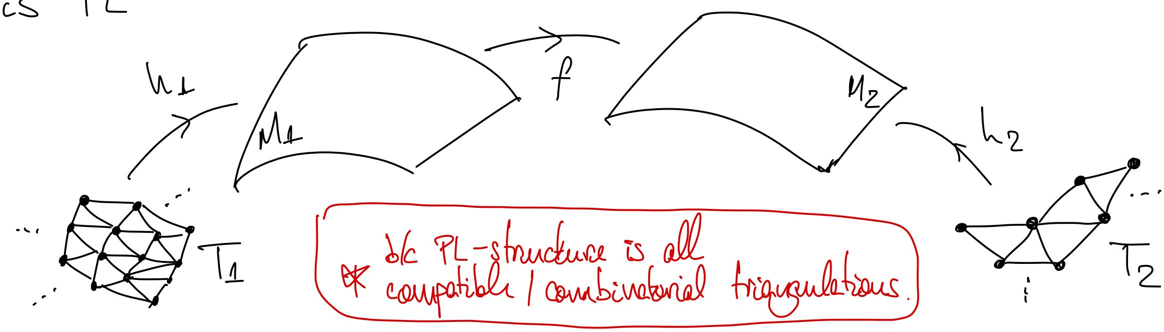
Def: (COMBINATORIAL TRIANGULATION) A triangulation (T, h) of an n-mnfd M is combinatorial provided that for each vertex v of T, $|lk(v, T)|$ is piecewise linearly homeo. to an $(n-1)$-sphere.

e.g.: the triangulations above are all combinatorial

Def: (PL-structure) A PL-structure on a mnfd M is a maximal, non-empty collection of compatible combinatorial triangulations of M.

Def: (PL-mnfd) Mnfd M together w/ a PL-structure on M.

Def: (PL-maps) If M_1, M_2 are PL-mnfds and we have $f: M_1 \to M_2$, then f is a PL-map provided that for some (hence any*) triangulations (T_i, h_i) of M_i, $i=1,2$, in the associated PL-structures, $h_2^{-1} \circ f \circ h_1 : |T_1| \to |T_2|$ is PL



* b/c PL-structure is all compatible / combinatorial triangulations.

Def: (Euler Characteristic) The Euler characteristic of a finite simplicial complex K of dimension d is

$$\chi(K) = \sum_{j=0}^{d} (-1)^j \,\#\{\text{simplices of dimension } j \text{ in } K\}.$$

e.g.: $\chi(S^n) = 1 + (-1)^n \qquad \chi(D^2) = 1$

$\chi(\Sigma_g) = 2 - 2g \qquad \chi(\text{Interval}) = 1$